Chapter 3

3.1 An examination of the standard temperature distribution through the atmosphere given in Figure 3.3 of the text shows that both 12 km and 18 km are in the same constant temperature region. Hence, the equations that apply are Eqs. (3.9) and (3.10) in the text. Since we are in the same isothermal region with therefore the same base values of p and ρ, these equations can be written as

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} = e^{-(g_0/RT)(h_2 - h_1)}$$

where points 1 and 2 are any two arbitrary points in the region. Hence, with $g_0 = 9.8$ m/sec^2 and $R = 287$ joule/kgK, and letting points 1 and 2 correspond to 12 km and 18 km altitudes, respectively,

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} = e^{\frac{9.8}{(287)(216.66)}(6000)} = 0.3884$$

Hence:

$$p_2 = (0.3884)(1.9399 \times 10^4) = 7.53 \times 10^3 \text{ N/m}^2$$

$$\rho_2 = (0.3884)(3.1194 \times 10^{-1}) = 0.121 \text{ kg/m}^3$$

and, of course,

$$T_2 = 216.66 \text{ K}$$

These answers check the results listed in Appendix A of the text within round-off error.

3.2 From Appendix A of the text, we see immediately that $p = 2.65 \times 10^4$ N/m^2 corresponds to 10,000 m, or 10 km, in the standard atmosphere. Hence,

pressure altitude = 10 km

The outside air density is

$$\rho = \frac{p}{RT} = \frac{2.65 \times 10^4}{(287)(220)} = 0.419 \text{ kg/m}^3$$

From Appendix A, this value of ρ corresponds to 9.88 km in the standard atmosphere. Hence,

density altitude = 9.88 km

3.3 At 35,000 ft, from Appendix B, we find that $p = 4.99 \times 10^2 = 499$ lb/ft^2.

3.4 From Appendix B in the text,

33,500 ft corresponds to $p = 535.89$ lb/ft^2

32,000 ft corresponds to $\rho = 8.2704 \times 10^{-4}$ slug/ft^3

Hence,

$$T = \frac{p}{\rho R} = \frac{535.89}{(8.2704 \times 10^{-4})(1716)} = 378 \, R$$

3.5 $$\frac{|h - h_G|}{h} = 0.02 = \left|1 - \frac{h_G}{h}\right|$$

From Eq. (3.6), the above equation becomes

$$\left|1 - \left(\frac{r + h_G}{r}\right)\right| = \left|1 - 1 - \frac{h_G}{r}\right| = 0.02$$

$$h_G = 0.02\, r = 0.02(6.357 \times 10^6)$$

$$h_G = 1.27 \times 10^5 \text{m} = 127 \text{ km}$$

3.6 $T = 15 - 0.0065h = 15 - 0.0065(5000) = -17.5°\text{C} = 255.5°\text{K}$

$$a = \frac{dT}{dh} = -0.0065$$

From Eq. (3.12)

$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{-g_0/aR} = \left(\frac{255.5}{288}\right)^{-(2.8)/(-0.0065)(287)} = 0.533$$

$$p = 0.533\, p_1 = 0.533\,(1.01 \times 10^5) = 5.38 \times 10^4 \text{N/m}^2$$

3.7 $$\ell n \frac{p}{p_1} = -\frac{g}{RT}(h - h_1)$$

$$h - h_1 = -\frac{1}{g} RT\, \ell n \frac{p}{p_1} = -\frac{1}{24.9}(4157)(150)\, \ell n\, 0.5$$

Letting $h_1 = 0$ (the surface)

h = 17,358 m = 17.358 km

3.8 A standard altitude of 25,000 ft falls within the first gradient region in the standard atmosphere. Hence, the variation of pressure and temperature are given by:

$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{-\frac{g}{aR}} \tag{1}$$

and

$$T = T_1 + a\,(h - h_1) \tag{2}$$

Differentiating Eq. (1) with respect to time:

$$\frac{1}{p_1}\frac{dp}{dt} = \left(\frac{1}{T_1}\right)^{-\frac{g}{aR}}\left(-\frac{g}{AR}\right)T^{\left(-\frac{g}{aR}-1\right)}\frac{dT}{dt} \tag{3}$$

Differentiating Eq. (2) with respect to time:

$$\frac{dT}{dt} = a\frac{dh}{dt} \tag{4}$$

Substitute Eq. (4) into (3)

$$\frac{dp}{dt} = -p_1(T_1)^{\frac{g}{aR}}\left(\frac{g}{R}\right)T^{-\left(\frac{g}{aR}+1\right)}\frac{dh}{dt} \tag{5}$$

In Eq. (5), dh/dt is the rate-of-climb, given by $dh/dt = 500$ ft/sec. Also, in the first gradient region, the lapse rate can be calculated from the tabulations in Appendix B. For example, take 0 ft and 10,000 ft, we find

$$a = \frac{T_2 - T_1}{h_2 - h_1} = \frac{483.04 - 518.69}{10,000 - 0} = -0.00357\frac{^\circ R}{\text{ft}}$$

