# CHAPTER 3

B-3-1. $$J = \tfrac{1}{2} mR^2 = \tfrac{1}{2} \times 100 \times 0.5^2 = 12.5 \text{ kg-m}^2$$

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B-3-2. Assume that the body of known moment of inertia $J_0$ is turned through a small angle θ about the vertical axis and then released. The equation of motion for the oscillation is

$$J_0 \ddot{\theta} = -k\theta$$

where k is the torsional spring constant of the string. This equation can be written as

$$\ddot{\theta} + \frac{k}{J_0}\theta = 0$$

or

$$\ddot{\theta} + \omega_n^2 \theta = 0$$

where

$$\omega_n = \sqrt{\frac{k}{J_0}}$$

The period $T_0$ of this oscillation is

$$T_0 = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{k}{J_0}}} \tag{1}$$

Next, we attach a rotating body of unknown moment of inertia J and measure the period T of oscillation. The equation for the period T is

$$T = \frac{2\pi}{\sqrt{\frac{k}{J}}} \tag{2}$$

By eliminating the unknown torsional spring constant k from Equations (1) and (2), we obtain

$$\frac{2\pi\sqrt{J_0}}{T_0} = \frac{2\pi\sqrt{J}}{T}$$

Hence

$$J = J_0\left(\frac{T}{T_0}\right)^2 \tag{3}$$

The unknown moment of inertia J can therefore be determined by measuring the period of oscillation T and substituting it into Equation (3).

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B-3-3. Define the vertical displacement of the ball as $x(t)$ with $x(0) = 0$. The positive direction is downward. The equation of motion for the system is

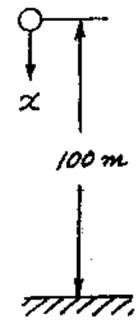
$$m\ddot{x} = mg$$

with initial conditions $x(0) = 0$ m and $\dot{x}(0) = 0$ m/s. So we have

$$\ddot{x} = g$$

$$\dot{x} = gt + \dot{x}(0) = gt$$

$$x = \tfrac{1}{2} gt^2 + x(0) = \tfrac{1}{2} gt^2$$




Assume that at $t = t_1$ the ball reaches the ground. Then

$$100 = \tfrac{1}{2} \times 9.807\ t_1{}^2$$

from which we obtain

$$t_1 = 4.516 \text{ s}$$

The ball reaches the ground in 4.516 s.

The velocity of the ball when it hits the ground is

$$\dot{x}(4.516) = 9.807 \times 4.516 = 44.288 \text{ m/s}$$

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B-3-4. Define the torque applied to the flywheel as T. The equation of motion for the system is

$$J\ddot{\theta} = T, \qquad \theta(0) = 0, \qquad \dot{\theta}(0) = 0$$

from which we obtain

$$\dot{\theta} = \frac{T}{J} t$$

By substituting numerical values into this equation, we have

$$20 \times 6.28 = \frac{T}{50} \times 5$$

Thus

$$T = 1256 \text{ N-m}$$

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B-3-5. $$J\ddot{\theta} = -T \qquad (T = \text{braking torque})$$

Integrating this equation,

$$\dot{\theta} = -\frac{T}{J} t + \dot{\theta}(0), \qquad \dot{\theta}(0) = 100 \text{ rad/s}$$

Substituting the given numerical values,

$$20 = -\frac{T}{J} \times 15 + 100$$

Solving for T/J, we obtain

$$\frac{T}{J} = 5.33$$

Hence, the deceleration given by the brake is 5.33 rad/s$^2$.

