Chapter 2

Section 2.1

1. $P(\text{does not fail}) = 1 - P(\text{fails}) = 1 - 0.12 = 0.88$

2. (a) $\{1, 2, 3\}$

   (b) $P(\text{odd number}) = P(1) + P(3) = 3/6 + 1/6 = 2/3$

   (c) No, the set of possible outcomes is still $\{1, 2, 3\}$.

   (d) Yes, a list of equally likely outcomes is then $\{1, 1, 1, 2, 2, 3, 3\}$, so $P(\text{odd}) = P(1) + P(3) = 3/7 + 2/7 = 5/7$.

3. (a) The outcomes are the 16 different strings of 4 true-false answers. These are {TTTT, TTTF, TTFT, TTFF, TFTT, TFTF, TFFT, TFFF, FTTT, FTTF, FTFT, FTFF, FFTT, FFTF, FFFT, FFFF}.

   (b) There are 16 equally likely outcomes. The answers are all the same in two of them, TTTT and FFFF. Therefore the probability is $2/16$ or $1/8$.

   (c) There are 16 equally likely outcomes. There are four of them, TFFF, FTFF, FFTF, and FFFT, for which exactly one answer is "True." Therefore the probability is $4/16$ or $1/4$.

   (d) There are 16 equally likely outcomes. There are five of them, TFFF, FTFF, FFTF, FFFT, and FFFF, for which at most one answer is "True." Therefore the probability is $5/16$.

4. (a) The outcomes are the 27 different strings of 3 chosen from conforming (C), downgraded (D), and scrap (S). These are {CCC, CCD, CCS, CDC, CDD, CDS, CSC, CSD, CSS, DCC, DCD, DCS, DDC, DDD, DDS, DSC, DSD, DSS, SCC, SCD, SCS, SDC, SDD, SDS, SSC, SSD, SSS}.

   (b) A = {CCC, DDD, SSS}

   (c) B = {CDS, CSD, DCS, DSC, SCD, SDC}

   (d) C = {CCD, CCS, CDC, CSC, DCC, SCC, CCC}

   (e) The only outcome common to A and C is CCC. Therefore $A \cap C = \{\text{CCC}\}$.

(f) The set $A \cup B$ contains the outcomes that are either in A, in B, or in both. Therefore $A \cup B = \{$CCC, DDD, SSS, CDS, CSD, DCS, DSC, SCD, SDC$\}$.

(g) C^c contains the outcomes that are not in C. $A \cap C^c$ contains the outcomes that are in A but not in C. Therefore $A \cap C^c = \{$DDD, SSS$\}$.

(h) A^c contains the outcomes that are not in A. $A^c \cap C$ contains the outcomes that are in C but not in A. Therefore $A^c \cap C = \{$CCD, CCS, CDC, CSC, DCC, SCC$\}$.

(i) No. They both contain the outcome CCC.

(j) Yes. They have no outcomes in common.

5. (a) The outcomes are the sequences of candidates that end with either #1 or #2. These are $\{1, 2, 31, 32, 41, 42, 341, 342, 431, 432\}$.

(b) $A = \{1, 2\}$

(c) $B = \{341, 342, 431, 432\}$

(d) $C = \{31, 32, 341, 342, 431, 432\}$

(e) $D = \{1, 31, 41, 341, 431\}$

(f) A and E are mutually exclusive because they have no outcomes in commom.
B and E are not mutually exclusive because they both contain the outcomes 341, 342, 431, and 432.
C and E are not mutually exclusive because they both contain the outcomes 341, 342, 431, and 432.
D and E are not mutually exclusive because they both contain the outcomes 41, 341, and 431.

6. (a) The equally likely outcomes are the sequences of two distinct candidates. These are $\{12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43\}$.

(b) Of the 12 equally likely outcomes, there are 2 (12 and 21) for which both candidates are qualified. The probability is therefore $2/12$ or $1/6$.

(c) Of the 12 equally likely outcomes, there are 8 (13, 14, 23, 24, 31, 32, 41, and 42) for which exactly one candidate is qualified. The probability is therefore $8/12$ or $2/3$.

7. (a)
$$\begin{aligned} P(\text{living room or den}) &= P(\text{living room}) + P(\text{den}) \\ &= 0.26 + 0.22 \\ &= 0.48 \end{aligned}$$

(b)
$$\begin{aligned} P(\text{not bedroom}) &= 1 - P(\text{bedroom}) \\ &= 1 - 0.37 \\ &= 0.63 \end{aligned}$$

8. (a) 0.7

(b) $P(\text{not poor risk}) = 1 - P(\text{poor risk}) = 1 - 0.1 = 0.9$

9. (a) The events of having a major flaw and of having only minor flaws are mutually exclusive. Therefore
$$P(\text{major flaw or minor flaw}) = P(\text{major flaw}) + P(\text{only minor flaws}) = 0.15 + 0.05 = 0.20.$$

(b) $P(\text{no major flaw}) = 1 - P(\text{major flaw}) = 1 - 0.05 = 0.95.$

10. (a) False

(b) True

(c) True. This is the definition of probability.

11. (a) False

(b) True

12. (a)
$$\begin{aligned} P(V \cap W) &= P(V) + P(W) - P(V \cup W) \\ &= 0.15 + 0.05 - 0.17 \\ &= 0.03 \end{aligned}$$

(b) $P(V^c \cap W^c) = 1 - P(V \cup W) = 1 - 0.17 = 0.83$.

(c) We need to find $P(V \cap W^c)$. Now $P(V) = P(V \cap W) + P(V \cap W^c)$ (this can be seen from a Venn diagram). We know that $P(V) = 0.15$, and from part (a) we know that $P(V \cap W) = 0.03$. Therefore $P(V \cap W^c) = 0.12$.

13. (a)

$$\begin{aligned} P(S \cup C) &= P(S) + P(C) - P(S \cap C) \\ &= 0.4 + 0.3 - 0.2 \\ &= 0.5 \end{aligned}$$

(b) $P(S^c \cap C^c) = 1 - P(S \cup C) = 1 - 0.5 = 0.5$.

(c) We need to find $P(S \cap C^c)$. Now $P(S) = P(S \cap C) + P(S \cap C^c)$ (this can be seen from a Venn diagram). Now

$$\begin{aligned} P(S \cap C) &= P(S) + P(C) - P(S \cup C) \\ &= 0.4 + 0.3 - 0.5 \\ &= 0.2 \end{aligned}$$

Since $P(S) = 0.4$ and $P(S \cap C) = 0.2$, $P(S \cap C^c) = 0.2$.

14. (a) Since 562 stones were neither cracked nor discolored, 38 stones were cracked, discolored, or both. The probability is therefore $38/600 = 0.0633$.

(b) Let A be the event that the stone is cracked and let B be the event that the stone is discolored. We need to find $P(A \cap B)$. We know that $P(A) = 15/600 = 0.025$ and $P(B) = 27/600 = 0.045$. From part (a) we know that $P(A \cup B) = 38/600$.

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Substituting, we find that $38/600 = 15/600 + 27/600 - P(A \cap B)$. It follows that $P(A \cap B) = 4/600 = 0.0067$.

(c) We need to find $P(A \cap B^c)$. Now $P(A) = P(A \cap B) + P(A \cap B^c)$ (this can be seen from a Venn diagram). We know that $P(A) = 15/600$ and $P(A \cap B) = 4/600$. Therefore $P(A \cap B^c) = 11/600 = 0.0183$.

15. (a) Let R be the event that a student is proficient in reading, and let M be the event that a student is proficient in mathematics. We need to find $P(R^c \cap M)$. Now $P(M) = P(R \cap M) + P(R^c \cap M)$ (this can be seen from a Venn diagram). We know that $P(M) = 0.78$ and $P(R \cap M) = 0.65$. Therefore $P(R^c \cap M) = 0.13$.

(b) We need to find $P(R \cap M^c)$. Now $P(R) = P(R \cap M) + P(R \cap M^c)$ (this can be seen from a Venn diagram). We know that $P(R) = 0.85$ and $P(R \cap M) = 0.65$. Therefore $P(R \cap M^c) = 0.20$.

(c) First we compute $P(R \cup M)$:
$P(R \cup M) = P(R) + P(M) - P(R \cap M) = 0.85 + 0.78 - 0.65 = 0.98$.
Now $P(R^c \cap M^c) = 1 - P(R \cup M) = 1 - 0.98 = 0.02$.

16.
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.95 + 0.90 - 0.88 \\ &= 0.97 \end{aligned}$$

17.
$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.98 + 0.95 - 0.99 \\ &= 0.94 \end{aligned}$$

18. (a)
$$\begin{aligned} P(O) &= 1 - P(\text{not } O) \\ &= 1 - [P(A) + P(B) + P(AB)] \\ &= 1 - [0.35 + 0.10 + 0.05] \\ &= 0.50 \end{aligned}$$

(b)
$$\begin{aligned} P(\text{does not contain } B) &= 1 - P(\text{contains } B) \\ &= 1 - [P(B) + P(AB)] \\ &= 1 - [0.10 + 0.05] \\ &= 0.85 \end{aligned}$$

19. (a) False

(b) True

(c) False

(d) True

20. (a) A and B are mutually exclusive, since it is impossible for both events to occur.

(b) If bolts #5 and #8 are torqued correctly, but bolt #3 is not torqued correctly, then events B and D both occur. Therefore B and D are not mutually exclusive.

(c) If bolts #5 and #8 are torqued correctly, but exactly one of the other bolts is not torqued correctly, then events C and D both occur. Therefore C and D are not mutually exclusive.

(d) If the #3 bolt is the only one not torqued correctly, then events B and C both occur. Therefore B and C are not mutually exclusive.

Section 2.2

1. (a) $(4)(4)(4) = 64$

(b) $(2)(2)(2) = 8$

(c) $(4)(3)(2) = 24$

2. $(4)(2)(3) = 24$

3. $\binom{8}{4} = \frac{8!}{4!4!} = 70$

4. $\binom{18}{9} = \frac{18!}{9!9!} = 48{,}620$

5. (a) $(8)(7)(6) = 336$

(b) $\binom{8}{3} = \frac{8!}{3!5!} = 56$

6. $(10)(9)(8) = 720$

7. $(2^{10})(4^5) = 1,048,576$

8. (a) $(26^3)(10^3) = 17,576,000$

(b) $(26)(25)(24)(10)(9)(8) = 11,232,000$

(c) $\frac{11,232,000}{17,576,000} = 0.6391$

9. (a) $36^8 = 2.8211 \times 10^{12}$

(b) $36^8 - 26^8 = 2.6123 \times 10^{12}$

(c) $\frac{36^8 - 26^8}{36^8} = 0.9260$

10. $\binom{15}{6,5,4} = \frac{15!}{6!5!4!} = 630,630$

11. $$\begin{aligned} P(\text{match}) &= P(BB) + P(WW) \\ &= (8/14)(4/6) + (6/14)(2/6) \\ &= 44/84 = 0.5238 \end{aligned}$$

12. $$\begin{aligned} P(\text{match}) &= P(RR) + P(GG) + P(BB) \\ &= (6/12)(5/11) + (4/12)(3/11) + (2/12)(1/11) \\ &= 1/3 \end{aligned}$$

Section 2.3

1. A and B are independent if $P(A \cap B) = P(A)P(B)$. Therefore $P(B) = 0.25$.

2. A and B are independent if $P(A \cap B) = P(A)P(B)$. Now $P(A) = P(A \cap B) + P(A \cap B^c)$. Since $P(A) = 0.5$ and $P(A \cap B^c) = 0.4$, $P(A \cap B) = 0.1$. Therefore $0.1 = 0.5P(B)$, so $P(B) = 0.2$.

3. (a) $5/15$

(b) Given that the first resistor is 50Ω, there are 14 resistors remaining of which 5 are 100Ω. Therefore $P(\text{2nd is } 100\Omega | \text{1st is } 50\Omega) = 5/14$.

(c) Given that the first resistor is 100Ω, there are 14 resistors remaining of which 4 are 100Ω. Therefore $P(\text{2nd is } 100\Omega | \text{1st is } 100\Omega) = 4/14$.

4. (a) $(10/15)(9/14) = 3/7$

(b)
$$\begin{aligned} P(\text{2 resistors selected}) &= P(\text{1st is } 50\Omega \text{ and 2nd is } 100\Omega) \\ &= (10/15)(5/14) \\ &= 5/21 \end{aligned}$$

(c)
$$\begin{aligned} P(\text{more than 3 resistors selected}) &= P(\text{1st 3 resistors are all } 50\Omega) \\ &= (10/15)(9/14)(8/13) \\ &= 24/91 \end{aligned}$$

5. Given that a student is an engineering major, it is almost certain that the student took a calculus course. Therefore $P(B|A)$ is close to 1. Given that a student took a calculus course, it is much less certain that the student is an engineering major, since many non-engineering majors take calculus. Therefore $P(A|B)$ is much less than 1, so $P(B|A) > P(A|B)$.

6. (0.056)(0.027) = 0.001512

7. Let A represent the event that the biotechnology company is profitable, and let B represent the event that the information technology company is profitable. Then $P(A) = 0.2$ and $P(B) = 0.15$.

(a) $P(A \cap B) = P(A)P(B) = (0.2)(0.15) = 0.03$.

(b) $P(A^c \cap B^c) = P(A^c)P(B^c) = (1-0.2)(1-0.15) = 0.68$.

(c)
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)P(B) \\ &= 0.2 + 0.15 - (0.2)(0.15) \\ &= 0.32 \end{aligned}$$

8. Let M denote the event that the main parachute deploys, and let B denote the event that backup parachute deploys. Then $P(M) = 0.99$ and $P(B|M^c) = 0.98$.

(a)
$$\begin{aligned} P(M \cup B) &= 1 - P(M^c \cap B^c) \\ &= 1 - P(B^c | M^c) P(M^c) \\ &= 1 - (1 - P(B|M^c))(1 - P(M)) \\ &= 1 - (1 - 0.98)(1 - 0.99) \\ &= 0.9998 \end{aligned}$$

(b) The backup parachute does not deploy if the main parachute deploys. Therefore

$$P(B) = P(B \cap M^c) = P(B | M^c) P(M^c) = (0.98)(0.01) = 0.0098$$

9. Let V denote the event that a person buys a hybrid vehicle, and let T denote the event that a person buys a hybrid truck. Then

$$\begin{aligned} P(T | V) &= \frac{P(T \cap V)}{P(V)} \\ &= \frac{P(T)}{P(V)} \\ &= \frac{0.05}{0.12} \\ &= 0.417 \end{aligned}$$

10. Let A denote the event that the allocation sector is damaged, and let N denote the event that a non-allocation sector is damaged. Then $P(A \cap N^c) = 0.20$, $P(A^c \cap N) = 0.7$, and $P(A \cap N) = 0.10$.

(a) $P(A) = P(A \cap N^c) + P(A \cap N) = 0.3$

(b) $P(N) = P(A^c \cap N) + P(A \cap N) = 0.8$

(c) $$\begin{aligned} P(N|A) &= \frac{P(A \cap N)}{P(A)} \\ &= \frac{P(A \cap N)}{P(A \cap N) + P(A \cap N^c)} \\ &= \frac{0.10}{0.10 + 0.20} \\ &= 1/3 \end{aligned}$$

(d) $$\begin{aligned} P(A|N) &= \frac{P(A \cap N)}{P(N)} \\ &= \frac{P(A \cap N)}{P(A \cap N) + P(A^c \cap N)} \\ &= \frac{0.10}{0.10 + 0.70} \\ &= 1/8 \end{aligned}$$

(e) $$\begin{aligned} P(N^c|A) &= \frac{P(A \cap N^c)}{P(A)} \\ &= \frac{P(A \cap N^c)}{P(A \cap N^c) + P(A \cap N)} \\ &= \frac{0.20}{0.20 + 0.10} \\ &= 2/3 \end{aligned}$$

Equivalently, one can compute $P(N^c|A) = 1 - P(N|A) = 1 - 1/3 = 2/3$

(f) $$\begin{aligned} P(A^c|N) &= \frac{P(A^c \cap N)}{P(N)} \\ &= \frac{P(A^c \cap N)}{P(A^c \cap N) + P(A \cap N)} \\ &= \frac{0.70}{0.70 + 0.10} \\ &= 7/8 \end{aligned}$$

Equivalently, one can compute $P(A^c|N) = 1 - P(A|N) = 1 - 1/8 = 7/8$

11. Let OK denote the event that a valve meets the specification, let R denote the event that a valve is reground, and let S denote the event that a valve is scrapped. Then $P(OK \cap R^c) = 0.7$, $P(R) = 0.2$, $P(S \cap R^c) = 0.1$, $P(OK|R) = 0.9$, $P(S|R) = 0.1$.

(a) $P(R^c) = 1 - P(R) = 1 - 0.2 = 0.8$

(b) $P(S|R^c) = \dfrac{P(S \cap R^c)}{P(R^c)} = \dfrac{0.1}{0.8} = 0.125$

(c)
$$\begin{aligned} P(S) &= P(S \cap R^c) + P(S \cap R) \\ &= P(S \cap R^c) + P(S|R)P(R) \\ &= 0.1 + (0.1)(0.2) \\ &= 0.12 \end{aligned}$$

(d)
$$\begin{aligned} P(R|S) &= \frac{P(S \cap R)}{P(S)} \\ &= \frac{P(S|R)P(R)}{P(S)} \\ &= \frac{(0.1)(0.2)}{0.12} \\ &= 0.167 \end{aligned}$$

(e)
$$\begin{aligned} P(OK) &= P(OK \cap R^c) + P(OK \cap R) \\ &= P(OK \cap R^c) + P(OK|R)P(R) \\ &= 0.7 + (0.9)(0.2) \\ &= 0.88 \end{aligned}$$

(f)
$$\begin{aligned} P(R|OK) &= \frac{P(R \cap OK)}{P(OK)} \\ &= \frac{P(OK|R)P(R)}{P(OK)} \\ &= \frac{(0.9)(0.2)}{0.88} \\ &= 0.205 \end{aligned}$$

$$\text{(g) } P(R^c|OK) = \frac{P(R^c \cap OK)}{P(OK)} = \frac{0.7}{0.88} = 0.795$$

12. Let S denote Sarah's score, and let T denote Thomas's score.

(a) $P(S > 175 \cap T > 175) = P(S > 175)P(T > 175) = (0.4)(0.2) = 0.08.$

(b) $P(T > 175 \cap S > T) = P(T > 175)P(S > T \,|\, T > 175) = (0.2)(0.3) = 0.06.$

13. Let $T1$ denote the event that the first device is triggered, and let $T2$ denote the event that the second device is triggered. Then $P(T1) = 0.9$ and $P(T2) = 0.8$.

$$\begin{aligned}\text{(a) } P(T1 \cup T2) &= P(T1) + P(T2) - P(T1 \cap T2) \\ &= P(T1) + P(T2) - P(T1)P(T2) \\ &= 0.9 + 0.8 - (0.9)(0.8) \\ &= 0.98\end{aligned}$$

(b) $P(T1^c \cap T2^c) = P(T1^c)P(T2^c) = (1 - 0.9)(1 - 0.8) = 0.02$

(c) $P(T1 \cap T2) = P(T1)P(T2) = (0.9)(0.8) = 0.72$

(d) $P(T1 \cap T2^c) = P(T1)P(T2^c) = (0.9)(1 - 0.8) = 0.18$

14. Let L denote the event that Laura hits the target, and let Ph be the event that Philip hits the target. Then $P(L) = 0.5$ and $P(Ph) = 0.3$.

(a)
$$\begin{aligned} P(L \cup Ph) &= P(L) + P(Ph) - P(L \cap Ph) \\ &= P(L) + P(Ph) - P(L)P(Ph) \\ &= 0.5 + 0.3 - (0.5)(0.3) \\ &= 0.65 \end{aligned}$$

(b)
$$\begin{aligned} P(\text{exactly one hit}) = P(L \cap Ph^c) + P(L^c \cap Ph) \\ &= P(L)P(Ph^c) + P(L^c)P(Ph) \\ &= (0.5)(1-0.3) - (1-0.5)(0.3) \\ &= 0.5 \end{aligned}$$

(c)
$$\begin{aligned} P(L|\text{exactly one hit}) &= \frac{P(L \cap \text{exactly one hit})}{P(\text{exactly one hit})} \\ &= \frac{P(L \cap Ph^c)}{P(\text{exactly one hit})} \\ &= \frac{P(L)P(Ph^c)}{P(\text{exactly one hit})} \\ &= \frac{(0.5)(1-0.3)}{0.5} \\ &= 0.7 \end{aligned}$$

15. (a) $\dfrac{88}{88+12} = 0.88$

(b) $\dfrac{88}{88+165+260} = 0.1715$

(c) $\dfrac{88+165}{88+65+260} = 0.4932$

(d) $\dfrac{88+165}{88+12+165+35} = 0.8433$

16. $P(E_2) = \dfrac{88+165+260}{600} = \dfrac{513}{600} = 0.855$. From Problem 15(a), $P(E_2|E_1) = 0.88$. Since $P(E_2|E_1) \neq P(E_1)$, E_1 and E_2 are not independent.

17. (a) $\dfrac{56+24}{100} = 0.80$

(b) $\dfrac{56+14}{100} = 0.70$

(c)
$$\begin{aligned} P(\text{Gene 2 dominant} \mid \text{Gene 1 dominant}) &= \frac{P(\text{Gene 1 dominant} \cap \text{Gene 2 dominant})}{P(\text{Gene 1 dominant})} \\ &= \frac{56/100}{0.8} \\ &= 0.7 \end{aligned}$$

(d) Yes. $P(\text{Gene 2 dominant} \mid \text{Gene 1 dominant}) = P(\text{Gene 2 dominant})$

18. (a) $\dfrac{71}{102+71+33+134} = \dfrac{71}{340}$

(b) $\dfrac{86}{102+86+26} = \dfrac{43}{107}$

(c) $\dfrac{22}{26+32+22+40} = \dfrac{11}{60}$

(d) $\dfrac{22}{33+36+22} = \dfrac{22}{91}$

(e) $\dfrac{86+63+36+26+32+22}{86+63+36+105+26+32+22+40} = \dfrac{53}{82}$

19. Let R, D, and I denote the events that the senator is a Republian, Democrat, or Independent, respectively, and let M and F denote the events that the senator is male or female, respectively.

