Chapter 2

The Solution of Nonlinear Equations $f(x) = 0$

2.1 Iteration for Solving $x = g(x)$

1. (a) Clearly, $g(x) \in C[0,1]$. Since $g'(x) = -x/2 < 0$ on the interval $[0,1]$, the function $g(x)$ is strictly decreasing on the interval $[0,1]$. If g is strictly decreasing on $[0,1]$, then $g(0) = 1$ and $g(1) = 0$ imply that $g([0,1]) = [0,1] \subseteq [0,1]$. Thus, by Theorem 2.2, the function $g(x)$ has a fixed point on the interval $[0,1]$.

   In addition: $|f'(x)| = |-x/2| = x/2 \leq 1/2 < 1$ on the interval $[0,1]$. Thus, by Theorem 2.2, the function $g(x)$ has a unique fixed point on the interval $[0,1]$.

   (b) Clearly, $g(x) \in C[0,1]$. Since $g'(x) = -\ln(2)2^{-x} < 0$ on the interval $[0,1]$, the function $g(x)$ is strictly decreasing on the interval $[0,1]$. If g is strictly decreasing on $[0,1]$, then $g(0) = 1$ and $g(1) = 1/2$ imply that $g([0,1]) = [1/2,1] \subset [0,1]$. Thus, by Theorem 2.2,the function $g(x)$ has a fixed point on the interval $[0,1]$.

   In addition: $|g'(x)| = |-\ln(2)2^{-x}| = \ln(2)2^{-x} < \ln(2) < \ln(e) = 1$ on the interval $[0,1]$. Thus, by Theorem 2.2, the function $g(x)$ has an unique fixed point on the interval $[0,1]$.

   (c) Clearly $g(x)$ is continuous on $[0.5, 5.2]$ and $g([0.5, 5.2]) \not\subseteq [0.5, 5.2]$. But, $g([0.5, 2]) \subseteq [0.5, 2]$. Thus, the hypotheses of the first part of Theorem 2.2 are satisfied and g has a fixed point in $[0.5, 2]$. While $(1,1)$ is the unique fixed point in $[0.5, 2]$, $|f'(1)| = 1 \not< 1$, thus the hypothesee in part (4) of Theorem 2.2 cannot be satisfied.

2. (a)

$$\begin{aligned} g(x) &= x \\ -4 + 4x - \tfrac{1}{2}x^2 &= 0 \\ x^2 - 6x + 8 &= 0 \\ x &= 2, 4 \end{aligned}$$

and

$$\begin{aligned} g(2) &= -4 + 8 - 2 = 2 \\ g(4) &= -4 + 16 - 8 = 4 \end{aligned}$$

(b)

$$\begin{aligned} p_0 &= 1.9 \\ p_1 &= 1.795 \\ p_2 &= 1.5689875 \\ p_3 &= 1.04508911 \end{aligned}$$

(c)

$$\begin{aligned} p_0 &= 3.8 \\ p_1 &= 3.98 \\ p_2 &= 3.9998 \\ p_3 &= 3.99999998 \end{aligned}$$

(d) For part (b)

$$\begin{aligned} E_0 &= 0.1 & R_0 &= 0.95 \\ E_1 &= 0.205 & R_1 &= 0.1025 \\ E_2 &= 0.43110125 & R_2 &= 0.21550625 \\ E_3 &= 0.95491089 & R_3 &= 0.477455414 \end{aligned}$$

(e) The sequence in part (b) does not converge to $P = 2$. The sequence in part (c) converges to $P = 4$.

3. (a) $p_1 = \sqrt{13}$, $p_2 = \sqrt{6 + \sqrt{13}}$, converges
   (b) $p_1 = \frac{3}{2}$, $p_2 = \frac{7}{3}$, converges
   (c) $p_1 = 4.083333$, $p_2 = 5.537869$, diverges
   (d) $p_1 = -5.5$, $p_2 = -69.5$, diverges

4. The fixed points are $P = 2$ and $P = -2$. Since $g'(2) = 5$ and $g'(-2) = -3$, fixed- point iteration will not converge to $P = 2$ and $P = -2$, respectively.

5.

$$\begin{aligned} x &= x\cos(x) \\ x(1 - \cos(x)) &= 0 \\ x &= 2n\pi \end{aligned}$$

Thus $g(x)$ has infinitely many fixed points: $P = 2n\pi$, where $n \in Z$. Note:

$$|g'(2n\pi)| = |\cos(2n) - 2n\pi \sin(2n\pi)| = 1.$$

Thus Theorem 2.3 may not be used to find the fixed points of $g(x)$.