Also from Appendix B, $p_1 = 2116.2$ lb/ft^2 at sea level, and $T = 429.64\ ^\circ R$ at 25,000 ft. Thus,

$$\frac{g}{aR} = \frac{32.2}{(-0.00357)(1716)} = -5.256$$

Hence, from Eq. (5)

$$\frac{dp}{dt} = -(2116.2)(518.69)^{-5.256}\left(\frac{32.2}{1716}\right)(429.64)^{4.256}(500)$$

$$\frac{dp}{dt} = -17.17\frac{\text{lb}}{\text{ft}^2\text{sec}}$$

3.9 From the hydrostatic equation, Eq. (3.2) or (3.3),

$$dp = -\rho g_0 dh$$

or

$$\frac{dp}{dt} = -\rho\, g_0 \frac{dh}{dt}$$

The upward speed of the elevator is dh/dt, which is

$$\frac{dp}{dt} = \frac{dp/dt}{-\rho\, g_0}$$

At sea level, $\rho = 1.225$ kg/m^3. Also, a one-percent change in pressure per minute starting from sea level is

$$\frac{dp}{dt} = -(1.01 \times 10^5)(0.01) = -1.01 \times 10^3 \text{ N/m}^2 \text{ per minute}$$

Hence,

$$\frac{dh}{dt} = \frac{-1.01 \times 10^3}{(1.225)(9.8)} = 84.1 \text{ meter per minute}$$

3.10 From Appendix B:

At 35,500 ft: $p = 535.89$ lb/ft^2

At 34,000 ft: $p = 523.47$ lb/ft^2

For a pressure of 530 lb/ft^2, the pressure altitude is

$$33{,}500 + 500\left(\frac{535.89 - 530}{535.89 - 523.47}\right) = 33737 \text{ ft}$$

The density at the altitude at which the airplane is flying is

$$\rho = \frac{p}{RT} = \frac{530}{(1716)(390)} = 7.919 \times 10^{-4} \text{ slug/ft}^3$$

From Appendix B:

$$\text{At 33,000 ft: } \rho = 7.9656 \times 10^{-4} \text{ slug/ft}^3$$

$$\text{At 33,500 ft: } \rho = 7.8165 \times 10^{-4} \text{ slug/ft}^3$$

Hence, the density altitude is

$$33{,}000 + 500\left(\frac{7.9656 - 7.919}{7.9656 - 7.8165}\right) = 33{,}156 \text{ ft}$$

3.11 Let ℓ be the length of one wall of the tank, $\ell = 30$ ft. Let d be the depth of the pool, $d = 10$ ft. At the water surface, the pressure is atmospheric pressure, p_a. The water pressure increases with increasing depth; the pressure as a function of distance below the surface, h, is given by the hydrostatic equation

$$dp = \rho g\, dh \tag{1}$$

Note: The hydrostatic equation given by Eq. (3.2) in the text has a minus sign because h_G is measured positive in the upward direction. In Eq. (1), h is measured positive in the downward direction, with $h = 0$ at the surface of the water. Hence, no minus sign appears in Eq. (1); as h increases (as we go deeper into the water), p increases. Eq. (1) is consistent with this fact. Integrating Eq. (1) from $h = 0$ where $p = p_a$ to some local depth h where the pressure is p, and noting that ρ is constant for water, we have

$$\int_{p_a}^{\rho} dp = \rho g \int_0^h dh$$

or, $\quad p - p_a = \rho\, g\, h$

or, $$p = \rho g h + p_a \tag{2}$$

Eq. (2) gives the water pressure exerted on the wall at an arbitrary depth h.

Consider an elementary small sliver of wall surface of length ℓ and height dh.

The water force on this sliver of area is

$$dF = p\, \ell\, dh$$

Total force, F, on the wall is

$$F = \int_0^F dF = \int_0^d p\, \ell\, dh \tag{3}$$

where p is given by Eq. (2). Inserting Eq. (2) into (3),

$$F = \int_0^d (\rho g h + p_a)\, \ell\, dh$$

or, $$F = \rho g \ell \left(\frac{d^2}{2} \right) + p_a \ell d \qquad (4)$$

In Eq. (4), the product ρg is the specific weight (weight per unit volume) of water; $\rho g = 62.4$ lb/ft^3. From Eq. (4),

$$F = (62.4)(30)\frac{(10)^2}{2} + (2116)(30)(10)$$

$$F = 93{,}600 + 634{,}800 = \boxed{728{,}400\,\text{lb}}$$

Note: This force is the combined effect of the force due to the weight of the water, 93,600 lb, and the force due to atmospheric pressure transmitted through the water, 634,800 Ib. In this example, the latter is the larger contribution to the force on the wall. If the wall were freestanding with atmospheric pressure exerted on the opposite side, then the net force exerted on the wall would be that due to the weight of the water only, i.e., 93,600 Ib. In tons, the force on the side of the wall in contact with the water is

$$F = \frac{728{,}400}{2000} = \boxed{364.2\,\text{tons}}$$

In the case of a freestanding wall, the net force, that due only to the weight of the water, is

$$F = \frac{93{,}600}{2000} = \boxed{46.8\,\text{tons}}$$

3.12 For the exponential atmosphere model,

$$\frac{\rho}{\rho_0} = e^{-g_0 h/(RT)}$$

$$\frac{\rho}{\rho_0} = e^{-(9.8)(45{,}000)/(287)(240)} = e^{-6.402}$$

Hence,

$$\rho = \rho_0 e^{-6.402} = (1225)(1.6575 \times 10^{-3}) = \boxed{2.03 \times 10^{-3}\ \text{kg/m}^3}$$