The total angle rotated in 15-second period is obtained from

$$\theta(t) = -\frac{T}{J}\frac{t^2}{2} + \dot{\theta}(0)t + \theta(0), \qquad \theta(0) = 0, \qquad \dot{\theta}(0) = 100$$

as follows;

$$\theta(15) = -5.33 \times \frac{15^2}{2} + 100 \times 15 = 900 \text{ rad}$$

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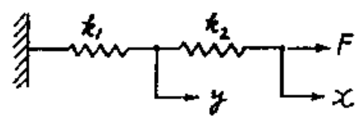
B-3-6. The equations for the system are

$$F = k_2(x - y)$$

$$k_2(x - y) = k_1 y$$




Eliminating y from the two equations gives

$$F = k_2\left(x - \frac{k_2 x}{k_1 + k_2}\right) = \frac{k_1 k_2}{k_1 + k_2} x = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} x$$

The equivalent spring constant $k_{eq}$ is then obtained as

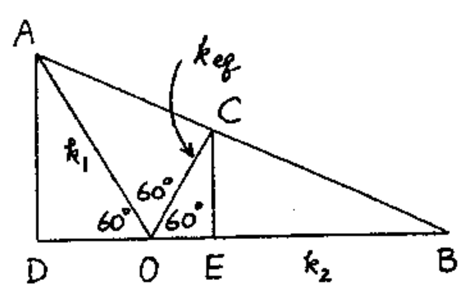
$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

Next, consider the figure shown below. Note that $\Delta$ABD and $\Delta$CBE are similar. So we have

$$\frac{\overline{CE}}{\overline{AD}} = \frac{\overline{BE}}{\overline{BD}}$$

which can be rewritten as

$$\frac{\overline{OC}\frac{\sqrt{3}}{2}}{\overline{OA}\frac{\sqrt{3}}{2}} = \frac{\overline{OB} - \overline{OC}\frac{1}{2}}{\overline{OB} + \overline{OA}\frac{1}{2}}$$




or

$$\overline{OC}(\overline{OB} + \tfrac{1}{2}\overline{OA}) = \overline{OA}(\overline{OB} - \tfrac{1}{2}\overline{OC})$$

Solving for $\overline{OC}$, we obtain

$$\overline{OC} = \frac{\overline{OA}\ \overline{OB}}{\overline{OA} + \overline{OB}} = \frac{1}{\frac{1}{\overline{OA}} + \frac{1}{\overline{OB}}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = k_{eq}$$

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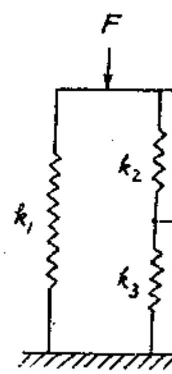
**B-3-7.** Assume that we apply force F to the spring system. Then

$$F = k_1 x + k_2(x - y)$$

$$k_2(x - y) = k_3 y$$

Eliminating y from the preceding equations, we obtain




$$F = \frac{k_1(k_2 + k_3) + k_2 k_3}{k_2 + k_3}\ x$$

$$= \left( k_1 + \frac{1}{\frac{1}{k_2} + \frac{1}{k_3}} \right) x$$

Hence the equivalent spring constant $k_{eq}$ is given by

$$k_{eq} = k_1 + \frac{1}{\frac{1}{k_2} + \frac{1}{k_3}}$$

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**B-3-8.**

(a) The force f due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x})$$

In terms of the equivalent viscous friction coefficient $b_{eq}$, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

Hence

$$b_{eq} = b_1 + b_2$$

(b) The force f due to the dampers is

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \qquad (1)$$

where z is the displacement of a point between damper $b_1$ and damper $b_2$. (Note that the same force is transmitted through the shaft.) From Equation (1), we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x}) \qquad (2)$$

In terms of the equivalent viscous friction coefficient $b_{eq}$, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

By substituting Equation (2) into Equation (1), we have

$$f = b_2(\dot{y} - \dot{z}) = b_2[\dot{y} - \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})]$$

$$= \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Thus,

$$f = b_{eq}(\dot{y} - \dot{x}) = b_2(\dot{y} - \dot{z}) = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Hence,

