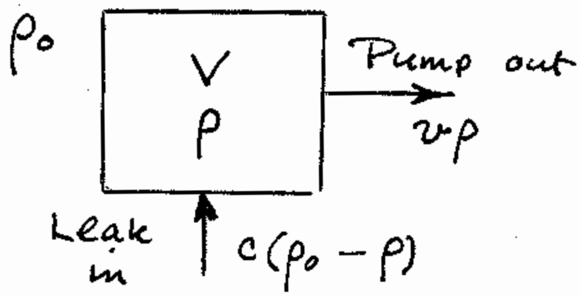
Evacuation of Leaking Tank



Transient Mass Balance

$$-v\rho + c(\rho_o - \rho) = \frac{d}{dt} V\rho = V\frac{d\rho}{dt}$$

Separate variables and integrate:

$$\int_0^t dt = V\int_{\rho_o}^{\rho} \frac{d\rho}{c\rho_o - (c+v)\rho} \quad \text{or} \quad t = \frac{-V}{c+v} \ln\left[c\rho_o - (c+v)\rho\right]\Big|_{\rho_o}^{\rho}$$

$$t = \frac{V}{c+v} \ln \frac{-v\rho_o}{c\rho_o - (c+v)\rho} = \frac{V}{c+v} \ln \frac{v\rho_o}{(c+v)\rho - c\rho_o}$$

$$= \frac{V}{c+v} \ln \frac{1}{\left(1+\frac{c}{v}\right)\frac{p}{p_o} - \frac{c}{v}} \qquad \text{since } \rho = \frac{Mp}{RT}$$

and T is constant

Lowest attainable pressure

As $t \to \infty$ $\quad \underbrace{\left(1+\frac{c}{v}\right)}_{1.01} \frac{p^*}{p_o} = \underbrace{\frac{c}{v}}_{0.01}$ $\qquad c = 10^{-5}\ m^3/s$, $\quad v = 10^{-3}\ m^3/s$

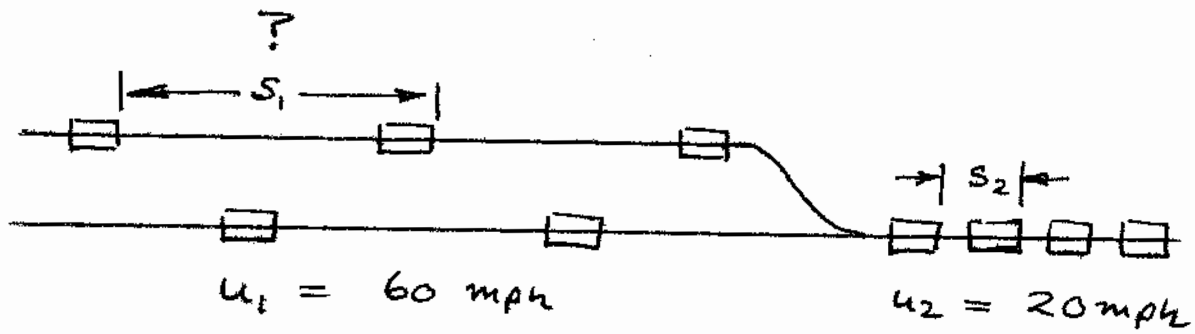
$$p^* = \frac{0.01}{1.01} \times 1 = \underline{\underline{0.00990\ \text{bar}}}$$

Time to fall half way to p^*, to $p = \dfrac{1 + 0.00990}{2} = 0.50495$ bar

$$t = \frac{1}{0.00101} \ln \frac{1}{1.01 \times 0.50495 - 0.01}$$

$$= \underline{\underline{686.3\ \text{s}}}$$

2.2
Slowed Traffic



Continuity of Cars

$$\text{"}m\text{"} = 2 u_1 \rho_1 = u_2 \rho_2$$

where $\rho_2 = \frac{1}{S_2} = \frac{1}{25} = 0.04 \ \frac{\text{cars}}{\text{ft}}$

Hence $\rho_1 = \frac{1}{2} \frac{u_2}{u_1} \rho_2 = \frac{1}{2} \frac{20}{60} \times 0.04 = 0.00667$

$$S_1 = \frac{1}{\rho_1} = \frac{1}{0.00667} = \underline{\underline{150 \text{ ft}}}$$

"Flow rate" of cars

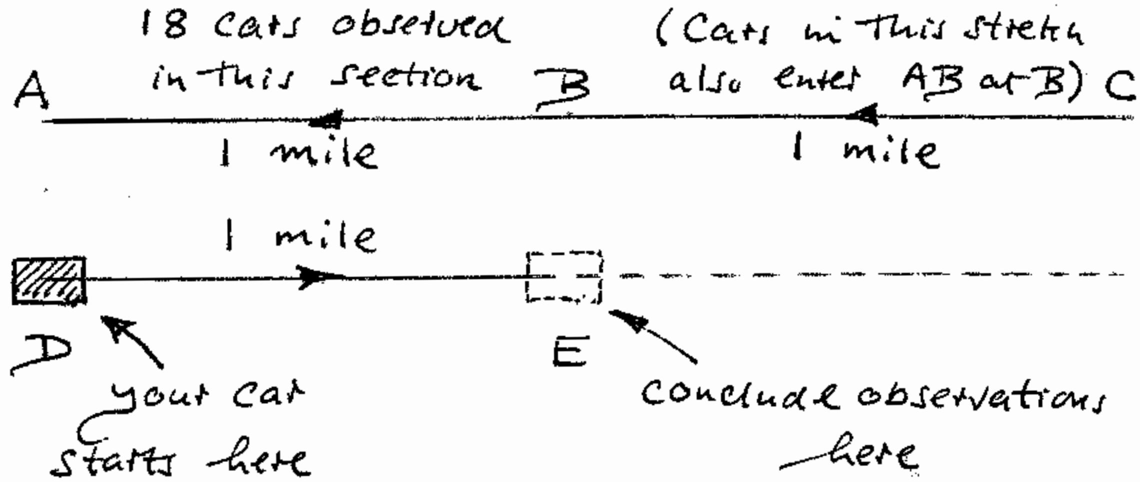
$$m = u_2 \rho_2 = 20 \times 0.04 \times 5280 \ \frac{\text{mile}}{\text{hr}} \frac{\text{cars}}{\text{ft}} \frac{\text{ft}}{\text{mile}}$$

$$= 4{,}224 \ \frac{\text{cars}}{\text{hr}}$$

$$= \underline{\underline{1.173 \ \frac{\text{cars}}{\text{sec}}}}$$

2.3

"Density" of Cars



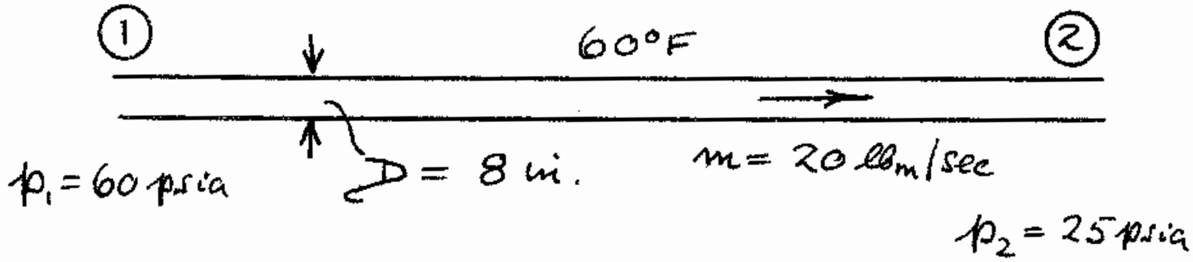
The cars you see in the mile between AB includes those in AB when you started at D <u>plus</u> those crossing into AB from BC.

Assume average speed for opposite lane is also 65 mph. Hence BC is also one mile, and a car starting at C reaches B just as you reach E.

Thus the 18 cars you observe originate from <u>two</u> miles of highway (AB + BC)

Hence car density $= \frac{18}{2} = $ <u>9 per mile</u>

Ethylene Pipeline



Ethylene C_2H_4 $MW = 28$

Cross-Sectional Area $A = \frac{\pi}{4}\left(\frac{8}{12}\right)^2 = 0.349$ ft²

Densities

$$\rho = \frac{Mp}{RT} \quad \begin{cases} \rho_1 = \dfrac{28 \times 60}{10.73 \times 520} = 0.301 \text{ lbm/ft}^3 \\[2ex] \rho_2 = \dfrac{25}{60} \times 0.301 = 0.125 \end{cases}$$

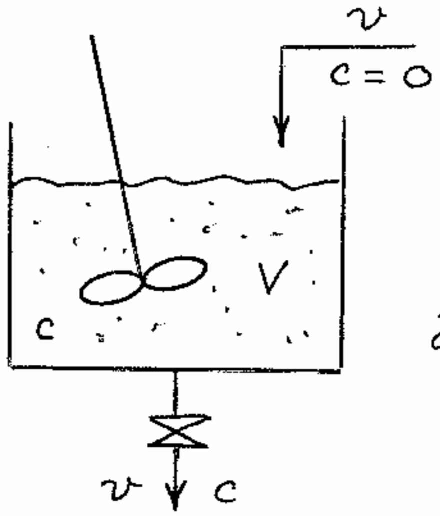
Flow rate $m = \rho \underbrace{uA}_{Q}$

Velocities $u_1 = \dfrac{m}{\rho_1 A} = \dfrac{20}{0.301 \times 0.349} = \underline{\underline{190.4 \text{ ft/sec}}}$

$$u_2 = \frac{m}{\rho_2 A} = \frac{20}{0.125 \times 0.349} = \underline{\underline{458.4 \text{ ft/sec}}}$$

Pressure falls because of pipe friction (for subsonic flow, at any rate).

Transient Behavior of a Stirred Tank



$V = 2\ m^3$ (constant)

$v = 0.01\ \frac{m^3}{s}$ (constant)

Unsteady-state mass balance for sodium chloride on the tank as system

Inlet − Exit

$v \times 0 - vc$

= Rate of accumulation

$$\frac{d}{dt}(Vc) = V\frac{dc}{dt}$$

Separate variables and integrate

$$\int_{c_0}^{c} \frac{dc}{c} = -\frac{v}{V}\int_0^t dt \quad \text{or} \quad \ln\frac{c}{c_0} = -\frac{vt}{V}$$

$$c = c_0\, e^{-vt/V}$$

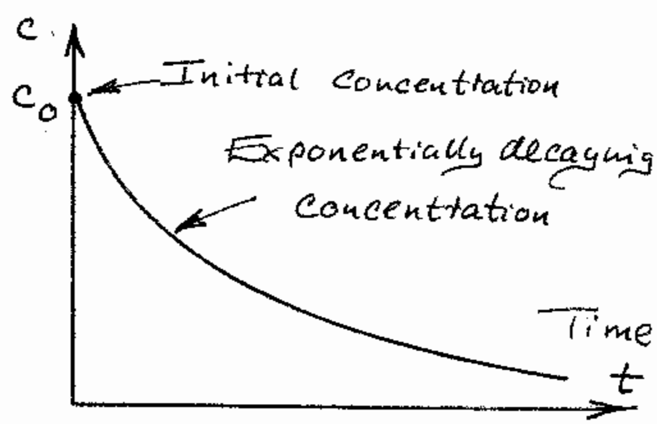
Required time

$c_0 = 1\ kg/m^3$

$c_f = 0.0001\ kg/m^3$

$$t_f = -\frac{V}{v}\ln\frac{c_f}{c_0}$$

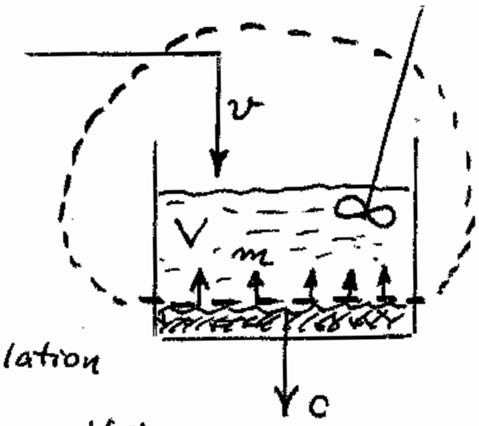
$$= -\frac{2}{0.01}\ln\frac{0.0001}{1}$$



$t_f = 1{,}842\ s = \underline{\underline{30\ m\ 42\ s}}$

Stirred Tank with Crystal Dissolution

Choose the system to exclude the solid salt in the tank. Then perform a rate balance for mass of salt



Entering − leaving = accumulation

$$m \quad - \quad vc \quad = \frac{d}{dt}(Vc) = V\frac{dc}{dt}$$

since V is constant.

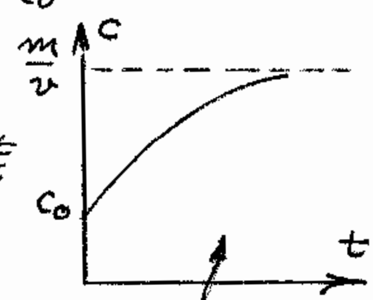
Separate variables and integrate

$$\int_{c_0}^{c} \frac{dc}{\frac{m}{v} - c} = \frac{v}{V}\int_0^t dt$$

$$-\ln\left(\frac{m}{v} - c\right)\Big|_{c_0}^{c} = \frac{vt}{V} \quad \text{or} \quad \ln\frac{m - vc}{m - vc_0} = -\frac{vt}{V}$$

$$m - vc = (m - vc_0)\, e^{-\frac{vt}{V}}$$

$$\underline{\underline{c = \frac{m}{v} - \left(\frac{m}{v} - c_0\right) e^{-\frac{vt}{V}}}}$$

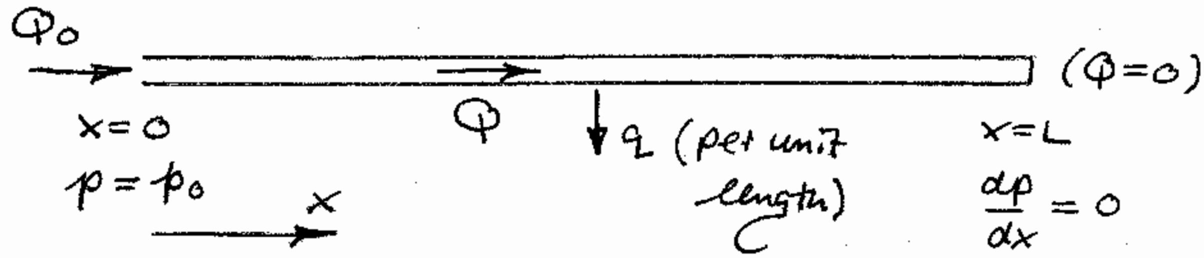


c is its steady-state value m/v, less a deviation that decreases exponentially with time.