6. $|p_2 - p_1| = |g(p_1) - g(p_0)| = |g'(c_0)(p_1 - p_0)| < K|p_1 - p_0|$

7. $|E_1| = |P - p_1| = |g(P) - g(p_0)| = |g'(c_0)(P - p_0)| > |P - p_0| = |E_0|$

8. (a) By way of contradiction assume there exists k such that $p_{k+1} = g(p_k) \geq p_k$. It follows that:

$$\begin{aligned} -0.0001p_k^2 + p_k &\geq p_k \\ 0.0001p_k^2 &> 0 \\ p_k &= 0 \end{aligned}$$

Thus $p_{k-1} = 0$ or $p_{k-1} = 10,000$. Clearly, $p_{k-1} \neq 10,000$, since the maximum value of $g(x)$ is 2500. Thus, if $p_k = 0$, then $p_{k-1}, \ldots, p_1 = 0$. A contradiction to the hypothesis $p_0 = 1$. Therefore, $p_0 > p_1 > \cdots > p_n > p_{n-1} > \cdots$.

(b) By way of contradiction assume there exists k such that $p_j \leq 0$. It follows that:

$$\begin{aligned} g(p_{j-1}) &\leq 0 \\ -0.0001p_{j-1}^2 + p_{j-1} &\leq 0 \\ (-0.0001p_{j-1} + 1)p_{j-1} &\leq 0 \end{aligned}$$

From part (a); if $p_{j-1} = 0$, then $p_1 \neq 0$. Thus $p_{j-1} \neq 0$. If $p_{j-1} < 0$, then

$$\begin{aligned} -0.0001p_{j-1} + 1 &\geq 0 \\ p_{j-1} &\geq 10,000, \end{aligned}$$

a contradiction. If $p_{j-1} > 0$, then

$$\begin{aligned} -0.0001p_{j-1} + 1 &\geq 0 \\ p_{j-1} &\leq 10,000, \end{aligned}$$

a contradiction. Therefore, $p_n > 0$ for all n.

(c) $\lim_{n\to\infty} p_n = 0$

9. (a) $g(3) = (0.5)(3) + 1.5 = 3$

(b) $|P - p_n| = |3 - 1.5 - 0.5p_{n-1}| = |1.5 - 0.5p_{n-1}| = \frac{1}{2}|3 - p_{n-1}| = \frac{1}{2}|P - p_{n-1}|$

(c) Using mathematical induction we note that $|P - p_1| = \frac{1}{2}|P - p_0|$ and assume that $|P - p_k| = \frac{1}{2^k}|P - p_0|$. Thus

$$\begin{aligned} |P - p_{k+1}| &= \frac{|P-p_k|}{2} \\ &= \frac{|P-p_0|}{2(2^k)} \\ &= \frac{|P-p_0|}{2^{k+1}} \end{aligned}$$

10. (a) Note: $p_1 = p_0/2,\ p_2 = p_0/2^2, \ldots, p_{k+1} = p_0/2^{k+1}, \ldots$. Thus

$$\frac{|p_{k+1} - p_k|}{|p_{k+1}|} = \frac{|2^{-k-1} - 2^{-k}|}{|2^{-k-1}|} = \frac{2^{-k}(1 - 2^{-1})}{2^{-k}2^{-1}} = 1$$

(b) Clearly, the stopping criteria will (theoretically) never be satisfied.

11. In inequality (11): $|P - p_n| \le K^n|P - p_0|$, where $|g'(x)| \le K < 1$. Therefore, the smaller the value of K the faster fixed-point iteration converges.

2.2 Bracketing Methods for Locating a Root

1.

$$\begin{aligned} I_0 &= (0.11 + 0.12)/2 = 0.115 & A(0.115) &= 254,403 \\ I_1 &= (0.11 + 0.115)/2 = 0.1125 & A(0.1125) &= 246,072 \\ I_2 &= (0.1125 + .125)/2 = 0.11375 & A(0.11375) &= 250,198 \end{aligned}$$

2.

$$\begin{aligned} I_0 &= (0.13 + 0.14)/2 = 0.135 & A(0.135) &= 394,539 \\ I_1 &= (0.135 + 0.14)/2 = 0.1375 & A(0.1375) &= 408,435 \\ I_2 &= (0.135 + 0.1375)/2 = 0.13625 & A(0.13625) &= 401,420 \end{aligned}$$

3. (a) $f(-3) > 0$, $f(0) < 0$, and $f(3) > 0$; thus roots lie in the intervals $[-3, 0]$ and $[0, 3]$.