From the standard atmosphere, at 45 km, $\rho = 2.02 \times 10^{-3}$ kg/m^3. The exponential atmosphere model gives a remarkably accurate value for the density at 45 km when a value of 240 K is used for the temperature.

3.13 At 3 km, from App. A, $T = 268.67$ K, $p = 7.0121 \times 10^4$ N/m^2, and $\rho = 0.90926$ kg/m^3. At 3.1 km, from App. A, $T = 268.02$ K, $p = 6.9235 \times 10^4$ N/m^2, and $\rho = 0.89994$ kg/m^3. At $h = 3.035$ km, using linear interpolation:

$$T = 268.67 - (268.67 - 268.02)\left(\frac{0.035}{0.1}\right) = \boxed{268.44\ \text{K}}$$

$$p = 7.0121 \times 10^4 - (7.0121 \times 10^4 - 6.9235 \times 10^4)\left(\frac{0.035}{0.1}\right)$$

$$= \boxed{6.98099 \times 10^4\ \frac{\text{N}}{\text{m}^2}}$$

$$\rho = 0.09026 - (0.90926 - 0.89994)\left(\frac{0.035}{0.1}\right) = \boxed{0.906\ \frac{\text{kg}}{\text{m}^3}}$$

3.14 The altitude of 3.035 km is in the gradient region. From Example 3.1, $a = -0.0065$ K/m. From Eq. (3.14),

$$T = T_1 + a(h - h_1) = T_s + a(h - 0) = 288.16 - (0.0065)(3035) = \boxed{268.43 \text{ K}}$$

From Eq. (3.12),

$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{-g_0/aR}$$

$$\frac{-g_0}{aR} = \frac{-9.8}{(-0.0065)(28.7)} = 5.25328$$

So

$$\frac{p}{p_s} = \left(\frac{T}{T_s}\right)^{5.25328} = \left(\frac{268.43}{288.16}\right)^{5.25328} = 0.688946$$

$$p = 0.688946(1.01325 \times 10^5) = 6.9807 \times 10^4 \ \frac{\text{N}}{\text{m}^2}$$

By comparison the approximate result from the solution of Prob. 3.13 differs from the exact result above by

$$\frac{6.980990 - 06.9807}{6.9807} = 4.15 \times 10^{-5} = \boxed{0.00415\%}$$

A very small amount.

From Eq. (3.13),

$$\frac{\rho}{\rho_1} = \left(\frac{T}{T_1}\right)^{-[(g_0/aR)+1]}$$

$$\frac{g_0}{aR} + 1 = -5.25328 + 1 = -4.25328$$

$$\frac{\rho}{\rho_s} = \left(\frac{T}{T_s}\right)^{4.2538} = \left(\frac{268.43}{288.16}\right)^{4.24328} = 0.73958$$

$$\rho = 0.73598 \ \ \rho_s = 0.73958 \ (1.2250) = \boxed{0.90599 \ \frac{\text{kg}}{\text{m}^3}}$$

3.15 We want to calculate h_G when

$$\frac{h_G - h}{h_G} = 0.01 = 1 - \frac{h}{h_G}$$

From Eq. (3.6)

$$h = \frac{rh_G}{r + h_G}, \text{ where } r = 6.356766 \times 10^6 \text{ m}$$

Thus,

$$0.01 = 1 - \frac{r}{r + h_G}$$

$$\frac{r}{r + h_G} = 1 - 0.01 = 0.99$$

$$0.99(r + h_G) = r$$

$$h_G = \left(\frac{1 - 0.99}{0.99}\right) r = 0.0101\ (6.356766 \times 10^5)$$

$$h_G = 64.21 \times 10^3 \text{ m} = \boxed{64.21 \text{ km}}$$

3.16 From Eq. (3.1),

$$g = g_0 \left(\frac{r}{r + h_G}\right)^2$$

where $r = 6.356766 \times 10^6$ m, $g_0 = 9.8$ m/sec^2, and

$$h_G = (70{,}000 \text{ ft})\left(\frac{0.3048 \text{ m}}{1 \text{ ft}}\right) = 21{,}336 \text{ m}$$

Thus,

$$g = 9.8\left(\frac{6.356766 \times 10^6}{6.356766 \times 10^6 + 21{,}336}\right) = 9.767 \ \frac{\text{m}}{\text{sec}^2}$$

$$\left(\frac{9.8 - 9.767}{98}\right) 100 = 0.34\%$$

Thus, at 70,000 ft, the acceleration of gravity has decreased only by 0.34% compared to its sea level value.