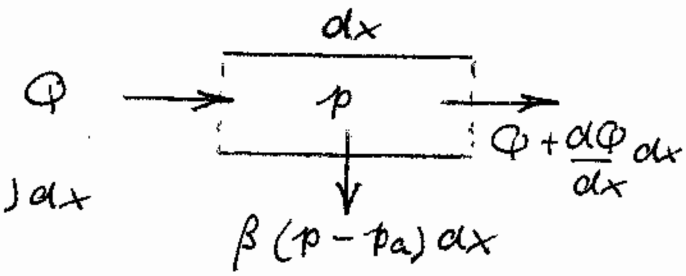
$m = 0.02$ kg/s
$v = 0.01\ m^3/s$
$V = 2\ m^3$
$c_0 = 1$ kg/m^3

t (sec)	c ($\frac{kg}{m^3}$)
0	1.000
10	1.049
100	1.393
∞	2.000

Soaker Garden Hose



Perform mass balance on element dx



$$Q = Q + \frac{dQ}{dx}dx + \beta(p-p_a)dx$$

$$-\frac{dQ}{dx} = \alpha\frac{d^2p}{dx^2} = \beta(p-p_a)$$

Substitute $P = p - p_a$, $\frac{\beta}{\alpha} = \gamma^2$ gives $\frac{d^2P}{dx^2} = \gamma^2 P$

Try $P = A\sinh\gamma x + B\cosh\gamma x$

$$\frac{dP}{dx} = \gamma A\cosh\gamma x + \gamma B\sinh\gamma x$$

$$\frac{d^2P}{dx^2} = \gamma^2 A\sinh\gamma x + \gamma^2 B\cosh\gamma x = \gamma^2 P \quad \underline{OK}$$

Boundary Conditions

$x=0$ $\quad P = p_o - p_a = B \quad (\sinh 0 = 0,\ \cosh 0 = 1)$

$x=L$ $\quad \frac{dP}{dx} = \gamma A\cosh\gamma L + \gamma\underbrace{(p_o-p_a)}_{B}\sinh\gamma L = 0$

Hence $A = -(p_o-p_a)\frac{\sinh\gamma L}{\cosh\gamma L}$

The pressure distribution as a function of x is then:

$$\frac{P}{p_0 - p_a} = \frac{p - p_a}{p_0 - p_a} = -\frac{\sinh \gamma L}{\cosh \gamma L} \sinh \gamma x + \cosh \gamma x$$

$$= \frac{\cosh \gamma (L - x)}{\cosh \gamma L}$$

Rate of loss of water can be obtained by integration

$$Q_{loss} = \int_0^L \beta (p - p_a)\, dx$$

since $p - p_a$ is known now as a function of x.

A somewhat easier approach is to note that $Q_{loss} = Q_0$, the water rate entering the hose:

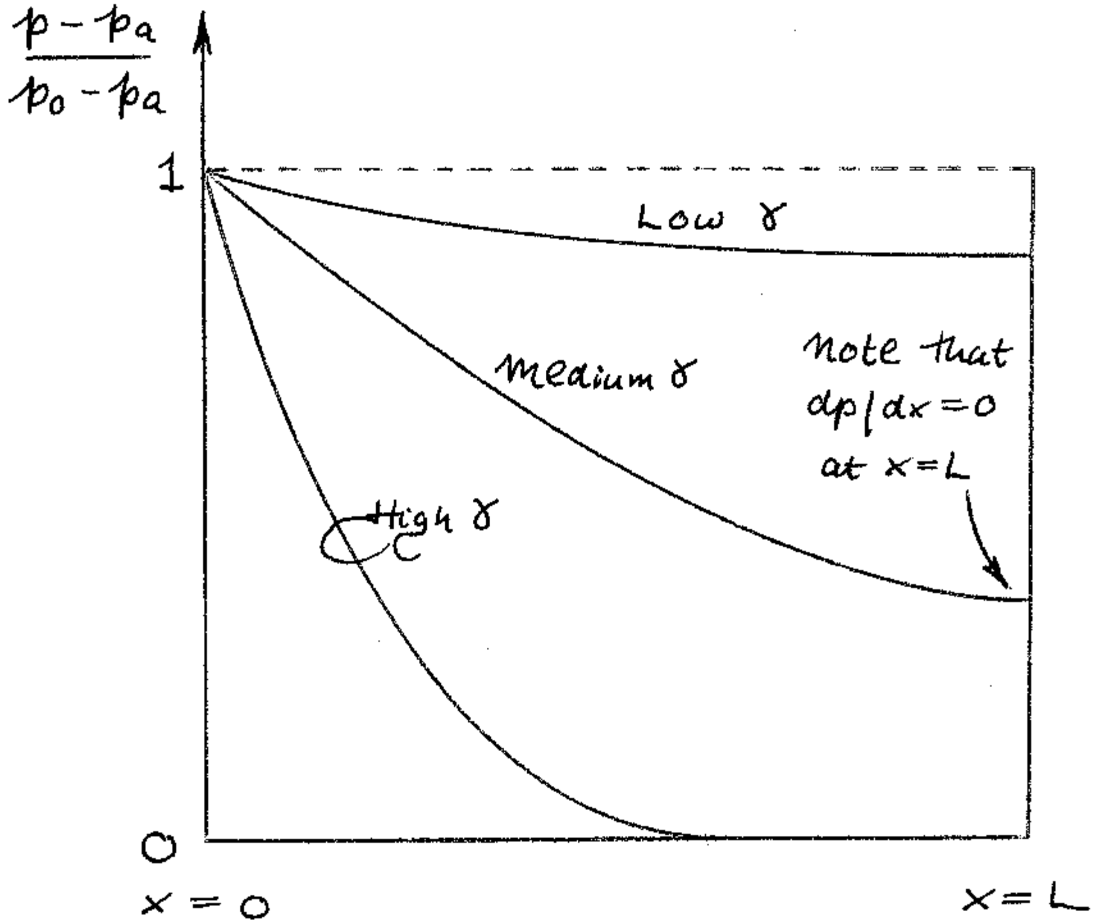
$$Q_0 = -\alpha \left(\frac{dp}{dx}\right)_{x=0} = -\alpha \gamma \left(A \underbrace{\cosh \gamma x}_{1} + B \underbrace{\sinh \gamma x}_{0}\right)_{x=0}$$

$$= -\alpha \gamma (-)(p_0 - p_a) \frac{\sinh \gamma L}{\cosh \gamma L}$$

$$Q_0 = \alpha \gamma (p_0 - p_a) \tanh \gamma L = Q_{loss}.$$

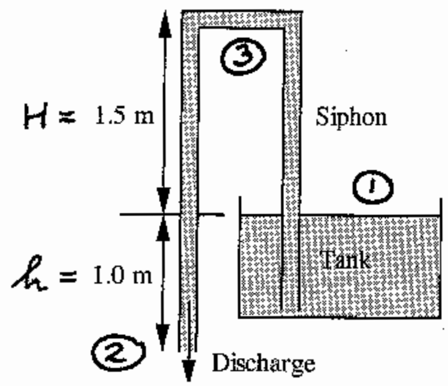
Sketch of pressure distribution

Note $\gamma = \sqrt{\beta / \alpha}$, so a low value of γ means a relatively low leakage rate, and a high value of γ means a relatively high leakage rate



2·8-1

Performance of a Siphon



Bernoulli ① → ② (ignoring friction)

$$\frac{u_1^2}{2} + \frac{p_1}{\rho} + g z_1 = \frac{u_2^2}{2} + \frac{p_2}{\rho} + g z_2$$

zero zero zero zero $-h$

$$u_2 = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 1.00} = \underline{\underline{4.43 \text{ m/s}}}$$

Lowest pressure occurs at point 3 at the top

Bernoulli ① → ③

$$\frac{u_1^2}{2} + \frac{p_1}{\rho} + g z_1 = 0 = \frac{u_3^2}{2} + \frac{p_3}{\rho} + gH$$

Continuity ③ → ②

$$A u_3 = A u_2$$

$$u_3 = u_2$$

Hence the lowest pressure is:

$$p_3 = -\rho\left(\frac{u_2^2}{2} + gH\right)$$

$$= -10^3\left(\frac{4.43^2}{2} + 9.81 \times 1.5\right)$$

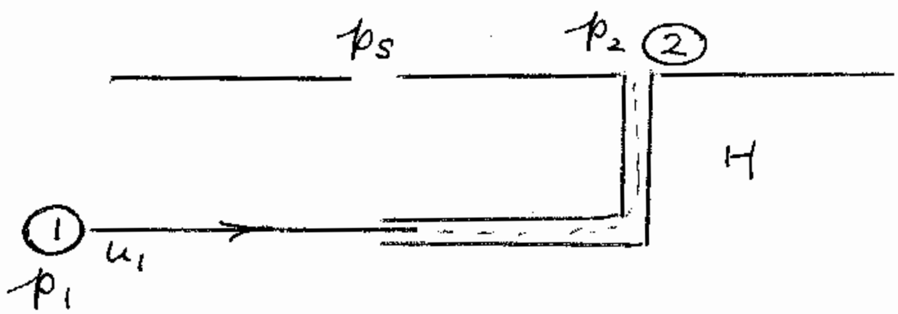
$$= -2.45 \times 10^4 \text{ Pa}$$

$$= \underline{\underline{-0.245 \text{ bar}}} \text{ (gauge)}$$

Yes, the answers are reasonable. The velocity of 4.43 m/s is typical for pipe flow, although in practice it would be reduced somewhat because of friction. Also, $p_3 \doteq 11.0$ psia, which is well above the vapor pressure of water at room temperature ($p_v = 0.363$ psia at 70°F), so vapor-lock will not occur.

The indicated draining time of $t = V/vA$ would only be true for constant velocity v. However, as the level in the tank falls, so does v, so $t = V/vA$ is not true.

2.9 Pitot Tube



Bernoulli $\quad \dfrac{u_1^2}{2} + \dfrac{p_1}{\rho} = \dfrac{p_2}{\rho} + gH$

But $p_1 = p_s + \rho g H$

$$\frac{u_1^2}{2} + \frac{p_s + \rho g H}{\rho} = \frac{p_2}{\rho} + gH$$

Hence $\quad u_1 = \sqrt{\dfrac{2(p_2 - p_s)}{\rho}}$

Sea Water

(a) $u_1 = \sqrt{\dfrac{2 \times 2.5 \times 144 \times 32.2}{64}} = 19.03$ ft/sec

$= \underline{\underline{12.98 \text{ mph}}}$

Fresh Water

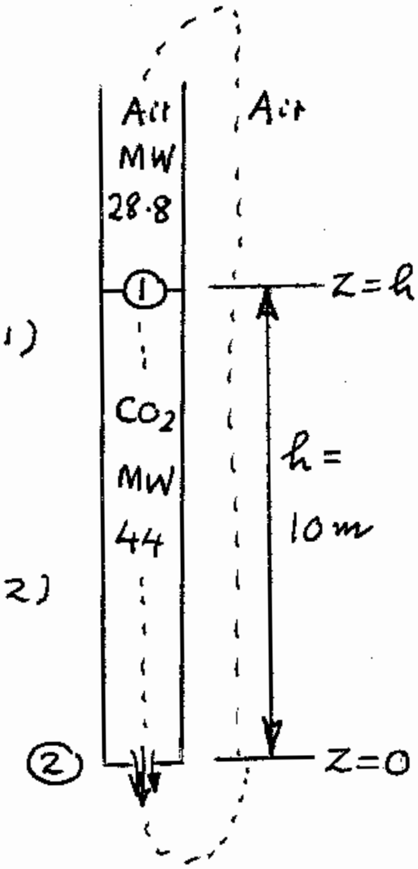
(b) $u_f = \sqrt{\dfrac{2 \times 2.5 \times 144 \times 32.2}{62.4}} = 19.28$ ft/sec

$= \underline{\underline{13.14 \text{ mph}}}$

2.10
Leaking Carbon Dioxide

Apply Bernoulli ① → ②

for two different paths



Carbon dioxide

$$\frac{u_1^2}{2} + \frac{p_1}{\rho_{CO_2}} + gh = \frac{u_2^2}{2} + \frac{p_2}{\rho_{CO_2}} \qquad (1)$$

Air

$$\frac{u_1^2}{2} + \frac{p_1}{\rho_A} + gh = 0 + \frac{p_2}{\rho_A} + 0 \qquad (2)$$

Neglect $u_1^2 \ll u_2^2$

From (1) $$\frac{u_2^2}{2} = \frac{1}{\rho_{CO_2}}(p_1 - p_2) + gh$$

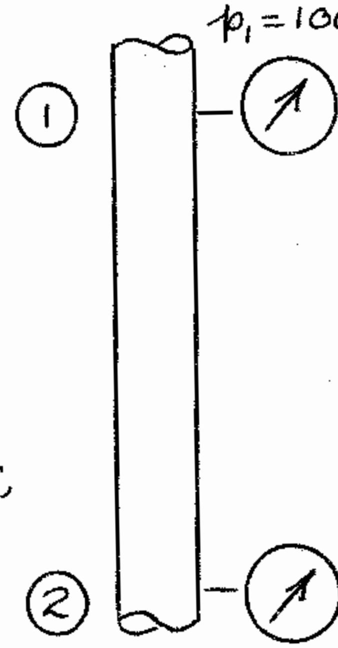
From (2) $$(p_1 - p_2) = -\rho_A g h$$

Hence $$\frac{u_2^2}{2} = gh\left(1 - \frac{\rho_A}{\rho_{CO_2}}\right)$$

$$u_2 = \sqrt{2 \times 9.81 \times 10 \left(1 - \frac{28.8}{44}\right)}$$

$$= \underline{\underline{8.23 \ m/s}}$$

Two Pressure Gauges



(a) If the water were not flowing, p_2 would be much higher than p_1, because of the hydrostatic effect.

However, this is not the case, therefore, the water must be flowing

(b) If the water were flowing upwards, say with $p_2 = 100$, then p_1 would be lower on two counts:

(i) Hydrostatic effect
(ii) Pipe friction.

However, $p_1 \neq p_2$. Therefore, the water cannot be flowing upwards, and must be flowing downwards.

