

# Linear Transformations and Matrices

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## 2.1 LINEAR TRANSFORMATIONS, NULL SPACES, AND RANGES

3. The nullity is 0, and the rank is 2. Thus  $T$  is one-to-one, but not onto.
6. The nullity is  $n - 1$ , and the rank is 1. Thus  $T$  is not one-to-one unless  $n = 1$ , and  $T$  is not onto unless  $n = 1$ .
11.  $T(8, 11) = (5, -3, 16)$
18.  $T(a, b) = (b, 0)$ .  $N(T) = \text{span}\{(1, 0)\} = R(T)$ .
19. Define  $T = I$  and  $U = 2I$ .
23. All of  $\mathbb{R}^3$  or a plane in  $\mathbb{R}^3$  through the origin
24. (a)  $T(a, b) = (0, b)$       (b)  $T(a, b) = (0, b - a)$
25. (b)  $T(a, b, c) = (0, 0, c)$
26. (a)  $T = I_V$       (d)  $T = T_0$
27. (b) See (a) and (b) of Exercise 22.
31. (c) Let  $V = P(F)$  and  $V = \text{span}\{1\}$ . Define  $T$  first on the standard basis of  $V$  by  $T(1) = T(x) = 0$ , and  $T(x^k) = x^{k-1}$  for  $k \geq 2$ . Now extend  $T$  to a linear transformation from  $V$  to  $V$ . Then  $N(T) = \text{span}\{1, x\}$ , and  $R(T) = \text{span}\{x^k : k \geq 1\}$ . So  $V = R(T) \oplus W$ , but  $W \neq N(T)$ .

## 2.2 THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

2. (b)  $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \end{pmatrix}$       (e)  $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$
4.  $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
5. (c)  $(1\ 0\ 0\ 1)$       (d)  $(1\ 2\ 4)$       (f)  $\begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$       (g) (a)

**2.3 COMPOSITION OF LINEAR TRANSFORMATIONS AND MATRIX MULTIPLICATION**

2. (a)  $(AB)D = \begin{pmatrix} 29 \\ -26 \end{pmatrix}$

(b)  $A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$ ,  $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$ ,  $CA = (20 \ 26)$

3. (b)  $[h]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ ,  $[U(h)]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

4. (b)  $\begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$  (d) (12)

9.  $T(a_1, a_2) = (0, a_1 + a_2)$ ,  $U(a_1, a_2) = (0, a_1)$ ,

$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ,  $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$ ,  $CA = (20 \ 26)$

20. (a)  $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ ,  $B^3 = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{pmatrix}$  There are no cliques.

(b)  $B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ ,  $B^3 = \begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 3 \\ 3 & 0 & 3 & 2 \end{pmatrix}$

Persons 1, 3, and 4 belong to a clique.

23.  $\frac{n^2 - n}{2}$

**2.4 INVERTIBILITY AND ISOMORPHISMS**

14.  $T \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c)$

**2.5 THE CHANGE OF COORDINATE MATRIX**

2. (b)  $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$  (d)  $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

3. (b)  $\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$  (d)  $\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$  (f)  $\begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}$

6. (b)  $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ,  $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$
- (d)  $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$ ,  $[L_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$
7. (b)  $T(x, y) = \frac{1}{m^2+1}(x + my, mx + m^2y)$

## 2.6 DUAL SPACES

3. (b)  $f_1(a + bx + cx^2) = a$ ,  $f_2(a + bx + cx^2) = b$ ,  $f_3(a + bx + cx^2) = c$
4. The basis for  $V$  is  $\{(4, -3, -1), (6, .3, .1), (.2, .1, -.3)\}$
6. (a)  $T^t(f)(x, y) = 7x + 4y$       (b)  $[T^t]_{\beta^*} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$
- (c)  $[T]_\beta = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$  and  $([T]_\beta)^t = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

## 2.7 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

3. (b)  $\{1, e^t\}$       (d)  $\{e^{-t}, te^{-t}\}$
4.  $\{t, te^t, t^2e^t\}$
16. (a)  $\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{l}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{l}}t\right)$
- (b)  $\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)$
17.  $y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$
18. (a) *Case 1:*  $r^2 = 4km$ .  $y(t) = e^{-(r/2m)t}[c_1 + c_2t]$   
*Case 2:*  $r^2 > 4km$ .  $y(t) = c_1e^{at} + c_2e^{bt}$ , where  

$$a = \frac{-r}{2m} + \frac{\sqrt{r^2 - 4mk}}{2m}, \quad b = \frac{-r}{2m} - \frac{\sqrt{r^2 - 4mk}}{2m}$$
  
*Case 3:*  $r^2 < 4km$ .  $y(t) = e^{at}[c_1 \cos bt + c_2 \sin bt]$ , where  

$$a = \frac{-r}{2m}, \quad b = \frac{\sqrt{4mk - r^2}}{2m}$$
- (b) Referring to the three cases listed in (a):  
*Case 1:*  $y(t) = v_0te^{-(r/2m)t}$   
*Case 2:*  $y(t) = \frac{v_0m}{\sqrt{r^2 - 4mk}}[e^{at} - e^{bt}]$   
*Case 3:*  $y(t) = \frac{v_0}{b}e^{at} \sin bt$