(b) $f(\pi/4) > 0$ and $f(\pi/2) < 0$; thus a root lies in the interval $[\pi/4, \pi/2]$.

(c) $f(3) < 0$ and $f(5) > 0$; thus a root lies in the interval $[3, 5]$.

(d) $f(3) > 0$, $f(5) < 0$, and $f(7) > 0$; thus roots lie in the intervals $[3, 5]$ and $[5, 7]$.

4. $[-2.4, -1.6], [-2.0, -1.6], [-2.0, -1.8], [-1.9, -1.8], [-1.85, -1.80]$

5. $[0.8, 1.6], [1.2, 1.6], [1.2, 1.4], [1.2, 1.3], [1.25, 1.30]$

6. $[3.2, 4.0], [3.6, 4.0], [3.6, 3.8], [3.6, 3.7], [3.65, 3.70]$

7. $[6.0, 6.8], [6.4, 6.8], [6.4, 6.6], [6.5, 6.5], [6.40, 6.45]$

8. (a) Starting with $a_0 < b_0$, then either $a_1 = a_0$ and $b_1 = \frac{a_0+b_0}{2}$, or $a_1 = \frac{a_0+b_0}{2}$ and $b_1 = b_0$. In either case we have $a_0 \le a_1 < b_1 \le b_0$. Now assume that the result is true for $n = 1, 2, \ldots, k$; in particular $a_0 \le a_1 \le \cdots \le a_k < b_k \le \cdots \le b_1 \le b_0$. Then either $a_{k+1} = a_k$ and $b_{k+1} = \frac{a_k+b_k}{2}$, or $a_{k+1} = \frac{a_k+b_k}{2}$ and $b_{k+1} = b_k$. In either case we have $a_k \le a_{k+1} < b_{k+1} \le b_k$. Hence $a_0 \le a_1 \le \cdots \le a_k \le a_{k+1} < b_{k+1} \le b_k \le \cdots \le b_1 \le b_0$. Thus by mathematical induction we have proven that $a_0 \le a_1 \le \cdots \le a_n < b_n \le \cdots \le b_1 \le b_0$ for all n.

   (b) From part (a) either $a_1 = a_0$, $b_1 = \frac{a_0+b_0}{2}$, and $b_1 - a_1 = \frac{b_0-a_0}{2}$ or $a_1 = \frac{a_0+b_0}{2}$, $b_1 = b_0$, and $b_1 - a_1 = \frac{b_0-a_0}{2}$. Now assume that the result is true for $n = 1, 2, \ldots, k$, in particular $b_k - a_k = \frac{b_0-a_0}{2^k}$. Then either $a_{k+1} = a_k$, $b_{k+1} = \frac{a_k+b_k}{2}$, and $b_{k+1} - a_{k+1} = \frac{b_k-a_k}{2} = \frac{b_0-a_0}{2^{k+1}}$ or $a_{k+1} = \frac{a_k+b_k}{2}$, $b_{k+1} = b_k$, and $b_{k+1} - a_{k+1} = \frac{b_k-a_k}{2} = \frac{b_0-a_0}{2^{k+1}}$. Thus by mathematical induction we have proven that $b_n - a_n = \frac{b_0-a_0}{2^n}$ for all n.

   (c) Using part (a) it follows that the sequence $\{a_n\}$ is non decreasing and bounded above by b_0, hence it is a convergent sequence and we write $\lim_{n\to\infty} a_n = L_1$. Similarly, the sequence $\{b_n\}$ is non-increasing and bounded below by a_0, hence it is a convergent sequence and we write $\lim_{n\to\infty} b_n = L_2$

   To show that the two limits are equal we observe that

$$\begin{aligned} L_2 &= \lim_{n\to\infty} b_n \\ &= \lim_{n\to\infty}(a_n + (b_n - a_n)) \\ &= \lim_{n\to\infty} a_n + \lim_{n\to\infty}(b_n - a_n) \\ &= L_1 + \lim_{n\to\infty} \tfrac{b_0-a_0}{2^n} \\ &= L_1 + 0 = L_1 \end{aligned}$$

   Since $a_n \le c_n \le b_n$ the squeeze principle for limits implies that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n$$

9. (a) The function does not change sign on the interval $[3, 7]$.

   (b) $\lim_{n\to\infty} a_n = 2 = \lim_{n\to\infty} b_n$, but $f(x)$ is undefined at 2.

10. (a) It will converge to the zero at $x = \pi$.

   (b) $\lim_{n\to\infty} a_n = \pi/2 = \lim_{n\to\infty} b_n$, but $f(x)$ is undefined at $\pi/2$.