(c) Check on conclusion of (b) by using overall energy balance from (1) to (2) with $w = 0$ and $\Delta(u^2/2) = 0$:

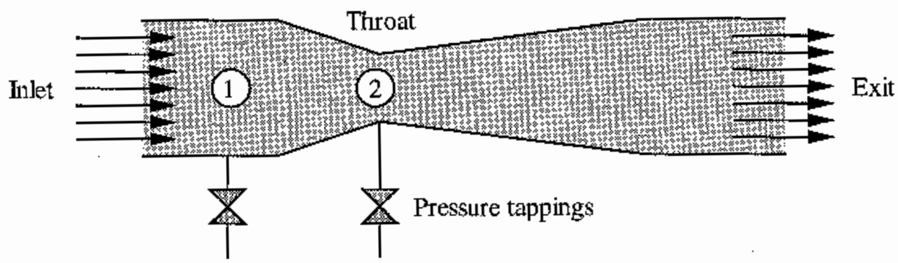
$$g(z_2 - z_1) + \frac{p_2 - p_1}{\rho} + \mathcal{F} = 0$$

Since $p_1 = p_2$, $\mathcal{F} = g(z_1 - z_2)$

Both sides are positive and equal. That is, the hydrostatic or potential energy is exactly consumed by pipe friction.

2.12

Venturi "Meter"



Bernoulli ① → ②
$$\frac{u_1^2}{2} + \frac{p_1}{\rho} = \frac{u_2^2}{2} + \frac{p_2}{\rho}$$

Continuity ① → ②
$$A u_1 = a u_2$$

Eliminate u_2 gives
$$\frac{p_1 - p_2}{\rho} = \frac{u_1^2}{2}\left(\frac{A^2}{a^2} - 1\right)$$

Flow rate
$$Q = A u_1 = C_D A \sqrt{\frac{2(p_1 - p_2)}{\rho\left(\frac{A^2}{a^2} - 1\right)}} = C_D A \sqrt{\frac{2(\rho_M - \rho_I) g \Delta h}{\rho_I\left(\frac{A^2}{a^2} - 1\right)}}$$

M = mercury I = isopentane

Substituting numbers, $A = \frac{\pi}{4}\left(\frac{6}{12}\right)^2 = 0.196\ ft^2$, $C_D = 0.98$

$$Q = 0.98 \times 0.196 \sqrt{\frac{2\,\overbrace{(13.57 \times 62.4 - 38.75) \times 32.2 \times \left(\frac{20}{12}\right)}^{\alpha}}{38.75\left(\frac{6^4}{3^4} - 1\right)}}$$

$$= 0.98 \times 0.196 \times 12.22 = \underline{\underline{2.35\ ft^3/sec}}$$

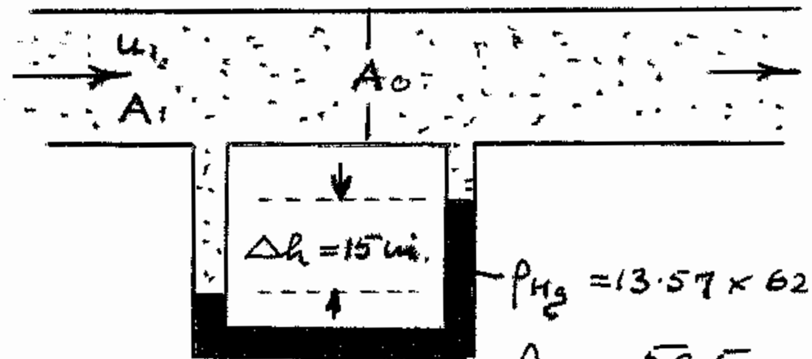
$$= \underline{\underline{1053\ gpm}}$$

$$p_1 - p_2 = \frac{\alpha}{32.2 \times 144} = 9.35\ psi$$

2.13

Orifice Plate

Use the orifice-plate equation $u_1 = C_D\sqrt{\dfrac{2(p_1-p_2)}{\rho\left(\dfrac{A_1^2}{A_0^2}-1\right)}}$



$\rho_{Hg} = 13.57 \times 62.4 = 847$

$\rho_K = 50.5$

$A_1 = \frac{\pi}{4}\left(\frac{2}{12}\right)^2 = 0.0218\ ft^2$ $\quad u_1 = \dfrac{560}{60 \times 50.5 \times 0.0218} = 8.48\ \frac{ft}{sec}$

For most orifice plates, $C_D \doteq 0.62$, which can be used as first trial.

$$8.48 = 0.62\sqrt{\dfrac{2 \times \frac{15}{12}(847-50.5)\times \overset{g}{32.2}}{50.5\left(\dfrac{A_1^2}{A_0^2}-1\right)}}$$

whence

$$\frac{A_1}{A_0} = \frac{D_1^2}{D_0^2} = 2.79 \qquad \frac{D_1}{D_0} = 1.67 \qquad \underline{\underline{D_0 = 1.2\ \text{in dia.}}}$$

Check on value of C_D

$$Re_0 = \frac{D_1}{D_0} Re_1 = \frac{\underset{D_1/D_0}{1.67} \times \overset{u_1}{8.48} \times \overset{\rho}{50.5} \times \overset{D_1}{\frac{2}{12}}}{\underset{\mu}{3.18} \times \frac{1}{3600}} = 1.35 \times 10^5$$

From chart, for $\dfrac{D_0}{D_1} = 0.60$, $Re = 1.35 \times 10^5$

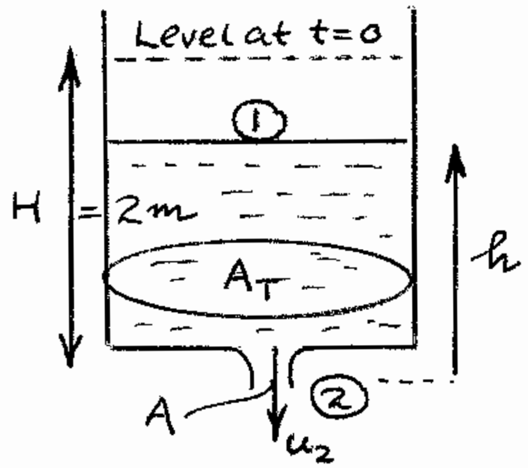
(page 56) $\quad$ $\underline{\underline{C_D = 0.62 \quad \text{Assumption was OK}}}$

2.14

Tank Draining

From lecture notes

$$u_2 = \sqrt{2gh}$$



Mass Balance on Acetone in the Tank

$$-\sqrt{2gh}\,A = \frac{d}{dt}(hA_T)$$

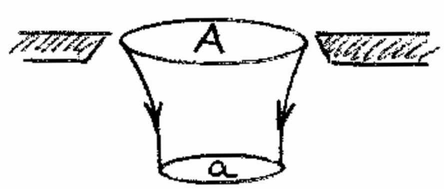
$$= A_T \frac{dh}{dt} \quad \text{or} \quad \int_0^t dt = -\frac{A_T}{A}\frac{1}{\sqrt{2g}}\int_H^0 \frac{dh}{\sqrt{h}}$$

Draining Time

$$t = +\frac{A_T}{A}\frac{1}{\sqrt{2g}}\,2\sqrt{h}\Big|_0^H = \frac{D_T^2}{D_0^2}\sqrt{\frac{2H}{g}}$$

$$t = \left(\frac{1}{0.02}\right)^2\sqrt{\frac{2\times 2}{9.81}} = 1{,}596\text{ s} = \underline{\underline{26\text{m } 36\text{s}}}$$

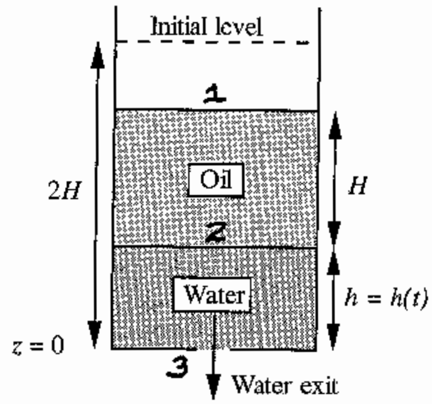
For sharp-edged orifice, nothing changes except the area ratio, which is now



$$\frac{A_T}{a} = \frac{1^2}{0.02^2\times 0.63},$$ so the time increases by a factor of $\frac{1}{0.63} = 1.59$

$$\underline{\underline{t = 2{,}533\text{ s} = 42\text{m } 13\text{s}}}$$

Draining Immiscible Liquids from a Tank



From hydrostatics,

$$p_2 = \rho_o g H$$

Bernoulli ② → ③

$$\frac{p_2}{\rho_w} + gh = \frac{u_3^2}{2}$$

Hence

$$u_3 = \sqrt{2g\left(\frac{\rho_o}{\rho_w}H + h\right)}$$

Unsteady-State Mass Balance on Tank (Rate balance)

$$-0.62\,a\,u_3\,\rho_w = -0.62\,a\,\rho_w\sqrt{2g\left(\frac{\rho_o}{\rho_w}H + h\right)} = A\rho_w\frac{dh}{dt}$$

Leaving through orifice

Rate of accumulation of water in tank (will be negative, but don't anticipate this with a minus sign)

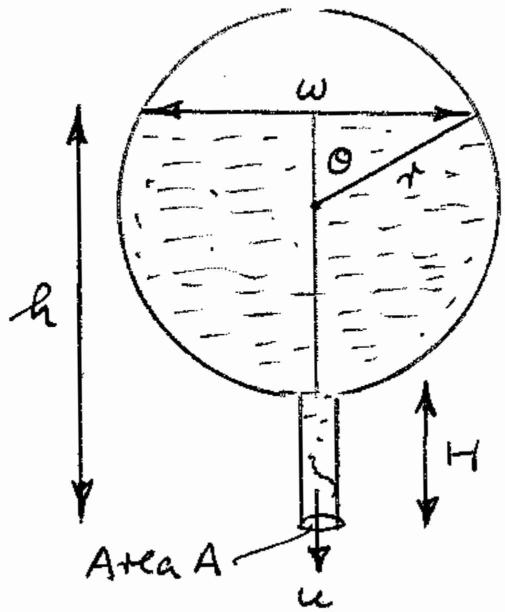
$$\int_0^t dt = -\frac{A}{0.62a\sqrt{2g}}\int_H^0 \frac{dh}{\sqrt{\frac{\rho_o}{\rho_w}H + h}}$$

$b = 1$

$$t = \frac{2A}{\sqrt{2g}\,0.62a}\left[\sqrt{\frac{\rho_o}{\rho_w}H + h}\right]_0^H$$

$$t = \frac{2A}{0.62a\sqrt{2g}}\left[\sqrt{H\left(\frac{\rho_o}{\rho_w}+1\right)} - \sqrt{\frac{\rho_o}{\rho_w}H}\right]$$

Draining a Horizontal Cylindrical Tank



From geometry

$$h = H + r(1+\cos\theta) \quad (1)$$

$$w = 2r\sin\theta \quad (2)$$

From Bernoulli

$$u = \sqrt{2gh} \quad (3)$$

Volume balance

In time dt, a volume dV leaves the tank

$$dV = uA\,dt = -wL\,dh \quad (4)$$

From (1), $dh = -r\sin\theta\, d\theta$ (5)

Hence, from (2), (3), (4), and (5):

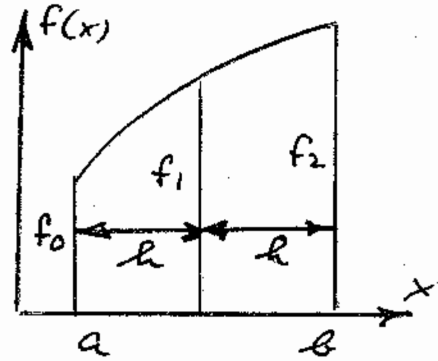
$$\sqrt{2g[H+r(1+\cos\theta)]}\,A\,dt = -2r\sin\theta\, L(-r\sin\theta\, d\theta)$$

Separate variables and integrate

$$\int_0^t dt = t = \frac{2r^2L}{A\sqrt{2g}} \underbrace{\int_0^{\pi} \frac{\sin^2\theta\, d\theta}{\sqrt{H+r(1+\cos\theta)}}}_{I}$$

Simpson's Rule

$$\int_a^b f(x)dx \doteq \frac{h}{3}(f_0 + 4f_1 + f_2)$$



Approximate I by two applications of Simpson's rule, $h = \pi/4$

$$I \doteq \frac{\pi/4}{3}\left[0 + 4\frac{\sin^2 \pi/4}{\sqrt{2+\cos \pi/4}} + 2\frac{\sin^2 \pi/2}{\sqrt{2+\cos \pi/2}} + 4\frac{\sin^2 3\pi/4}{\sqrt{2+\cos 3\pi/4}} + 0\right]$$

$$= \frac{\pi}{12}(1.216 + 1.414 + 1.759) = 1.149 \ m^{-1/2}$$

(One application of Simpson's rule gives $I \doteq 1.48$)

Required area is

$$A = \frac{2r^2 L I}{t\sqrt{2g}} = \frac{2 \times 1^2 \times 5 \times 1.149}{3600 \times \sqrt{2 \times 9.81}} = \underline{\underline{0.000721 \ m^2}}$$

Diameter of tail pipe $A = \frac{\pi d^2}{4}$

$$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4 \times 0.000721}{\pi}} = 0.0303 \ m$$

$$= \underline{\underline{3.03 \ cm}}$$

Is this answer reasonable?