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}$$

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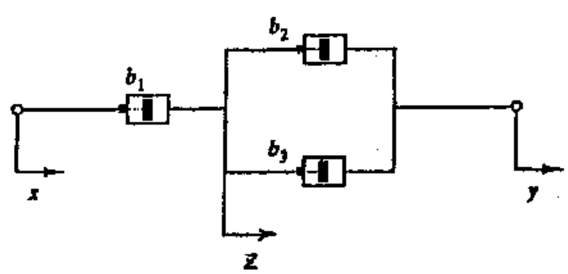
B-3-9. Since the same force transmits the shaft, we have

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) + b_3(\dot{y} - \dot{z}) \qquad (1)$$

where displacement z is defined in the figure below.




In terms of the equivalent viscous friction coefficient, the force f is given by

$$f = b_{eq}(\dot{y} - \dot{x}) \qquad (2)$$

From Equation (1) we have

$$b_1\dot{z} + b_2\dot{z} + b_3\dot{z} = b_1\dot{x} + b_2\dot{y} + b_3\dot{y}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2 + b_3}[b_1\dot{x} + (b_2 + b_3)\dot{y}] \qquad (3)$$

By substituting Equation (3) into Equation (1), we have

$$f = b_1(\dot{z} - \dot{x}) = b_1 \left\{ \frac{1}{b_1 + b_2 + b_3} [b_1\dot{x} + (b_2 + b_3)\dot{y}] - \dot{x} \right\}$$

$$= b_1 \frac{b_2 + b_3}{b_1 + b_2 + b_3} (\dot{y} - \dot{x}) \qquad (4)$$

Hence, by comparing Equations (2) and (4), we obtain

$$b_{eq} = b_1 \left( \frac{b_2 + b_3}{b_1 + b_2 + b_3} \right) = \frac{1}{\frac{1}{b_2 + b_3} + \frac{1}{b_1}}$$

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B-3-10. The equation for the system is

$$m\ddot{x} = - (k_1 + k_2)x - k_3x$$

or

$$m\ddot{x} + (k_1 + k_2 + k_3)x = 0$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_1 + k_2 + k_3}{m}}$$

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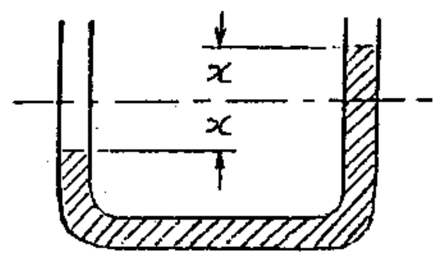
B-3-11. The density $\rho$ of the liquid is

$$\rho = \frac{m}{LA}$$

where A is the cross-sectional area of the inside of the glass tube. The mathematical model for the system is

$$m\ddot{x} = -\rho Ag\ 2x$$

or




$$\ddot{x} + \frac{2g}{L} x = 0$$

The natural frequency is

$$\omega_n = \sqrt{\frac{2g}{L}}$$

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B-3-12. For a small displacement x, the torque balance equation for the system is

$$m\ddot{x}(2a) = -k(\frac{1}{2}x)a$$

or

$$m\ddot{x} + \frac{k}{4}x = 0$$

The natural frequency is

$$\omega_n = \frac{1}{2}\sqrt{\frac{k}{m}} = \frac{1}{2}\sqrt{\frac{400}{\frac{5}{9.81}}} = 14.01 \text{ rad/sec}$$

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B-3-13. Applying Newton's second law to the system, we obtain

$$J\ddot{\theta}_o = -b\dot{\theta}_o - k(\theta_o - \theta_i)$$

Hence

$$J\ddot{\theta}_o + b\dot{\theta}_o + k\theta_o = k\theta_i$$

This is a mathematical model for the system.