11. Solve:

$$\begin{aligned} \tfrac{7-2}{2^N} &< 5\times 10^{-9} \\ \ln(5) - N\ln(2) &< \ln(5\times 10^{-9}) \\ N &> \tfrac{\ln(5) - \ln(5\times 10^{-9})}{\ln(2)} \\ N &> 29.89735 \end{aligned}$$

Thus $N = 30$.

12.

$$
\begin{aligned}
c_n &= b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)} \\
&= \frac{b_n(f(b_n) - f(a_n)) - f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)} \\
&= \frac{-b_n f(a_n) + a_n f(b_n)}{f(b_n) - f(a_n)} \\
&= \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}
\end{aligned}
$$

13.

$$
\begin{aligned}
\frac{|b - a|}{2^{N+1}} &< \delta \\
\ln\left(\frac{|b - a|}{2^{N+1}}\right) &< \ln(\delta),
\end{aligned}
$$

since ln is a strictly increasing function. Thus

$$
\begin{aligned}
\ln(b - a) - (N+1)\ln(2) &< \ln(\delta) \\
\frac{\ln(b - a) - \ln(\delta)}{\ln(2)} &< N + 1 \\
N &> \frac{\ln(b - a) - \ln(\delta)}{\ln(2)} - 1
\end{aligned}
$$

Therefore, the smallest value of N is

$$N = int\left(\frac{\ln(b - a) - \ln(\delta)}{\ln(2)}\right)$$

14. The bisection method can't converge to $x = 2$, unless $c_n = 2$ for some $n \geq 1$.

15. We refer the reader to "Which Root Does the Bisection Algorithm Find?" by George Corliss, Mathematical Modeling: Classroom Notes in Applied Mathematics, Murray Klankin Ed., SIAM, 1987.

2.3 Initial Approximation and Convergence Criteria

1. Approximate root location -0.7. Computed root -0.7034674225.

2. Approximate root location 0.7. Computed root 0.7390851332.

3. Approximate root locations -1.0 and 0.6. Computed roots -1.002966954 and 0.6348668712.

4. Approximate root locations ± 1.8. Computed roots ± 1.807375379.

5. Approximate root locations 1.4 and 3.0. Computed roots 1.412391172 and 3.057103550.

6. Approximate root locations ± 1.2 and 0.

2.4 Newton-Raphson and Secant Methods

1. (a) $p_k = p_{k-1} - \frac{p_{k-1}^2 - p_{k-1} + 2}{2p_{k-1} - 1}$

   (b)

$$
\begin{aligned}
p_0 &= -1.5 \\
p_1 &= -0.0625 \\
p_2 &= 1.7743 \\
p_3 &= 0.4505
\end{aligned}
$$

2. (a) $p_k = p_{k-1} - \frac{p_{k-1}^2 - p_{k-1} - 3}{2p_{k-1} - 1}$

   (b)

$$
\begin{aligned}
p_0 &= 1.6 \\
p_1 &= 2.52727 \\
p_2 &= 2.31521 \\
p_3 &= 2.30282
\end{aligned}
$$

   (c)

$$
\begin{aligned}
p_0 &= 0.0 \\
p_1 &= -3.0 \\
p_2 &= -1.7143 \\
p_3 &= 1.3416 \\
p_4 &= -1.3410
\end{aligned}
$$

3. (a) $p_k = p_{k-1} - \frac{1}{4}(p_{k-1} - 1)$

   (b)

$$
\begin{aligned}
p_0 &= 2.1 \\
p_1 &= 2.075 \\
p_2 &= 2.05625 \\
p_3 &= 2.0421875 \\
p_4 &= 2.031640625
\end{aligned}
$$

   (c) Convergence is linear. The error is reduced by a factor of $\frac{3}{4}$ with each iteration.

4. (a) $p_k = p_{k-1} - \frac{p_{k-1}^3 - 3p_{k-1} - 2}{3p_{k-1}^2 - 3}$

   (b)

$$
\begin{aligned}
p_0 &= 2.1 \\
p_1 &= 2.00606062 \\
p_2 &= 2.00002434 \\
p_3 &= 2.00000000 \\
p_4 &= 2.00000000
\end{aligned}
$$

   (c) Convergence is quadratic. The number of accurate decimal places (roughly) doubles with each iteration.

5. (a) $p_k = p_{k-1} + \frac{1}{\tan(p_{k-1})}$

   (b) No, $p_0 = 3$, $p_1 = -4.01525$. The sequence $\{p_k\}$ converges to $-\frac{3\pi}{2}$.