(a) $P(R \cap M) = 0.41$

(b) $$\begin{aligned} P(D \cup F) &= P(D) + P(F) - P(D \cap F) \\ &= (0.37 + 0.16) + (0.16 + 0.04) - 0.16 \\ &= 0.57 \end{aligned}$$

(c) $$\begin{aligned} P(R) &= P(R \cap M) + P(R \cap F) \\ &= 0.41 + 0.04 \\ &= 0.45 \end{aligned}$$

(d) $P(R^c) = 1 - P(R) = 1 - 0.45 = 0.55$

(e) $$\begin{aligned} P(D) &= P(D \cap M) + P(D \cap F) \\ &= 0.37 + 0.16 \\ &= 0.53 \end{aligned}$$

(f) $$\begin{aligned} P(I) &= P(I \cap M) + P(I \cap F) \\ &= 0.02 + 0 \\ &= 0.02 \end{aligned}$$

(g) $P(D \cup I) = P(D) + P(I) = 0.53 + 0.02 = 0.55$

20. Let G denote the event that a customer is a good risk, let M denote the event that a customer is a medium risk, let P denote the event that a customer is a poor risk, and let C be the event that a customer has filed a claim. Then $P(G) = 0.7$, $P(M) = 0.2$, $P(P) = 0.1$, $P(C \mid G) = 0.005$, $P(C \mid M) = 0.01$, and $P(C \mid P) = 0.025$.

(a) $P(G \cap C) = P(G)P(C \mid G) = (0.70)(0.005) = 0.0035$

(b) $$\begin{aligned} P(C) &= P(G \cap C) + P(M \cap C) + P(P \cap C) \\ &= P(G)P(C \mid G) + P(M)P(C \mid M) + P(P)P(C \mid P) \\ &= (0.7)(0.005) + (0.2)(0.01) + (0.1)(0.025) \\ &= 0.008 \end{aligned}$$

(c) $P(G\,|\,C) = \dfrac{P(C\,|\,G)P(G)}{P(C)} = \dfrac{(0.005)(0.7)}{0.008} = 0.4375$.

21. (a) That the gauges fail independently.

(b) One cause of failure, a fire, will cause both gauges to fail. Therefore, they do not fail independently.

(c) Too low. The correct calculation would use $P(\text{second gauge fails}|\text{first gauge fails})$ in place of $P(\text{second gauge fails})$. Because there is a chance that both gauges fail together in a fire, the condition that the first gauge fails makes it more likely that the second gauge fails as well.
Therefore $P(\text{second gauge fails}|\text{first gauge fails}) > P(\text{second gauge fails})$.

22. No. $P(\text{both gauges fail}) = P(\text{first gauge fails})P(\text{second gauge fails}|\text{first gauge fails})$.
Since $P(\text{second gauge fails}|\text{first gauge fails}) \le 1$, $P(\text{both gauges fail}) \le P(\text{first gauge fails}) = 0.01$.

23. (a) $P(A) = 3/10$

(b) Given that A occurs, there are 9 components remaining, of which 2 are defective.
Therefore $P(B|A) = 2/9$.

(c) $P(A \cap B) = P(A)P(B|A) = (3/10)(2/9) = 1/15$

(d) Given that A^c occurs, there are 9 components remaining, of which 3 are defective.
Therefore $P(B|A) = 3/9$. Now $P(A^c \cap B) = P(A^c)P(B|A^c) = (7/10)(3/9) = 7/30$.

(e) $P(B) = P(A \cap B) + P(A^c \cap B) = 1/15 + 7/30 = 3/10$

(f) No. $P(B) \neq P(B|A)$ [or equivalently, $P(A \cap B) \neq P(A)P(B)$].

24. (a) $P(A) = 300/1000 = 3/10$

(b) Given that A occurs, there are 999 components remaining, of which 299 are defective.
Therefore $P(B|A) = 299/999$.

(c) $P(A \cap B) = P(A)P(B|A) = (3/10)(299/999) = 299/3330$

(d) Given that A^c occurs, there are 999 components remaining, of which 300 are defective.
Therefore $P(B|A) = 300/999$. Now $P(A^c \cap B) = P(A^c)P(B|A^c) = (7/10)(300/999) = 70/333$.

(e) $P(B) = P(A \cap B) + P(A^c \cap B) = 299/3330 + 70/333 = 3/10$

(f) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{299/3330}{3/10} = \dfrac{299}{999}$

(g) A and B are not independent, but they are very nearly independent. To see this note that $P(B) = 0.3$, while $P(B|A) = 0.2993$. So $P(B)$ is very nearly equal to $P(B|A)$, but not exactly equal. Alternatively, note that $P(A \cap B) = 0.0898$, while $P(A)P(B) = 0.09$. Therefore in most situations it would be reasonable to treat A and B as though they were independent.

25. $n = 10,000$. The two components are a simple random sample from the population. When the population is large, the items in a simple random sample are nearly independent.

26. Let E denote the event that a parcel is sent express (so E^c denotes the event that a parcel is sent standard), and let N denote the event that a parcel arrives the next day. Then $P(E) = 0.25$, $P(N|E) = 0.95$, and $P(N|E^c) = 0.80)$.

(a) $P(E \cap N) = P(E)P(N|E) = (0.25)(0.95) = 0.2375$.

$$
\text{(b) } P(N) = P(N|E)P(E) + P(N|E^c)P(E^c)
$$
$$
= (0.95)(0.25) + (0.80)(1-0.25)
$$
$$
= 0.8375
$$

$$
\text{(c) } P(E|N) = \frac{P(N|E)P(E)}{P(N|E)P(E) + P(N|E^c)P(E^c)}
$$
$$
= \frac{(0.95)(0.25)}{(0.95)(0.25) + (0.80)(1-0.25)}
$$
$$
= 0.2836
$$

27. Let R denote the event of a rainy day, and let C denote the event that the forecast is correct. Then $P(R) = 0.1$, $P(C|R) = 0.8$, and $P(C|R^c) = 0.9$.

$$
\text{(a) } P(C) = P(C|R)P(R) + P(C|R^c)P(R^c)
$$
$$
= (0.8)(0.1) + (0.9)(1-0.1)
$$
$$
= 0.89
$$

(b) A forecast of no rain will be correct on every non-rainy day. Therefore the probability is 0.9.

28. Let A denote the event that the flaw is found by the first inspector, and let B denote the event that the flaw is found by the second inspector.

(a) $P(A \cap B) = P(A)P(B) = (0.9)(0.7) = 0.63$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.9 + 0.7 - 0.63 = 0.97$

(c) $P(A^c \cap B) = P(A^c)P(B) = (1-0.9)(0.7) = 0.07$

29. Let F denote the event that an item has a flaw. Let A denote the event that a flaw is detected by the first inspector, and let B denote the event that the flaw is detected by the second inspector.

$$\text{(a) } P(F|A^c) = \frac{P(A^c|F)P(F)}{P(A^c|F)P(F)+P(A^c|F^c)P(F^c)} = \frac{(0.1)(0.1)}{(0.1)(0.1)+(1)(0.9)} = 0.011$$

$$\begin{aligned}
\text{(b) } P(F|A^c\cap B^c) &= \frac{P(A^c\cap B^c|F)P(F)}{P(A^c\cap B^c|F)P(F)+P(A^c\cap B^c|F^c)P(F^c)}\\
&= \frac{P(A^c|F)P(B^c|F)P(F)}{P(A^c|F)P(B^c|F)P(F)+P(A^c|F^c)P(B^c|F^c)P(F^c)}\\
&= \frac{(0.1)(0.3)(0.1)}{(0.1)(0.3)(0.1)+(1)(1)(0.9)}\\
&= 0.0033
\end{aligned}$$

30. Let D denote the event that a person has the disease, and let $+$ denote the event that the test is positive. Then $P(D) = 0.05$, $P(+|D) = 0.99$, and $P(+|D^c) = 0.01$.

$$\begin{aligned}
\text{(a) } P(D|+) &= \frac{P(+|D)P(D)}{P(+|D)P(D)+P(+|D^c)P(D^c)}\\
&= \frac{(0.99)(0.05)}{(0.99)(0.05)+(0.01)(0.95)}\\
&= 0.8390
\end{aligned}$$

$$\begin{aligned}
\text{(b) } P(D^c|-) &= \frac{P(-|D^c)P(D^c)}{P(-|D^c)P(D^c)+P(-|D)P(D)}\\
&= \frac{(0.99)(0.95)}{(0.99)(0.95)+(0.01)(0.05)}\\
&= 0.9995
\end{aligned}$$

31. (a) Each child has probability 0.25 of having the disease. Since the children are independent, the probability that both are disease-free is $0.75^2 = 0.5625$.

(b) Each child has probability 0.5 of being a carrier. Since the children are independent, the probability that both are carriers is $0.5^2 = 0.25$.

(c) Let D denote the event that a child has the disease, and let C denote the event that the child is a carrier. Then $P(D) = 0.25$, $P(C) = 0.5$, and $P(D^c) = 0.75$. We first compute $P(C \mid D^c)$, the probability that a child who does not have the disease is a carrier. First, $P(C \cap D^c) = P(C) = 0.5$. Now

$$P(C \mid D^c) = \frac{P(C \cap D^c)}{P(D^c)} = \frac{0.5}{0.75} = \frac{2}{3}$$

Since children are independent, the probability that both children are carriers given that neither has the disease is $(2/3)^2 = 4/9$.

(d) Let W_D denote the event that the woman has the disease, Let W_C denote the event that the woman is a carrier, and let W_F denote the event that the woman does not have the disease and is not a carrier. Then $P(W_D) = 0.25$, $P(W_C) = 0.5$, and $P(W_F) = 0.25$, Let C_D denote the event that the child has the disease.

$$\begin{aligned}
P(C_D) &= P(C_D \cap W_D) + P(C_D \cap W_C) + P(C_D \cap W_F) \\
&= P(C_D \mid W_D)P(W_D) + P(C_D \mid W_C)P(W_C) + P(C_D \mid W_F)P(W_F) \\
&= (0.5)(0.25) + (0.25)(0.5) + (0)(0.25) \\
&= 0.25
\end{aligned}$$

32. Let Fl denote the event that a bottle has a flaw. Let F denote the event that a bottle fails inspection. We are given $P(Fl) = 0.0002$, $P(F|Fl) = 0.995$, and $P(F^c|Fl^c) = 0.99$.

(a)
$$\begin{aligned}
P(Fl|F) &= \frac{P(F|Fl)P(Fl)}{P(F|Fl)P(Fl) + P(F|Fl^c)P(Fl^c)} \\
&= \frac{P(F|Fl)P(Fl)}{P(F|Fl)P(Fl) + [1 - P(F^c|Fl^c)]P(Fl^c)} \\
&= \frac{(0.995)(0.0002)}{(0.995)(0.0002) + (1 - 0.99)(0.9998)} \\
&= 0.01952
\end{aligned}$$

(b) i. Given that a bottle failed inspection, the probability that it had a flaw is only 0.01952.

(c)
$$\begin{aligned}
P(Fl^c|F^c) &= \frac{P(F^c|Fl^c)P(Fl^c)}{P(F^c|Fl^c)P(Fl^c)+P(F^c|Fl)P(Fl)} \\
&= \frac{P(F^c|Fl^c)P(Fl^c)}{P(F^c|Fl^c)P(Fl^c)+[1-P(F|Fl)]P(Fl)} \\
&= \frac{(0.99)(0.9998)}{(0.99)(0.9998)+(1-0.995)(0.0002)} \\
&= 0.999999
\end{aligned}$$

(d) ii. Given that a bottle passes inspection, the probability that is has no flaw is 0.999999.

(e) The small probability in part (a) indicates that some good bottles will be scrapped. This is not so serious. The important thing is that of the bottles that pass inspection, very few should have flaws. The large probability in part (c) indicates that this is the case.

33. Let D represent the event that the man actually has the disease, and let $+$ represent the event that the test gives a positive signal.

We are given that $P(D) = 0.005$, $P(+|D) = 0.99$, and $P(+|D^c) = 0.01$.

It follows that $P(D^c) = 0.995$, $P(-|D) = 0.01$, and $P(-|D^c) = 0.99$.

(a)
$$\begin{aligned}
P(D|-) &= \frac{P(-|D)P(D)}{P(-|D)P(D)+P(-|D^c)P(D^c)} \\
&= \frac{(0.01)(0.005)}{(0.01)(0.005)+(0.99)(0.995)} \\
&= 5.08\times 10^{-5}
\end{aligned}$$

(b) $P(++|D) = 0.99^2 = 0.9801$

(c) $P(++|D^c) = 0.01^2 = 0.0001$

(d)
$$\begin{aligned}
P(D|++) &= \frac{P(++|D)P(D)}{P(++|D)P(D)+P(++|D^c)P(D^c)} \\
&= \frac{(0.9801)(0.005)}{(0.9801)(0.005)+(0.0001)(0.995)} \\
&= 0.9801
\end{aligned}$$

34. $P(\text{system functions}) = P[(A\cap B)\cup(C\cup D)]$. Now $P(A\cap B) = P(A)P(B) = (1-0.10)(1-0.05) = 0.855$, and $P(C\cup D) = P(C)+P(D)-P(C\cap D) = (1-0.10)+(1-0.20)-(1-0.10)(1-0.20) = 0.98$.
Therefore

$$\begin{aligned} P[(A\cap B)\cup(C\cup D)] &= P(A\cap B)+P(C\cup D)-P[(A\cap B)\cap(C\cup D)] \\ &= P(A\cap B)+P(C\cup D)-P(A\cap B)P(C\cup D) \\ &= 0.855+0.98-(0.855)(0.98) \\ &= 0.9971 \end{aligned}$$

35. $P(\text{system functions}) = P[(A\cap B)\cap(C\cup D)]$. Now $P(A\cap B) = P(A)P(B) = (1-0.05)(1-0.03) = 0.9215$, and $P(C\cup D) = P(C)+P(D)-P(C\cap D) = (1-0.07)+(1-0.14)-(1-0.07)(1-0.14) = 0.9902$.
Therefore

$$\begin{aligned} P[(A\cap B)\cap(C\cup D)] &= P(A\cap B)P(C\cup D) \\ &= (0.9215)(0.9902) \\ &= 0.9125 \end{aligned}$$

36. (a) $P(A\cap B) = P(A)P(B) = (1-0.05)(1-0.03) = 0.9215$

(b) $P(A\cap B) = (1-p)^2 = 0.9$. Therefore $p = 1-\sqrt{0.9} = 0.0513$.

(c) $P(\text{three components all function}) = (1-p)^3 = 0.90$. Therefore $p = 1-(0.9)^{1/3} = 0.0345$.

37. Let C denote the event that component C functions, and let D denote the event that component D functions.

(a)
$$\begin{aligned} P(\text{system functions}) &= P(C\cup D) \\ &= P(C)+P(D)-P(C\cap D) \\ &= (1-0.08)+(1-0.12)-(1-0.08)(1-0.12) \\ &= 0.9904 \end{aligned}$$

Alternatively,

$$\begin{aligned} P(\text{system functions}) &= 1-P(\text{system fails}) \\ &= 1-P(C^c\cap D^c) \end{aligned}$$

$$\begin{aligned} &= 1 - P(C^c)P(D^c) \\ &= 1 - (0.08)(0.12) \\ &= 0.9904 \end{aligned}$$

(b) $P(\text{system functions}) = 1 - P(C^c \cap D^c) = 1 - p^2 = 0.99$. Therefore $p = \sqrt{1 - 0.99} = 0.1$.

(c) $P(\text{system functions}) = 1 - p^3 = 0.99$. Therefore $p = (1 - 0.99)^{1/3} = 0.2154$.

(d) Let n be the required number of components. Then n is the smallest integer such that $1 - 0.5^n \geq 0.99$. It follows that $n\ln(0.5) \leq \ln 0.01$, so $n \geq (\ln 0.01)(\ln 0.5) = 6.64$. Since n must be an integer, $n = 7$.

38. To show that A^c and B are independent, we show that $P(A^c \cap B) = P(A^c)P(B)$. Now $B = (A^c \cap B) \cup (A \cap B)$, and $(A^c \cap B)$ and $(A \cap B)$ are mutually exclusive. Therefore $P(B) = P(A^c \cap B) + P(A \cap B)$, from which it follows that $P(A^c \cap B) = P(B) - P(A \cap B)$. Since A and B are independent, $P(A \cap B) = P(A)P(B)$. Therefore $P(A^c \cap B) = P(B) - P(A)P(B) = P(B)[1 - P(A)] = P(A^c)P(B)$. To show that A and B^c are independent, it suffices to interchange A and B in the argument above. To show that A^c and B^c are independent, replace B with B^c in the argument above, and use the fact that A and B^c are independent.

Section 2.4

1. (a) Discrete

(b) Continuous

(c) Discrete

(d) Continuous

(e) Discrete

2. (a) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.4 + 0.3 + 0.15 = 0.85$.

(b) $P(X > 1) = P(X = 2) + P(X = 3) + P(X = 4) = 0.15 + 0.1 + 0.05 = 0.3$.

(c) $\mu_X = 0(0.4) + 1(0.3) + 2(0.15) + 3(0.1) + 4(0.05) = 1.1$

(d) $\sigma_X^2 = (0-1.1)^2(0.4) + (1-1.1)^2(0.3) + (2-1.1)^2(0.15) + (3-1.1)^2(0.1) + (4-1.1)^2(0.05) = 1.39$
Alternatively, $\sigma_X^2 = 0^2(0.4) + 1^2(0.3) + 2^2(0.15) + 3^2(0.1) + 4^2(0.05) - 1.1^2 = 1.39$

3. (a) $\mu_X = 1(0.4) + 2(0.2) + 3(0.2) + 4(0.1) + 5(0.1) = 2.3$

(b) $\sigma_X^2 = (1-2.3)^2(0.4) + (2-2.3)^2(0.2) + (3-2.3)^2(0.2) + (4-2.3)^2(0.1) + (5-2.3)^2(0.1) = 1.81$
Alternatively, $\sigma_X^2 = 1^2(0.4) + 2^2(0.2) + 3^2(0.2) + 4^2(0.1) + 5^2(0.1) - 2.3^2 = 1.81$

(c) $\sigma_X = \sqrt{1.81} = 1.345$

(d) $Y = 10X$. Therefore the probability density function is as follows.

y	10	20	30	40	50
$p(y)$	0.4	0.2	0.2	0.1	0.1

(e) $\mu_Y = 10(0.4) + 20(0.2) + 30(0.2) + 40(0.1) + 50(0.1) = 23$

(f) $\sigma_Y^2 = (10-23)^2(0.4) + (20-23)^2(0.2) + (30-23)^2(0.2) + (40-23)^2(0.1) + (50-23)^2(0.1) = 181$
Alternatively, $\sigma_Y^2 = 10^2(0.4) + 20^2(0.2) + 30^2(0.2) + 40^2(0.1) + 50^2(0.1) - 23^2 = 181$

(g) $\sigma_Y = \sqrt{181} = 13.45$

4. (a) $p_3(x)$ is the only probability mass function, because it is the only one whose probabilities sum to 1.

(b) $\mu_X = 0(0.1) + 1(0.2) + 2(0.4) + 3(0.2) + 4(0.1) = 2.0$,
$\sigma_X^2 = (0-2)^2(0.1) + (1-2)^2(0.2) + (2-2)^2(0.4) + (3-2)^2(0.2) + (4-2)^2(0.1) = 1.2$

5. (a)

x	1	2	3	4	5
p(x)	0.70	0.15	0.10	0.03	0.02

(b) $P(X \leq 2) = P(X = 1) + P(X = 2) = 0.70 + 0.15 = 0.85$

(c) $P(X > 3) = P(X = 4) + P(X = 5) = 0.03 + 0.02 = 0.05$

(d) $\mu_X = 1(0.70) + 2(0.15) + 3(0.10) + 4(0.03) + 5(0.02) = 1.52$

(e) $\sigma_X = \sqrt{1^2(0.70) + 2^2(0.15) + 3^2(0.10) + 4^2(0.03) + 5^2(0.02) - 1.52^2} = 0.9325$

6. (a) $\mu_X = 24(0.0825) + 25(0.0744) + 26(0.7372) + 27(0.0541) + 28(0.0518) = 25.9183$

(b) $\sigma_X^2 = (24-25.9183)^2(0.0825) + (25-25.9183)^2(0.0744) + (26-25.9183)^2(0.7372) + (27-25.9183)^2(0.0541) + (28-25.9183)^2(0.0518) = 0.659025$

Alternatively, $\sigma_X^2 = 24^2(0.0825) + 25^2(0.0744) + 26^2(0.7372) + 27^2(0.0541) + 28^2(0.0518) - 25.9183^2 = 0.659025$

$\sigma_X = \sqrt{0.659025} = 0.8118.$

7. (a) $\sum_{x=1}^{5} cx = 1$, so $c(1+2+3+4+5) = 1$, so $c = 1/15$.

(b) $P(X = 2) = c(2) = 2/15 = 0.2$

(c) $\mu_X = \sum_{x=1}^{5} xP(X = x) = \sum_{x=1}^{5} x^2/15 = (1^2 + 2^2 + 3^2 + 4^2 + 5^2)/15 = 11/3$

(d) $\sigma_X^2 = \sum_{x=1}^{5}(x-\mu_X)^2 P(X = x) = \sum_{x=1}^{5} x(x-11/3)^2/15 = (64/135) + 2(25/135) + 3(4/135) + 4(1/135) + 5(16/135) = 14/9$

Alternatively, $\sigma_X^2 = \sum_{x=1}^{5} x^2 P(X = x) - \mu_X^2 = \sum_{x=1}^{5} x^3/15 - (11/3)^2 = (1^3 + 2^3 + 3^3 + 4^3 + 5^3)/15 - (11/3)^2 = 14/9$

(e) $\sigma_X = \sqrt{14/9} = 1.2472$

8. (a) $P(X \leq 2) = F(2) = 0.83$

(b) $P(X > 3) = 1 - P(X \leq 3) = 1 - F(3) = 1 - 0.95 = 0.05$

(c) $P(X = 1) = P(X \leq 1) - P(X \leq 0) = F(1) - F(0) = 0.72 - 0.41 = 0.31$

(d) $P(X = 0) = P(X \leq 0) = F(0) = 0.41$

(e) For any integer x, $P(X = x) = F(x) - F(x-1)$ [note that $F(-1) = 0$]. The value of x for which this quantity is greatest is $x = 0$.