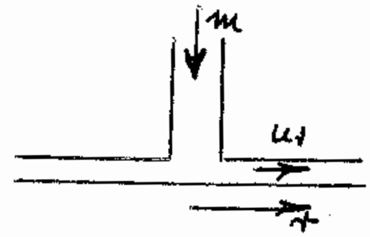
When $\theta = \pi/2$, $h = 2$, $u = \sqrt{2 \times 9.81 \times 2} = 6.26 \ m/s$

At this rate for 1 hr, $V = 6.26 \times 0.000721 \times 3600$

$$= 16.3 \ m^3$$

Tank volume $= \pi \left(\frac{2r}{4}\right)^2 L = \pi \times \left(\frac{2 \times 1}{4}\right)^2 \times 5 = 15.7 \ m^3$ $\Big\}$ OK

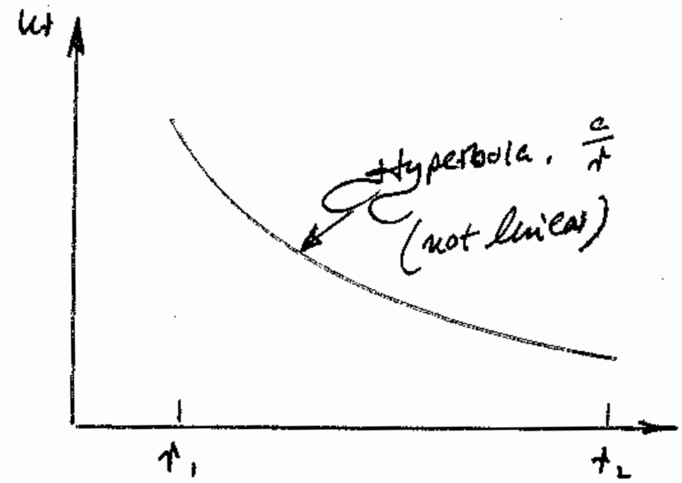
Lifting Silicon Wafers



Mass Balance

$$\dot{m} = 2\pi r H \rho u_r$$

$$u_r = \frac{\dot{m}}{2\pi r H \rho}$$

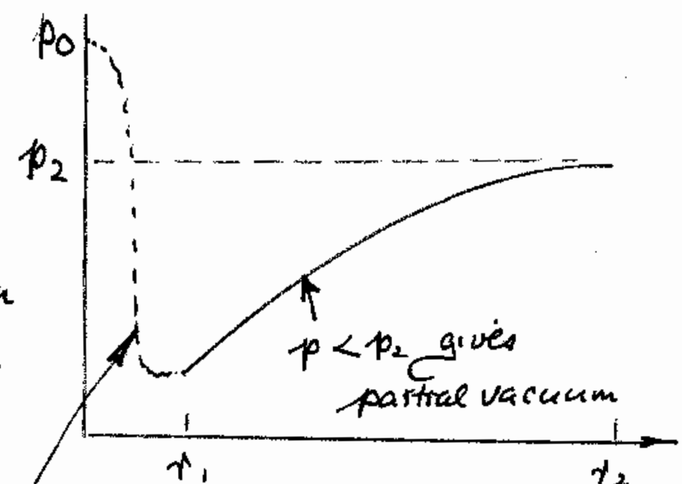


Bernoulli's Equation

$$\frac{u_r^2}{2} + \frac{p}{\rho} = \frac{u_2^2}{2} + \frac{p_2}{\rho}$$

$$p = p_2 - \frac{\rho}{2}\underbrace{\left(u_r^2 - u_2^2\right)}_{\text{positive}}$$

Hence $p < p_2$ and there is a partial vacuum between the flange and wafer, which is therefore picked up.



Reduction in pressure from p_0 as air accelerates into gap.

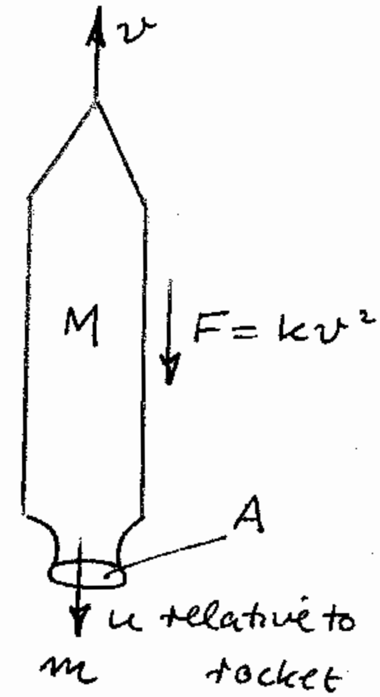
Total Force upwards on Wafer

$$F = \int_0^{r_2} (p_2 - p)\, 2\pi r\, dr.$$

Will be positive, so the device is OK for picking up the wafer

2.18

Rocket Performance



(a) Absolute downwards velocity of propellant

$$= u - v$$

(b) Momentum flux

$$= m\,(u - v)$$

(c) Mass M is decreasing:

$$\frac{dM}{dt} = -m \quad \text{or} \quad M = M_0 - mt$$

(M_0 is initial mass, and this equation holds until all the propellant has been ejected)

(d) Upwards momentum balance on moving rocket

$$\underset{\text{gravity}}{-Mg} \quad \underset{\text{drag}}{-kv^2} \quad \underset{\text{loss}}{\uparrow}\;\underset{\text{negative direction}}{\uparrow}\;\; m(u-v) = \frac{d}{dt}(Mv)$$

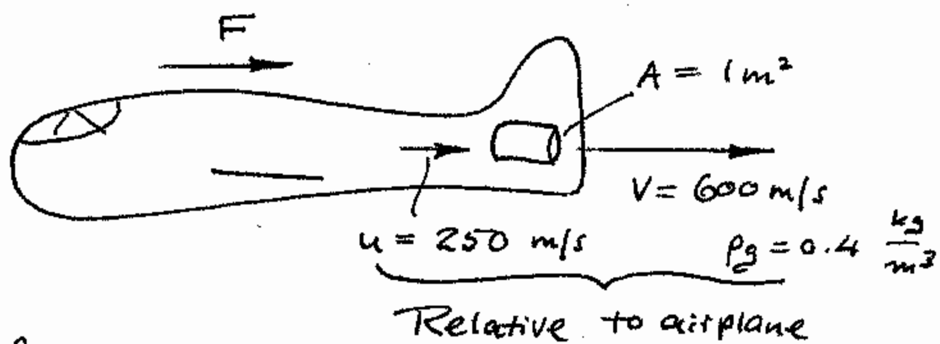
$$= M\frac{dv}{dt} + v\frac{dM}{dt}$$

$$= M\frac{dv}{dt} - mv$$

Hence

$$\frac{dv}{dt} = -g + \frac{1}{M_0 - mt}\left(mu - kv^2\right)$$

2.19
Jet Airplane



(a) Mass flow rate

$$\dot{m} = \rho_g V A = 0.4 \times 600 \times 1 = \underline{\underline{240 \text{ kg/s}}}$$

(b) $\overrightarrow{\text{Momentum}}$ balance (as seen by the pilot — this corresponds to a "stationary airplane with air flowing past it)

$$\underbrace{\dot{m}u}_{\text{Flow in}} - \underbrace{\dot{m}v}_{\text{Flow out}} + \underbrace{F}_{\text{Force}} = 0 \quad \left(\begin{array}{c}\text{Steady}\\ \text{State}\end{array}\right)$$

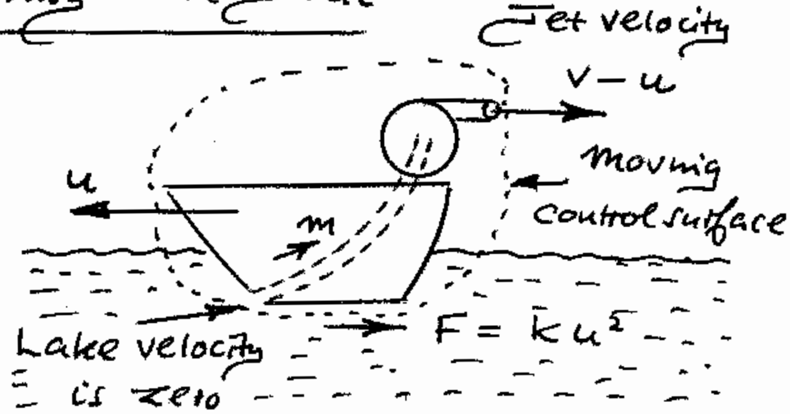
$$F = \dot{m}(v-u) = 240(600 - 250) = \underline{\underline{8.4 \times 10^4 \text{ N}}}$$

(c) Power may be obtained by different methods, the easiest of which is to note that the engine serves to increase the kinetic energy of the air flowing through it:

$$P = \frac{\dot{m}}{2}(v^2 - u^2) = \frac{240}{2}(600^2 - 250^2) = \underline{\underline{3.57 \times 10^7 \text{ W}}}$$

Jet-Propelled Boat

Control Surface Moving with Boat



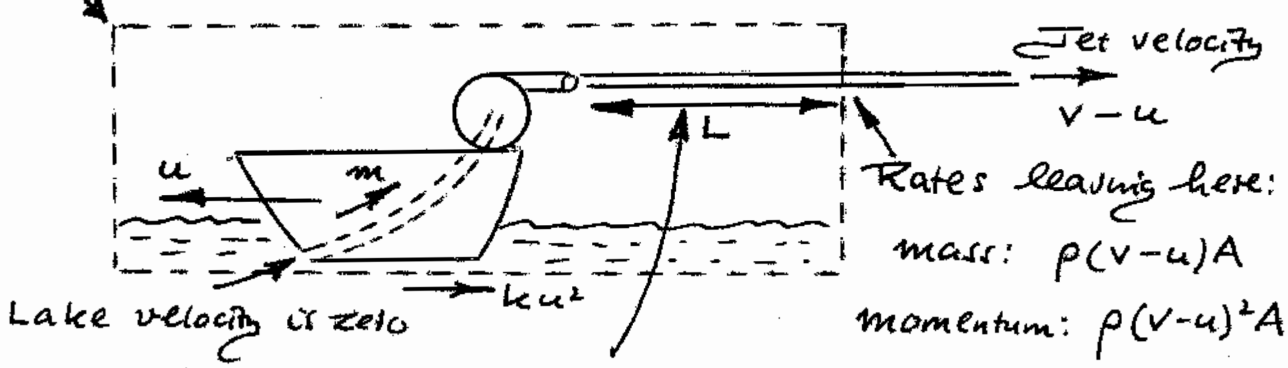
Momentum balance ←

$$m \times 0 - -m(v-u)$$

Input from lake — Loss — negative direction — Jet

$$-ku^2 = \frac{d}{dt}(Mu) = Ma \quad \text{or} \quad a = \frac{1}{M}\left[m(v-u) - ku^2\right]$$

Fixed Control Surface – Boat + Jet Moving Within it



Momentum balance ← ; Momentum of this portion of jet $= \rho AL(v-u)$ →

$$m \times 0 - -\rho(v-u)^2 A = \frac{d}{dt}Mu + \frac{d}{dt}\left[-\rho AL(v-u)\right]$$

From lake — Leaving with jet

$-ku^2$ (friction)

Also note $m = \rho A v$

$$\rho(v-u)^2 A - ku^2 = Ma - \rho A(v-u)\left(\frac{dL}{dt}\right)$$

This equals u

$$a = \frac{1}{M}\left[\rho A(v-u)(v-u+u) - ku^2\right] = \frac{1}{M}\left[m(v-u) - ku^2\right]$$

Numerical Values

At maximum velocity, there is no acceleration, so

$$m(v-u) = Ku^2$$

or $$K = \frac{m(30-20)}{20^2} = \frac{m}{40}$$

When $u = 10$ ft/sec,

$$a = \frac{1}{M}\left[m(v-u) - Ku^2\right] = \frac{m}{M}\left[(v-u) - \frac{u^2}{40}\right]$$

$$= \frac{374}{1000}\left[(30-10) - \frac{10^2}{40}\right] = 6.55 \text{ ft/sec}$$

Pumping Power for Propelling Boat

Observer on the boat sees a mass flowrate m with kinetic energy per unit mass of $\frac{u^2}{2}$ coming in and $\frac{V^2}{2}$ leaving. Hence

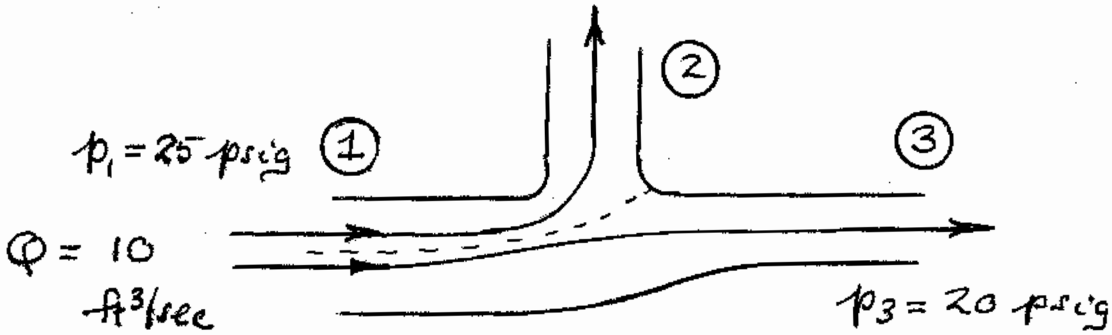
$$P = \frac{m}{2}(V^2 - u^2)$$

[The same result is obtained from the viewpoint of an observer on the shore, who sees power needed to overcome the drag force (Fu), to accelerate water from rest to $(v-u)$ ft/sec ($m\{v-u\}^2/2$), and to increase the kinetic energy of the boat ($\frac{d}{dt} M \frac{u^2}{2} = Mua$)]

Power decreases as u increases because the boat is accelerating less rapidly.

2.21

Branch Pipe



Areas $A_1 = 0.1364$, $A_2 = 0.0873$, $A_3 = 0.0491$ ft^2

$$u_1 = \frac{10}{0.1364} = 73.3 \text{ ft/sec}, \quad \rho = 0.8 \times 62.4 = 49.9 \ \frac{\text{lbm}}{\text{ft}^3}$$

Bernoulli 1 → 3 $\quad \frac{u_1^2}{2} + \frac{p_1}{\rho} = \frac{u_3^2}{2} + \frac{p_3}{\rho}$

$$\frac{73.3^2}{2} + \frac{25 \times 32.2 \times 144}{49.9} = \frac{u_3^2}{2} + \frac{20 \times 32.2 \times 144}{49.9}$$

Hence $\underline{\underline{u_3 = 79.4 \text{ ft/sec}}}$

Continuity (Mass Balance) (ρ constant)

$$10 = 0.0873\, u_2 + 0.0491 \times 79.4$$

$$\underline{\underline{u_2 = 69.9 \text{ ft/sec}}}$$

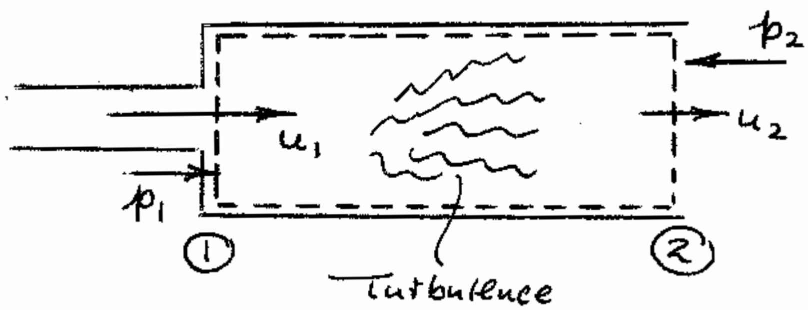
Bernoulli 1 → 2

$$\frac{73.3^2}{2} + \frac{25 \times 32.2 \times 144}{49.9} = \frac{69.9^2}{2} + \frac{p_2 \times 32.2 \times 144}{49.9}$$

Hence $\underline{\underline{p_2 = 27.6 \text{ psig}}}$.