   (c) Yes, $p_0 = 5$, $p_1 = 4.70149$. The sequence $\{p_k\}$ converges to $\frac{3\pi}{2}$.

6. (a) $p_k = p_{k-1} - (1 + p_{k-1}^2)\arctan(p_{k-1})$

   (b) i.

$$
\begin{aligned}
p_0 &= 1.0 \\
p_1 &= -0.570796327 \\
p_2 &= -0.116859904 \\
p_3 &= -0.0010061022 \\
p_4 &= 0.000000001
\end{aligned}
$$

   ii. $\lim_{n\to\infty} p_k = 0.0$

   (c) i.

$$
\begin{aligned}
p_0 &= 2.0 \\
p_1 &= -3.535743590 \\
p_2 &= 13.95095909 \\
p_3 &= -279.344667 \\
p_4 &= 122016.9990
\end{aligned}
$$

   ii. The sequence is a case of divergent oscillation.

7. (a) $p_k = p_{k-1} - \frac{p_{k-1}}{1 - p_{k-1}}$

   (b) i.

$$
\begin{aligned}
p_0 &= 0.20 \\
p_1 &= -0.05 \\
p_2 &= -0.002380952 \\
p_3 &= -0.000005655 \\
p_4 &= -0.000000000
\end{aligned}
$$

   ii. $\lim_{n\to\infty} p_k = 0.0$

(c) i.

$$\begin{aligned} p_0 &= 20.0 \\ p_1 &= 21.05263158 \\ p_2 &= 22.10250034 \\ p_3 &= 23.14988809 \\ p_4 &= 24.19503505 \end{aligned}$$

ii. $\lim_{n\to\infty} p_k = \infty$

(d) $f(p_4) \approx 0.00000000075155$

8. $p_2 = 2.419354484$, $p_3 = 2.414364464$

9. $p_2 = 2.463371308$, $p_3 = 2.270027831$

10. $p_2 = -1.52140264$, $p_3 = -1.52137968$

11. Following the procedure outlined in Corollary 2.2, we assume that A is a real number and find the Newton-Raphson iteration function $g(x)$ for the function $f(x) = x^3 - A$. Thus

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{x^3 - A}{3x^2} \\ &= x - \frac{x - \frac{A}{x^2}}{3} \\ &= \frac{2x + \frac{A}{x^2}}{3} \end{aligned}$$

Now let p_0 be an initial approximation to $\sqrt[3]{A}$. Thus the Newton–Raphson iteration is defined by

$$p_k = \frac{2p_{k-1} + A/p_{k-1}^2}{3}$$

for $k = 1, 2, \ldots$.

12. (a) $\sqrt[N]{A}$

(b) Following the procedure outlined in Corollary 2.2, we assume that A is an appropriate real number and find the Newton-Raphson iteration function $g(x)$ for the function $f(x) = x^N - A$. Thus

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{x^N - A}{Nx^{N-1}} \\ &= x - \frac{x - \frac{A}{x^{N-1}}}{N} \\ &= \frac{(N-1)x + \frac{A}{x^{N-1}}}{N} \end{aligned}$$

Now let p_0 be an initial approximation to $\sqrt[N]{A}$. Thus the Newton-Raphson iteration is defined by

$$p_k = \frac{(N-1)p_{k-1} + A/p_{k-1}^{N-1}}{N}$$

for $k = 1, 2, \ldots$.

13. No, because $f(x)$ has no real zeros.

14. No, because $f'(x)$ is not continuous at the root $x = 0$.

15. No, because $f(x)$ is not defined on an interval about the root $x = 0$.

16. From (12) and (13) we see that (11) is the Newton-Raphson recursive rule for the function $f(x) = x^2 - A$. The zeros of f are $\pm\sqrt{A}$. It follows from Theorem 2.5 that there is a p_0 such that (11) converges to $\sqrt{A}$.

17. (a) $g(p) = p - \frac{f(p)}{f'(p)} = p$ which implies that $-\frac{f(p)}{f'(p)} = 0$, which implies that $f(p) = 0$.

    (b) $g'(p) = 1 - \frac{(f'(p))^2 - f(p)f''(p)}{(f'(p))^2} = \frac{f(p)f''(p)}{(f'(p))^2} = \frac{0}{(f'(p))^2} = 0$. Since $g'(p) = 0$ and $g'(p)$ is a continuous function, choose $c = 1$. Then there exists an interval $(p-d, p+d)$ in which $|g'(x)| < c$ or $|g'(x)| < 1$. Therefore, Theorem 2.2 implies that $\lim_{n\to\infty} p_n = p$.