9. (a)

x	$p_1(x)$
0	0.2
1	0.16
2	0.128
3	0.1024
4	0.0819
5	0.0655

(b)

x	$p_2(x)$
0	0.4
1	0.24
2	0.144
3	0.0864
4	0.0518
5	0.0311

(c) $p_2(x)$ appears to be the better model. Its probabilities are all fairly close to the proportions of days observed in the data. In contrast, the probabilities of 0 and 1 for $p_1(x)$ are much smaller than the observed proportions.

(d) No, this is not right. The data are a simple random sample, and the model represents the population. Simple random samples generally do not reflect the population exactly.

10. Let A denote an acceptable chip, and U an unacceptable one.

(a) $P(A) = 0.9$

(b) $P(UA) = P(U)P(A) = (0.1)(0.9) = 0.09$

(c) $P(X=3)=P(UUA)=P(U)P(U)P(A)=(0.1)(0.1)(0.9)=0.009$

(d) $p(x)=\begin{cases}(0.9)(0.1)^{x-1} & x=1,2,3,\dots\\ 0 & \text{otherwise}\end{cases}$

11. Let A denote an acceptable chip, and U an unacceptable one.

(a) If the first two chips are both acceptable, then $Y=2$. This is the smallest possible value.

(b) $P(Y=2)=P(AA)=(0.9)^2=0.81$

(c) $P(Y=3|X=1)=\dfrac{P(Y=3 \text{ and } X=1)}{P(X=1)}$.
Now $P(Y=3 \text{ and } X=1)=P(AUA)=(0.9)(0.1)(0.9)=0.081$, and $P(X=1)=P(A)=0.9$.
Therefore $P(Y=3|X=1)=0.081/0.9=0.09$.

(d) $P(Y=3|X=2)=\dfrac{P(Y=3 \text{ and } X=2)}{P(X=2)}$.
Now $P(Y=3 \text{ and } X=2)=P(UAA)=(0.1)(0.9)(0.9)=0.081$, and
$P(X=2)=P(UA)=(0.1)(0.9)=0.09$.
Therefore $P(Y=3|X=2)=0.081/0.09=0.9$.

(e) If $Y=3$ the only possible values for X are $X=1$ and $X=2$.
Therefore

$$\begin{aligned}P(Y=3) &= P(Y=3|X=1)P(X=1)+P(Y=3|X=2)P(X=2)\\ &= (0.09)(0.9)+(0.9)(0.09)\\ &= 0.162\end{aligned}$$

12. (a) 0, 1, 2, 3

(b) $P(X=3)=P(SSS)=(0.8)^3=0.512$

(c) $P(FSS) = (0.2)(0.8)^2 = 0.128$

(d) $P(SFS) = P(SSF) = (0.8)^2(0.2) = 0.128$

(e) $P(X = 2) = P(FSS) + P(SFS) + P(SSF) = 0.384$

(f)
$$\begin{aligned} P(X = 1) &= P(SFF) + P(FSF) + P(FFS) \\ &= (0.8)(0.2)^2 + (0.8)(0.2)^2 + (0.8)(0.2)^2 \\ &= 0.096 \end{aligned}$$

(g) $P(X = 0) = P(FFF) = (0.2)^3 = 0.008$

(h) $\mu_X = 0(0.08) + 1(0.096) + 2(0.384) + 3(0.512) = 2.4$

(i) $\sigma_X^2 = (0 - 2.4)^2(0.08) + (1 - 2.4)^2(0.096) + (2 - 2.4)^2(0.384) + (3 - 2.4)^2(0.512) = 0.48$
Alternatively, $\sigma_X^2 = 0^2(0.08) + 1^2(0.096) + 2^2(0.384) + 3^2(0.512) - 2.4^2 = 0.48$

(j)
$$\begin{aligned} P(Y = 3) &= P(SSSF) + P(SSFS) + P(SFSS) + P(FSSS) \\ &= (0.8)^3(0.2) + (0.8)^3(0.2) + (0.8)^3(0.2) + (0.8)^3(0.2) \\ &= 0.4096 \end{aligned}$$

13. (a) $\displaystyle\int_{80}^{90} \frac{x - 80}{800}\,dx = \left.\frac{x^2 - 160x}{1600}\right|_{80}^{90} = 0.0625$

(b) $\displaystyle\int_{80}^{120} x\frac{x - 80}{800}\,dx = \left.\frac{x^3 - 120x}{2400}\right|_{80}^{120} = 320/3 = 106.67$

(c) $\displaystyle\sigma_X^2 = \int_{80}^{120} x^2\frac{x - 80}{800}\,dx - (320/3)^2 = \left.\frac{x^4}{3200} - \frac{x^3}{30}\right|_{80}^{120} - (320/3)^2 = 800/9$

$\sigma_X = \sqrt{800/9} = 9.428$

(d) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x < 80$, $F(x) = \int_{-\infty}^{80} 0\,dt = 0$

If $80 \le x < 120$, $F(x) = \int_{-\infty}^{80} 0\,dt + \int_{80}^{x} \frac{t-80}{800}\,dt = x^2/1600 - x/10 + 4.$

If $x \ge 120$, $F(x) = \int_{-\infty}^{80} 0\,dt + \int_{80}^{120} \frac{t-80}{800}\,dt + \int_{120}^{x} 0\,dt = 1.$

14. (a) $\int_{25}^{30} \frac{x}{250}\,dx = \left.\frac{x^2}{500}\right|_{25}^{30} = 0.55$

(b) $\mu = \int_{20}^{30} x\frac{x}{250}\,dx = \left.\frac{x^3}{750}\right|_{20}^{30} = 76/3 = 25.33$

(c) $\sigma_X^2 = \int_{20}^{30} x^2\frac{x}{250}\,dx - (76/3)^2 = \left.\frac{x^4}{1000}\right|_{20}^{30} - (76/3)^2 = 74/9$

(d) $\sigma_X = \sqrt{\sigma_X^2} = \sqrt{74/9} = 2.867$

(e) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x < 20$, $F(x) = \int_{-\infty}^{20} 0\,dt = 0$

If $20 \le x < 30$, $F(x) = \int_{-\infty}^{20} 0\,dt + \int_{20}^{x} \frac{t}{250}\,dt = x^2/500 - 4/5.$

If $x \ge 30$, $F(x) = \int_{-\infty}^{80} 0\,dt + \int_{20}^{30} \frac{t}{250}\,dt + \int_{30}^{x} 0\,dt = 1.$

(f) $\int_{28}^{30} \frac{x}{250}\,dx = \left.\frac{x^2}{500}\right|_{28}^{30} = 0.232$

15. (a) $\mu \quad = \quad \int_{0}^{\infty} 0.1te^{-0.1t}\,dt$

$$\begin{aligned}
&= -te^{-0.1t}\Big|_0^\infty - \int_0^\infty -e^{-0.1t}\,dt \\
&= 0 - 10e^{-0.1t}\Big|_0^\infty \\
&= 10
\end{aligned}$$

(b) $$\begin{aligned}
\sigma^2 &= \int_0^\infty 0.1t^2e^{-0.1t}\,dt - \mu^2 \\
&= -t^2e^{-0.1t}\Big|_0^\infty - \int_0^\infty -2te^{-0.1t}\,dt - 100 \\
&= 0 + 20\int_0^\infty 0.1te^{-0.1t}\,dt - 100 \\
&= 0 + 20(10) - 100 \\
&= 100
\end{aligned}$$

$$\sigma_X = \sqrt{100} = 10$$

(c) $F(x) = \int_{-\infty}^{x} f(t)\,dt$.

If $x \le 0$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$.

If $x > 0$, $F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^x 0.1e^{-0.1t}\,dt = 1 - e^{-0.1x}$.

(d) Let T represent the lifetime. $P(T < 12) = P(T \le 12) = F(12) = 1 - e^{-1.2} = 0.6988$.

16. (a) $\mu = \int_{9.75}^{10.25} 3x[1-16(x-10)^2]\,dx = -12x^4 + 320x^3 - 2398.5x^2\Big|_{9.75}^{10.25} = 10$

(b) $\sigma^2 = \int_{9.75}^{10.25} 3(x-10)^2[1-16(x-10)^2]\,dx = (x-10)^3 - 9.6(x-10)^5\Big|_{9.75}^{10.25} = 0.0125$.

$\sigma = \sqrt{0.0125} = 0.1118$

(c) $F(x) = \int_{-\infty}^{x} f(t)\,dt$.

If $x \le 9.75$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$.

If $9.75 < x < 10.25$,

$$F(x) = \int_{-\infty}^{x} 0\,dt + \int_{9.75}^{x} 3[1-16(t-10)^2]\,dt = 3t - 16(t-10)^3 \Big|_{9.75}^{x} = 3x - 16(x-10)^3 - 29.5$$

If $x \ge 10.25$, $F(x) = 1$.

(d) None of them. $F(9.75) = 0$.

(e) All of them. $F(10.25) = 1$, so all of the rings have diameters less than or equal to 10.25 cm. Since none of the rings have diameters less than 9.75 cm, all of them have diameters between 9.75 and 10.25 cm.

17. With this process, the probability that a ring meets the specification is

$$\int_{9.9}^{10.1} 15[1-25(x-10.05)^2]/4\,dx = \int_{-0.15}^{0.05} 15[1-25x^2]/4\,dx = 0.25(15x - 125x^3)\Big|_{-0.15}^{0.05} = 0.641.$$

With the process in Exercise 16, the probability is

$$\int_{9.9}^{10.1} 3[1-16(x-10)^2]\,dx = \int_{-0.1}^{0.1} 3[1-16x^2]\,dx = 3x - 16x^3\Big|_{-0.1}^{0.1} = 0.568.$$

Therefore this process is better than the one in Exercise 16.

18. (a) $P(X > 2) = \int_{2}^{\infty} \frac{64}{(x+2)^5}\,dx = -\frac{16}{(x+2)^4}\Big|_{2}^{\infty} = 1/16$

(b) $P(2 < X < 4) = \int_{2}^{4} \frac{64}{(x+2)^5}\,dx = -\frac{16}{(x+2)^4}\Big|_{2}^{4} = 65/1296$

(c) $\mu = \int_{0}^{\infty} x\frac{64}{(x+2)^5}\,dx = \int_{2}^{\infty} (u-2)\frac{64}{u^5}\,du = 64\int_{2}^{\infty} (u^{-4} - 2u^{-5})\,du = 64\left(-\frac{1}{3}u^{-3} + \frac{1}{2}u^{-4}\right)\Big|_{2}^{\infty} = 2/3$

(d) $\sigma^2 \quad = \quad \int_{0}^{\infty} x^2\frac{64}{(x+2)^5}\,dx - \mu^2$

$$
\begin{aligned}
&= \int_2^\infty (u-2)^2 \frac{64}{u^5}\,du - (2/3)^2 \\
&= 64\int_2^\infty (u^{-3} - 4u^{-4} + 4u^{-5})\,du \;-\; 4/9 \\
&= 64\left(-\frac{1}{2}u^{-2} + \frac{4}{3}u^{-3} - u^{-4}\right)\Bigg|_2^\infty -\; 4/9 \\
&= 8/9
\end{aligned}
$$

(e) $F(x) = \int_{-\infty}^{x} f(t)\,dt$.

If $x < 0$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$.

If $x \geq 0$, $F(x) = \int_0^x \frac{64}{(t+2)^5}\,dt = -\frac{16}{(t+2)^4}\Bigg|_0^x = 1 - \frac{16}{(x+2)^4}$

(f) The median x_m solves $F(x_m) = 0.5$. Therefore $1 - \frac{16}{(x_m+2)^4} = 0.5$, so $x_m = 0.3784$.

(g) The 60th percentile x_{60} solves $F(x_{60}) = 0.6$. Therefore $1 - \frac{16}{(x_{60}+2)^4} = 0.6$, so $x_{60} = 0.5149$.

19. (a) $P(X > 3) = \int_3^4 (3/64)x^2(4-x)\,dx = \left(\frac{x^3}{16} - \frac{3x^4}{256}\right)\Bigg|_3^4 = 67/256$

(b) $P(2 < X < 3) = \int_2^3 (3/64)x^2(4-x)\,dx = \left(\frac{x^3}{16} - \frac{3x^4}{256}\right)\Bigg|_2^3 = 109/256$

(c) $\mu = \int_0^4 (3/64)x^3(4-x)\,dx = \left(\frac{3x^4}{64} - \frac{3x^5}{320}\right)\Bigg|_0^4 = 2.4$

(d) $\sigma^2 = \int_0^4 (3/64)x^4(4-x)\,dx - \mu^2 = \left(\frac{3x^5}{80} - \frac{x^6}{128}\right)\Bigg|_0^4 - 2.4^2 = 0.64$

(e) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x \le 0$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$

If $0 < x < 4$, $F(x) = \int_{0}^{x} (3/64)t^2(4-t)\,dt = (16x^3 - 3x^4)/256$

If $x \ge 3$, $F(x) = \int_{0}^{4} (3/64)t^2(4-t)\,dt = 1$

20. (a) $P(X < 0.02) = \int_{0}^{0.02} 625x\,dx = \left.\frac{625x^2}{2}\right|_{0}^{0.02} = 0.125$

(b) $\mu = \int_{0}^{0.04} 625x^2\,dx + \int_{0.04}^{0.08} (50x - 625x^2)\,dx = \left.\frac{625x^3}{3}\right|_{0}^{0.04} + \left.\left(25x^2 - \frac{625x^3}{3}\right)\right|_{0.04}^{0.08} = 0.04$

(c) The variance is

$$\begin{aligned} \sigma^2 &= \int_{0}^{0.04} 625x^3\,dx + \int_{0.04}^{0.08} (50x^2 - 625x^3)\,dx - \mu^2 \\ &= \left.\frac{625x^4}{4}\right|_{0}^{0.04} + \left.\left(\frac{50x^3}{3} - \frac{625x^4}{4}\right)\right|_{0.04}^{0.08} - 0.04^2 \\ &= 0.0002667 \end{aligned}$$

The standard deviation is $\sigma = \sqrt{0.0002667} = 0.01633$.

(d) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x \le 0$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$

If $0 < x \le 0.04$, $F(x) = \int_{0}^{x} 625t\,dt = 625x^2/2$

If $0.04 < x \le 0.08$, $F(x) = \int_{0}^{0.04} 625t\,dt + \int_{0.04}^{x} (50 - 625t)\,dt = 50x - \frac{625}{2}x^2 - 1$

If $x > 0.08$, $F(x) = \int_{0}^{0.04} 625t\,dt + \int_{0.04}^{0.08} (50 - 625t)\,dt = 1$

(e) The median x_m solves $F(x_m) = 0.5$. Since $F(x)$ is described by different expressions for $x \le 0.04$ and $x > 0.04$, we compute $F(0.04)$. $F(0.04) = 625(0.04)^2/2 = 0.5$ so $x_m = 0.04$.

(f) $P(0.015 < X < 0.063) = \int_{0.015}^{0.04} 625x\,dx + \int_{0.04}^{0.063} (50 - 625x)\,dx = \left.\dfrac{625x^2}{2}\right|_0^{0.04} + \left.\left(50x - \dfrac{625x^2}{2}\right)\right|_{0.04}^{0.063} = 0.9097$

21. (a) $P(X < 0.2) = \int_0^{0.2} 12(x^2 - x^3)\,dx = \left. 4x^3 - 3x^4 \right|_0^{0.2} = 0.0272$

(b) $\mu = \int_0^1 12x(x^2 - x^3)\,dx = \left. 3x^4 - \dfrac{12}{5}x^5 \right|_0^1 = 0.6$

(c) The variance is

$$\begin{aligned} \sigma^2 &= \int_0^1 12x^2(x^2 + x^3)\,dx - \mu^2 \\ &= \left.\left(\frac{12}{5}x^5 + 2x^6\right)\right|_0^1 - 0.6^2 \\ &= 0.04 \end{aligned}$$

(d) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x \le 0$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$

If $0 < x \le 1$, $F(x) = \int_0^x 12(t^2 - t^3)\,dt = \left. 4t^3 - 3t^4 \right|_0^x = 4x^3 - 3x^4$

If $x > 1$, $F(x) = \int_0^1 12(t^2 - t^3)\,dt = 1$

(e) $P(0 < X < 0.8) = \int_0^{0.8} 12(x^2 - x^3)\,dx = \left.\left(4x^3 - 3x^4\right)\right|_0^{0.8} = 0.81922$

22. (a) $P(X > 0.5) = \int_{0.5}^{1} \dfrac{2e^{-2x}}{1 - e^{-2}}\,dx = \left.\left(\dfrac{1}{1 - e^{-2}}\right)(-e^{-2x})\right|_{0.5}^{1} = 0.2689$

(b)
$$
\begin{aligned}
\mu &= \int_0^1 \frac{2xe^{-2x}}{1-e^{-2}}\,dx \\
&= \left(\frac{e^2}{e^2-1}\right)\int_0^1 2xe^{-2x}\,dx \\
&= \left(\frac{e^2}{2e^2-2}\right)\int_0^2 ue^{-u}\,du \\
&= \left(\frac{e^2}{2e^2-2}\right)\left(-ue^{-u}\Big|_0^2 + \int_0^2 e^{-u}\,du\right) \\
&= \left(\frac{e^2}{2e^2-2}\right)\left(-2e^{-2} - e^{-u}\Big|_0^2\right) \\
&= \frac{e^2-3}{2e^2-2} \\
&= 0.34348
\end{aligned}
$$

(c) Let X denote the concentration. The mean is $\mu_X = 0.34348$.

$$
\begin{aligned}
P(\mu - 0.1 < X < \mu + 0.1) &= \int_{0.24348}^{0.44348} \frac{2e^{-2x}}{1-e^{-2}}\,dx \\
&= \frac{-e^{-2x}}{1-e^{-2}}\Bigg|_{0.24348}^{0.44348} \\
&= \frac{e^{-0.48696} - e^{-0.88696}}{1-e^{-2}} \\
&= 0.23429
\end{aligned}
$$

(d) The variance is σ^2
$$
\begin{aligned}
\sigma^2 &= \int_0^1 \frac{2x^2e^{-2x}}{1-e^{-2}}\,dx - \mu^2 \\
&= \left(\frac{e^2}{e^2-1}\right)\int_0^1 2x^2e^{-2x}\,dx - \mu^2 \\
&= \left(\frac{e^2}{4e^2-4}\right)\int_0^2 u^2e^{-u}\,du - \mu^2 \\
&= \left(\frac{e^2}{4e^2-4}\right)\left(-u^2e^{-u}\Big|_0^2 + \int_0^2 2ue^{-u}\,du\right) - \left(\frac{e^2-3}{2e^2-2}\right)^2 \\
&= \left(\frac{e^2}{4e^2-4}\right)\left[-4e^{-2} + 2\left(-ue^{-u}\Big|_0^2 + \int_0^2 e^{-u}\,du\right)\right] - \left(\frac{e^2-3}{2e^2-2}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{e^2}{4e^2-4}\right)\left[-4e^{-2}+2\left(-2e^{-2}-e^{-u}\Big|_0^2\right)\right]-\left(\frac{e^2-3}{2e^2-2}\right)^2 \\
&= \left(\frac{e^2}{4e^2-4}\right)\left[-4e^{-2}+2\left(1-3e^{-2}\right)\right]-\left(\frac{e^2-3}{2e^2-2}\right)^2 \\
&= \frac{e^2-5}{2e^2-2}-\left(\frac{e^2-3}{2e^2-2}\right)^2 \\
&= 0.0689845
\end{aligned}
$$

The standard deviation is $\sigma = \sqrt{0.0689845} = 0.26265$.

(e) Let X denote the concentration. The mean is $\mu_X = 0.34348$. The standard deviation is $\sigma = 0.26265$.

$$
\begin{aligned}
P(\mu-\sigma < X < \mu+\sigma) &= \int_{0.08083}^{0.60613} \frac{2e^{-2x}}{1-e^{-2}}\,dx \\
&= \left.\frac{-e^{-2x}}{1-e^{-2}}\right|_{0.08083}^{0.60613} \\
&= \frac{e^{-0.16167}-e^{-1.21226}}{1-e^{-2}} \\
&= 0.63979
\end{aligned}
$$

(f) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x \le 0$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$

If $0 < x < 1$, $F(x) = \int_0^x \frac{2e^{-2t}}{1-e^2}\,dt = \frac{1-e^{-2x}}{1-e^{-2}}$

If $x \ge 1$, $F(x) = \int_0^1 \frac{e^{-2t}}{1-e^{-2}}\,dt = 1$

23. (a) $P(X < 2.5) = \int_2^{2.5} (3/52)x(6-x)\,dx = (9x^2-x^3)/52\Big|_2^{2.5} = 0.2428$

(b) $P(2.5 < X < 3.5) = \int_{2.5}^{3.5} (3/52)x(6-x)\,dx = \left.\frac{9x^2-x^3}{52}\right|_{2.5}^{3.5} = 0.5144$

(c) $\mu = \int_2^4 (3/52)x^2(6-x)\,dx = \left.\dfrac{24x^3 - 3x^4}{208}\right|_2^4 = 3$

(d) The variance is

$$\begin{aligned}\sigma^2 &= \int_2^4 (3/52)x^3(6-x)\,dx - \mu^2 \\ &= \left.\frac{9x^4}{104} - \frac{3x^5}{260}\right|_2^4 - 3^2 \\ &= 0.3230769\end{aligned}$$

The standard deviation is $\sigma = \sqrt{0.3230769} = 0.5684$.

(e) Let X represent the thickness. Then X is within $\pm\sigma$ of the mean if $2.4316 < X < 3.5684$.