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B-3-14. A mathematical model for the system is

$$m\ddot{x} = -k_1x - b_1\dot{x} - k_2x - b_2\dot{x}$$

or

$$m\ddot{x} + (b_1 + b_2)\dot{x} + (k_1 + k_2)x = 0$$

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B-3-15. The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting the given numerical values for m, b, and k into this equation, we obtain

$$2\ddot{x} + 4\dot{x} + 20x = 0 \qquad (1)$$

where $x(0) = 0.1$ and $\dot{x}(0) = 0$. The response to the given initial condition can be obtained by taking the Laplace transform of this equation, solving the resulting equation for X(s), and finding the inverse Laplace transform of X(s). The Laplace transform of Equation (1) is

$$2[s^2X(s) - sx(0) - \dot{x}(0)] + 4[sX(s) - x(0)] + 20\,X(s) = 0$$

By substituting the given initial conditions into this last equation, we get

$$2[s^2X(s) - 0.1s] + 4[sX(s) - 0.1] + 20\ X(s) = 0$$

Solving this equation for X(s) gives

$$X(s) = \frac{0.2s + 0.4}{2s^2 + 4s + 20} = \frac{0.1 + 0.1(s + 1)}{(s + 1)^2 + 3^2}$$

The inverse Laplace transform of this last equation gives

$$x(t) = 0.1(\frac{1}{3} e^{-t} \sin 3t + e^{-t} \cos 3t)$$

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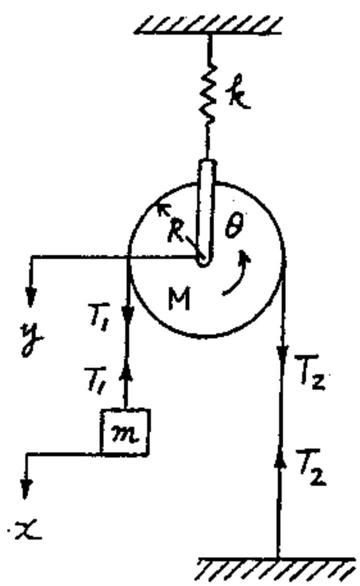
B-3-16. The equations of motion for the system are

$$J\ddot{\theta} = (T_1 - T_2)R$$

$$m\ddot{x} = - T_1$$

$$M\ddot{y} = - ky + T_1 + T_2$$




Noting that $x = 2y$, $R\theta = x - y = y$, and $J = \frac{1}{2}MR^2$, the three equations can be rewritten as

$$\frac{1}{2} MR^2\ddot{\theta} = \frac{1}{2} MR\ddot{y} = (T_1 - T_2)R$$

$$m\ddot{x} = - T_1$$

$$M\ddot{y} + ky = T_1 + T_2$$

Eliminating $T_2$ from the preceding equations gives

$$\frac{1}{2} M\ddot{y} + M\ddot{y} + ky = 2T_1 = - 2m\ddot{x}$$

By changing y into x,

$$\frac{3}{2} M \frac{\ddot{x}}{2} + k \frac{x}{2} = - 2m\ddot{x}$$

or

$$(m + \frac{3}{8} M)\ddot{x} + \frac{1}{4} kx = 0$$

The natural frequency is

$$\omega_n = \sqrt{\frac{2k}{8m + 3M}}$$

If mass m is pulled down a distance $x_o$ and released with zero initial velocity, the motion of mass m is

$$x(t) = x_o \cos \sqrt{\frac{2k}{8m + 3M}}\, t$$

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B-3-17. Referring to the figure below, we have

$$m\ddot{x} = -T \qquad (1)$$

where T is the tension in the wire. (Note that since x is measured from the static equilibrium position, the term mg does not enter the equation.) For the rotational motion of the pulley, we have

$$J\ddot{\theta} = -k_2(y + R_2\theta)R_2 + k_2(y - R_2\theta)R_2 + TR_1 - k_1R_1x$$

or

$$J\ddot{\theta} = -k_2R_2^2\theta - k_2R_2^2\theta + TR_1 - k_1R_1x \qquad (2)$$