18. (a) Given

$$0 = f(p_k) + f'(p_k)(p - p_k) + \frac{1}{2}f''(c_k)(p - p_k)^2$$

then

$$\begin{aligned} f(p_k) + f'(p_k)(p - p_k) &= -\frac{1}{2}f''(c_k)(p - p_k)^2 \\ (p - p_k) + \frac{f(p_k)}{f'(p_k)} &= -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2 \end{aligned}$$

    (b) The last expression in part (a) can be written as:

$$\begin{aligned} p - \left(p_k - \frac{f(p_k)}{f'(p_k)}\right) &= -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2 \\ p - p_{k+1} &= -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2 \end{aligned}$$

Assuming $f'(p_k) \approx f'(p)$ and $f''(c_k) \approx f''(p)$ when k is sufficiently large yields

$$\begin{aligned} p - p_{k+1} &\approx -\frac{f''(p)}{2f'(p)}(p - p_k)^2 \\ E_{k+1} &\approx -\frac{f''(p)}{2f'(p)}E_k \\ |E_{k+1}| &\approx \frac{|f''(p)|}{2|f'(p)|}|E_k| \end{aligned}$$

19. (a) If $1/4 \le q < 1$, then

$$-2 < \log_2(q) < 0$$
$$-2 + 2m \le log_2(q) + 2m < 2m$$
$$\frac{1}{4}(2^{2m}) \le q \times 2^{2m} < 2^{2m}$$

By the Squeeze or Sandwich Theorem $lim_{m\to-\infty} q \times 2^{2m} = 0$ and $lim_{m\to\infty} q \times 2^{2m} = \infty$. Therefore, if $A \in \Re^+$, then there exists $m \in Z$ and $q \in [1/4, 1)$ such that $A = q \times 2^{2m}$.

(b) If $A \in \Re^+$ then $\sqrt{A} = \sqrt{q \times 2^{2m}} = q^{1/2} \times 2^m$.

20. (a)

$$\begin{aligned} p_{k+1} &= p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})} \\ &= \frac{p_k(f(p_k) - f(p_{k-1})) - f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})} \\ &= \frac{-p_k f(p_{k-1}) + p_{k-1} f(p_k)}{f(p_k) - f(p_{k-1})} \\ &= \frac{p_{k-1} f(p_k) - p_k f(p_{k-1})}{f(p_k) - f(p_{k-1})} \end{aligned}$$

(b) As the number of iterations increases the precision of the difference in the numerator can lead to a reduction in the precision of p_{k+1}.

21. If p is a root of multiplicity $M = 2$, then $f(x) = (x-p)^2 q(x)$ and $q(p) \neq 0$. Consider

$$\begin{aligned} h(x) &= x - \frac{2f(x)}{f'(x)} \\ &= x - \frac{2(x-p)^2 q(x)}{((x-p)^2 q(x))'} \\ &\approx x - \frac{2(x-p)q(x)}{(2(x-p)q(x))'} \\ &= x - \frac{k(x)}{k'(x)} \end{aligned}$$

Since p is a root of multiplicity $M - 1$ of $k(x)$ it follows that the Newton-Raphson method

$$p_k = p_{k-1} - \frac{2f(p_{k-1})}{f'(p_{k-1})}$$

converges quadratically.

22. (a) Halley's formula for finding $\sqrt{A}$ is:

$$g(x) = x - \frac{x^2 - A}{2x}\left(1 - \frac{(x^2 - A)(2)}{2(2x)^2}\right)^{-1} = \frac{x(x^2 + 3A)}{3x^2 + A}$$

When $A = 5$, Halley's iteration formula becomes

$$p_k = \frac{p_{k-1}^3 + 15p_{k-1}}{3p_{k-1}^2 + 5}$$

and $p_1 = 2.2352941176$, $p_2 = 2.2360679775$, and $p_3 = 2.2360679775$.

(b) Halley's formula for $f(x) = x^3 - 3x + 2$ is $g(x) = \frac{x^3+2x^2+4x+2}{2x^2+4x+3}$ and $p_1 = -2.0130081301$, $p_2 = -2.0000007211$, and $p_3 = -2.0000000000$.