$P(2.4316 < X < 3.5684) = \int_{2.4316}^{3.5684} (3/52)x(6-x)\,dx = \left.\dfrac{9x^2 - x^3}{52}\right|_{2.4316}^{3.5684} = 0.5832$

(f) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x \le 2$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$.

If $2 < x < 4$, $F(x) = \int_{-\infty}^{2} 0\,dt + \int_2^x (3/52)t(6-t)\,dt = \dfrac{9x^2 - x^3 - 28}{52}$.

If $x \ge 4$, $F(x) = \int_{-\infty}^{2} 0\,dt + \int_2^4 (3/52)t(6-t)\,dt + \int_4^x 0\,dt = 1$.

24. (a) c solves the equation $\int_1^\infty c/x^3\,dx = 1$. Therefore $\left.-0.5c/x^2\right|_1^\infty = 1$, so $c = 2$.

(b) $\mu_X = \int_1^\infty cx/x^3\,dx = \int_1^\infty 2/x^2\,dx = \left.-\dfrac{2}{x}\right|_1^\infty = 2$

(c) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x < 1$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$.

If $x \geq 1$, $F(x) = \int_{-\infty}^{1} 0\,dt + \int_{1}^{x} 2/t^3\,dt = -\frac{1}{t^2}\Big|_1^x = 1 - 1/x^2$.

(d) The median x_m solves $F(x_m) = 0.5$. Therefore $1 - 1/x_m^2 = 0.5$, so $x_m = 1.414$.

(e) $P(X \leq 10) = F(10) = 1 - 1/10^2 = 0.99$

(f) $P(X \leq 2.5) = F(2.5) = 1 - 1/2.5^2 = 0.84$

(g) $P(X \leq 2.5 | X \leq 10) = \dfrac{P(X \leq 2.5 \text{ and } X \leq 10)}{P(X \leq 10)} = \dfrac{P(X \leq 2.5)}{P(X \leq 10)} = \dfrac{0.84}{0.99} = 0.85$

25. (a) $P(X < 2) = \int_0^2 xe^{-x}\,dx = \left(-xe^{-x}\Big|_0^2 + \int_0^2 e^{-x}\,dx\right) = \left(-2e^{-2} - e^{-x}\Big|_0^2\right) = 1 - 3e^{-2} = 0.5940$

(b) $P(1.5 < X < 3) = \int_{1.5}^{3} xe^{-x}\,dx = \left(-xe^{-x}\Big|_{1.5}^{3} + \int_{1.5}^{3} e^{-x}\,dx\right) = \left(-3e^{-3} + 1.5e^{-1.5} - e^{-x}\Big|_{1.5}^{3}\right)$
$= 2.5e^{-1.5} - 4e^{-3} = 0.3587$

(c) $\mu = \int_0^\infty x^2 e^{-x}\,dx = -x^2 e^{-x}\Big|_0^\infty + \int_0^\infty 2xe^{-x}\,dx = 0 + 2xe^{-x}\Big|_0^\infty = 2$

(d) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x < 0$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$

If $x > 0$, $F(x) = \int_0^x te^{-t}\,dt = 1 - (x+1)e^{-x}$

26. (a) $P(X < 12.5) = \int_{12}^{12.5} 6(x-12)(13-x)\,dx = -2x^3 + 75x^2 - 936x\Big|_{12}^{12.5} = 0.5$

(b) $\mu = \int_{12}^{13} 6x(x-12)(13-x)\,dx = -\frac{3x^4}{2} + 50x^3 - 468x^2\Big|_{12}^{13} = 12.5$

(c) $\sigma^2 = \int_{12}^{13} 6x^2(x-12)(13-x)\,dx - \mu^2 = -\frac{6x^5}{5} + \frac{75x^4}{2} - 312x^3 \Big|_{12}^{13} - 12.5^2 = 0.05$

The standard deviation is $\sigma = \sqrt{0.05} = 0.2236$.

(d) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x \le 12$, $F(x) = \int_{-\infty}^{x} 0\,dt = 0$

If $12 < x < 13$, $F(x) = \int_{12}^{x} 6(t-12)(13-t)\,dt = -2x^3 + 75x^2 - 936x + 3888$

If $x \ge 13$, $F(x) = \int_{12}^{13} 6(t-12)(13-t)\,dt = 1$

Section 2.5

1. (a) $\mu_{3X} = 3\mu_X = 3(9.5) = 28.5$

$\sigma_{3X} = 3\sigma_X = 3(0.4) = 1.2$

(b) $\mu_{Y-X} = \mu_Y - \mu_X = 6.8 - 9.5 = -2.7$

$\sigma_{Y-X} = \sqrt{\sigma_Y^2 + \sigma_X^2} = \sqrt{0.1^2 + 0.4^2} = 0.412$

(c) $\mu_{X+4Y} = \mu_X + 4\mu_Y = 9.5 + 4(6.8) = 36.7$

$\sigma_{X+4Y} = \sqrt{\sigma_X^2 + 4^2\sigma_Y^2} = \sqrt{0.4^2 + 16(0.1^2)} = 0.566$

2. (a) Let H denote the height. Then $\mu_V = \mu_{10H} = 10\mu_H = 10(5) = 50$

(b) $\sigma_V = \sigma_{10H} = 10\sigma_H = 10(0.1) = 1$

3. Let $X_1, \ldots, X_4$ be the lifetimes of the four transistors. Let $S = X_1 + \cdots + X_4$ be the total lifetime.

$$\mu_S = \sum \mu_{X_i} = 4(900) = 3600$$
$$\sigma_S = \sqrt{\sum \sigma^2_{X_i}} = \sqrt{4(30^2)} = 60$$

4. (a) $\mu_V = \mu_{V_1} + \mu_{V_2} = 12 + 6 = 18$

(b) $\sigma_R = \sqrt{\sigma^2_{R_1} + \sigma^2_{R_2}} = \sqrt{1^2 + 0.5^2} = 1.118$

5. Let $X_1, \ldots, X_5$ be the thicknesses of the five layers. Let $S = X_1 + \cdots + X_5$ be the total thickness.

(a) $\mu_S = \sum \mu_{X_i} = 5(1.2) = 6.0$

(b) $\sigma_S = \sqrt{\sum \sigma^2_{X_i}} = \sqrt{5(0.04^2)} = 0.0894$

6. Let X_1 and X_2 be the two measurements. The average is $\overline{X}$.

$$\sigma_{\overline{X}} = \sigma_{X_i}/\sqrt{2} = \frac{7 \times 10^{-15}}{\sqrt{2}} = 4.95 \times 10^{-15}.$$

7. (a) $\mu_M = \mu_{X+1.5Y} = \mu_X + 1.5\mu_Y = 0.125 + 1.5(0.350) = 0.650$

(b) $\sigma_M = \sigma_{X+1.5Y} = \sqrt{\sigma^2_X + 1.5^2\sigma^2_Y} = \sqrt{0.05^2 + 1.5^2(0.1^2)} = 0.158$

8. Let $X_1,\ldots,X_{24}$ be the volumes of the 24 bottles. Let $S = X_1 + \cdots + X_{24}$ be the total weight. The average volume per bottle is $\overline{X}$.

(a) $\mu_S = \sum \mu_{X_i} = 24(20.01) = 480.24$

(b) $\sigma_S = \sqrt{\sum \sigma^2_{X_i}} = \sqrt{24(0.02^2)} = 0.098$

(c) $\mu_{\overline{X}} = \mu_{X_i} = 20.01$

(d) $\sigma_{\overline{X}} = \sigma_{X_i}/\sqrt{24} = 0.02/\sqrt{24} = 0.00408$

(e) Let n be the required number of bottles. Then $0.02/\sqrt{n} = 0.0025$, so $n = 64$.

9. Let X_1 and X_2 denote the lengths of the pieces chosen from the population with mean 30 and standard deviation 0.1, and let Y_1 and Y_2 denote the lengths of the pieces chosen from the population with mean 45 and standard deviation 0.3.

(a) $\mu_{X_1+X_2+Y_1+Y_2} = \mu_{X_1} + \mu_{X_2} + \mu_{Y_1} + \mu_{Y_2} = 30 + 30 + 45 + 45 = 150$

(b) $\sigma_{X_1+X_2+Y_1+Y_2} = \sqrt{\sigma^2_{X_1} + \sigma^2_{X_2} + \sigma^2_{Y_1} + \sigma^2_{Y_2}} = \sqrt{0.1^2 + 0.1^2 + 0.3^2 + 0.3^2} = 0.447$

10. The daily revenue is $R = 2.60X_1 + 2.75X_2 + 2.90X_3$.

(a) $\mu_R = 2.60\mu_1 + 2.75\mu_2 + 2.90\mu_3 = 2.60(1500) + 2.75(500) + 2.90(300) = 6145$

(b) $\sigma_R = \sqrt{2.60^2\sigma_1^2 + 2.75^2\sigma_2^2 + 2.90^2\sigma_3^2} = \sqrt{2.60^2(180^2) + 2.75^2(90^2) + 2.90^2(40^2)} = 541.97$

11. (a) The number of passenger-miles is 8000(210) = 1,680,000. Let X be the number of gallons of fuel used. Then $\mu_X = 1{,}680{,}000(0.15) = 252{,}000$ gallons.

(b) $\sigma_X = \sqrt{(1{,}680{,}000)(0.01)} = 12.9615$

(c) $\mu_{X/1{,}680{,}000} = (1/1{,}680{,}000)\mu_X = (1/1{,}680{,}000)(252{,}000) = 0.15$.

(d) $\sigma_{X/1{,}680{,}000} = (1/1{,}680{,}000)\sigma_X = (1/1{,}680{,}000)(12.9615) = 7.7152 \times 10^{-6}$

12. Let X denote the number of mismatches and let Y denote the number of gaps. Then $\mu_X = 5$, $\sigma_X = 2$, $\mu_Y = 2$, and $\sigma_Y = 1$. Let S denote the Needleman-Wunsch score. Then $S = X + 3Y$.

(a) $\mu_S = \mu_{X+3Y} = \mu_X + 3\mu_Y = 5 + 3(2) = 11$

(b) $\sigma_S^2 = \sigma_{X+3Y}^2 = \sigma_X^2 + 9\sigma_Y^2 = 2^2 + 9(1^2) = 13$

13. (a) $\mu = 0.0695 + \dfrac{1.0477}{20} + \dfrac{0.8649}{20} + \dfrac{0.7356}{20} + \dfrac{0.2171}{30} + \dfrac{2.8146}{60} + \dfrac{0.5913}{15} + \dfrac{0.0079}{10} + 5(0.0006) = 0.2993$

(b) $\sigma = \sqrt{0.0018^2 + (\frac{0.0269}{20})^2 + (\frac{0.0225}{20})^2 + (\frac{0.0113}{20})^2 + (\frac{0.0185}{30})^2 + (\frac{0.0284}{60})^2 + (\frac{0.0031}{15})^2 + (\frac{0.0006}{10})^2 + 5^2(0.0002)^2} = 0.00288$

14. (a) $\mu_X = \mu_{O+2N+2C/3} = \mu_O + 2\mu_N + (2/3)\mu_C = 0.1668 + 2(0.0255) + (2/3)(0.0247) = 0.2343$

(b) $\sigma_X = \sigma_{O+2N+2C/3} = \sqrt{\sigma_O^2 + 4\sigma_N^2 + (2/3)^2\sigma_C^2} = \sqrt{0.0340^2 + 4(0.0194)^2 + (2/3)^2(0.0131)^2} = 0.05232$

15. (a) $P(X < 9.98) = \displaystyle\int_{9.95}^{9.98} 10\,dx = 10x\Big|_{9.95}^{9.98} = 0.3$

(b) $P(Y > 5.01) = \displaystyle\int_{5.01}^{5.1} 5\,dy = 5y\Big|_{5.01}^{5.1} = 0.45$

(c) Since X and Y are independent,
$P(X < 9.98 \text{ and } Y > 5.01) = P(X < 9.98)P(Y > 5.01) = (0.3)(0.45) = 0.135$

(d) $\mu_X = \int_{9.95}^{10.05} 10x\,dx = 5x^2 \Big|_{9.95}^{10.05} = 10$

(e) $\mu_Y = \int_{4.9}^{5.1} 5y\,dy = 2.5y^2 \Big|_{4.9}^{5.1} = 5$

16. (a) $\mu_X = \int_4^6 \left(\frac{3}{4}x - \frac{3x(x-5)^2}{4}\right) dx = \left(-9x^2 + \frac{5x^3}{2} - \frac{3x^4}{16}\right) \Bigg|_4^6 = 5$

(b) $\sigma_X^2 = \int_4^6 \left(\frac{3(x-5)^2}{4} - \frac{3(x-5)^4}{4}\right) dx = \left(\frac{(x-5)^3}{4} - \frac{3(x-5)^5}{20}\right) \Bigg|_4^6 = 0.2$

(c) $\mu_Y = \mu_{0.0394X} = 0.0394\mu_X = 0.0394(5) = 0.197$
$\sigma_Y^2 = \sigma_{0.0394X}^2 = (0.0394)^2\sigma_X^2 = (0.0394)^2(0.2) = 0.00031$

(d) Let X_1, X_2, and X_3 be the three thicknesses, in millimeters.
Then $S = X_1 + X_2 + X_3$ is the total thickness.
$\mu_S = \mu_{X_1} + \mu_{X_2} + \mu_{X_3} = 3(5) = 15.$
$\sigma_S^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 = 3(0.2) = 0.6.$

17. (a) Let $\mu = 40.25$ be the mean SiO_2 content, and let $\sigma = 0.36$ be the standard deviation of the SiO_2 content, in a randomly chosen rock. Let $\overline{X}$ be the average content in a random sample of 10 rocks.

Then $\mu_{\overline{X}} = \mu = 40.25$, and $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{10}} = \frac{0.36}{\sqrt{10}} = 0.11.$

(b) Let n be the required number of rocks. Then $\frac{\sigma}{\sqrt{n}} = \frac{0.36}{\sqrt{n}} = 0.05.$

Solving for n yields $n = 51.84$. Since n must be an integer, take $n = 52$.

18. (a) Yes. Let X_i denote the number of bytes downloaded in the ith second. The total number of bytes is $X_1 + X_2 + X_3 + X_4 + X_5$. The mean of the total number is the sum of the five means, each of which is equal to 10^5.

(b) No. $X_1, \ldots, X_5$ are not independent, so the standard deviation of the sum depends on the covariances between the X_i.

Section 2.6

1. (a) 0.17

(b) $P(X \geq 1 \text{ and } Y < 2) = P(1,0) + P(1,1) + P(2,0) + P(2,1) = 0.17 + 0.23 + 0.06 + 0.14 = 0.60$

(c) $P(X < 1) = P(X = 0) = P(0,0) + P(0,1) + P(0,2) = 0.10 + 0.11 + 0.05 = 0.26$

(d) $P(Y \geq 1) = 1 - P(Y = 0) = 1 - P(0,0) - P(1,0) - P(2,0) = 1 - 0.10 - 0.17 - 0.06 = 0.67$

(e) $P(X \geq 1) = 1 - P(X = 0) = 1 - P(0,0) - P(0,1) - P(0,2) = 1 - 0.10 - 0.11 - 0.05 = 0.74$

(f) $P(Y = 0) = P(0,0) + P(1,0) + P(2,0) = 0.10 + 0.17 + 0.06 = 0.33$

(g) $P(X = 0 \text{ and } Y = 0) = 0.10$

2. (a) The marginal probability mass function $p_X(x)$ is found by summing along the rows of the joint probability mass function.

		y		
x	0	1	2	$p_X(x)$
0	0.10	0.11	0.05	0.26
1	0.17	0.23	0.08	0.48
2	0.06	0.14	0.06	0.26
$p_Y(y)$	0.33	0.48	0.19	

$p_X(0) = 0.26$, $p_X(1) = 0.48$, $p_X(2) = 0.26$, $p_X(x) = 0$ if $x \neq 0, 1,$ or 2

(b) The marginal probability mass function $p_Y(y)$ is found by summing down the columns of the joint probability mass function. So $p_Y(0) = 0.33$, $p_Y(1) = 0.48$, $p_Y(2) = 0.19$, $p_Y(y) = 0$ if $y \neq 0, 1,$ or 2

(c) $\mu_X = 0p_X(0) + 1p_X(1) + 2p_X(2) = 0(0.26) + 1(0.48) + 2(0.26) = 1.00$

(d) $\mu_Y = 0p_Y(0) + 1p_Y(1) + 2p_Y(2) = 0(0.33) + 1(0.48) + 2(0.19) = 0.86$

(e) $\sigma_X^2 = 0^2 p_X(0) + 1^2 p_X(1) + 2^2 p_X(2) - \mu_X^2 = 0^2(0.26) + 1^2(0.48) + 2^2(0.26) - 1.00^2 = 0.5200$
$\sigma_X = \sqrt{0.5200} = 0.7211$

(f) $\sigma_Y^2 = 0^2 p_Y(0) + 1^2 p_Y(1) + 2^2 p_Y(2) - \mu_Y^2 = 0^2(0.33) + 1^2(0.48) + 2^2(0.19) - 0.86^2 = 0.5004$
$\sigma_Y = \sqrt{0.5004} = 0.7074$

(g) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X \mu_Y$

$$\begin{aligned}
\mu_{XY} &= (0)(0)p_{X,Y}(0,0) + (0)(1)p_{X,Y}(0,1) + (0)(2)p_{X,Y}(0,2) + (1)(0)p_{X,Y}(1,0) + (1)(1)p_{X,Y}(1,1) \\
&\quad + (1)(2)p_{X,Y}(1,2) + (2)(0)p_{X,Y}(2,0) + (2)(1)p_{X,Y}(2,1) + (2)(2)p_{X,Y}(2,2) \\
&= (0)(0)(0.10) + (0)(1)(0.11) + (0)(2)(0.05) + (1)(0)(0.17) + (1)(1)(0.23) \\
&\quad + (1)(2)(0.08) + (2)(0)(0.06) + (2)(1)(0.14) + (2)(2)(0.06) \\
&= 0.91
\end{aligned}$$

$\mu_X = 1.00, \quad \mu_Y = 0.86$
$\text{Cov}(X,Y) = 0.91 - (1.00)(0.86) = 0.0500$

(h) $\rho_{X,Y} = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \dfrac{0.0500}{(0.7211)(0.7074)} = 0.0980$

(i) No. The joint probability density function is not equal to the product of the marginals. For example, $P(X = 0 \text{ and } Y = 0) = 0.10$, but $P(X = 0)P(Y = 0) = (0.26)(0.33) = 0.0858$.

3. (a) $p_{Y|X}(0\,|\,0) = \dfrac{p_{X,Y}(0,0)}{p_X(0)} = \dfrac{0.10}{0.26} = 0.3846$

$p_{Y|X}(1\,|\,0) = \dfrac{p_{X,Y}(0,1)}{p_X(0)} = \dfrac{0.11}{0.26} = 0.4231$

$$p_{Y|X}(2\,|\,0) = \frac{p_{X,Y}(0,2)}{p_X(0)} = \frac{0.05}{0.26} = 0.1923$$

(b) $$p_{X|Y}(0\,|\,1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{0.11}{0.48} = 0.2292$$

$$p_{X|Y}(1\,|\,1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{0.23}{0.48} = 0.4792$$

$$p_{X|Y}(2\,|\,1) = \frac{p_{X,Y}(2,1)}{p_Y(1)} = \frac{0.14}{0.48} = 0.2917$$

(c) $E(Y\,|\,X=0) = 0p_{Y|X}(0\,|\,0) + 1p_{Y|X}(1\,|\,0) + 2p_{Y|X}(2\,|\,0) = 0.8077$

(d) $E(X\,|\,Y=1) = 0p_{X|Y}(0\,|\,1) + 1p_{X|Y}(1\,|\,1) + 2p_{X|Y}(2\,|\,1) = 1.0625$

4. (a) The marginal probability mass function $p_X(x)$ is found by summing along the rows of the joint probability mass function.

	y				
x	0	1	2	3	$p_X(x)$
0	0.15	0.12	0.11	0.10	0.48
1	0.09	0.07	0.05	0.04	0.25
2	0.06	0.05	0.04	0.02	0.17
3	0.04	0.03	0.02	0.01	0.10
$p_Y(y)$	0.34	0.27	0.22	0.17	

$p_X(0) = 0.48$, $p_X(1) = 0.25$, $p_X(2) = 0.17$, $p_X(3) = 0.10$, $p_X(x) = 0$ if $x \neq 0, 1, 2,$ or 3

(b) The marginal probability mass function $p_Y(y)$ is found by summing down the columns of the joint probability mass function. So $p_Y(0) = 0.34$, $p_Y(1) = 0.27$, $p_Y(2) = 0.22$, $p_Y(3) = 0.17$, $p_Y(y) = 0$ if $y \neq 0, 1, 2,$ or 3

(c) No, X and Y are not independent. For example, $P(X = 0 \text{ and } Y = 0) = 0.15$, but $P(X = 0)P(Y = 0) = (0.48)(0.34) = 0.1632 \neq 0.15$.