Eliminating T from Equations (1) and (2), we obtain

$$J\ddot{\theta} + 2k_2R_2^2\theta + mR_1\ddot{x} + k_1R_1x = 0$$

Since $x = R_1\theta$, we have

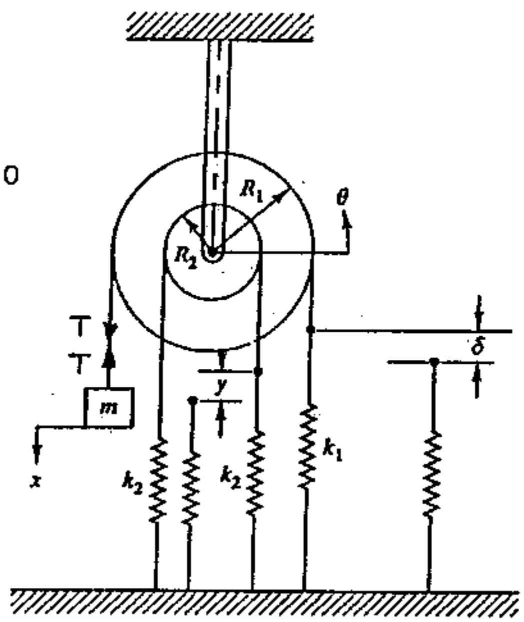
$$J\ddot{\theta} + 2k_2R_2^2\theta + mR_1^2\ddot{\theta} + k_1R_1^2\theta = 0$$

or

$$(J + mR_1^2)\ddot{\theta} + (2k_2R_2^2 + k_1R_1^2)\theta = 0$$

or

$$\ddot{\theta} + \frac{2k_2R_2^2 + k_1R_1^2}{J + mR_1^2}\theta = 0$$




This last equation is a mathematical model of the system. The natural frequency of the system is

$$\omega_n = \sqrt{\frac{2k_2R_2^2 + k_1R_1^2}{J + mR_1^2}}$$

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B-3-18. Torque = T = 50 x 0.5 = 25 N-m

Power = T $\omega$ = 25 x 100 = 2500 N-m/s = 2500 W

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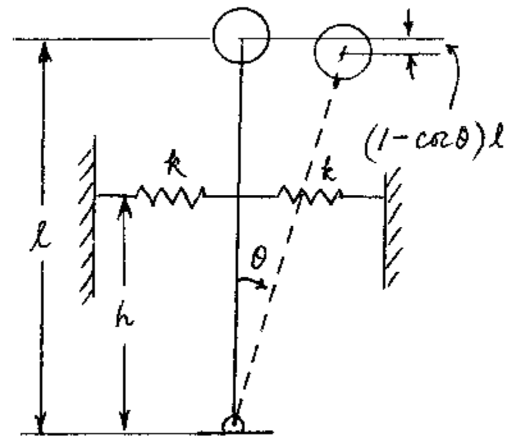
B-3-19. The potential energy of the system consists of the potential energy of the mass m and that of the springs. Choose the datum line such that the potential energy at the equilibrium position (where the pendulum is vertical) to be zero. Then the potential energy, when the pendulum is vibrating, is




$$U = -(1-\cos\theta)\ell mg + \tfrac{1}{2}k(\theta h)^2 + \tfrac{1}{2}k(\theta h)^2$$

$$= -(1-\cos\theta)\ell mg + k\theta^2h^2$$

Since the expression for U involves $\theta^2$ but does not involve $\theta$, it is necessary to expand cos $\theta$ into the following infinite series:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

and retain only the first two terms, or

$$\cos\theta = 1 - \tfrac{1}{2}\theta^2$$

The potential energy U for small $\theta$ can then be written as

$$U = -(1 - 1 + \tfrac{1}{2}\theta^2)\ell mg + k\theta^2h^2$$

$$= (kh^2 - \tfrac{1}{2}\ell mg)\theta^2$$

The kinetic energy T when the pendulum is vibrating is

$$T = \tfrac{1}{2}m(\ell\dot{\theta})^2 = \tfrac{1}{2}m\ell^2\dot{\theta}^2$$

When the pendulum is vibrating, we can assume that

$$\theta = A \sin\omega t$$

where A is the amplitude of vibration. By substituting the preceding expression for $\theta$ into the potential energy U and kinetic energy T, we obtain