23. (a)

$$\begin{aligned} h(x) &= \frac{f(x)}{f'(x)} \\ &= \frac{(x-p)^M q(x)}{M(x-p)^{M-1}q(x) + (x-p)^M q'(x)} \\ &= \frac{(x-p)^M q(x)}{(x-p)^{M-1}\left(Mq(x) + (x-p)q'(x)\right)} \\ &= (x-p)\left(\frac{q(x)}{Mq(x) + (x-p)q'(x)}\right) \\ &= (x-p)s(x) \end{aligned}$$

Note

$$s(p) = \frac{q(p)}{Mq(p)} = \frac{1}{M} \neq 0.$$

Therefore, $h(x)$ has a simple root at p.

(b) From (5) the Newton-Raphson iterative function for $h(x)$ is

$$g(x) = x - \frac{h(x)}{h'(x)}.$$

Making the substitution $h(x) = f(x)/f'(x)$ yields

$$\begin{aligned} g(x) &= x - \frac{f(x)/f'(x)}{\left(\frac{f(x)}{f'(x)}\right)'} \\ &= x - \frac{f(x)/f'(x)}{\frac{f'(x)f'(x) - f(x)f''}{(f'(x))^2}} \\ &= x - \frac{f(x)f'(x)}{(f'(x))^2 - f(x)f''(x)} \end{aligned}$$

(c) The iteration function g in part (b) is the Newton-Raphson iterative function of a function h with a simple root at p. Therefore, by Theorem 2.6, iteration using g in part (b) converges quadratically to p

(d) $p_1 = 0.782537832237921$, $p_2 = 0.265581322223138$, $p_3 = 0.000186285551512$

24. It appears that the error in each successive iteration is proportional to the cube of the error in the previous iteration: $E_{n+1} \approx AE_n^3$, i.e.; $R = 3$. The value $A = 3/4$ is a reasonable estimate for the proportionality constant.

2.5 Aitken's Process and Steffensen's and Muller's Methods

1. (a) $\Delta p_n = 0$

   (b) $\Delta p_n = 6(n+1) + 2 - 6n - 2 = 6$

   (c) $\Delta p_n = (n+1)(n+2) - n(n+1) = 2(n+1)$

2. (a) $\Delta^2 p_n = \Delta(\Delta p_n) = \Delta(2(n+1)^2 + 1 - 2n^2 - 1) = \Delta(4n+2) = 4$

   (b) $\Delta^3 p_n = \Delta(\Delta^2 p_n) = \Delta(4) = 0$

   (c) $\Delta^4 p_n = \Delta(\Delta^3 p_n) = \Delta(0) = 0$

3.

$$\Delta p_n = \Delta(1/2^{n+1} - 1/2^n) = -1/2^{n+1}$$

and

$$\Delta^2 p_n = \Delta(\Delta p_n) = \Delta(-1/2^{n+1}) = -\frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} = \frac{1}{2^{n+2}}$$

thus

$$q_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} = \frac{1}{2^n} - \frac{1/2^{2n+2}}{1/2^{n+2}} = \frac{1}{2^n} - \frac{1}{2^n} = 0$$

4.

$$\Delta p_n = \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)}$$

and

$$\begin{aligned} \Delta p_n^2 &= \Delta(\Delta p_n) \\ &= \Delta\left(-\tfrac{1}{n(n+1)}\right) \\ &= -\tfrac{1}{(n+1)(n+2)} + \tfrac{1}{n(n+1)} \\ &= \tfrac{2}{n(n+1)(n+2)} \end{aligned}$$

thus

$$q_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} = \frac{1}{n} - \frac{\left(-\frac{1}{n(n+1)}\right)^2}{\frac{2}{n(n+1)(n+2)}} = \frac{1}{2(n+1)}$$

5.

$$
\begin{aligned}
q_n &= p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} \\
&= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= \frac{p_n p_{n+2} - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= \frac{\frac{1}{2^n-1}\frac{1}{2^{n+2}-1} - \frac{1}{(2^{n+1}-1)^2}}{\frac{1}{2^{n+2}-1} - \frac{2}{2^{n+1}-1} + \frac{1}{2^n-1}} \\
&= \frac{2^n}{2^n(2^{n+1}-1)(2^{n+1}+1)} \\
&= \frac{1}{(2^2)^{n+1} - 1} = \frac{1}{4^{n+1} - 1}
\end{aligned}
$$

6. $p_n = 1/(4^n + 4^{-n})$

n	p_n	q_n Aitken's
0	0.5	-0.26437542
1	0.23529412	-0.00158492
2	0.06225681	0.00002390
3	0.01562119	0.00000037
4	0.00390619	
5	0.00097656	

7. $g(x) = (6+x)^{1/2}$

n	p_n	q_n Aitken's
0	2.5	3.00024351
1	2.91547595	3.00000667
2	2.98587943	3.00000018
3	2.99764565	3.00000001
4	2.99960758	
5	2.99993460	