(page 66)

Sudden Expansion in a Pipe



Bernoulli cannot be applied between 1 and 2 because of the turbulence and energy losses that are characteristic of an expanding jet.

Continuity ① → ② $$A_1 u_1 = A_2 u_2 \qquad (1)$$

Momentum ① → ②

$$p_1 A_2 - p_2 A_2 + \rho A_1 u_1^2 - \rho A_2 u_2^2 = 0 \qquad (2)$$

note

Substitute $u_1 = u_2 \frac{A_2}{A_1}$ from (1) into (2)

$$p_2 - p_1 = \rho u_2^2 \left(\frac{A_2}{A_1} - 1 \right), \qquad (3)$$

which is always positive, so pressure increases. Increase in pressure energy is at the expense of kinetic energy

Overall Energy Balance

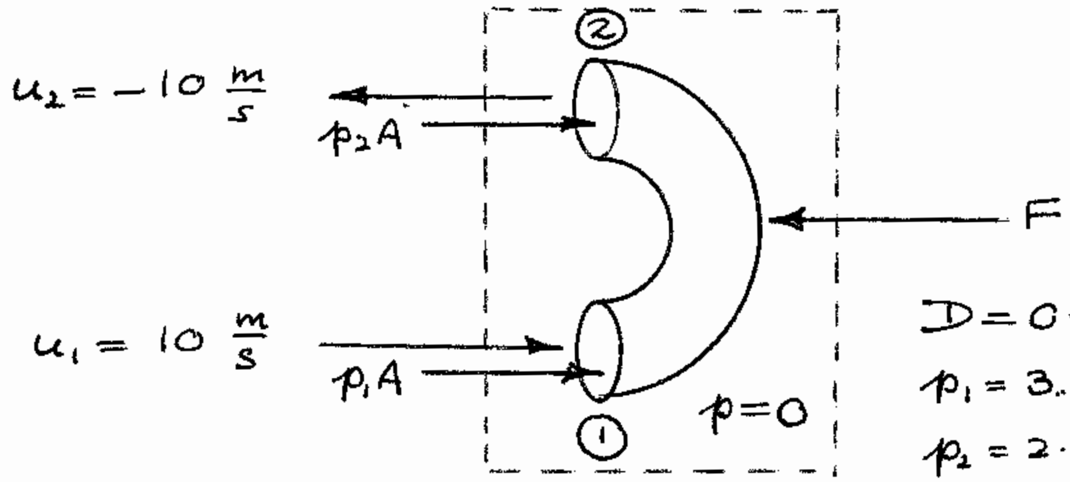
$$\Delta\left(\frac{u^2}{2}\right) + \frac{\Delta p}{\rho} + \cancel{g\Delta z}^{\text{zero}} + \cancel{w}^{\text{zero}} + \mathcal{F} = 0$$

$$\mathcal{F} = \frac{u_1^2}{2} - \frac{u_2^2}{2} + \frac{p_1 - p_2}{\rho} = \frac{u_1^2}{2} - \frac{u_2^2}{2} + \rho u_2^2\left(1 - \frac{A_2}{A_1}\right)$$

$$= \frac{u_2^2}{2}\left(\frac{A_2^2}{A_1^2} - 1 + 2 - 2\frac{A_2}{A_1}\right) = \frac{u_2^2}{2}\left(\frac{A_2}{A_1} - 1\right)^2,$$

which is always positive.

Force on Return Elbow



$D = 0.1\ m$

$p_1 = 3.0$ } bar
$p_2 = 2.5$ } (gauge)

$$A = \frac{\pi D^2}{4} = \frac{\pi (0.1)^2}{4} = 0.00785\ m^2$$

$$m = \rho A u_1 = \underset{\frac{kg}{m^3}}{1{,}000} \times \underset{m^2}{0.00785} \times \underset{\frac{m}{s}}{10} = 78.54\ \frac{kg}{s}$$

Momentum Balance → on Return Elbow

$$\underbrace{+\ m u_1 \quad - m u_2}_{\text{Addition Loss By flow}} + \underbrace{p_1 A + p_2 A}_{\text{Both } p_1 \text{ and } p_2 \text{ tend to push elbow to right}} \underbrace{- F}_{\text{Elbow stays put.}} = 0$$

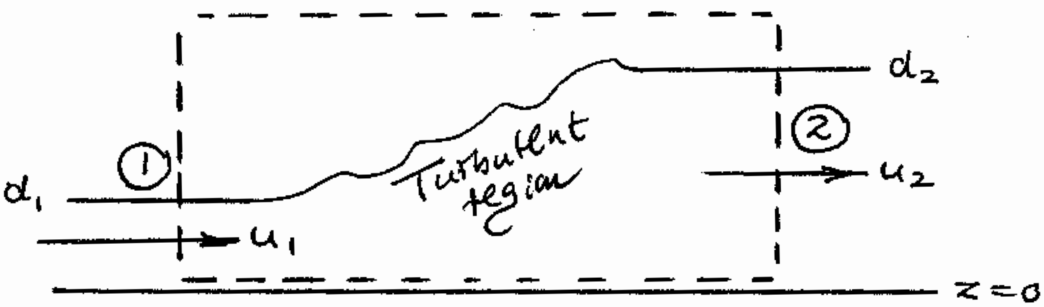
$$F = (p_1 + p_2) A + 2 m u_1$$

$$= (3.0 + 2.5) \times \underset{\frac{N/m^2}{bar}}{10^5} \times 0.00785 + 2 \times 78.54 \times 10$$

$$= 4{,}317 + 1{,}571 = \underline{\underline{5{,}888\ N}}$$

Hydraulic Jump

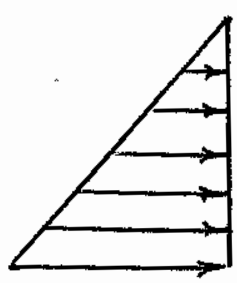
Abrupt, irreversible transition from shallow, rapid stream to deep, tranquil stream.



Ignore shear stress exerted by bottom on liquid

Take unit distance normal to the plane of the diagram

Consider pressure at ① acting on control surface



From earlier notes, total force acting to right is

mean pressure × depth

$$= \frac{0 + \rho g d_1}{2} \times d_1 = \frac{\rho g d_1^2}{2}$$

<u>Continuity</u> $\quad u_1 d_1 = u_2 d_2 = \frac{m}{\rho} \qquad (1)$

<u>$\overrightarrow{\text{Momentum}}$ Balance</u>

$$m u_1 + \frac{\rho g d_1^2}{2} - m u_2 - \frac{\rho g d_2^2}{2} = 0 \qquad (2)$$

Eliminating m and u_2

$$\rho u_1^2 d_1 \left(1 - \frac{d_1}{d_2}\right) + \frac{\rho g}{2}\left(d_1^2 - d_2^2\right) = 0$$

Either $d_1 = d_2$ (no change) or

$$-\frac{u_1^2 d_1}{d_2}\cancel{(d_1 - d_2)} + \frac{g}{2}\cancel{(d_1 - d_2)}(d_1 + d_2) = 0$$

$$d_2^2 + d_1 d_2 - \frac{2u_1^2 d_1}{g} = 0 \qquad (3)$$

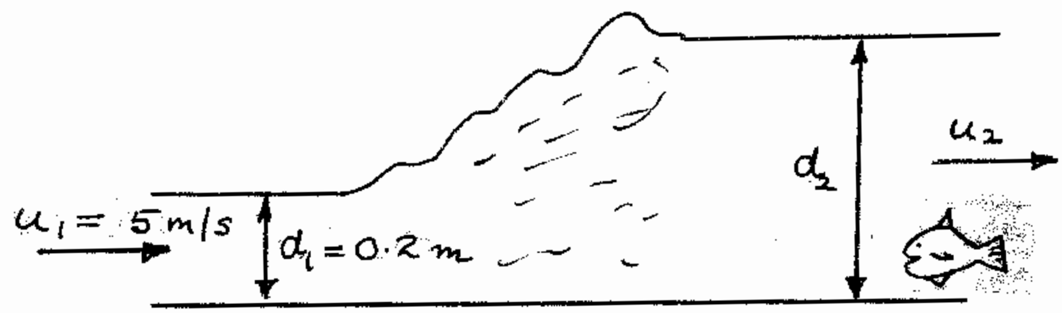
$$d_2 = \frac{-d_1 \underset{(-)}{+} \sqrt{d_1^2 + \frac{8u_1^2 d_1}{g}}}{2} \qquad (4)$$

The above equations are satisfied for flow <u>either</u> from L to R <u>or</u> R to L.

$\mathcal{I}$ may be determined and is positive for rapid shallow flow becoming deep tranquil flow only, and <u>not</u> <u>vice</u> <u>versa</u>.

Hydraulic jump is similar to a <u>shock</u> <u>wave</u> in a flowing compressible gas (supersonic flow suddenly becomes subsonic, with sudden increase in pressure).

Numbers – Hydraulic Jump



From text:

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{d_1^2}{4} + \frac{2u_1^2 d_1}{g}}$$

$$d_2 = -\frac{0.2}{2} + \sqrt{\frac{0.2^2}{4} + \frac{2 \times 5^2 \times 0.2}{9.81}} = \underline{\underline{0.915\,m}}$$

Continuity

$$u_1 d_1 = u_2 d_2$$

$$u_2 = u_1 \frac{d_1}{d_2} = 5 \times \frac{0.2}{0.915}$$

$$= \underline{\underline{1.09\ m/s}}$$

Demonstration that $\mathcal{F}$ is positive for $d_2 > d_1$

Overall Energy Balance

$$\underbrace{\frac{gd_2}{2} - \frac{gd_1}{2}}_{\frac{\Delta \bar{p}}{\rho}} + \frac{u_2^2}{2} - \frac{u_1^2}{2} + \underbrace{\frac{gd_2}{2} - \frac{gd_1}{2}}_{g\Delta \bar{z}} + \mathcal{F} = 0$$

Hence
$$\mathcal{F} = g(d_1 - d_2) + \frac{1}{2}(u_1^2 - u_2^2)$$
$$= g(d_1 - d_2) + \frac{u_1^2}{2}\left(1 - \frac{d_1^2}{d_2^2}\right)$$

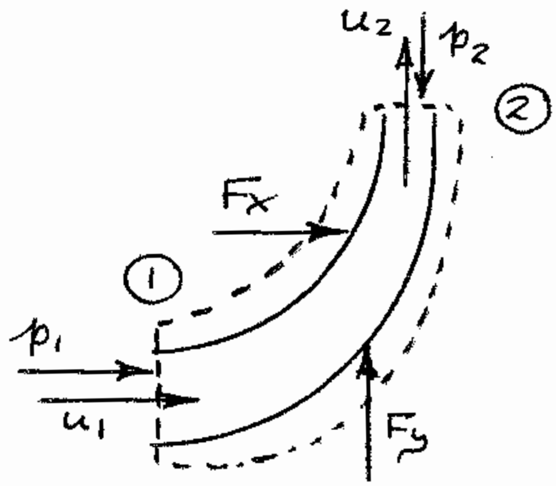
But from equation (3),
$$\frac{u_1^2}{2} = \frac{g}{4}\,\frac{d_2}{d_1}(d_2 + d_1)$$

Eliminating u_1^2 and dividing by g,
$$\frac{\mathcal{F}}{g} = (d_1 - d_2) + \frac{d_2}{4d_1}(d_2 + d_1)\frac{(d_2^2 - d_1^2)}{d_2^2}$$
$$= (d_1 - d_2) + \frac{1}{4d_1 d_2}(d_1 + d_2)^2(d_2 - d_1)$$
$$= (d_2 - d_1)\left[\frac{(d_1 + d_2)^2}{4d_1 d_2} - 1\right]$$
$$= \frac{(d_2 - d_1)}{4d_1 d_2}\left(\begin{matrix} d_1^2 + 2d_1 d_2 + d_2^2 \\ - 4d_1 d_2 \end{matrix}\right) = \frac{(d_2 - d_1)^3}{4d_1 d_2}$$

Thus, $\mathcal{F}/g$ is always positive provided that d_2 is greater than d_1 (and <u>not</u> vice versa)

2.25

90° Reducing Elbow



Continuity (ρ constant)

$$A_1 u_1 = A_2 u_2 \qquad (1)$$

Energy (Bernoulli)

$$\frac{u_1^2}{2} + \frac{p_1}{\rho} = \frac{u_2^2}{2} + \frac{p_2}{\rho} \qquad (2)$$

x – Momentum

$$\rho A_1 u_1^2 + p_1 A_1 + F_x = 0 \qquad (3)$$

y – Momentum

$$-\rho A_2 u_2^2 - p_2 A_2 + F_y = 0 \qquad (4)$$

Hence

$$F_x = -(\rho u_1^2 + p_1) A_1$$

$$F_y = (\rho u_2^2 + p_2) A_2 = \left(\rho u_1^2 \frac{A_1^2}{A_2^2} + p_2\right) A_2$$

But $\quad p_2 = p_1 + \frac{\rho u_1^2}{2}\left(1 - \frac{A_1^2}{A_2^2}\right)$

so $\quad F_y = A_2 \left[p_1 + \frac{\rho u_1^2}{2}\left(1 + \frac{A_1^2}{A_2^2}\right)\right]$

Numerically, with $p_1 = 1.5 \times 10^5 \ N/m^2$, $\quad u_1 = 5.0 \ m/s$

$A_1 = \frac{\pi}{4}(0.20)^2 = 0.03142 \ m^2$

$A_2 = \frac{\pi}{4}(0.15)^2 = 0.01767 \ m^2$

$\frac{A_1^2}{A_2^2} = 3.160 \qquad \rho = 1{,}000 \ \frac{kg}{m^3}$

$$F_x = -5{,}498 \ N$$

$$F_y = 3{,}569 \ N$$

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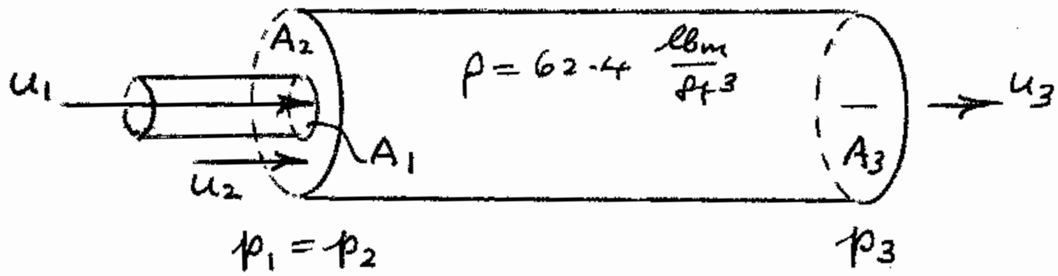
Jet-Ejector Pump