(d) $\mu_X = 0p_X(0) + 1p_X(1) + 2p_X(2) + 3p_X(3) = 0(0.48) + 1(0.25) + 2(0.17) + 3(0.10) = 0.89$

$\mu_Y = 0p_Y(0) + 1p_Y(1) + 2p_Y(2) + 3p_Y(3) = 0(0.34) + 1(0.27) + 2(0.22) + 3(0.17) = 1.22$

(e) $\sigma_X^2 = 0^2 p_X(0) + 1^2 p_X(1) + 2^2 p_X(2) + 3^2 p_X(3) - \mu_X^2 = 0^2(0.48) + 1^2(0.25) + 2^2(0.17) + 3^2(0.10) - 0.89^2 = 1.0379$

$\sigma_X = \sqrt{1.0379} = 1.0188$

$\sigma_Y^2 = 0^2 p_Y(0) + 1^2 p_Y(1) + 2^2 p_Y(2) + 3^2 p_Y(3) - \mu_Y^2 = 0^2(0.34) + 1^2(0.27) + 2^2(0.22) + 3^2(0.17) - 1.22^2 = 1.1916$

$\sigma_Y = \sqrt{1.1916} = 1.0916$

(f) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X \mu_Y$

$$\begin{aligned}
\mu_{XY} &= (0)(0)p_{X,Y}(0,0) + (0)(1)p_{X,Y}(0,1) + (0)(2)p_{X,Y}(0,2) + (0)(3)p_{X,Y}(0,3) + (1)(0)p_{X,Y}(1,0) \\
&\quad + (1)(1)p_{X,Y}(1,1) + (1)(2)p_{X,Y}(1,2) + (1)(3)p_{X,Y}(1,2) + (2)(0)p_{X,Y}(2,0) + (2)(1)p_{X,Y}(2,1) \\
&\quad + (2)(2)p_{X,Y}(2,2) + (2)(3)p_{X,Y}(2,2) + (3)(0)p_{X,Y}(3,0) + (3)(1)p_{X,Y}(3,1) + (3)(2)p_{X,Y}(3,3) \\
&\quad + (3)(3)p_{X,Y}(3,3) \\
&= (0)(0)(0.15) + (0)(1)(0.12) + (0)(2)(0.11) + (0)(3)(0.10) \\
&+ (1)(0)(0.09) + (1)(1)(0.07) + (1)(2)(0.05) + (1)(3)(0.04) \\
&+ (2)(0)(0.06) + (2)(1)(0.05) + (2)(2)(0.04) + (2)(3)(0.02) \\
&+ (3)(0)(0.04) + (3)(1)(0.03) + (3)(2)(0.02) + (4)(3)(0.01) \\
&= 0.97
\end{aligned}$$

$\mu_X = 0.89$, $\mu_Y = 1.22$

$\text{Cov}(X,Y) = 0.97 - (0.89)(1.22) = -0.1158$

(g) $\rho_{X,Y} = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \dfrac{-0.1158}{(1.0188)(1.0196)} = -0.1041$

5. (a) $\mu_{X+Y} = \mu_X + \mu_Y = 0.89 + 1.22 = 2.11$

(b) $\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\text{Cov}(X,Y)} = \sqrt{1.0379 + 1.1916 + 2(-0.1158)} = 1.4135$

(c) $P(X+Y=3) = P(0,3) + P(1,2) + P(2,1) + P(3,0) = 0.10 + 0.05 + 0.05 + 0.04 = 0.24$

6. (a) $p_{Y|X}(0\,|\,1) = \dfrac{p_{X,Y}(1,0)}{p_X(1)} = \dfrac{0.09}{0.25} = 0.36$

$p_{Y|X}(1\,|\,1) = \dfrac{p_{X,Y}(1,1)}{p_X(1)} = \dfrac{0.07}{0.25} = 0.28$

$p_{Y|X}(2\,|\,1) = \dfrac{p_{X,Y}(1,2)}{p_X(1)} = \dfrac{0.05}{0.25} = 0.20$

$p_{Y|X}(3\,|\,1) = \dfrac{p_{X,Y}(1,3)}{p_X(1)} = \dfrac{0.04}{0.25} = 0.16$

(b) $p_{X|Y}(0\,|\,2) = \dfrac{p_{X,Y}(0,2)}{p_Y(2)} = \dfrac{0.11}{0.22} = 1/2$

$p_{X|Y}(1\,|\,2) = \dfrac{p_{X,Y}(1,2)}{p_Y(2)} = \dfrac{0.05}{0.22} = 5/22$

$p_{X|Y}(2\,|\,2) = \dfrac{p_{X,Y}(2,2)}{p_Y(2)} = \dfrac{0.04}{0.22} = 2/11$

$p_{X|Y}(3\,|\,2) = \dfrac{p_{X,Y}(3,2)}{p_Y(2)} = \dfrac{0.02}{0.22} = 1/11$

(c) $E(Y\,|\,X=1) = 0p_{Y|X}(0\,|\,1) + 1p_{Y|X}(1\,|\,1) + 2p_{Y|X}(2\,|\,1) + 3p_{Y|X}(3\,|\,1)$
$= 0(0.36) + 1(0.28) + 2(0.20) + 3(0.16) = 1.16$

(d) $E(X\,|\,Y=2) = 0p_{X|Y}(0\,|\,2) + 1p_{X|Y}(1\,|\,2) + 2p_{X|Y}(2\,|\,2) + 3p_{X|Y}(3\,|\,2)$
$= 0(1/2) + 1(5/22) + 2(2/11) + 3(1/11) = 19/22$

7. (a) $2X + 3Y$

(b) $\mu_{2X+3Y} = 2\mu_X + 3\mu_Y = 2(0.89) + 3(1.22) = 5.44$

(c)
$$\begin{aligned}
\sigma_{2X+3Y} &= \sqrt{2^2\sigma_X^2 + 3^2\sigma_Y^2 + 2(2)(3)\mathrm{Cov}(X,Y)} \\
&= \sqrt{2^2(1.0379) + 3^2(1.1916) - 2(2)(3)(-0.1158)} \\
&= 3.6724
\end{aligned}$$

8. (a) $P(X = 2 \text{ and } Y = 2) = P(X = 2)P(Y = 2 \mid X = 2)$. Now $P(X = 2) = 0.30$. Given that $X = 2$, $Y = 2$ if and only if each of the two customers purchases one item. Therefore $P(Y = 2 \mid X = 2) = 0.05^2 = 0.0025$, so $P(X = 2 \text{ and } Y = 2) = (0.30)(0.0025) = 0.00075$.

   (b) $P(X = 2 \text{ and } Y = 6) = P(X = 2)P(Y = 6 \mid X = 2)$. Now $P(X = 2) = 0.30$. Given that $X = 2$, Let Y_1 be the number of items purchased by the first customer, and let Y_2 be the number of items purchased by the second customer. Then the event $Y = 6$ is equivalent to the event $\{Y_1 = 5 \text{ and } Y_2 = 1\}$ or $\{Y_1 = 4 \text{ and } Y_2 = 2\}$ or $\{Y_1 = 3 \text{ and } Y_2 = 3\}$ or $\{Y_1 = 2 \text{ and } Y_2 = 4\}$ or $\{Y_1 = 1 \text{ and } Y_2 = 5\}$.

   Therefore

$$\begin{aligned} P(Y = 6 \mid X = 2) &= P(Y_1 = 5 \text{ and } Y_2 = 1) + P(Y_1 = 4 \text{ and } Y_2 = 2) + P(Y_1 = 3 \text{ and } Y_2 = 3) \\ &\quad + P(Y_1 = 2 \text{ and } Y_2 = 4) + P(Y_1 = 1 \text{ and } Y_2 = 5) \\ &= (0.15)(0.05) + (0.30)(0.15) + (0.25)(0.25) + (0.15)(0.30) + (0.05)(0.15) \\ &= 0.1675 \end{aligned}$$

   $P(X = 2 \text{ and } Y = 6) = (0.30)(0.1675) = 0.05025$

   (c) $P(Y = 2) = P(X = 1 \text{ and } Y = 2) + P(X = 2 \text{ and } Y = 2)$.

   $P(X = 1 \text{ and } Y = 2) = P(X = 1)P(Y = 2 \mid X = 1) = (0.25)(0.15) = 0.0375$.

   From part (a), $P(X = 2 \text{ and } Y = 2) = 0.00075$.

   $P(Y = 2) = 0.0375 + 0.00075 = 0.03825$.

9. (a) The marginal probability mass function $p_X(x)$ is found by summing along the rows of the joint probability mass function.

			y			
x	0	1	2	3	4	$p_X(x)$
0	0.06	0.03	0.01	0.00	0.00	0.10
1	0.06	0.08	0.04	0.02	0.00	0.20
2	0.04	0.05	0.12	0.06	0.03	0.30
3	0.00	0.03	0.07	0.09	0.06	0.25
4	0.00	0.00	0.02	0.06	0.07	0.15
$p_Y(y)$	0.16	0.19	0.26	0.23	0.16	

   $p_X(0) = 0.10$, $p_X(1) = 0.20$, $p_X(2) = 0.30$, $p_X(3) = 0.25$, $p_X(4) = 0.15$, $p_X(x) = 0$ if $x \neq 0, 1, 2, 3,$ or 4.

   (b) The marginal probability mass function $p_Y(y)$ is found by summing down the columns of the joint probability mass function. So $p_Y(0) = 0.16$, $p_Y(1) = 0.19$, $p_Y(2) = 0.26$, $p_Y(3) = 0.23$, $p_Y(4) = 0.16$, $p_Y(y) = 0$ if $y \neq 0, 1, 2, 3,$ or 4.

(c) No. The joint probability mass function is not equal to the product of the marginals. For example, $p_{X,Y}(0,0) = 0.06 \neq p_X(0)p_Y(0)$.

(d) $\mu_X = 0p_X(0)+1p_X(1)+2p_X(2)+3p_X(3)+4p_X(4) = 0(0.10)+1(0.20)+2(0.30)+3(0.25)+4(0.15) = 2.15$
$\mu_Y = 0p_Y(0)+1p_Y(1)+2p_Y(2)+3p_Y(3)+4p_Y(4) = 0(0.16)+1(0.19)+2(0.26)+3(0.23)+4(0.16) = 2.04$

(e)
$$\begin{aligned}\sigma_X^2 &= 0^2p_X(0)+1^2p_X(1)+2^2p_X(2)+3^2p_X(3)+4^2p_X(4)-\mu_X^2\\ &= 0^2(0.10)+1^2(0.20)+2^2(0.30)+3^2(0.25)+4^2(0.15)-2.15^2\\ &= 1.4275\end{aligned}$$

$\sigma_X = \sqrt{1.4275} = 1.1948$

$$\begin{aligned}\sigma_Y^2 &= 0^2p_Y(0)+1^2p_Y(1)+2^2p_Y(2)+3^2p_Y(3)+4^2p_Y(4)-\mu_Y^2\\ &= 0^2(0.16)+1^2(0.19)+2^2(0.26)+3^2(0.23)+4^2(0.16)-2.04^2\\ &= 1.6984\end{aligned}$$

$\sigma_Y = \sqrt{1.6984} = 1.3032$

(f) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y$.

$$\begin{aligned}\mu_{XY} = \; & (0)(0)p_{X,Y}(0,0)+(0)(1)p_{X,Y}(0,1)+(0)(2)p_{X,Y}(0,2)+(0)(3)p_{X,Y}(0,3)+(0)(4)p_{X,Y}(0,4)\\ & +(1)(0)p_{X,Y}(1,0)+(1)(1)p_{X,Y}(1,1)+(1)(2)p_{X,Y}(1,2)+(1)(3)p_{X,Y}(1,3)+(1)(4)p_{X,Y}(1,4)\\ & +(2)(0)p_{X,Y}(2,0)+(2)(1)p_{X,Y}(2,1)+(2)(2)p_{X,Y}(2,2)+(2)(3)p_{X,Y}(2,3)+(2)(4)p_{X,Y}(2,4)\\ & +(3)(0)p_{X,Y}(3,0)+(3)(1)p_{X,Y}(3,1)+(3)(2)p_{X,Y}(3,2)+(3)(3)p_{X,Y}(3,3)+(3)(4)p_{X,Y}(3,4)\\ & +(4)(0)p_{X,Y}(4,0)+(4)(1)p_{X,Y}(4,1)+(4)(2)p_{X,Y}(4,2)+(4)(3)p_{X,Y}(4,3)+(4)(4)p_{X,Y}(4,4)\\ = \; & (0)(0)(0.06)+(0)(1)(0.03)+(0)(2)(0.01)+(0)(3)(0.00)+(0)(4)(0.00)\\ & +(1)(0)(0.06)+(1)(1)(0.08)+(1)(2)(0.04)+(1)(3)(0.02)+(1)(4)(0.00)\\ & +(2)(0)(0.04)+(2)(1)(0.05)+(2)(2)(0.12)+(2)(3)(0.06)+(2)(4)(0.03)\\ & +(3)(0)(0.00)+(3)(1)(0.03)+(3)(2)(0.07)+(3)(3)(0.09)+(3)(4)(0.06)\\ & +(4)(0)(0.00)+(4)(1)(0.00)+(4)(2)(0.02)+(4)(3)(0.06)+(4)(4)(0.07)\\ = \; & 5.44\end{aligned}$$

$\mu_X = 2.15$, $\mu_Y = 2.04$
$\text{Cov}(X,Y) = 5.44 - (2.15)(2.04) = 1.0540$

(g) $\rho_{X,Y} = \dfrac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \dfrac{1.0540}{(1.1948)(1.3032)} = 0.6769$

10. (a) $\mu_{X+Y} = \mu_X + \mu_Y = 2.15 + 2.04 = 4.19$

(b) $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\mathrm{Cov}(X,Y) = 1.4275 + 1.6984 + 2(1.0540) = 5.23$

(c) $P(X+Y=5) = P(X=1 \text{ and } Y=4) + P(X=2 \text{ and } Y=3) + P(X=3 \text{ and } Y=2) + P(X=4 \text{ and } Y=1) = 0.13$

11. (a) $p_{Y|X}(0\,|\,4) = \dfrac{p_{X,Y}(4,0)}{p_X(4)} = \dfrac{0.00}{0.15} = 0$

$$p_{Y|X}(1\,|\,4) = \frac{p_{X,Y}(4,1)}{p_X(4)} = \frac{0.00}{0.15} = 0$$

$$p_{Y|X}(2\,|\,4) = \frac{p_{X,Y}(4,2)}{p_X(4)} = \frac{0.02}{0.15} = 2/15$$

$$p_{Y|X}(3\,|\,4) = \frac{p_{X,Y}(4,3)}{p_X(4)} = \frac{0.06}{0.15} = 2/5$$

$$p_{Y|X}(4\,|\,4) = \frac{p_{X,Y}(4,4)}{p_X(4)} = \frac{0.07}{0.15} = 7/15$$

(b) $p_{X|Y}(0\,|\,3) = \dfrac{p_{X,Y}(0,3)}{p_Y(3)} = \dfrac{0.00}{0.23} = 0$

$$p_{X|Y}(1\,|\,3) = \frac{p_{X,Y}(1,3)}{p_Y(3)} = \frac{0.02}{0.23} = 2/23$$

$$p_{X|Y}(2\,|\,3) = \frac{p_{X,Y}(2,3)}{p_Y(3)} = \frac{0.06}{0.23} = 6/23$$

$$p_{X|Y}(3\,|\,3) = \frac{p_{X,Y}(3,3)}{p_Y(3)} = \frac{0.09}{0.23} = 9/23$$

$$p_{X|Y}(4\,|\,3) = \frac{p_{X,Y}(4,3)}{p_Y(3)} = \frac{0.06}{0.23} = 6/23$$

(c) $E(Y\,|\,X=4) = 0p_{Y|X}(0\,|\,4) + 1p_{Y|X}(1\,|\,4) + 2p_{Y|X}(2\,|\,4) + 3p_{Y|X}(3\,|\,4) + 4p_{Y|X}(4\,|\,4) = 3.33$

(d) $E(X\,|\,Y=3) = 0p_{X|Y}(0\,|\,3) + 1p_{X|Y}(1\,|\,3) + 2p_{X|Y}(2\,|\,3) + 3p_{X|Y}(3\,|\,3) + 4p_{X|Y}(4\,|\,3) = 2.83$

12. (a) The marginal probability mass function $p_X(x)$ is found by summing along the rows of the joint probability mass function.

	y				
x	0	1	2	3	$p_X(x)$
0	0.13	0.10	0.07	0.03	0.33
1	0.12	0.16	0.08	0.04	0.40
2	0.02	0.06	0.08	0.04	0.20
3	0.01	0.02	0.02	0.02	0.07
$p_Y(y)$	0.28	0.34	0.25	0.13	

$p_X(0) = 0.33$, $p_X(1) = 0.40$, $p_X(2) = 0.20$, $p_X(3) = 0.07$, $p_X(x) = 0$ if $x \neq 0, 1, 2,$ or 3

(b) The marginal probability mass function $p_Y(y)$ is found by summing down the columns of the joint probability mass function. So $p_Y(0) = 0.28$, $p_Y(1) = 0.34$, $p_Y(2) = 0.25$, $p_Y(3) = 0.13$, $p_Y(y) = 0$ if $y \neq 0, 1, 2,$ or 3.

(c) $\mu_X = 0p_X(0) + 1p_X(1) + 2p_X(2) + 3p_X(3) = 0(0.33) + 1(0.40) + 2(0.20) + 3(0.07) = 1.01$

(d) $\mu_Y = 0p_Y(0) + 1p_Y(1) + 2p_Y(2) = 0(0.28) + 1(0.34) + 2(0.25) + 3(0.13) = 1.23$

(e)
$$\begin{aligned} \sigma_X^2 &= 0^2 p_X(0) + 1^2 p_X(1) + 2^2 p_X(2) + 3^2 p_X(3) - \mu_X^2 \\ &= 0^2(0.33) + 1^2(0.40) + 2^2(0.20) + 3^2(0.07) - 1.01^2 \\ &= 0.8099 \end{aligned}$$

$\sigma_X = \sqrt{0.8099} = 0.8999$

(f) $\sigma_Y^2 = 0^2 p_Y(0) + 1^2 p_Y(1) + 2^2 p_Y(2) - \mu_Y^2 = 0^2(0.28) + 1^2(0.34) + 2^2(0.25) + 3^2(0.13) - 1.23^2 = 0.9971$.
$\sigma_Y = \sqrt{0.9971} = 0.9985$

(g) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X \mu_Y$

$$\begin{aligned} \mu_{XY} &= (0)(0)p_{X,Y}(0,0) + (0)(1)p_{X,Y}(0,1) + (0)(2)p_{X,Y}(0,2) + (0)(3)p_{X,Y}(0,3) \\ &\quad + (1)(0)p_{X,Y}(1,0) + (1)(1)p_{X,Y}(1,1) + (1)(2)p_{X,Y}(1,2) + (1)(3)p_{X,Y}(1,3) \\ &\quad + (2)(0)p_{X,Y}(2,0) + (2)(1)p_{X,Y}(2,1) + (2)(2)p_{X,Y}(2,2) + (2)(3)p_{X,Y}(2,3) \\ &\quad + (3)(0)p_{X,Y}(3,0) + (3)(1)p_{X,Y}(3,1) + (3)(2)p_{X,Y}(3,2) + (3)(3)p_{X,Y}(3,3) \\ &= (0)(0)(0.13) + (0)(1)(0.10) + (0)(2)(0.07) + (0)(3)(0.03) \\ &\quad + (1)(0)(0.12) + (1)(1)(0.16) + (1)(2)(0.08) + (1)(3)(0.04) \\ &\quad + (2)(0)(0.02) + (2)(1)(0.06) + (2)(2)(0.08) + (2)(3)(0.04) \\ &\quad + (3)(0)(0.01) + (3)(1)(0.02) + (3)(2)(0.02) + (3)(3)(0.02) \\ &= 1.48 \end{aligned}$$

$\mu_X = 1.01$, $\mu_Y = 1.23$
$\text{Cov}(X,Y) = 1.48 - (1.01)(1.23) = 0.2377$

(h) $\rho_{X,Y} = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \dfrac{0.2377}{(0.8999)(0.9985)} = 0.2645$

13. (a) $\mu_Z = \mu_{X+Y} = \mu_X + \mu_Y = 1.01 + 1.23 = 2.24$

(b) $\sigma_Z = \sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\text{Cov}(X,Y)} = \sqrt{0.8099 + 0.9971 + 2(0.2377)} = 1.511$

(c)
$$\begin{aligned}
P(Z=2) &= P(X+Y=2) \\
&= P(X=0 \text{ and } Y=2) + P(X=1 \text{ and } Y=1) + P(X=2 \text{ and } Y=0) \\
&= 0.07 + 0.16 + 0.02 \\
&= 0.25
\end{aligned}$$

14. (a) $T = 50X + 100Y$, so $\mu_T = \mu_{50X+100Y} = 50\mu_X + 100\mu_Y = 50(1.01) + 100(1.23) = 173.50$.

(b)
$$\begin{aligned}
\sigma_T &= \sigma_{50X+100Y} \\
&= \sqrt{50^2\sigma_X^2 + 100^2\sigma_Y^2 + 2(50)(100)\text{Cov}(X,Y)} \\
&= \sqrt{50^2(0.8099) + 100^2(0.9971) + 2(50)(100)(0.2377)} \\
&= 119.9
\end{aligned}$$

(c) $P(T=250) = P(X=1 \text{ and } Y=2) + P(X=3 \text{ and } Y=1) = 0.08 + 0.02 = 0.10$

15. (a) $p_{Y|X}(0\,|\,3) = \dfrac{p_{X,Y}(3,0)}{p_X(3)} = \dfrac{0.01}{0.07} = 0.1429$

$p_{Y|X}(1\,|\,3) = \dfrac{p_{X,Y}(3,1)}{p_X(3)} = \dfrac{0.02}{0.07} = 0.2858$

$$p_{Y|X}(2\,|\,3) = \frac{p_{X,Y}(3,2)}{p_X(3)} = \frac{0.02}{0.07} = 0.2858$$

$$p_{Y|X}(3\,|\,3) = \frac{p_{X,Y}(3,3)}{p_X(3)} = \frac{0.02}{0.07} = 0.2858$$

(b) $$p_{X|Y}(0\,|\,1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{0.10}{0.34} = 0.2941$$

$$p_{X|Y}(1\,|\,1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{0.16}{0.34} = 0.4706$$

$$p_{X|Y}(2\,|\,1) = \frac{p_{X,Y}(2,1)}{p_Y(1)} = \frac{0.06}{0.34} = 0.1765$$

$$p_{X|Y}(3\,|\,1) = \frac{p_{X,Y}(3,1)}{p_Y(1)} = \frac{0.02}{0.34} = 0.0588$$

(c) $E(Y\,|\,X=3) = 0p_{Y|X}(0\,|\,3) + 1p_{Y|X}(1\,|\,3) + 2p_{Y|X}(2\,|\,3) + 3p_{Y|X}(3\,|\,3) = 1.71.$

(d) $E(X\,|\,Y=1) = 0p_{X|Y}(0\,|\,1) + 1p_{X|Y}(1\,|\,1) + 2p_{X|Y}(2\,|\,1) + 3p_{X|Y}(3\,|\,1) = 1$

16. (a)
$$\begin{aligned}
P(X > 1 \text{ and } Y > 1) &= \int_1^\infty \int_1^\infty xe^{-(x+xy)}\,dy\,dx \\
&= \int_1^\infty \left(-e^{-(x+xy)} \Big|_1^\infty \right) dx \\
&= \int_1^\infty e^{-2x}\,dx \\
&= -0.5e^{-2x} \Big|_1^\infty \\
&= 0.0676676
\end{aligned}$$

(b) $f_X(x) = \int_0^\infty f(x,y)\,dy.$

If $x \le 0$ then $f(x,y) = 0$ for all y so $f_X(x) = 0$.