$$U = (kh^2 - \tfrac{1}{2}\ell mg)A^2 \sin^2\omega t$$

$$T = \tfrac{1}{2}m\ell^2A^2\omega^2\cos^2\omega t$$

Hence, $U_{max}$ and $T_{max}$ can be obtained as

$$U_{max} = (kh^2 - \tfrac{1}{2}\ell mg)A^2, \qquad T_{max} = \tfrac{1}{2}m\ell^2A^2\omega^2$$

By equating $U_{max}$ and $T_{max}$, we have

$$(kh^2 - \tfrac{1}{2}\ell mg)A^2 = \tfrac{1}{2}m\ell^2 A^2 \omega^2$$

which can be simplified to

$$\omega^2 = \frac{2kh^2 - \ell mg}{m\ell^2} = \frac{2kh^2}{m\ell^2} - \frac{g}{\ell}$$

Hence, the frequency of vibration is

$$\omega = \sqrt{\frac{2kh^2}{m\ell^2} - \frac{g}{\ell}}$$

Note that this expression is valid for small θ such that cos θ can be approximated by $1 - \tfrac{1}{2}\theta^2$.

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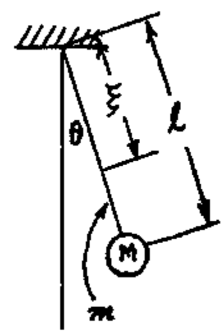
B-3-20. **The kinetic energy T is**

$$T = \frac{1}{2} M\ell^2\dot{\theta}^2 + \frac{1}{2}\int_0^\ell \frac{m}{\ell}\dot{\theta}^2\xi^2\,d\xi = \frac{1}{2}\left(M + \frac{m}{3}\right)\ell^2\dot{\theta}^2$$




The potential energy U is

$$U = Mg\ell(1 - \cos\theta) + \int_0^\ell \frac{m}{\ell} g\xi(1 - \cos\theta)\,d\xi$$

$$= \left(M + \frac{m}{2}\right)g\ell(1 - \cos\theta)$$

**Since the system is conservative, we have**

$$T + U = \frac{1}{2}\left(M + \frac{m}{3}\right)\ell^2\dot{\theta}^2 + \left(M + \frac{m}{2}\right)g\ell(1 - \cos\theta)$$

$$= \text{constant}$$

Noting that $d(T + U)/dt = 0$, we obtain

$$\left(M + \frac{m}{3}\right)\ell^2\dot{\theta}\ddot{\theta} + \left(M + \frac{m}{2}\right)g\ell\sin\theta\,\dot{\theta} = 0$$

or

$$\left[\left(M + \frac{m}{3}\right)\ell^2\ddot{\theta} + \left(M + \frac{m}{2}\right)g\ell\sin\theta\right]\dot{\theta} = 0$$

Since θ is not identically zero, we have

$$\left(M + \frac{m}{3}\right)\ell^2\ddot{\theta} + \left(M + \frac{m}{2}\right)g\ell\sin\theta = 0$$

**Rewriting,**

$$\ddot{\theta} + \frac{M + \frac{m}{2}}{M + \frac{m}{3}} \frac{g}{\ell} \sin \theta = 0$$

For small values of θ,

$$\ddot{\theta} + \frac{M + \frac{m}{2}}{M + \frac{m}{3}} \frac{g}{\ell} \theta = 0$$

So the natural frequency is

$$\omega_n = \sqrt{\frac{M + \frac{m}{2}}{M + \frac{m}{3}} \frac{g}{\ell}}$$

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