Areas: $A_1 = 0.05, \quad A_2 = 0.5, \quad A_3 = 0.55 \ ft^2$

Suction Line $\quad Q_2 = 10 = A_2 u_2$ or $\underline{u_2 = 20 \ ft/s}$

$$-5 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} \quad \text{or} \quad p_2 = p_1 = \underline{-2.253 \times 10^4} \ (ft, lbm, s \text{ units})$$

Jet Line $\quad 100 = \frac{p_1}{\rho g} + \frac{u_1^2}{2g}$ or $\underline{u_1 = 84.63 \ ft/s}$

Continuity $\quad u_1 A_1 + u_2 A_2 = u_3 A_3$

$$u_3 = \frac{1}{0.55}\Big(\underbrace{\underbrace{84.63 \times 0.05}_{Q_1 = 4.23} + \underbrace{20 \times 0.5}_{Q_2 = 10}}_{Q_3 = 14.23 \ ft^3/s}\Big) = \underline{25.88 \ ft/s}$$

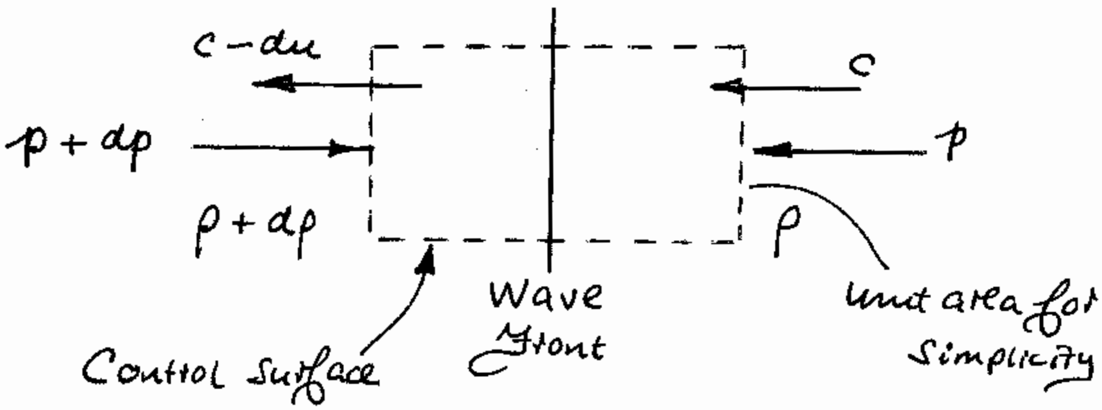
Momentum

$$(p_1 - p_3)A_3 + \rho\,\underbrace{(Q_1 u_1 + Q_2 u_2 - Q_3 u_3)}_{1.184 \times 10^4} = 0$$

$$p_3 = -1003$$

$$h_3 = \underbrace{\frac{u_3^2}{2g}}_{10.40} + \underbrace{\frac{p_3}{\rho g}}_{-0.50} = \underline{\underline{9.90 \ ft}}.$$

Speed of a Sound Wave



Mass Balance

$$c\rho = (c - du)(\rho + d\rho) = c\rho + c\,d\rho - \rho\,du - \underbrace{du\,d\rho}_{\text{small}}$$

Hence $\quad c\,d\rho = \rho\,du \qquad (1)$

Momentum Balance ←

$$\rho c^2 - \underbrace{(\rho + d\rho)(c - du)^2}_{c\rho(c-du)\text{ from mass balance}} + p - (p + dp) = 0$$

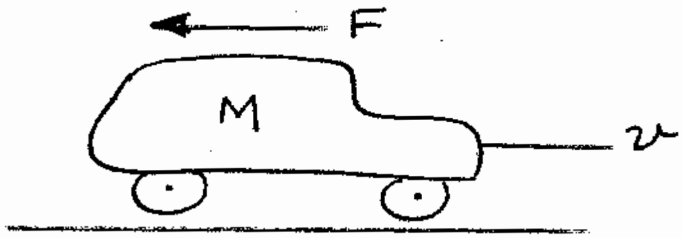
$$\underbrace{\cancel{\rho c^2} - \cancel{\rho c^2}}_{\text{cancel}} + \rho c\,du - dp = 0 \qquad (2)$$

Substitute $\rho\,du = c\,d\rho$ from (1) into (2):

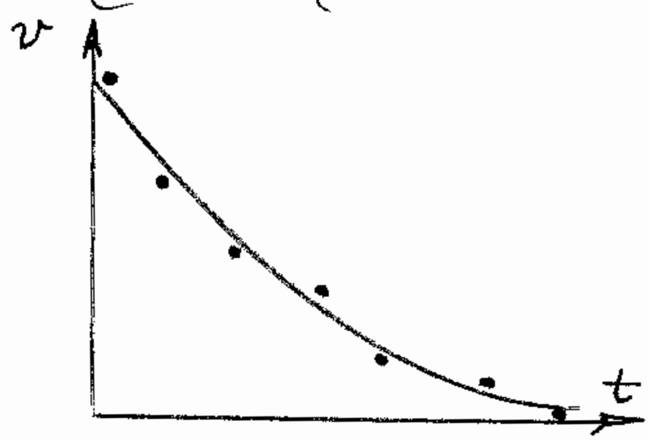
$$c^2\,d\rho - dp = 0$$

$$c = \sqrt{\frac{dp}{d\rho}}$$

Car Retarding Force



Accelerate to maximum speed (eg 65 mph), turn off engine after putting transmission into neutral, and ask a friend to record speed v as a function of time t:



Use CricketGraph least squares to fit a polynomial to the data, eg

$$v = a_0 + a_1 t + a_2 t^2$$

Momentum Balance on Car (to left ←) gives

$$F = -M \frac{dv}{dt} \doteq -M(a_1 + 2a_2 t)$$

Hence F can be obtained as a function of time and then as a function of speed v.

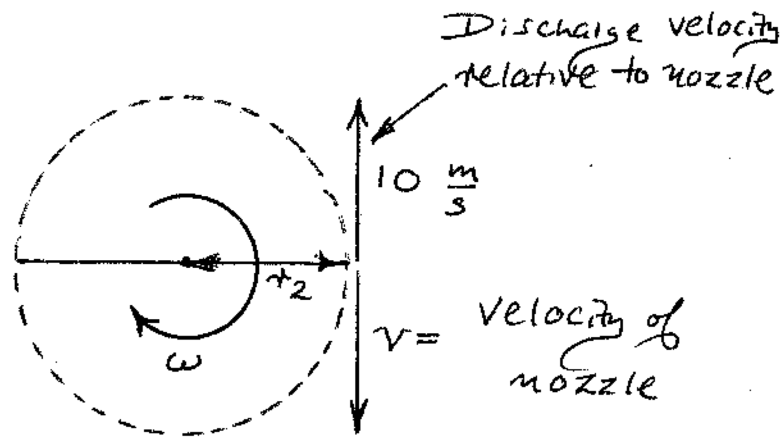
2.29

Garden Sprinkler – Arms Rotating

Since the arm is free to rotate, there is no applied torque. Now

$$T = m\left[(ru)_2 - \underbrace{\cancel{(ru)_1}}_{\text{small or zero}}\right] \doteq m r_2 u_2,$$

which means that the discharge velocity u_2 is zero as seen by a stationary observer.



Discharge velocity relative to nozzle

$$= \frac{Q}{2A}$$

$$= \frac{0.0001}{2 \times 5 \times 10^{-6}}$$

$$= 10 \ \frac{m}{s}$$

$r_2 = 20\ cm = 0.2\ m$

Thus, $u_2 = \underbrace{10}_{\text{velocity of water relative to nozzle}} - \underbrace{v}_{\text{velocity of nozzle}} = 0$

$$v = 10 \ \frac{m}{s}$$

Angular Velocity

$$\omega = \frac{v}{r_2} = \frac{10}{0.2} = 50 \ \frac{rad}{s} = \frac{50}{2\pi} = 7.96 \ rps$$

2.30

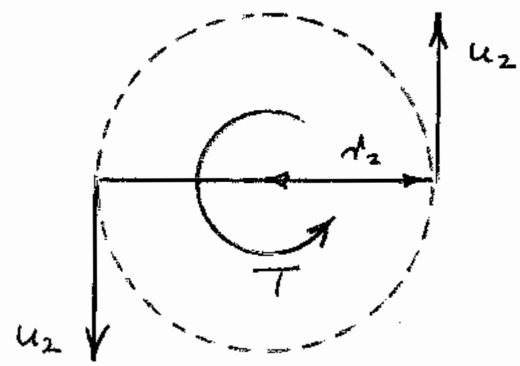
Garden Sprinkler – Arms Fixed

Total mass flow rate

$$\dot{m} = \rho Q = 1{,}000 \times 0.0001 = 0.1 \ \frac{kg}{s}$$

Discharge velocity from each of two nozzles

$$u_2 = \frac{Q}{2A} = \frac{0.0001}{2 \times 5 \times 10^{-6}} = 10 \ \frac{m}{s}$$



$r_2 = 20$ cm
$= 0.2$ m

Angular momentum balance

zero or small

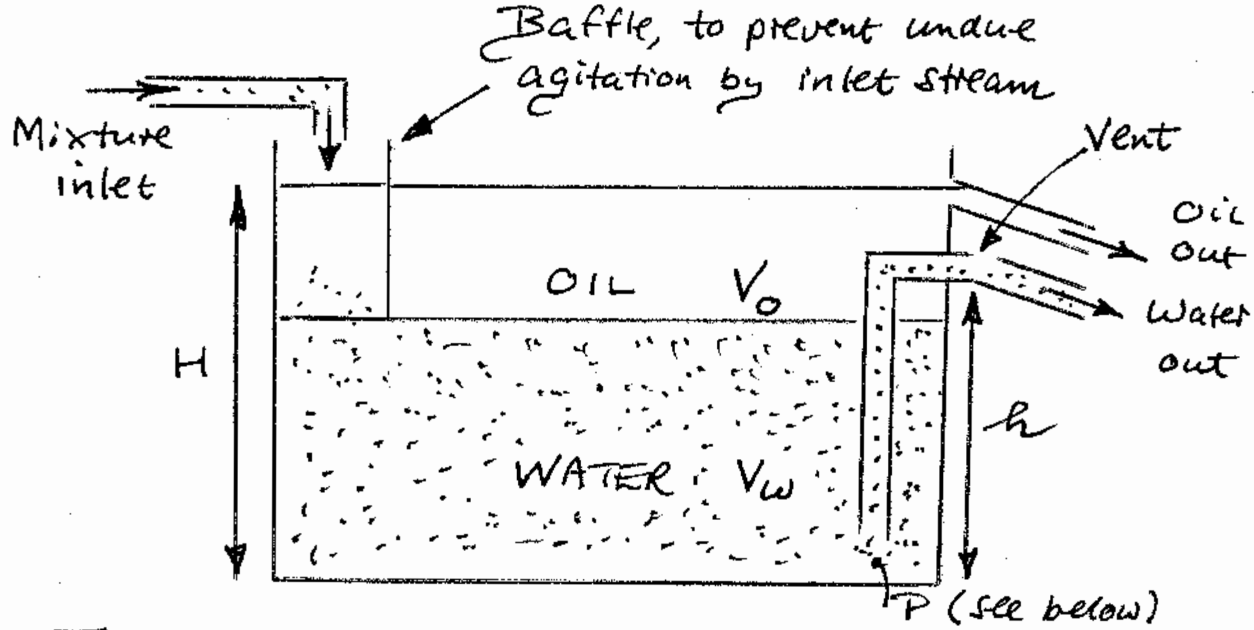
$$T = \dot{m}\left[(ru)_2 - \cancel{(ru)_1}\right]$$

$$= 0.1 \times 0.2 \times 10$$

$$T = 0.2 \ \text{N m}$$

2.31

Oil/Water Separation



To insure 20 min residence times at the highest flow rates, the necessary volumes are

$$V_O = \underset{gpm}{10} \times \underset{min}{20} = 200 \text{ gal} \qquad V_W = 40 \times 20 = 800 \text{ gal}$$

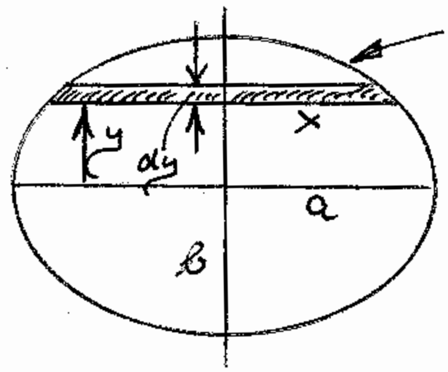
For tank of rectangular cross-section, hydrostatics gives

$$\rho_w g h = \rho_w g (0.8H) + \rho_o g (0.2H)$$

Assume $\dfrac{\rho_o}{\rho_w} = 0.8$ Hence $h = 0.96H$

A key feature of the design is that the interface level is self-regulating and will remain closely at 80% of the total depth. If, for example, the water level should rise, it would cause an imbalance of pressures at point P, so the level would automatically decline.

Slowing Down of the Earth



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

For disc of radius x,

$$dM = \pi x^2 \rho \, dy$$

$$dI = dM \frac{x^2}{2} = \frac{1}{2}\pi\rho \, x^4 dy$$

Integrate to get M and I for whole ellipsoid (note there are top and bottom halves)

Mass $\quad M = 2\int_0^b \pi\rho \, a^2\left(1-\frac{y^2}{b^2}\right)dy = \frac{4}{3}\pi\rho a^2 b$

Moment of inertia $\quad I = 2\int_0^b \frac{1}{2}\pi\rho a^4\left(1-\frac{y^2}{b^2}\right)^2 dy = \frac{8}{15}\pi\rho a^4 b$

$$= \frac{2}{5} M a^2$$

M ← constant

Conservation of Angular Momentum

$$I_s \omega_s = I_e \omega_e$$

I_s: Sphere of radius R $\qquad$ I_e: Ellipsoid

$$\frac{\omega_s}{\omega_e} = \frac{I_e}{I_s} = \frac{a^2}{R^2}, \quad \frac{\omega_s}{\omega_e} = \frac{365\times24\times3600 + \frac{16}{19}}{365\times24\times3600} = \frac{T_e}{T_s}$$

$$= 1 + 2.67\times10^{-8}$$

Since $\omega = \frac{2\pi}{T}$

$$\frac{a^2}{R^2} = (1+\varepsilon)^2 \doteq 1 + 2\varepsilon = 1 + 2.67\times10^{-8}$$

ε: fractional increase

Hence $\underline{\underline{\varepsilon = 1.34\times10^{-8}}}$

Let fractional decrease at poles $= \delta$.