If $x > 0$ then $f_X(x) = \int_0^\infty xe^{-(x+xy)}\,dy = -e^{-(x+xy)} \Big|_0^\infty = e^{-x}.$

$f_Y(y) = \int_0^\infty f(x,y)\,dx.$

If $y \le 0$ then $f(x,y) = 0$ for all x so $f_Y(y) = 0$.

If $y > 0$ then $f_Y(y) = \int_0^{\infty} xe^{-(x+xy)}\,dx = \dfrac{1}{(1+y^2)}$.

(c) NO, $f(x,y) \ne f_X(x)f_Y(y)$.

17. (a) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y$

$$
\begin{aligned}
\mu_{XY} &= \int_1^2 \int_4^5 \frac{1}{6}xy(x+y)\,dy\,dx \\
&= \int_1^2 \frac{1}{6}\left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)\Bigg|_4^5 dx \\
&= \int_1^2 \frac{1}{6}\left(\frac{9x^2}{2} + \frac{61x}{3}\right) dx \\
&= \frac{1}{6}\left(\frac{3x^3}{2} + \frac{61x^2}{6}\right)\Bigg|_1^2 \\
&= \frac{41}{6}
\end{aligned}
$$

$f_X(x) = \dfrac{1}{6}\left(x + \dfrac{9}{2}\right)$ for $1 \le x \le 2$ (see Example 2.55).

$\mu_X = \displaystyle\int_1^2 \frac{1}{6}x\left(x+\frac{9}{2}\right)dx = \frac{1}{6}\left(\frac{x^3}{3} + \frac{9x^2}{4}\right)\Bigg|_1^2 = \frac{109}{72}$.

$f_Y(y) = \dfrac{1}{6}\left(y + \dfrac{3}{2}\right)$ for $4 \le y \le 5$ (see Example 2.55).

$\mu_Y = \displaystyle\int_4^5 \frac{1}{6}y\left(y+\frac{3}{2}\right)dy = \frac{1}{6}\left(\frac{y^3}{3} + \frac{3y^2}{4}\right)\Bigg|_4^5 = \frac{325}{72}$.

$\text{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y = \dfrac{41}{6} - \left(\dfrac{109}{72}\right)\left(\dfrac{325}{72}\right) = -0.000193$.

(b) $\rho_{X,Y} = \dfrac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}$.

$\sigma_X^2 = \displaystyle\int_1^2 \frac{1}{6}x^2\left(x+\frac{9}{2}\right)dx - \mu_X^2 = \frac{1}{6}\left(\frac{x^4}{4} + \frac{3x^3}{2}\right)\Bigg|_1^2 - \left(\frac{109}{72}\right)^2 = 0.08314$.

$$\sigma_Y^2 = \int_4^5 \frac{1}{6}y^2\left(y+\frac{3}{2}\right)dx - \mu_Y^2 = \frac{1}{6}\left(\frac{y^4}{4}+\frac{y^3}{2}\right)\Bigg|_4^5 - \left(\frac{325}{72}\right)^2 = 0.08314.$$

$$\rho_{X,Y} = \frac{-0.000193}{\sqrt{(0.08314)(0.08314)}} = -0.00232.$$

18. (a)

$$\begin{aligned} P(X>0.5 \text{ and } Y>0.5) &= \int_{0.5}^1\int_{0.5}^1 \frac{3(x^2+y^2)}{2}\,dy\,dx \\ &= \int_{0.5}^1 \left(\frac{3x^2y}{2}+\frac{y^3}{2}\right)\Bigg|_{0.5}^1 dx \\ &= \int_{0.5}^1 \left(\frac{3x^2}{4}+\frac{21}{48}\right)dx \\ &= \frac{x^3}{4}+\frac{21x}{48}\Bigg|_{0.5}^1 \\ &= \frac{21}{48} \end{aligned}$$

(b) For $0<x<1$, $f_X(x) = \int_0^1 \frac{3(x^2+y^2)}{2}\,dy = 0.5+1.5x^2$. For $x \le 0$ and $x \ge 1$, $f_X(x) = 0$.

For $0<y<1$, $f_Y(y) = \int_0^1 \frac{3(x^2+y^2)}{2}\,dx = 0.5+1.5y^2$. For $y \le 0$ and $y \ge 1$, $f_Y(y) = 0$.

(c) No. $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$.

19. (a) $\mathrm{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y$.

$$\begin{aligned} \mu_{XY} &= \int_0^1\int_0^1 xy\frac{3(x^2+y^2)}{2}\,dx\,dy \\ &= \int_0^1 \left(\frac{3x^4y}{8}+\frac{3x^2y^3}{4}\right)\Bigg|_0^1 dy \\ &= \int_0^1 \left(\frac{3y}{8}+\frac{3y^3}{4}\right)dy \end{aligned}$$

$$= \left(\frac{3y^2}{16} + \frac{3y^4}{16}\right)\Bigg|_0^1 dy$$

$$= \frac{3}{8}$$

$$\mu_X = \int_0^1 x\frac{1+3x^2}{2}\,dx = \left(\frac{x^2}{4} + \frac{3x^4}{8}\right)\Bigg|_0^1 = \frac{5}{8}.$$

$$\mu_Y = \int_0^1 y\frac{1+3y^2}{2}\,dy = \left(\frac{y^2}{4} + \frac{3y^4}{8}\right)\Bigg|_0^1 = \frac{5}{8}.$$

$$\mathrm{Cov}(X,Y) = \frac{3}{8} - \left(\frac{5}{8}\right)^2 = -\frac{1}{64}.$$

(b) $\rho_{X,Y} = \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}$.

$$\sigma_X^2 = \int_0^1 x^2\frac{1+3x^2}{2}\,dx - \mu_X^2 = \left(\frac{x^3}{6} + \frac{3x^5}{10}\right)\Bigg|_0^1 - \left(\frac{5}{8}\right)^2 = \frac{73}{960}.$$

$$\sigma_Y^2 = \int_0^1 y^2\frac{1+3y^2}{2}\,dy - \mu_Y^2 = \left(\frac{y^3}{6} + \frac{3y^5}{10}\right)\Bigg|_0^1 - \left(\frac{5}{8}\right)^2 = \frac{73}{960}.$$

$$\rho_{X,Y} = \frac{-1/64}{\sqrt{73/960}\sqrt{73/960}} = -\frac{15}{73}.$$

(c) $f_{Y\,|\,X}(y\,|\,0.5) = \dfrac{f_{X,Y}(0.5,y)}{f_X(0.5)}$.

For $0 < y < 1$, $f_{X,Y}(0.5,y) = \dfrac{3+12y^2}{8}$, $f_X(0.5) = \dfrac{7}{8}$.

So for $0 < y < 1$, $f_{Y\,|\,X}(y\,|\,0.5) = \dfrac{3+12y^2}{7}$

(d) $$E(Y\,|\,X=0.5) = \int_0^1 y f_{Y\,|\,X}(y\,|\,0.5)\,dy = \int_0^1 y\frac{3+12y^2}{7}\,dy = \frac{3y^2+6y^4}{14}\Bigg|_0^1 = \frac{9}{14}$$

20. (a) $$P(X > 1 \text{ and } Y > 1) = \int_1^\infty\int_1^\infty 4xye^{-(2x+y)}\,dy\,dx$$

$$
\begin{aligned}
&= -\int_1^{\infty}\left(4x(1+y)e^{-(2x+y)}\Big|_1^{\infty}\right)dx \\
&= \int_1^{\infty} 8xe^{-(2x+1)}\,dx = -2(1+2x)e^{-(2x+1)}\Big|_1^{\infty} \\
&= 6e^{-3}
\end{aligned}
$$

(b) For $x \le 0$, $f_X(x) = 0$.

For $x > 0$, $f_X(x) = \int_0^{\infty} 4xye^{-(2x+y)}\,dy = -4x(1+y)e^{-(2x+y)}\Big|_0^{\infty} = 4xe^{-2x}$.

Therefore $f_X(x) = \begin{cases} 4xe^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$

For $y \le 0$, $f_Y(y) = 0$.

For $y > 0$, $f_Y(y) = \int_0^{\infty} 4xye^{-(2x+y)}\,dx = -y(1+2x)e^{-(2x+y)}\Big|_0^{\infty} = ye^{-y}$.

Therefore $f_Y(y) = \begin{cases} ye^{-y} & y > 0 \\ 0 & y \le 0 \end{cases}$

(c) Yes, $f(x,y) = f_X(x)f_Y(y)$.

21. (a) The probability mass function of Y is the same as that of X, so $f_Y(y) = e^{-y}$ if $y > 0$ and $f_Y(y) = 0$ if $y \le 0$. Since X and Y are independent, $f(x,y) = f_X(x)f_Y(y)$.

Therefore $f(x,y) = \begin{cases} e^{-x-y} & x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$

(b)
$$
\begin{aligned}
P(X \le 1 \text{ and } Y > 1) &= P(X \le 1)P(Y > 1) \\
&= \left(\int_0^1 e^{-x}\,dx\right)\left(\int_1^{\infty} e^{-y}\,dy\right) \\
&= \left(-e^{-x}\Big|_0^1\right)\left(-e^{-y}\Big|_1^{\infty}\right) \\
&= (1-e^{-1})(e^{-1}) \\
&= e^{-1} - e^{-2} \\
&= 0.2325
\end{aligned}
$$

(c) $\mu_X = \int_0^\infty xe^{-x}\,dx = -xe^{-x}\Big|_0^\infty - \int_0^\infty e^{-x}\,dx = 0 - (-e^{-x})\Big|_0^\infty = 0 + 1 = 1$

(d) Since X and Y have the same probability mass function, $\mu_Y = \mu_X = 1$.
Therefore $\mu_{X+Y} = \mu_X + \mu_Y = 1 + 1 = 2$.

(e)
$$\begin{aligned}
P(X+Y \le 2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{2-x} f(x,y)\,dy\,dx \\
&= \int_0^2\int_0^{2-x} e^{-x-y}\,dy\,dx \\
&= \int_0^2 e^{-x}\left(-e^{-y}\Big|_0^{2-x}\right)dx \\
&= \int_0^2 e^{-x}(1-e^{x-2})\,dx \\
&= \int_0^2 (e^{-x}-e^{-2})\,dx \\
&= (-e^{-x}-xe^{-2})\Big|_0^2 \\
&= 1-3e^{-2} \\
&= 0.5940
\end{aligned}$$

22. (a) $P(X = 1) = 1/3$, $P(Y = 1) = 2/3$, $P(X = 1 \text{ and } Y = 1) = 1/3 \ne P(X = 1)P(Y = 1)$.

(b) $\mu_{XY} = (-1)(1/3) + (0)(1/3) + (1)(1/3) = 0$.
Now $\mu_X = (-1)(1/3) + (0)(1/3) + (1)(1/3) = 0$, and $\mu_Y = (0)(1/3) + (1)(2/3) = 2/3$.
So $\mathrm{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y = 0 - (0)(2/3) = 0$.
Since $\mathrm{Cov}(X,Y) = 0$, $\rho_{X,Y} = 0$.

23. (a) $R = 0.3X + 0.7Y$

(b) $\mu_R = \mu_{0.3X+0.7Y} = 0.3\mu_X + 0.7\mu_Y = (0.3)(6) + (0.7)(6) = 6$.

The risk is $\sigma_R = \sigma_{0.3X+0.7Y} = \sqrt{0.3^2\sigma_X^2 + 0.7^2\sigma_Y^2 + 2(0.3)(0.7)\text{Cov}(X,Y)}$.

$\text{Cov}(X,Y) = \rho_{X,Y}\sigma_X\sigma_Y = (0.3)(3)(3) = 2.7$.

Therefore $\sigma_R = \sqrt{0.3^2(3^2) + 0.7^2(3^2) + 2(0.3)(0.7)(2.7)} = 2.52$.

(c) $\mu_R = \mu_{(0.01K)X+(1-0.01K)Y} = (0.01K)\mu_X + (1-0.01K)\mu_Y = (0.01K)(6) + (1-0.01K)(6) = 6$.

$\sigma_R = \sqrt{(0.01K)^2\sigma_X^2 + (1-0.01K)^2\sigma_Y^2 + 2(0.01K)(1-0.01K)\text{Cov}(X,Y)}$.

Therefore $\sigma_R = \sqrt{(0.01K)^2(3^2) + (1-0.01K)^2(3^2) + 2(0.01K)(1-0.01K)(2.7)} = 0.03\sqrt{1.4K^2 - 140K + 10,000}$.

(d) σ_R is minimized when $1.4K^2 - 140K + 10000$ is minimized.

Now $\dfrac{d}{dK}(1.4K^2 - 140K + 10000) = 2.8K - 140$, so $\dfrac{d}{dK}(1.4K^2 - 140K + 10000) = 0$ if $K = 50$.

σ_R is minimized when $K = 50$.

(e) For any correlation ρ, the risk is $0.03\sqrt{K^2 + (100-K)^2 + 2\rho K(100-K)}$.

If $\rho \neq 1$ this quantity is minimized when $K = 50$.

24.
$$\begin{aligned}
\mu_V &= \int_{19}^{21}\int_{5}^{6} 3\pi r^2 h(h-20)^2(r-5)\,dr\,dh \\
&= 3\pi\int_{19}^{21} h(h-20)^2\,dh \int_{5}^{6} r^2(r-5)\,dr \\
&= (3\pi)\left(\frac{h^4}{4} - \frac{40h^3}{3} + 200h^2\right)\Bigg|_{19}^{21}\left(\frac{r^4}{4} - \frac{5r^3}{3}\right)\Bigg|_{5}^{6} \\
&= 2021.0913\text{cm}^3
\end{aligned}$$

25. (a) $\sigma_{M_1} = \sqrt{\sigma_{M1}^2} = \sqrt{\sigma_{R+E_1}^2} = \sqrt{\sigma_R^2 + \sigma_{E1}^2} = \sqrt{2^2 + 1^2} = 2.2361$. Similarly, $\sigma_{M_2} = 2.2361$.

(b) $\mu_{M_1M_2} = \mu_{R^2+E_1R+E_2R+E_1E_2} = \mu_{R^2} + \mu_{E_1}\mu_R + \mu_{E_2}\mu_R + \mu_{E_1}\mu_{E_2} = \mu_{R^2}$

(c) $\mu_{M_1}\mu_{M_2} = \mu_{R+E_1}\mu_{R+E_2} = (\mu_R + \mu_{E_1})(\mu_R + \mu_{E_2}) = \mu_R\mu_R = \mu_R^2$

(d) $\text{Cov}(M_1, M_2) = \mu_{M_1M_2} - \mu_{M_1}\mu_{M_2} = \mu_{R^2} - \mu_R^2 = \sigma_R^2$

(e) $\rho_{M_1,M_2} = \dfrac{\mathrm{Cov}(M_1,M_2)}{\sigma_{M_1}\sigma_{M_2}} = \dfrac{\sigma_R^2}{\sigma_{M_1}\sigma_{M_2}} = \dfrac{4}{(2.2361)(2.2361)} = 0.8$

26. $\mathrm{Cov}(X,X) = \mu_{X\cdot X} - \mu_X\mu_X = \mu_{X^2} - (\mu_X)^2 = \sigma_X^2.$

27. (a) $\mathrm{Cov}(aX,bY) = \mu_{aX\cdot bY} - \mu_{aX}\mu_{bY} = \mu_{abXY} - a\mu_X b\mu_Y = ab\mu_{XY} - ab\mu_X\mu_Y$
$= ab(\mu_{XY} - \mu_X\mu_Y) = ab\mathrm{Cov}(X,Y).$

(b) $\rho_{aX,bY} = \mathrm{Cov}(aX,bY)/(\sigma_{aX}\sigma_{bY}) = ab\mathrm{Cov}(X,Y)/(ab\sigma_X\sigma_Y) = \mathrm{Cov}(X,Y)/(\sigma_X\sigma_Y) = \rho_{X,Y}.$

28.
$$\begin{aligned}
\mathrm{Cov}(X+Y,Z) &= \mu_{(X+Y)Z} - \mu_{X+Y}\mu_Z \\
&= \mu_{XZ+YZ} - (\mu_X+\mu_Y)\mu_Z \\
&= \mu_{XZ} + \mu_{YZ} - \mu_X\mu_Z - \mu_Y\mu_Z \\
&= \mu_{XZ} - \mu_X\mu_Z + \mu_{YZ} - \mu_Y\mu_Z \\
&= \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)
\end{aligned}$$

29. (a)
$$\begin{aligned}
V(X-(\sigma_X/\sigma_Y)Y) &= \sigma_X^2 + (\sigma_X/\sigma_Y)^2\sigma_Y^2 - 2(\sigma_X/\sigma_Y)\mathrm{Cov}(X,Y) \\
&= 2\sigma_X^2 - 2(\sigma_X/\sigma_Y)\mathrm{Cov}(X,Y)
\end{aligned}$$

(b)
$$\begin{aligned}
V(X-(\sigma_X/\sigma_Y)Y) &\geq 0 \\
2\sigma_X^2 - 2(\sigma_X/\sigma_Y)\mathrm{Cov}(X,Y) &\geq 0 \\
2\sigma_X^2 - 2(\sigma_X/\sigma_Y)\rho_{X,Y}\sigma_X\sigma_Y &\geq 0 \\
2\sigma_X^2 - 2\rho_{X,Y}\sigma_X^2 &\geq 0 \\
1 - \rho_{X,Y} &\geq 0 \\
\rho_{X,Y} &\leq 1
\end{aligned}$$

(c)
$$\begin{aligned}
V(X+(\sigma_X/\sigma_Y)Y) &\geq 0\\
2\sigma_X^2+2(\sigma_X/\sigma_Y)\text{Cov}(X,Y) &\geq 0\\
2\sigma_X^2+2(\sigma_X/\sigma_Y)\rho_{X,Y}\sigma_X\sigma_Y &\geq 0\\
2\sigma_X^2+2\rho_{X,Y}\sigma_X^2 &\geq 0\\
1+\rho_{X,Y} &\geq 0\\
\rho_{X,Y} &\geq -1
\end{aligned}$$

30. (a)
$$\begin{aligned}
\mu_X &= \mu_{1.12C+2.69N+O-0.21Fe}\\
&= 1.12\mu_C+2.69\mu_N+\mu_O-0.21\mu_{Fe}\\
&= 1.12(0.0247)+2.69(0.0255)+0.1668-0.21(0.0597)\\
&= 0.2505
\end{aligned}$$

(b) $\text{Cov}(C,N)=\rho_{C,N}\sigma_C\sigma_N=-0.44(0.0131)(0.0194)=-1.118\times10^{-4}$
$\text{Cov}(C,O)=\rho_{C,O}\sigma_C\sigma_O=0.58(0.0131)(0.0340)=2.583\times10^{-4}$
$\text{Cov}(C,Fe)=\rho_{C,Fe}\sigma_C\sigma_{Fe}=0.39(0.0131)(0.0413)=2.110\times10^{-4}$
$\text{Cov}(N,O)=\rho_{N,O}\sigma_N\sigma_O=-0.32(0.0194)(0.0340)=-2.111\times10^{-4}$
$\text{Cov}(N,Fe)=\rho_{N,Fe}\sigma_N\sigma_{Fe}=0.09(0.0194)(0.0413)=7.211\times10^{-5}$
$\text{Cov}(O,Fe)=\rho_{O,Fe}\sigma_O\sigma_{Fe}=-0.35(0.0340)(0.0413)=-4.915\times10^{-4}$

(c)
$$\begin{aligned}
\sigma_X^2 &= \sigma^2_{1.12C+2.69N+O-0.21Fe}\\
&= 1.12^2\sigma_C^2+2.69^2\sigma_N^2+\sigma_O^2+0.21^2\sigma_{Fe}^2+2(1.12)(2.69)\text{Cov}(C,N)+2(1.12)\text{Cov}(C,O)-2(1.12)(0.21)\text{Cov}(C,Fe)\\
&\quad +2(2.69)\text{Cov}(N,O)-2(2.69)(0.21)\text{Cov}(N,Fe)-2(0.21)\text{Cov}(O,Fe)\\
&= 1.12^2(0.0131)^2+2.69^2(0.0194)^2+0.0340^2+0.21^2(0.0413)^2+2(1.12)(2.69)(-0.0001118)\\
&\quad +2(1.12)(0.0002583)-2(1.12)(0.21)(0.0002110)+2(2.69)(-0.0002111)-2(2.69)(0.21)(0.00007211)\\
&\quad -2(0.21)(0.0004915)\\
&= 0.0029648
\end{aligned}$$

$\sigma=\sqrt{0.0029648}=0.05445$

31.
$$\mu_Y = \mu_{7.84C+11.44N+O-1.58Fe}$$

$$
\begin{aligned}
&= 7.84\mu_C + 11.44\mu_N + \mu_O - 1.58\mu_{Fe} \\
&= 7.84(0.0247) + 11.44(0.0255) + 0.1668 - 1.58(0.0597) \\
&= 0.5578
\end{aligned}
$$

$$
\begin{aligned}
\sigma_Y^2 &= \sigma^2_{7.84C+11.44N+O-1.58Fe} \\
&= 7.84^2\sigma_C^2 + 11.44^2\sigma_N^2 + \sigma_O^2 + 1.58^2\sigma_{Fe}^2 + 2(7.84)(11.44)\mathrm{Cov}(C,N) + 2(7.84)\mathrm{Cov}(C,O) - 2(7.84)(1.58)\mathrm{Cov}(C,Fe) \\
&\quad +2(11.44)\mathrm{Cov}(N,O) - 2(11.44)(1.58)\mathrm{Cov}(N,Fe) - 2(1.58)\mathrm{Cov}(O,Fe) \\
&= 7.84^2(0.0131)^2 + 11.44^2(0.0194)^2 + 0.0340^2 + 1.58^2(0.0413)^2 + 2(7.84)(11.44)(-0.0001118) \\
&\quad +2(7.84)(0.0002583) - 2(7.84)(1.58)(0.0002110) + 2(11.44)(-0.0002111) - 2(11.44)(1.58)(0.00007211) \\
&\quad -2(1.58)(0.0004915) \\
&= 0.038100
\end{aligned}
$$

$\sigma = \sqrt{0.038100} = 0.1952$

32. (a) Let $c = \int_{-\infty}^{\infty} h(y)\,dy$.