8. $g(x) = \ln(x+2)$

n	p_n	q_n Aitken's
0	3.14	3.14619413
1	3.14422280	3.14619331
2	3.14556674	3.14619323
3	3.14599408	3.14619322
4	3.14612992	
5	3.14617310	

9. $\cos(x) - 1 = 0$

n	p_n Steffenson's
0	0.5
1	0.24465808
2	0.12171517
3	0.00755300
4	0.00377648
5	0.00188824
6	0.00000003

10. In formula (4) let $p_n = S_n$ and $q_n = T_n$, then $\Delta S_n = S_{n+1} - S_n = A_{n+1}$ and $\Delta^2 S_n = \Delta(\Delta S_n) = \Delta A_{n+1} = A_{n+2} - A_{n+1}$. Substituting into formula (4) yields $T_n = S_n - \frac{A_{n+1}^2}{A_{n+2} - A_{n+1}}$.

11. The sum of the series is 99.

n	S_n	T_n
1	0.99	98.9999988
2	1.9701	99.0000017
3	2.940399	98.9999988
4	3.90099501	98.9999992
5	4.85198506	
6	5.79346521	

12. The sum is $S \approx 0.31838039$

n	S_n	T_n
1	0.23529412	0.31840462
2	0.29755093	0.31838076
3	0.31317211	0.31838039
4	0.31707830	0.31838039
5	0.31805487	
6	0.31829901	

13. The sum of the series if 4.

n	S_n	T_n
1	1.0	5.0
2	2.0	4.25
3	2.75	4.08333333
4	3.25	4.031215
5	3.5625	4.0125
6	3.75	4.00520833
7	3.859375	4.00223215
8	3.921875	4.0097656

14. The sum of the series is ln(2).

n	S_n	T_n
1	0.5	0.6875
2	0.625	0.69166667
3	0.66666667	0.69270833
4	0.68229167	0.69300595
5	0.68854167	0.69309896
6	0.69114583	0.69312996
7	0.69226191	0.69314081
8	0.69275019	0.69314476

15. $f(x) = x^3 - x - 3$.

n	p_n	$f(p_n)$
0	1.0	-2.0
1	1.2	1.472
2	1.4	-0.656
3	1.52495614	0.02131598
4	1.52135609	-0.00014040
5	1.52137971	0.00000001

16. $f(x) = 4x^2 - e^x$.

n	p_n	$f(p_n)$
0	4.0	9.40184997
1	4.1	6.89971240
2	4.2	3.87366896
3	4.30844335	-0.07396483
4	4.30657286	0.00047140
5	4.30658473	0.00000005

17. (a)

$$\begin{aligned}\Delta(p_n + q_n) &= (p_{n+1} + q_{n+1}) - (p_n + q_n)\\ &= (p_{n+1} - p_n) + (q_{n+1} - q_n)\\ &= \Delta p_n + \Delta q_n\end{aligned}$$

(b)

$$\begin{aligned}\Delta(p_n q_n) &= p_{n+1}q_{n+1} - p_n q_n\\ &= p_{n+1}q_{n+1} - p_{n+1}q_n + p_{n+1}q_n - p_n q_n\\ &= p_{n+1}\Delta q_n + q_n \Delta p_n\end{aligned}$$

18.

$$
\begin{aligned}
p &\approx \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= p_{n+2} - p_{n+2} + \frac{p_{n+2}p_n - p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= p_{n+2} + \frac{p_{n+2}p_n - p_{n+1}^2 - p_{n+2}^2 + 2p_{n+2}p_{n+1} - p_n p_{n+2}}{p_{n+2} - 2p_{n+1} + p_n} \\
&= p_{n+2} - \frac{p_{n+2}^2 - 2p_{n+2}p_{n+1} + p_{n+1}^2}{p_{n+2} - 2p_{n+1} + p_n} \\
&= p_{n+2} - \frac{(p_{n+2} - p_{n+1})^2}{p_{n+2} - 2p_{n+1} + p_n}
\end{aligned}
$$

19. (a) $E_N = K^N E_0$

(b) From part (a), if $E_N = K^N E_0$, then

$$
\begin{aligned}
|E_N| &= |K^N E_0| \\
|K|^N &< \frac{10^{-8}}{|E_0|} \\
N &> \frac{-8 - \log_{10}|E_0|}{\log_{10}|K|} \\
N &= int\left(\frac{-8 - \log_{10}|E_0|}{\log_{10}|K|}\right) + 1
\end{aligned}
$$