Since M is constant,

$$a^2 b = R^3$$

$$\frac{a^2}{R^2} \times \frac{b}{R} = 1$$

$$(1 + 2\epsilon)(1-\delta) = 1$$

$$\delta = 2\varepsilon = 2.67 \times 10^{-8}$$

Actual Distances

Bulge in radius at equator

$$= 1.34 \times 10^{-8} \times 6.37 \times 10^{6} = 0.0854 \text{ m}$$

$$= 8.54 \text{ cm}$$

Contraction in radius at poles

$$= 2.67 \times 10^{-8} \times 6.37 \times 10^{6} = 0.1701 \text{ m}$$

$$= 17.1 \text{ cm}$$

2.33

Head for a Real Liquid

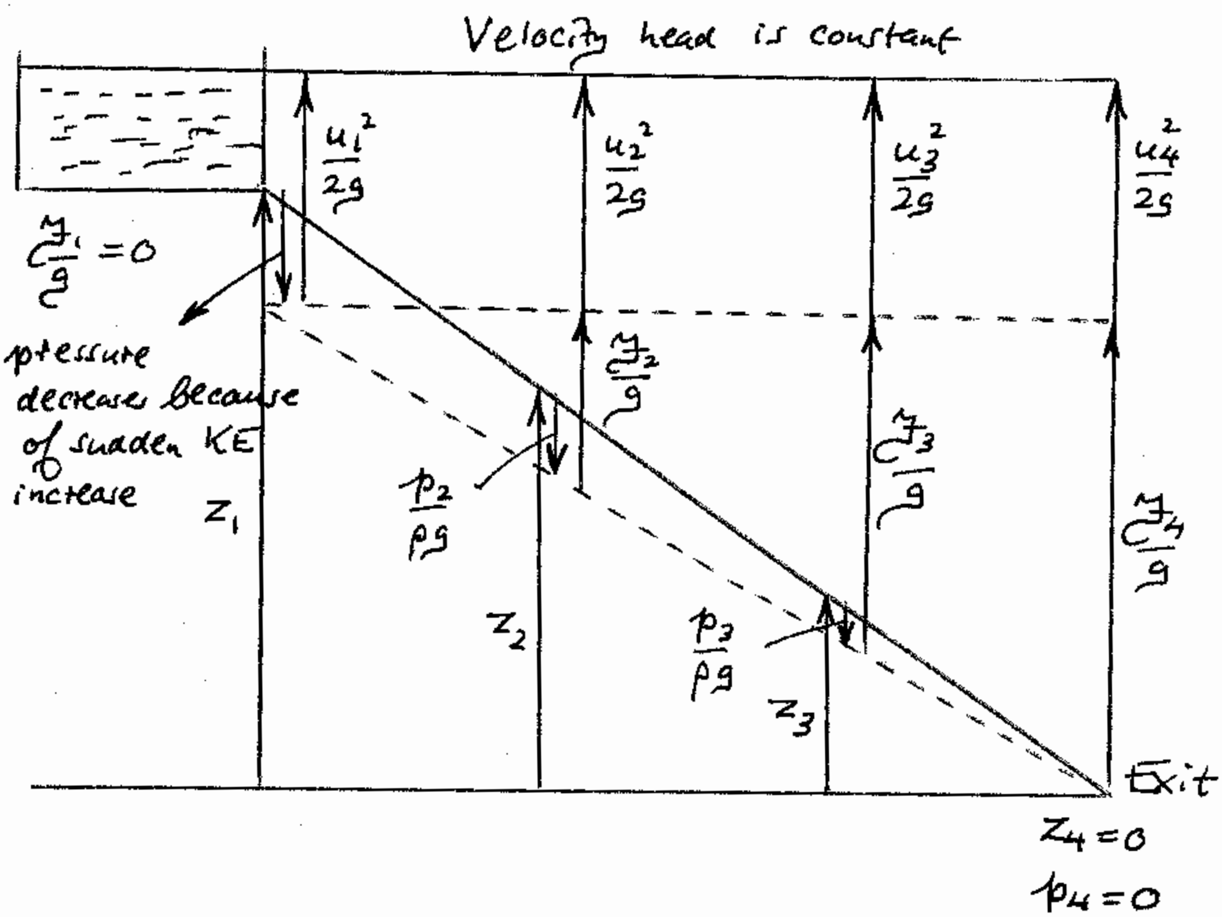
$$\frac{u^2}{2g} + \frac{p}{\rho g} + z + \frac{\mathcal{Y}}{g} = H \text{ (constant)}$$

$\frac{u^2}{2g}$: Constant

$\frac{p}{\rho g}$: increases to $p_4 = 0$ at exit, so must be negative before

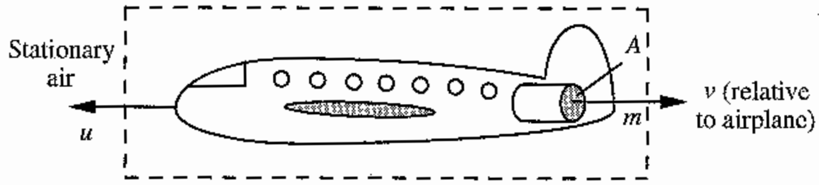
z: decreases towards exit

$\frac{\mathcal{Y}}{g}$: Increases linearly towards exit



2.34

Acceleration of a Jet Airplane



Control volume moving with airplane

(a) Mass flow rate of hot gases

$$m = \rho_g A v$$

(b) Exhaust gas velocity is $v - u$ to the right

(c) Momentum balance ←

$$\underset{\text{input}}{m \times 0} \; \underset{\text{out}}{-} \; m\,[\underset{\text{negative direction}}{-}(v-u)] \; \underset{\text{drag}}{-\,cu^2} = \frac{d}{dt}(Mu) = M\frac{du}{dt} = Ma$$

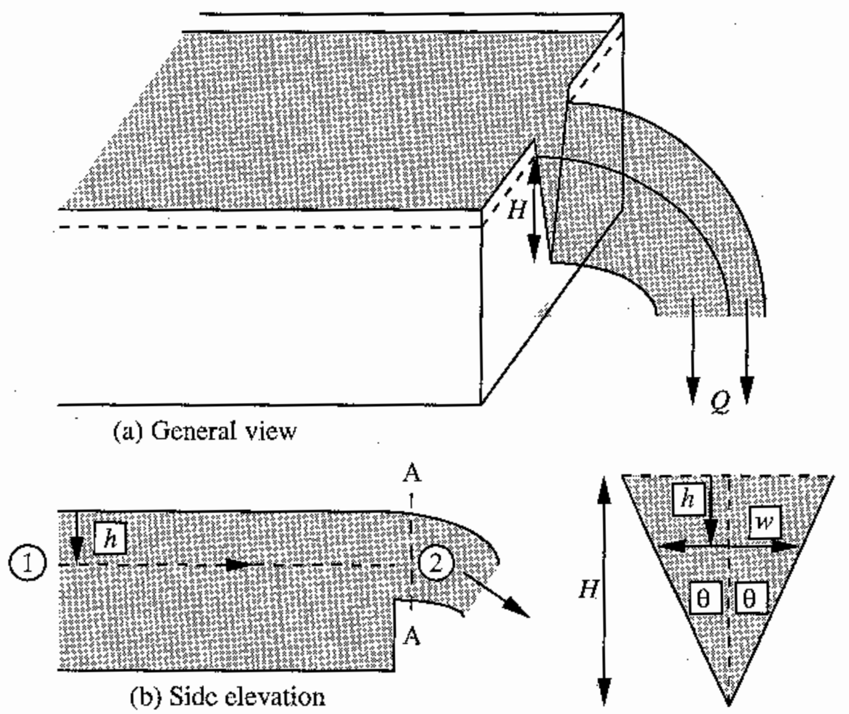
Hence $$a = \frac{1}{M}\left[\underbrace{\rho_g A v}_{m}(v-u) - cu^2\right]$$

(d) Numerical values

$$a = \frac{1}{10^4}\left[\underbrace{0.4 \times 2 \times 300(300-100)}_{48,000} - \underbrace{0.60 \times 100^2}_{6,000}\right]$$

$$a = 4.2 \ \frac{m}{s^2}$$

Performance of a "V" Notch



(a) General view

(b) Side elevation

(c) End elevation of notch (enlarged)

Bernoulli 1 → 2

$$\underbrace{\frac{u_1^2}{2}}_{\text{zero}} + \frac{p_1}{\rho} + g z_1 = \frac{u_2^2}{2} + \underbrace{\frac{p_2}{\rho}}_{\text{zero}} + g z_2$$

z_1 and z_2: Equal

Hydrostatics

$$p_1 = \rho g h$$

Hence

$$u_2 = \sqrt{\frac{2 \rho g h}{\rho}} = \sqrt{2 g h}$$

Width of notch $\quad \omega = 2(H-h)\tan\theta$

Flow rate is obtained by integration

$$Q = \int_{h=0}^{H} u_2 \omega \, dh = \int_0^H \sqrt{2gh}\; 2(H-h)\tan\theta \, dh$$

$$Q = 2\sqrt{2g}\tan\theta \underbrace{\int_0^H (H\sqrt{h} - h^{3/2})\,dh}_{\left[\frac{2}{3}Hh^{3/2} - \frac{2}{5}h^{5/2}\right]_0^H}$$

$$= 2\sqrt{2g}\tan\theta\; H^{5/2} \underbrace{\left(\frac{2}{3} - \frac{2}{5}\right)}_{\frac{4}{15}}$$

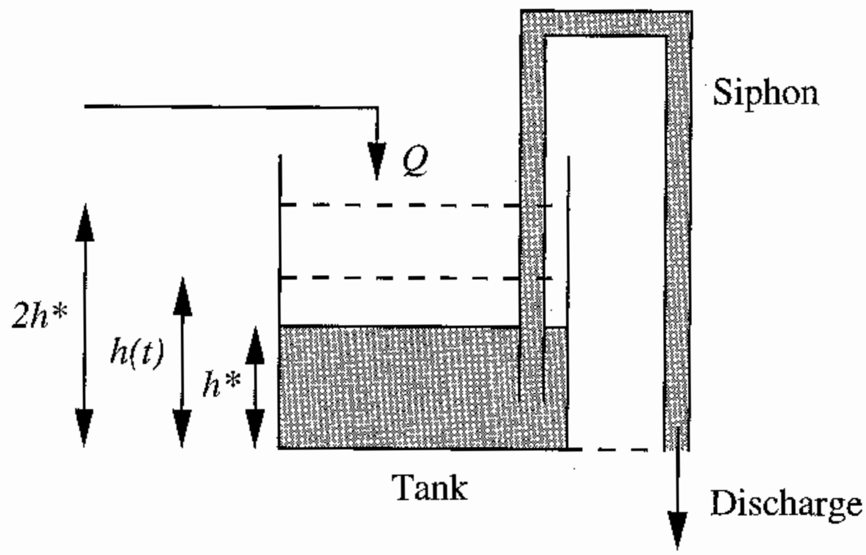
Hence

$$Q = \frac{8}{15}\sqrt{2g}\tan\theta\; H^{5/2}$$

More accurately

$$Q = \frac{8}{15}\sqrt{2g}\; C_D \tan\theta\; H^{5/2}$$

Tank Draining with a Siphon



Since the density is constant, volumetric balances can be substituted for mass balances

(a) Steady-state balance

$$Q - \underbrace{a\sqrt{2gh^*}}_{\text{Bernoulli}} = 0$$

Hence $\quad h^* = \dfrac{Q^2}{2ga^2} \quad$ and $\quad \sqrt{h^*} = \dfrac{Q}{a\sqrt{2g}}$ (needed below)

(b) Transient balance

$$2Q - a\sqrt{2gh} = \frac{d}{dt}(Ah) = A\frac{dh}{dt}$$

<u>Separate variables and integrate:</u>

$$A\int_{h^*}^{2h^*} \frac{dh}{\underbrace{2Q}_{\text{"a"}} \underbrace{-a\sqrt{2gh}}_{\text{"b"}}} = \int_0^t dt = t$$

Use $$\int \frac{dh}{a+b\sqrt{h}} = \frac{2}{b^2}\left(b\sqrt{h} - a\ln|a+b\sqrt{h}|\right)$$

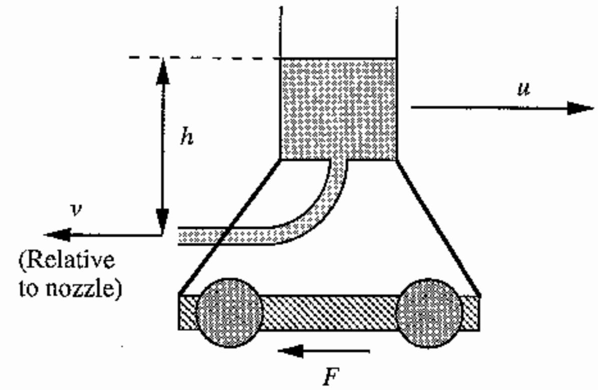
$$t = \frac{2A}{a^2 2g}\left[-a\sqrt{2g}\sqrt{h} - 2Q\ln|2Q - a\sqrt{2g}\sqrt{h}|\right]_{h^*}^{2h^*}$$

$$= \frac{A}{g a^2}\left[\underbrace{-a\sqrt{2g}(\sqrt{2h^*}-\sqrt{h^*})}_{-Q(\sqrt{2}-1)} - 2Q\ln\frac{|2Q - a\sqrt{2g}\,\frac{\sqrt{2}Q}{a\sqrt{2g}}|}{|2Q - a\sqrt{2g}\,\frac{Q}{a\sqrt{2g}}|}\right]$$

$$t = \frac{QA}{g a^2}\left[\underbrace{1-\sqrt{2} - 2\ln\frac{2-\sqrt{2}}{2-1}}_{1.070}\right] = \frac{0.655\,QA}{g a^2}$$

[Note that t increases as Q increases. The reason is that h^* is proportional to Q^2; therefore, there is a larger height differential, $2h^* - h^*$, at larger Q.]