Now $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} g(x)h(y)\,dy = g(x)\int_{-\infty}^{\infty} h(y)\,dy = cg(x)$.

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} g(x)h(y)\,dx = h(y)\int_{-\infty}^{\infty} g(x)\,dx = (1/c)h(y)\int_{-\infty}^{\infty} cg(x)\,dx$

$= (1/c)h(y)\int_{-\infty}^{\infty} f_X(x)\,dx = (1/c)h(y)(1) = (1/c)h(y)$.

(b) $f(x,y) = g(x)h(y) = cg(x)(1/c)h(y) = f_X(x)g_Y(y)$. Therefore X and Y are independent.

33. (a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_c^d\int_a^b k\,dx\,dy = k\int_c^d\int_a^b dx\,dy = k(d-c)(b-a) = 1$.

Therefore $k = \dfrac{1}{(b-a)(d-c)}$.

(b) $f_X(x) = \int_c^d k\,dy = \dfrac{d-c}{(b-a)(d-c)} = \dfrac{1}{b-a}$

(c) $f_Y(y) = \int_a^b k\,dx = \dfrac{b-a}{(b-a)(d-c)} = \dfrac{1}{d-c}$

(d) $f(x,y) = \dfrac{1}{(b-a)(d-c)} = \left(\dfrac{1}{b-a}\right)\left(\dfrac{1}{d-c}\right) = f_X(x)f_Y(y)$

Supplementary Exercises for Chapter 2

1. Let A be the event that component A functions, let B be the event that component B functions, let C be the event that component C functions, and let D be the event that component D functions. Then $P(A) = 1-0.1 = 0.9$, $P(B) = 1-0.2 = 0.8$, $P(C) = 1-0.05 = 0.95$, and $P(D) = 1-0.3 = 0.7$. The event that the system functions is $(A\cup B)\cup(C\cup D)$.

   $P(A\cup B) = P(A)+P(B)-P(A\cap B) = P(A)+P(B)-P(A)P(B) = 0.9+0.8-(0.9)(0.8) = 0.98$.

   $P(C\cup D) = P(C)+P(D)-P(C\cap D) = P(C)+P(D)-P(C)P(D) = 0.95+0.7-(0.95)(0.7) = 0.985$.

   $P[(A\cup B)\cup(C\cup D)] = P(A\cup B)+P(C\cup D)-P(A\cup B)P(C\cup D) = 0.98+0.985-(0.98)(0.985) = 0.9997$.

2. $P(\text{more than 3 tosses necessary}) = P(\text{first 3 tosses are tails}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

3. Let A denote the event that the resistance is above specification, and let B denote the event that the resistance is below specification. Then A and B are mutually exclusive.

   (a) P(doesn't meet specification) = $P(A\cup B) = P(A)+P(B) = 0.05+0.10 = 0.15$

   (b) $P[B\,|\,(A\cup B)] = \dfrac{P[(B\cap(A\cup B)]}{P(A\cup B)} = \dfrac{P(B)}{P(A\cup B)} = \dfrac{0.10}{0.15} = 0.6667$

4. Let D denote the event that the bag is defective, let L_1 denote the event that the bag came from line 1, and let L_2 denote them event that the bag came from line 2. Then $P(D\,|\,L_1) = 1/100$ and $P(D\,|\,L_2) = 3/100$.

   (a) $P(L_1) = \dfrac{2}{3}$

   (b) $P(D) = P(D\,|\,L_1)P(L_1)+P(D\,|\,L_2)P(L_2) = (1/100)(2/3)+(3/100)(1/3) = \dfrac{1}{60}$

   (c) $P(L_1\,|\,D) = \dfrac{P(D\,|\,L_1)P(L_1)}{P(D)} = \dfrac{(1/100)(2/3)}{1/60} = \dfrac{2}{5}$

(d) $P(L_1\,|\,D^c) = \dfrac{P(D^c\,|\,L_1)P(L_1)}{P(D^c)} = \dfrac{[1-P(D\,|\,L_1)]P(L_1)}{P(D^c)} = \dfrac{(1-1/100)(2/3)}{59/60} = \dfrac{198}{295} = 0.6712$

5. Let R be the event that the shipment is returned. Let B_1 be the event that the first brick chosen meets the specification, let B_2 be the event that the second brick chosen meets the specification, let B_3 be the event that the third brick chosen meets the specification, and let B_4 be the event that the fourth brick chosen meets the specification. Since the sample size of 4 is a small proportion of the population, it is reasonable to treat these events as independent, each with probability 0.9.
$P(R) = 1 - P(R^c) = 1 - P(B_1 \cap B_2 \cap B_3 \cap B_4) = 1 - (0.9)^4 = 0.3439$.

6. (a) $(0.99)^{10} = 0.904$

(b) Let p be the required probability. Then $p^{10} = 0.95$. Solving for p, $p = 0.9949$.

7. Let A be the event that the bit is reversed at the first relay, and let B be the event that the bit is reversed at the second relay. Then $P(\text{bit received is the same as the bit sent}) = P(A^c \cap B^c) + P(A \cap B) = P(A^c)P(B^c) + P(A)P(B) = 0.9^2 + 0.1^2 = 0.82$.

8. (a) $\int_{-1}^{1} k(1-x^2)\,dx = k\int_{-1}^{1}(1-x^2)\,dx = 1$. Since $\int_{-1}^{1}(1-x^2)\,dx = \left(x - \dfrac{x^3}{3}\right)\Big|_{-1}^{1} = \dfrac{4}{3}$, $k = \dfrac{3}{4} = 0.75$.

(b) $\int_{0}^{1} 0.75(1-x^2)\,dx = 0.75\left(x - \dfrac{x^3}{3}\right)\Big|_{0}^{1} = 0.5$

(c) $\int_{-0.25}^{0.25} 0.75(1-x^2)\,dx = 0.75\left(x - \frac{x^3}{3}\right)\Bigg|_{-0.25}^{0.25} = 0.3672$

(d) $\mu = \int_{-1}^{1} 0.75x(1-x^2)\,dx = 0.75\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{-1}^{1} = 0$

(e) The median x_m solves $\int_{-1}^{x_m} 0.75(1-x^2)\,dx = 0.5$. Therefore $0.75\left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^{x_m} = 0.5$, so $x_m = 0$

(f)
$$\begin{aligned}\sigma^2 &= \int_{-1}^{1} 0.75x^2(1-x^2)\,dx - \mu^2\\ &= 0.75\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_{-1}^{1} - 0^2\\ &= 0.2\end{aligned}$$

$\sigma = \sqrt{0.2} = 0.4472$

9. Let A be the event that two different numbers come up, and let B be the event that one of the dice comes up 6. Then A contains 30 equally likely outcomes (6 ways to choose the number for the first die times 5 ways to choose the number for the second die). Of these 30 outcomes, 10 belong to B, specifically (1,6), (2,6), (3,6), (4,6), (5,6), (6,1), (6,2), (6,3), (6,4), and (6,5). Therefore $P(B\,|\,A) = 10/30 = 1/3$.

10. Let A be the event that the first component is defective and let B be the event that the second component is defective.

(a) $P(X=0) = P(A^c \cap B^c) = P(A^c)P(B^c|A^c) = \left(\frac{8}{10}\right)\left(\frac{7}{9}\right) = 0.6222$

(b)
$$\begin{aligned}P(X=1) &= P(A \cap B^c) + P(A^c \cap B)\\ &= P(A)P(B^c|A) + P(A^c)P(B|A^c)\\ &= \left(\frac{2}{10}\right)\left(\frac{8}{9}\right) + \left(\frac{8}{10}\right)\left(\frac{2}{9}\right)\\ &= 0.3556\end{aligned}$$

(c) $P(X=2)=P(A\cap B)=P(A)P(B|A)=\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)=0.0222$

(d) $p_X(0)=0.6222$, $p_X(1)=0.3556$, $p_X(2)=0.0222$, $p_X(x)=0$ if $x\neq 0,1,$ or 2.

(e) $\mu_X=0pX(0)+1p_X(1)+2p_X(2)=0(0.6222)+1(0.3556)+2(0.0222)=0.4$

(f) $\sigma_X=\sqrt{0^2pX(0)+1^2p_X(1)+2^2p_X(2)-\mu_X^2}=\sqrt{0^2(0.6222)+1^2(0.3556)+2^2(0.0222)-0.4^2}=0.5333$

11. (a)
$$\begin{aligned}
P(X\leq 2 \text{ and } Y\leq 3) &= \int_0^2\int_0^3 \frac{1}{6}e^{-x/2-y/3}\,dy\,dx\\
&= \int_0^2 \frac{1}{2}e^{-x/2}\left(-e^{-y/3}\Big|_0^3\right)dx\\
&= \int_0^2 \frac{1}{2}e^{-x/2}(1-e^{-1})\,dx\\
&= (e^{-1}-1)e^{-x/2}\Big|_0^2\\
&= (1-e^{-1})^2\\
&= 0.3996
\end{aligned}$$

(b)
$$\begin{aligned}
P(X\geq 3 \text{ and } Y\geq 3) &= \int_3^\infty\int_3^\infty \frac{1}{6}e^{-x/2-y/3}\,dy\,dx\\
&= \int_3^\infty \frac{1}{2}e^{-x/2}\left(-e^{-y/3}\Big|_3^\infty\right)dx\\
&= \int_3^\infty \frac{1}{2}e^{-x/2}e^{-1}\,dx\\
&= -e^{-1}e^{-x/2}\Big|_3^\infty\\
&= e^{-5/2}\\
&= 0.0821
\end{aligned}$$

(c) If $x\leq 0$, $f(x,y)=0$ for all y so $f_X(x)=0$.

If $x > 0$, $f_X(x) = \int_0^\infty \frac{1}{6}e^{-x/2-y/3}\,dy = \frac{1}{2}e^{-x/2}\left(-e^{-y/3}\Big|_0^\infty\right) = \frac{1}{2}e^{-x/2}$.

Therefore $f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2} & x > 0 \\ 0 & x \le 0 \end{cases}$

(d) If $y \le 0$, $f(x,y) = 0$ for all x so $f_Y(y) = 0$.

If $y > 0$, $f_Y(y) = \int_3^\infty \frac{1}{6}e^{-x/2-y/3}\,dx = \frac{1}{3}e^{-y/3}\left(-e^{-x/2}\Big|_0^\infty\right) = \frac{1}{3}e^{-y/3}$.

Therefore $f_Y(y) = \begin{cases} \frac{1}{3}e^{-y/3} & y > 0 \\ 0 & y \le 0 \end{cases}$

(e) Yes, $f(x,y) = f_X(x)f_Y(y)$.

12. (a) A and B are mutually exclusive if $P(A \cap B) = 0$, or equivalently, if $P(A \cup B) = P(A) + P(B)$.

So if $P(B) = P(A \cup B) - P(A) = 0.7 - 0.3 = 0.4$, then A and B are mutually exclusive.

(b) A and B are independent if $P(A \cap B) = P(A)P(B)$. Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

So A and B are independent if $P(A \cup B) = P(A) + P(B) - P(A)P(B)$, that is, if $0.7 = 0.3 + P(B) - 0.3P(B)$. This equation is satisfied if $P(B) = 4/7$.

13. Let D denote the event that a snowboard is defective, let E denote the event that a snowboard is made in the eastern United States, let W denote the event that a snowboard is made in the western United States, and let C denote the event that a snowboard is made in Canada. Then $P(E) = P(W) = 10/28$, $P(C) = 8/28$, $P(D|E) = 3/100$, $P(D|W) = 6/100$, and $P(D|C) = 4/100$.

(a)
$$\begin{aligned} P(D) &= P(D|E)P(E) + P(D|W)P(W) + P(D|C)P(C) \\ &= \left(\frac{10}{28}\right)\left(\frac{3}{100}\right) + \left(\frac{10}{28}\right)\left(\frac{6}{100}\right) + \left(\frac{8}{28}\right)\left(\frac{4}{100}\right) \\ &= \frac{122}{2800} = 0.0436 \end{aligned}$$

(b) $P(D \cap C) = P(D|C)P(C) = \left(\frac{8}{28}\right)\left(\frac{4}{100}\right) = \frac{32}{2800} = 0.0114$

(c) Let U be the event that a snowboard was made in the United States.
Then $P(D \cap U) = P(D) - P(D \cap C) = \frac{122}{2800} - \frac{32}{2800} = \frac{90}{2800}$.
$P(U|D) = \frac{P(D \cap U)}{P(D)} = \frac{90/2800}{122/2800} = \frac{90}{122} = 0.7377$.

14. (a) Discrete. The possible values are 10, 60, and 80.

(b) $\mu_X = 10p_X(10) + 60p_X(60) + 80p_X(80) = 10(0.40) + 60(0.50) + 80(0.10) = 42$

(c) $\sigma_X^2 = 10^2 p_X(10) + 60^2 p_X(60) + 80^2 p_X(80) - \mu_X^2 = 10^2(0.40) + 60^2(0.50) + 80^2(0.10) - 42^2 = 716$
$\sigma_X = \sqrt{716} = 26.76$

(d) $P(X > 50) = P(X = 60) + P(X = 80) = 0.5 + 0.1 = 0.6$

15. The total number of pairs of cubicles is $\binom{6}{2} = \frac{6!}{2!4!} = 15$. Each is equally likely to be chosen. Of these pairs, five are adjacent (1 and 2, 2 and 3, 3 and 4, 4 and 5, 5 and 6). Therefore the probability that an adjacent pair of cubicles is selected is $5/15$, or $1/3$.

16. The total number of combinations of four shoes that can be selected from eight is $\binom{8}{4} = \frac{8!}{4!4!} = 70$. The four shoes will contain no pair if exactly one shoe is selected from each pair. Since each pair contains two shoes, the number of ways to select exactly one shoe from each pair is $2^4 = 16$. Therefore the probability that the four shoes contain no pair is $16/70$, or $8/35$.

17. (a) $\mu_{3X} = 3\mu_X = 3(2) = 6, \quad \sigma_{3X}^2 = 3^2\sigma_X^2 = (3^2)(1^2) = 9$

(b) $\mu_{X+Y} = \mu_X + \mu_Y = 2 + 2 = 4, \quad \sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y = 1^2 + 3^2 = 10$

(c) $\mu_{X-Y} = \mu_X - \mu_Y = 2 - 2 = 0, \quad \sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y = 1^2 + 3^2 = 10$

(d) $\mu_{2X+6Y} = 2\mu_X + 6\mu_Y = 2(2) + 6(2) = 16, \quad \sigma^2_{2X+6Y} = 2^2\sigma^2_X + 6^2\sigma^2_Y = (2^2)(1^2) + (6^2)(3^2) = 328$

18. $\text{Cov}(X,Y) = \rho_{X,Y}\sigma_X\sigma_Y = (0.5)(2)(1) = 1$

(a) $\mu_{X+Y} = \mu_X + \mu_Y = 1 + 3 = 4, \quad \sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\text{Cov}(X,Y) = 2^2 + 1^2 + 2(1) = 7$

(b) $\mu_{X-Y} = \mu_X - \mu_Y = 1 - 3 = -2, \quad \sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y - 2\text{Cov}(X,Y) = 2^2 + 1^2 - 2(1) = 3$

(c) $\mu_{3X+2Y} = 3\mu_X + 2\mu_Y = 3(1) + 2(3) = 9$,
$\sigma^2_{3X+2Y} = 3^2\sigma^2_X + 2^2\sigma^2_Y + 2(3)(2)\text{Cov}(X,Y) = (3^2)(2^2) + (2^2)(1^2) + 2(3)(2)(1) = 52$

(d) $\mu_{5Y-2X} = 5\mu_Y - 2\mu_X = 5(3) - 2(1) = 13$,
$\sigma^2_{5Y-2X} = 5^2\sigma^2_Y + (-2)^2\sigma^2_X + 2(5)(-2)\text{Cov}(X,Y) = 5^2(1^2) + (-2)^2(2^2) + 2(5)(-2)(1) = 21$

19. The marginal probability mass function $p_X(x)$ is found by summing along the rows of the joint probability mass function.

	y			
x	100	150	200	$p_X(x)$
0.02	0.05	0.06	0.11	0.22
0.04	0.01	0.08	0.10	0.19
0.06	0.04	0.08	0.17	0.29
0.08	0.04	0.14	0.12	0.30
$p_Y(y)$	0.14	0.36	0.50	

(a) For additive concentration (X): $p_X(0.02) = 0.22$, $p_X(0.04) = 0.19$, $p_X(0.06) = 0.29$, $p_X(0.08) = 0.30$, and $p_X(x) = 0$ for $x \neq 0.02, 0.04, 0.06$, or 0.08.

For tensile strength (Y): The marginal probability mass function $p_Y(y)$ is found by summing down the columns of the joint probability mass function. Therefore $p_Y(100) = 0.14$, $p_Y(150) = 0.36$, $p_Y(200) = 0.50$, and $p_Y(y) = 0$ for $y \neq 100, 150$, or 200.

(b) No, X and Y are not independent. For example $P(X = 0.02 \cap Y = 100) = 0.05$, but $P(X = 0.02)P(Y = 100) = (0.22)(0.14) = 0.0308$.

(c)
$$\begin{aligned}
P(Y \geq 150 \,|\, X = 0.04) &= \frac{P(Y \geq 150 \text{ and } X = 0.04)}{P(X = 0.04)} \\
&= \frac{P(Y = 150 \text{ and } X = 0.04) + P(Y = 200 \text{ and } X = 0.04)}{P(X = 0.04)} \\
&= \frac{0.08 + 0.10}{0.19} \\
&= 0.947
\end{aligned}$$

(d)
$$\begin{aligned}
P(Y > 125 \,|\, X = 0.08) &= \frac{P(Y > 125 \text{ and } X = 0.08)}{P(X = 0.08)} \\
&= \frac{P(Y = 150 \text{ and } X = 0.08) + P(Y = 200 \text{ and } X = 0.08)}{P(X = 0.08)} \\
&= \frac{0.14 + 0.12}{0.30} \\
&= 0.867
\end{aligned}$$

(e) The tensile strength is greater than 175 if $Y = 200$. Now

$$P(Y = 200 \,|\, X = 0.02) = \frac{P(Y = 200 \text{ and } X = 0.02)}{P(X = 0.02)} = \frac{0.11}{0.22} = 0.500,$$

$$P(Y = 200 \,|\, X = 0.04) = \frac{P(Y = 200 \text{ and } X = 0.04)}{P(X = 0.04)} = \frac{0.10}{0.19} = 0.526,$$

$$P(Y = 200 \,|\, X = 0.06) = \frac{P(Y = 200 \text{ and } X = 0.06)}{P(X = 0.06)} = \frac{0.17}{0.29} = 0.586,$$

$$P(Y = 200 \,|\, X = 0.08) = \frac{P(Y = 200 \text{ and } X = 0.08)}{P(X = 0.08)} = \frac{0.12}{0.30} = 0.400.$$

The additive concentration should be 0.06.

20. (a)
$$\begin{aligned}
\mu_X &= 0.02p_X(0.02) + 0.04p_X(0.04) + 0.06p_X(0.06) + 0.08p_X(0.08) \\
&= 0.02(0.22) + 0.04(0.19) + 0.06(0.29) + 0.08(0.30) \\
&= 0.0534
\end{aligned}$$

(b) $\mu_Y = 100p_Y(100) + 150p_Y(150) + 200p_Y(200) = 100(0.14) + 150(0.36) + 200(0.50) = 168$

(c)
$$\begin{aligned}\sigma_X^2 &= 0.02^2 p_X(0.02) + 0.04^2 p_X(0.04) + 0.06^2 p_X(0.06) + 0.08^2 p_X(0.08) - \mu_X^2 \\ &= 0.02^2(0.22) + 0.04^2(0.19) + 0.06^2(0.29) + 0.08^2(0.30) - 0.0534^2 \\ &= 0.00050444\end{aligned}$$

$\sigma_X = \sqrt{0.00050444} = 0.02246$

(d)
$$\begin{aligned}\sigma_Y^2 &= 100^2 p_Y(100) + 150^2 p_Y(150) + 200^2 p_Y(200) - \mu_Y^2 \\ &= 100^2(0.14) + 150^2(0.36) + 200^2(0.50) - 168^2 \\ &= 1276\end{aligned}$$

$\sigma_Y = \sqrt{1276} = 35.721$

(e) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y$.

$$\begin{aligned}\mu_{XY} &= (0.02)(100)P(X = 0.02 \text{ and } Y = 100) + (0.02)(150)P(X = 0.02 \text{ and} Y = 150) \\ &\quad + (0.02)(200)P(X = 0.02 \text{ and } Y = 200) + (0.04)(100)P(X = 0.04 \text{ and } Y = 100) \\ &\quad + (0.04)(150)P(X = 0.04 \text{ and } Y = 150) + (0.04)(200)P(X = 0.04 \text{ and } Y = 200) \\ &\quad + (0.06)(100)P(X = 0.06 \text{ and } Y = 100) + (0.06)(150)P(X = 0.06 \text{ and } Y = 150) \\ &\quad + (0.06)(200)P(X = 0.06 \text{ and } Y = 200) + (0.08)(100)P(X = 0.08 \text{ and } Y = 100) \\ &\quad + (0.08)(150)P(X = 0.08 \text{ and } Y = 150) + (0.08)(200)P(X = 0.08 \text{ and } Y = 200) \\ &= (0.02)(100)(0.05) + (0.02)(150)(0.06) + (0.02)(200)(0.11) + (0.04)(100)(0.01) \\ &\quad + (0.04)(150)(0.08) + (0.04)(200)(0.10) + (0.06)(100)(0.04) + (0.06)(150)(0.08) \\ &\quad + (0.06)(200)(0.17) + (0.08)(100)(0.04) + (0.08)(150)(0.14) + (0.08)(200)(0.12) \\ &= 8.96\end{aligned}$$

$\text{Cov}(X,Y) = 8.96 - (0.0534)(168) = -0.0112$

(f) $\rho_{X,Y} = \dfrac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \dfrac{-0.0112}{(0.02246)(35.721)} = -0.01396$

21. (a) $p_{Y|X}(100\,|\,0.06) = \dfrac{p(0.06,100)}{p_X(0.06)} = \dfrac{0.04}{0.29} = \dfrac{4}{29} = 0.138$

$p_{Y|X}(150\,|\,0.06) = \dfrac{p(0.06,150)}{p_X(0.06)} = \dfrac{0.08}{0.29} = \dfrac{8}{29} = 0.276$

$$p_{Y|X}(200\,|\,0.06) = \frac{p(0.06, 200)}{p_X(0.06)} = \frac{0.17}{0.29} = \frac{17}{29} = 0.586$$

(b) $$p_{X|Y}(0.02\,|\,100) = \frac{p(0.02, 100)}{p_Y(100)} = \frac{0.05}{0.14} = \frac{5}{14} = 0.357$$

$$p_{X|Y}(0.04\,|\,100) = \frac{p(0.04, 100)}{p_Y(100)} = \frac{0.01}{0.14} = \frac{1}{14} = 0.071$$

$$p_{X|Y}(0.06\,|\,100) = \frac{p(0.06, 100)}{p_Y(100)} = \frac{0.04}{0.14} = \frac{4}{14} = 0.286$$

$$p_{X|Y}(0.08\,|\,100) = \frac{p(0.08, 100)}{p_Y(100)} = \frac{0.04}{0.14} = \frac{4}{14} = 0.286$$

(c) $$\begin{aligned} E(Y\,|\,X = 0.06) &= 100p_{Y|X}(100\,|\,0.06) + 150p_{Y|X}(150\,|\,0.06) + 200p_{Y|X}(200\,|\,0.06) \\ &= 100(4/29) + 150(8/29) + 200(17/29) \\ &= 172.4 \end{aligned}$$

(d) $$\begin{aligned} E(X\,|\,Y = 100) &= 0.02p_{X|Y}(0.02\,|\,100) + 0.04p_{X|Y}(0.04\,|\,100) + 0.06p_{X|Y}(0.06\,|\,100) + 0.08p_{X|Y}(0.08\,|\,100) \\ &= 0.02(5/14) + 0.04(1/14) + 0.06(4/14) + 0.08(4/14) \\ &= 0.0500 \end{aligned}$$

22. Let D denote the event that an item is defective, let S_1 denote the event that an item is produced on the first shift, let S_2 denote the event that an item is produced on the second shift, and let S_3 denote the event that an item is produced on the third shift. Then $P(S_1) = 0.50$, $P(S_2) = 0.30$, $P(S_3) = 0.20$, $P(D|S_1) = 0.01$, $P(D|S_2) = 0.02$, and $P(D|S_3) = 0.03$.

(a) $$\begin{aligned} P(S_1|D) &= \frac{P(D|S_1)P(S_1)}{P(D|S_1)P(S_1) + P(D|S_2)P(S_2) + P(D|S_3)P(S_3)} \\ &= \frac{(0.01)(0.50)}{(0.01)(0.50) + (0.02)(0.30) + (0.03)(0.20)} \\ &= 0.294 \end{aligned}$$

(b)
$$\begin{aligned}
P(S_3|D^c) &= \frac{P(D^c|S_3)P(S_3)}{P(D^c|S_1)P(S_1)+P(D^c|S_2)P(S_2)+P(D^c|S_3)P(S_3)} \\
&= \frac{[1-P(D|S_3)]P(S_3)}{[1-P(D|S_1)]P(S_1)+[1-P(D|S_2)]P(S_2)+[1-P(D|S_3)]P(S_3)} \\
&= \frac{(1-0.03)(0.20)}{(1-0.01)(0.50)+(1-0.02)(0.30)+(1-0.03)(0.20)} \\
&= 0.197
\end{aligned}$$

23. (a) Under scenario A:

$\mu = 0(0.65)+5(0.2)+15(0.1)+25(0.05) = 3.75$

$\sigma = \sqrt{0^2(0.65)+5^2(0.2)+15^2(0.1)+25^2(0.05)-3.75^2} = 6.68$

(b) Under scenario B:

$\mu = 0(0.65)+5(0.24)+15(0.1)+20(0.01) = 2.90$

$\sigma = \sqrt{0^2(0.65)+5^2(0.24)+15^2(0.1)+20^2(0.01)-2.90^2} = 4.91$

(c) Under scenario C:

$\mu = 0(0.65)+2(0.24)+5(0.1)+10(0.01) = 1.08$

$\sigma = \sqrt{0^2(0.65)+2^2(0.24)+5^2(0.1)+10^2(0.01)-1.08^2} = 1.81$

(d) Let L denote the loss.

Under scenario A, $P(L<10) = P(L=0)+P(L=5) = 0.65+0.2 = 0.85$.

Under scenario B, $P(L<10) = P(L=0)+P(L=5) = 0.65+0.24 = 0.89$.

Under scenario C, $P(L<10) = P(L=0)+P(L=2)+P(L=5) = 0.65+0.24+0.1 = 0.99$.

24. Let L denote the loss.

(a) $P(A\cap L=5) = P(L=5|A)P(A) = (0.20)(0.20) = 0.040$

(b)
$$\begin{aligned}
P(L=5) &= P(L=5|A)P(A)+P(L=5|B)P(B)+P(L=5|C)P(C) \\
&= (0.20)(0.20)+(0.30)(0.24)+(0.50)(0.1) \\
&= 0.162
\end{aligned}$$

(c) $P(A|L=5) = \dfrac{P(A \cap L=5)}{P(L=5)} = \dfrac{0.040}{0.162} = 0.247$

25. (a) $p(0,0) = P(X=0 \text{ and } Y=0) = \left(\dfrac{3}{10}\right)\left(\dfrac{2}{9}\right) = \dfrac{1}{15} = 0.0667$

$p(1,0) = P(X=1 \text{ and } Y=0) = \left(\dfrac{4}{10}\right)\left(\dfrac{3}{9}\right) + \left(\dfrac{3}{10}\right)\left(\dfrac{4}{9}\right) = \dfrac{4}{15} = 0.2667$

$p(2,0) = P(X=2 \text{ and } Y=0) = \left(\dfrac{4}{10}\right)\left(\dfrac{3}{9}\right) = \dfrac{2}{15} = 0.1333$

$p(0,1) = P(X=0 \text{ and } Y=1) = \left(\dfrac{3}{10}\right)\left(\dfrac{3}{9}\right) + \left(\dfrac{3}{10}\right)\left(\dfrac{3}{9}\right) = \dfrac{3}{15} = 0.2000$

$p(1,1) = P(X=1 \text{ and } Y=1) = \left(\dfrac{4}{10}\right)\left(\dfrac{3}{9}\right) + \left(\dfrac{3}{10}\right)\left(\dfrac{4}{9}\right) = \dfrac{4}{15} = 0.2667$

$p(0,2) = P(X=0 \text{ and } Y=2) = \left(\dfrac{3}{10}\right)\left(\dfrac{2}{9}\right) = \dfrac{1}{15} = 0.0667$

$p(x,y) = 0$ for all other pairs (x,y).

The joint probability mass function is

x \ y	0	1	2
0	0.0667	0.2000	0.0667
1	0.2667	0.2667	0
2	0.1333	0	0

(b) The marginal probability density function of X is:

$p_X(0) = p(0,0) + p(0,1) + p(0,2) = \dfrac{1}{15} + \dfrac{3}{15} + \dfrac{1}{15} = \dfrac{1}{3}$

$p_X(1) = p(1,0) + p(1,1) = \dfrac{4}{15} + \dfrac{4}{15} = \dfrac{8}{15}.$

$p_X(2) = p(2,0) = \dfrac{2}{15}$

$\mu_X = 0p_X(0) + 1p_X(1) + 2p_X(2) = 0\left(\dfrac{1}{3}\right) + 1\left(\dfrac{8}{15}\right) + 2\left(\dfrac{2}{15}\right) = \dfrac{12}{15} = 0.8$

(c) The marginal probability density function of Y is:

$p_Y(0) = p(0,0) + p(1,0) + p(2,0) = \dfrac{1}{15} + \dfrac{4}{15} + \dfrac{2}{15} = \dfrac{7}{15}$

$p_Y(1) = p(0,1) + p(1,1) = \dfrac{3}{15} + \dfrac{4}{15} = \dfrac{7}{15}.$

$$p_Y(2) = p(0,2) = \frac{1}{15}$$

$$\mu_Y = 0p_Y(0) + 1p_Y(1) + 2p_Y(2) = 0\left(\frac{7}{15}\right) + 1\left(\frac{7}{15}\right) + 2\left(\frac{1}{15}\right) = \frac{9}{15} = 0.6$$

(d)
$$\begin{aligned}\sigma_X &= \sqrt{0^2 p_X(0) + 1^2 p_X(1) + 2^2 p_X(2) - \mu_X^2} \\ &= \sqrt{0^2(1/3) + 1^2(8/15) + 2^2(2/15) - (12/15)^2} \\ &= \sqrt{96/225} = 0.6532\end{aligned}$$

(e)
$$\begin{aligned}\sigma_Y &= \sqrt{0^2 p_Y(0) + 1^2 p_Y(1) + 2^2 p_Y(2) - \mu_Y^2} \\ &= \sqrt{0^2(7/15) + 1^2(7/15) + 2^2(1/15) - (9/15)^2} \\ &= \sqrt{84/225} = 0.6110\end{aligned}$$

(f) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y$.

$$\begin{aligned}\mu_{XY} &= (0)(0)p(0,0) + (1)(0)p(1,0) + (2)(0)p(2,0) + (0)(1)p(0,1) + (1)(1)p(1,1) + (0)(2)p(0,2) \\ &= (1)(1)\frac{4}{15} = \frac{4}{15}\end{aligned}$$

$$\text{Cov}(X,Y) = \frac{4}{15} - \left(\frac{12}{15}\right)\left(\frac{9}{15}\right) = -\frac{48}{225} = -0.2133$$

(g) $$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{-48/225}{\sqrt{96/225}\sqrt{84/225}} = -0.5345$$

26. (a) The constant c solves $\int_0^1\int_0^1 c(x+y)^2\,dx\,dy = 1$.

Since $\int_0^1\int_0^1 (x+y)^2\,dx\,dy = \frac{7}{6}$, $c = \frac{6}{7}$.

(b) For $0 < x < 1$, $f_X(x) = \int_0^1 \frac{6}{7}(x+y)^2\,dy = \left.\frac{2(x+y)^3}{7}\right|_0^1 = \frac{6x^2+6x+2}{7}$.

For $x \le 0$ or $x \ge 1$ $f(x,y) = 0$ for all y, so $f_X(x) = 0$.

Therefore $f_X(x) = \begin{cases} \dfrac{6x^2+6x+2}{7} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(c) $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$. If $x \le 0$ or $x \ge 1$ $f_X(x) = 0$, so $f_{Y|X}(y|x)$ is undefined.

Now assume $0 < x < 1$. If $y \le 0$ or $y \ge 1$ then $f(x,y) = 0$ for all x so $f_{Y|X}(y|x) = 0$.

If $0 < y < 1$ then

$$f_{Y|X}(y|x) = \frac{(6/7)(x+y)^2}{(6x^2+6x+2)/7} = \frac{3(x+y)^2}{3x^2+3x+1}.$$

Therefore $f_{Y|X}(y|x) = \begin{cases} \dfrac{3(x+y)^2}{3x^2+3x+1} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

(d)
$$\begin{aligned} E(Y|X=0.4) &= \int_{-\infty}^{\infty} y\,f_{Y|X}(y|0.4)\,dy \\ &= \int_0^1 \frac{3y(0.4+y)^2}{3(0.4)^2+3(0.4)+1}\,dy \\ &= \left.\frac{0.24y^2+0.8y^3+0.75y^4}{2.68}\right|_0^1 \\ &= 0.6679 \end{aligned}$$

(e) No, $f_{Y|X}(y|x) \ne f_Y(y)$.

27. (a) $\mu_X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\frac{6x^2+6x+2}{7}\,dx = \left.\frac{1}{14}(3x^4+4x^3+2x^2)\right|_0^1 = \frac{9}{14} = 0.6429$

(b) $\sigma_X^2 = \int_0^1 x^2\frac{6x^2+6x+2}{7}\,dx - \mu_X^2 = \left.\frac{1}{7}\left(\frac{6x^5}{5}+\frac{3x^4}{2}+\frac{2x^3}{3}\right)\right|_0^1 - \left(\frac{9}{14}\right)^2 = \frac{199}{2940} = 0.06769$

(c) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X \mu_Y$.

$$\begin{aligned}
\mu_{XY} &= \int_0^1 \int_0^1 xy\left(\frac{6}{7}\right)(x+y)^2\,dx\,dy \\
&= \int_0^1 \frac{6}{7}y\left(\frac{x^4}{4} + \frac{2x^3y}{3} + \frac{x^2y^2}{2}\Bigg|_0^1\right)dy \\
&= \int_0^1 \frac{6}{7}\left(\frac{y^3}{2} + \frac{2y^2}{3} + \frac{y}{4}\right)dy \\
&= \frac{6}{7}\left(\frac{y^2}{8} + \frac{2y^3}{9} + \frac{y^4}{8}\right)\Bigg|_0^1 \\
&= \frac{17}{42}
\end{aligned}$$

$\mu_X = \dfrac{9}{14}$, computed in part (a). To compute μ_Y, note that the joint density is symmetric in x and y, so the marginal density of Y is the same as that of X. It follows that $\mu_Y = \mu_X = \dfrac{9}{14}$.

$\text{Cov}(X,Y) = \dfrac{17}{42} - \left(\dfrac{9}{14}\right)\left(\dfrac{9}{14}\right) = \dfrac{-5}{588} = -0.008503$.

(d) Since the marginal density of Y is the same as that of X, $\sigma_Y^2 = \sigma_X^2 = \dfrac{199}{2940}$.

Therefore $\rho_{X,Y} = \dfrac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \dfrac{-5/588}{\sqrt{199/2940}\sqrt{199/2940}} = \dfrac{-25}{199} = -0.1256$.

28. Since the coin is fair, $P(H) = P(T) = 1/2$. The tosses are independent.
So $P(HTTHH) = P(H)P(T)P(T)P(H)P(H) = (1/2)^5 = 1/32$, and
$P(HHHHH) = P(H)P(H)P(H)P(H)P(H) = (1/2)^5 = 1/32$ as well.

29. (a) $p_X(0) = 0.6$, $p_X(1) = 0.4$, $p_X(x) = 0$ if $x \neq 0$ or 1.

(b) $p_Y(0) = 0.4$, $p_Y(1) = 0.6$, $p_Y(y) = 0$ if $y \neq 0$ or 1.

(c) Yes. It is reasonable to assume that knowledge of the outcome of one coin will not help predict the outcome of the other.

(d) $p(0,0) = p_X(0)p_Y(0) = (0.6)(0.4) = 0.24$, $p(0,1) = p_X(0)p_Y(1) = (0.6)(0.6) = 0.36$,
$p(1,0) = p_X(1)p_Y(0) = (0.4)(0.4) = 0.16$, $p(1,1) = p_X(1)p_Y(1) = (0.4)(0.6) = 0.24$,
$p(x,y) = 0$ for other values of (x,y).

30. The probability mass function of X is $p_X(x) = 1/6$ for $x = 1,2,3,4,5,$ or 6, and $p_X(x) = 0$ for other values of x. Therefore $\mu_X = 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6) = 3.5$. The probability mass function of Y is the same as that of X, so $\mu_Y = \mu_X = 3.5$. Since X and Y are independent, $\mu_{XY} = \mu_X\mu_Y = (3.5)(3.5) = 12.25$.

31. (a) The possible values of the pair (X,Y) are the ordered pairs (x,y) where each of x and y is equal to 1, 2, or 3. There are nine such ordered pairs, and each is equally likely. Therefore $p_{X,Y}(x,y) = 1/9$ for $x = 1,2,3$ and $y = 1,2,3$, and $p_{X,Y}(x,y) = 0$ for other values of (x,y).

(b) Both X and Y are sampled from the numbers $\{1,2,3\}$, with each number being equally likely.
Therefore $p_X(1) = p_X(2) = p_X(3) = 1/3$, and $p_X(x) = 0$ for other values of x. p_Y is the same.

(c) $\mu_X = \mu_Y = 1(1/3) + 2(1/3) + 3(1/3) = 2$

(d) $\mu_{XY} = \sum_{x=1}^{3}\sum_{y=1}^{3} xy\,p_{X,Y}(x,y) = \dfrac{1}{9}\sum_{x=1}^{3}\sum_{y=1}^{3} xy = \dfrac{1}{9}(1+2+3)(1+2+3) = 4$.
Another way to compute μ_{XY} is to note that X and Y are independent, so $\mu_{XY} = \mu_X\mu_Y = (2)(2) = 4$.

(e) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y = 4 - (2)(2) = 0$

32. (a) The values of X and Y must both be integers between 1 and 3 inclusive, and may not be equal. There are six possible values for the ordered pair (X,Y), specifically $(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)$. Each of these six ordered pairs is equally likely.

Therefore $p_{X,Y}(x,y) = 1/6$ for $(x,y) = (1,2),(1,3),(2,1),(2,3),(3,1),(3,2)$, and $p_{X,Y}(x,y) = 0$ for other values of (x,y).

(b) The value of X is chosen from the integers 1, 2, 3 with each integer being equally likely.

Therefore $p_X(1) = p_X(2) = p_X(3) = 1/3$. The marginal probability mass function p_Y is the same.

To see this, compute

$p_Y(1) = p_{X,Y}(2,1) + p_{X,Y}(3,1) = 1/6 + 1/6 = 1/3$
$p_Y(2) = p_{X,Y}(1,2) + p_{X,Y}(3,2) = 1/6 + 1/6 = 1/3$
$p_Y(3) = p_{X,Y}(1,3) + p_{X,Y}(2,3) = 1/6 + 1/6 = 1/3$

(c) $\mu_X = \mu_Y = 1(1/3) + 2(1/3) + 3(1/3) = 2$

(d)
$$\begin{aligned}\mu_{XY} &= (1)(2)p_{X,Y}(1,2) + (1)(3)p_{X,Y}(1,3) + (2)(1)p_{X,Y}(2,1) \\ &\quad + (2)(3)p_{X,Y}(2,3) + (3)(1)p_{X,Y}(3,1) + (3)(2)p_{X,Y}(3,2) \\ &= [(1)(2) + (1)(3) + (2)(1) + (2)(3) + (3)(1) + (3)(2)](1/6) \\ &= \frac{11}{3}\end{aligned}$$

(e) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y = \dfrac{11}{3} - (2)(2) = -\dfrac{1}{3}$

33. (a) $\mu_X = \int_{-\infty}^{\infty} x f(x)\,dx$. Since $f(x) = 0$ for $x \le 0$, $\mu_X = \int_0^{\infty} x f(x)\,dx$.

(b) $\mu_X = \int_0^{\infty} x f(x)\,dx \ge \int_k^{\infty} x f(x)\,dx \ge \int_k^{\infty} k f(x)\,dx = kP(X \ge k)$

(c) $\mu_X/k \ge kP(X \ge k)/k = P(X \ge k)$

(d) $\mu_X = \mu_{(Y-\mu_Y)^2} = \sigma_Y^2$

(e) $P(|Y - \mu_Y| \ge k\sigma_Y) = P((Y-\mu_Y)^2 \ge k^2\sigma_Y^2) = P(X \ge k^2\sigma_Y^2)$

(f) $P(|Y - \mu_Y| \ge k\sigma_Y) = P(X \ge k^2\sigma_Y^2) \le \mu_X/(k^2\sigma_Y^2) = \sigma_Y^2/(k^2\sigma_Y^2) = 1/k^2$

34. $\mu_A = \pi\mu_{R^2}$. We now find μ_R^2.
$\sigma_R^2 = \mu_{R^2} - \mu_R^2$. Substituting, we obtain $1 = \mu_{R^2} - 10^2$. Therefore $\mu_{R^2} = 101$, and $\mu_A = 101\pi$.

35. (a) If the pooled test is negative, it is the only test performed, so $X = 1$. If the pooled test is positive, then n additional tests are carried out, one for each individual, so $X = n+1$. The possible values of X are therefore 1 and $n+1$.

(b) The possible values of X are 1 and 5. Now $X = 1$ if the pooled test is negative. This occurs if none of the individuals has the disease. The probability that this occurs is $(1-0.1)^4 = 0.6561$. Therefore $P(X = 1) = 0.6561$. It follows that $P(X = 5) = 0.3439$.
So $\mu_X = 1(0.6561) + 5(0.3439) = 2.3756$.

(c) The possible values of X are 1 and 7. Now $X = 1$ if the pooled test is negative. This occurs if none of the individuals has the disease. The probability that this occurs is $(1-0.2)^6 = 0.262144$. Therefore $P(X = 1) = 0.262144$. It follows that $P(X = 7) = 0.737856$.
So $\mu_X = 1(0.262144) + 7(0.737856) = 5.4271$.

(d) The possible values of X are 1 and $n+1$. Now $X = 1$ if the pooled test is negative. This occurs if none of the individuals has the disease. The probability that this occurs is $(1-p)^n$. Therefore $P(X = 1) = (1-p)^n$. It follows that $P(X = n+1) = 1-(1-p)^n$.
So $\mu_X = 1(1-p)^n + (n+1)(1-(1-p)^n) = n+1-n(1-p)^n$.

(e) The pooled method is more economical if $11-10(1-p)^{10} < 10$. Solving for p yields $p < 0.2057$.