Fluid "Jetset"



Bernoulli's equation between surface of water and nozzle exit (same as tank draining in text) gives:

$$v = \sqrt{2gh}, \text{ so mass flow rate is } m = \rho A\sqrt{2gh}$$

Momentum balance $\longrightarrow$ A stationary observer sees the jet moving to the left with velocity $(v-u)$

$$\underset{\text{loss}}{\overset{-}{\uparrow}} m \left[\underset{\text{to left}}{\overset{-}{\uparrow}}(v-u)\right] - F = \frac{d}{dt}(Mu) = M\frac{du}{dt} + u\frac{dM}{dt}$$

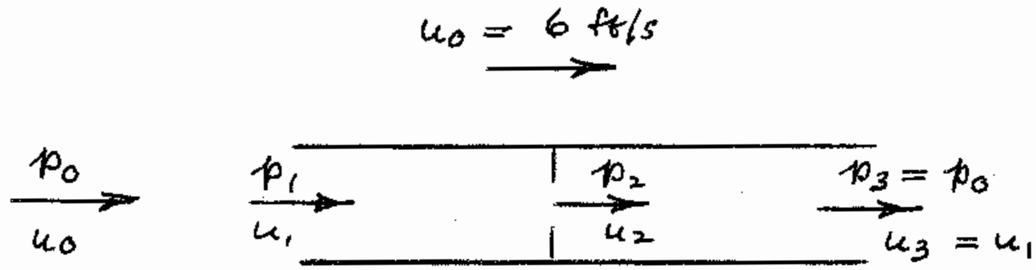
But $$\frac{dM}{dt} = -m$$

Hence $$mv - mu - F = M\frac{du}{dt} - mu$$

$$a = \frac{du}{dt} = \frac{1}{M}\left(\underbrace{\rho A\sqrt{2gh}}_{m}\ \underbrace{\sqrt{2gh}}_{v} - F\right)$$

$$a = \frac{1}{M}(2\rho A g h - F)$$

Force to Hold Orifice Pipe in River



From text, the overall pressure drop for an orifice plate is:

$$p_1 - p_3 = p_1 - p_0 = \frac{\rho u_1^2}{2}\left(\frac{A_1}{A_2} - 1\right)^2 \qquad (1)$$

Bernoulli from upstream to pipe entrance

$$\frac{p_0}{\rho} + \frac{u_0^2}{2} = \frac{p_1}{\rho} + \frac{u_1^2}{2} \qquad (2)$$

Using (1) & (2):

$$p_1 - p_0 = \frac{\rho}{2}\left(u_0^2 - u_1^2\right) \overset{\text{from (1)}}{=} \frac{\rho u_1^2}{2}\left(\frac{A_1}{A_2} - 1\right)^2$$

$$\frac{u_0^2}{u_1^2} - 1 = \left(\frac{A_1}{A_2} - 1\right)^2 \quad \text{or} \quad \frac{u_0^2}{u_1^2} = \left(\frac{A_1}{A_2} - 1\right)^2 + 1 \qquad (3)$$

Numerical values

$$\frac{A_1}{A_2} = \frac{\pi D_1^2/4}{0.62\,\pi D_2^2/4} = \frac{D_1^2}{0.62\,D_2^2} = \frac{4^2}{0.62 \times 3^2} = 2.87$$

From (3), $\dfrac{u_0^2}{u_1^2} = (2.87-1)^2 + 1 = 4.49$

$$\frac{u_0}{u_1} = \sqrt{4.49} = 2.12 \quad \text{or} \quad u_1 = \frac{6}{2.12} = 2.83 \ \frac{ft}{s}$$

<u>Overall pressure drop</u>, from (1):

$$p_1 - p_0 = \frac{\rho u_1^2}{2}\left(\frac{A_1}{A_2} - 1\right)^2 = \frac{62.3 \times 2.83^2 (2.87-1)^2}{144 \times 32.2 \times 2}$$

$$= 0.188 \ \frac{lb_f}{in^2}$$

<u>Flow rate through pipe</u> $\quad A_1 = \dfrac{\pi 4^2}{4} = 12.6 \ in^2$

$$Q = u_1 A_1 = 2.83 \times \frac{12.6}{144} = \underline{\underline{0.247 \ \frac{ft^3}{s}}}$$

<u>Force to hold pipe in stream</u>

Since $u_1 = u_3$, convected momentum terms cancel.

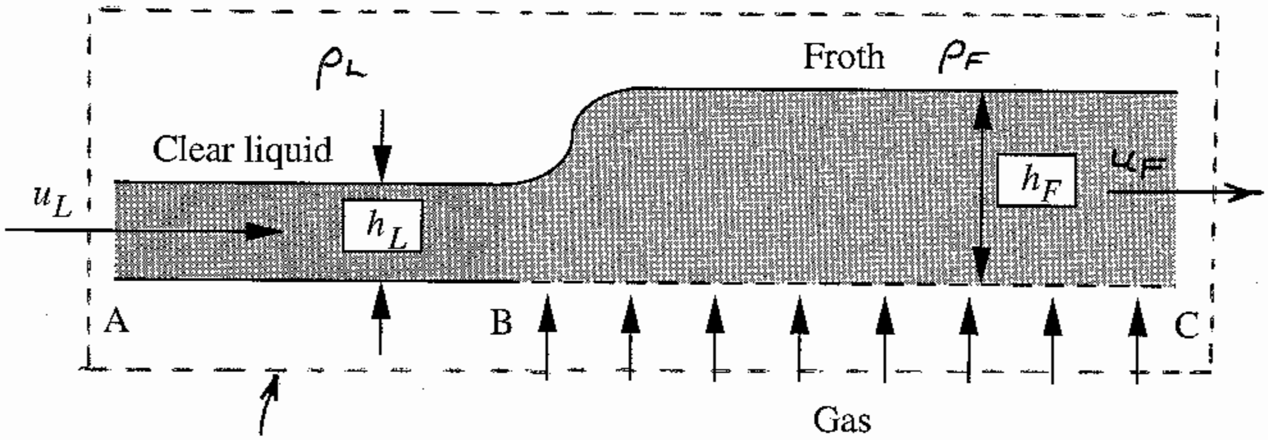
Momentum balance $\rightarrow$ gives

$$(p_1 - p_0) A_1 - F = 0$$

$$F = (p_1 - p_0) A_1 = 0.188 \times 12.6$$

$$\underline{\underline{F = 2.37 \ lb_f}}$$

Froth on Sieve Tray



Control volume (unit distance normal to plane of diagram)

Continuity $\quad u_L h_L \rho_L = u_F h_F \rho_F \quad$ or $u_F = \dfrac{u_L h_L \rho_L}{h_F \rho_F}$

Momentum balance $\longrightarrow$

$$\underbrace{(u_L h_L \rho_L)}_{\text{mass flow rate}}(\underset{\text{flow in}}{u_L} - \underset{\text{flow out}}{u_F}) + \frac{1}{2} g \underbrace{(\rho_L h_L^2 - \rho_F h_F^2)}_{\text{pressure forces at left and right}} = 0$$

Substitute for u_F from continuity

$$(u_L h_L \rho_L) u_L \left(1 - \frac{h_L \rho_L}{h_F \rho_F}\right) + \frac{1}{2} g (\rho_L h_L^2 - \rho_F h_F^2) = 0$$

Multiply through by $\dfrac{2 h_F}{\rho_L g h_L^3}$

$$\frac{2}{g} u_L^2 \frac{h_F}{h_L^2}\left(1 - \frac{h_L \rho_L}{h_F \rho_F}\right) + \frac{h_F}{h_L} - \frac{\rho_F}{\rho_L}\frac{h_F^3}{h_L^3} = 0$$

Rearrange

$$\frac{\rho_F}{\rho_L}\left(\frac{h_F}{h_L}\right)^3 - \left(1 + \frac{2u_L^2}{g h_L}\right)\frac{h_F}{h_L} + \frac{2 u_L^2 \rho_L}{g h_L \rho_F} = 0$$

Numerical Values

Substitute $\dfrac{\rho_F}{\rho_L} = 0.5$, $u_L = 0.5$ ft/s, $h_L = 0.2$ ft
$g = 32.2$ ft/s²

gives

$$\left(\frac{h_F}{h_L}\right)^3 - 2.155 \frac{h_F}{h_L} + 0.3106 = 0$$

Solution (Excel Solver) is $\dfrac{h_F}{h_L} = 1.39$ or $\underline{\underline{h_F = 0.278 \text{ ft}}}$

Relative pressures at base of liquid and froth

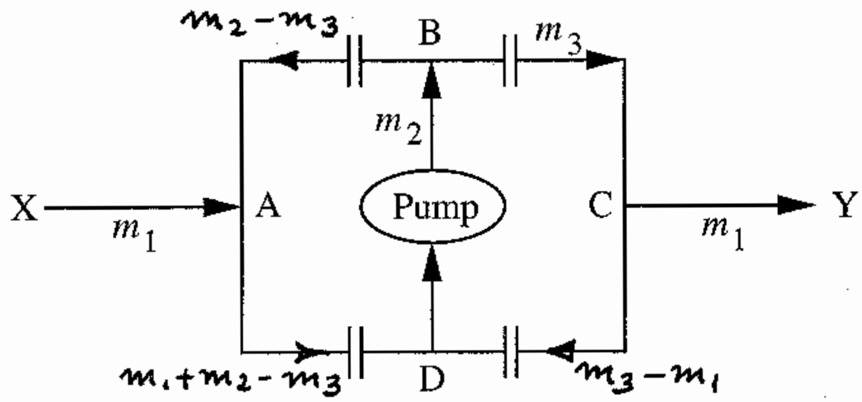
$$p_L = \rho_L g h_L = 0.2 \rho_L g$$

$$p_F = \rho_F g h_F = \rho_L \frac{\rho_F}{\rho_L} g h_F = 0.5 \times 0.278 \rho_L g$$

$$= 0.139 \rho_L g$$

$$\frac{p_F}{p_L} = \frac{0.139}{0.2} = \underline{\underline{0.695}}$$

Multiple Orifice Plates



<u>Pressure recovery</u> downstream of an orifice plate occurs when kinetic energy is partly converted into pressure energy.

<u>Overall pressure drop across an orifice plate.</u>

From text, $\quad p_1 - p_3 = \dfrac{\rho u_1^2}{2}\left(\dfrac{A_1}{A_2} - 1\right)^2$

$$p_1 - p_3 = (\underbrace{\rho u_1 A_1}_{m})(\underbrace{u_1 A_1}_{Q})\underbrace{\left[\frac{1}{2}\left(\frac{1}{A_2} - \frac{1}{A_1}\right)\right]^2}_{k} = kQm$$

Since k essentially depends only on the geometry (assuming reasonably that the contraction coefficient is constant), it is effectively constant.

Since $Q = m/\rho$, the pressure drop can be rewritten as cm^2, where $c = k/\rho$.

Consider the "loop" A-B-C-D-A

The sum of the pressure drops around the complete circuit must equal zero. Omitting the constant c, we have:

$$-\underset{A \to B}{(m_2 - m_3)^2} + \underset{B \to C}{m_3^2} + \underset{C \to D}{(m_3 - m_1)^2}$$

$$-\underset{D \to A}{(m_1 + m_2 - m_3)^2} = 0$$

Expanding terms

$$-m_2^2 + 2m_2 m_3 - m_3^2 + m_3^2 + m_3^2 - 2m_1 m_3 + m_1^2$$

$$- m_1^2 - m_2^2 - m_3^2 - 2m_1 m_2 + 2m_1 m_3 + 2m_2 m_3 = 0$$

Cancellation of terms gives $-2m_2^2 + 4m_2 m_3 - 2m_1 m_3 = 0$

Hence $$m_3 = \frac{m_1 + m_2}{2}$$

Overall pressure drop becomes

$$p_A - p_C = -c(m_2 - m_3)^2 + cm_3^2$$

$$= -c\left(m_2 - \frac{m_1 + m_2}{2}\right)^2 + c\left(\frac{m_1 + m_2}{2}\right)^2$$

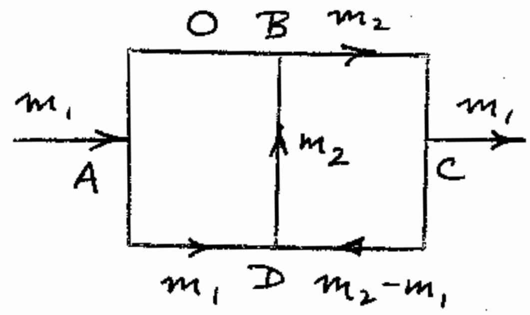
$$= \frac{c}{4}\left[-(m_2 - m_1)^2 + (m_1 + m_2)^2\right]$$

Thus, $$p_A - p_C = \frac{c}{4}\begin{pmatrix} -m_2^2 + 2m_1m_2 - m_1^2 \\ + m_1^2 + 2m_1m_2 + m_2^2 \end{pmatrix}$$

or $$p_A - p_C = c\,m_1 m_2,$$

which is directly proportional to m_1.

<u>Critical pump flow rate</u>



The above scheme requires a positive flow from B to A, and will fail if this is reduced to zero, in which case the loop ABCDA gives

$$0 + m_2^2 + (m_2 - m_1)^2 - m_1^2 = 0$$

$$m_2^2 + m_2^2 - 2m_1m_2 + m_1^2 - m_1^2 = 0$$

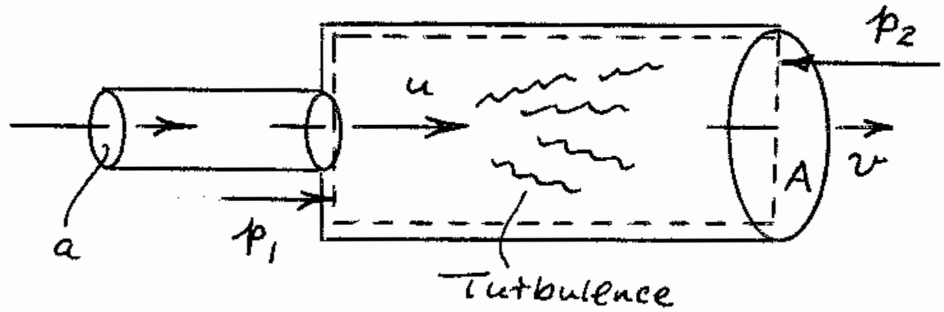
That is $$m_2 = m_1,$$

in which case there is also zero flow from C to D, and the overall pressure drop is

$$p_A - p_C = 2c\,m_1^2,$$

no longer directly proportional to m_1. Therefore, m_2 must at least equal m_1 for it to work.

Pressure Recovery in Sudden Expansion



Continuity $\quad ua = vA \qquad (1)$

Momentum $\quad p_1 A - p_2 A + \rho a u^2 - \rho A v^2 = 0 \qquad (2)$

note

Substitute $u = v\dfrac{A}{a}$ from (1) into (2):

$$p_2 - p_1 = \rho v^2\left(\frac{A}{a} - 1\right) = \rho u^2 \frac{a^2}{A^2}\left(\frac{A}{a} - 1\right)$$

This pressure recovery will be a maximum when

$$\frac{d}{dA}(p_2 - p_1) = \rho u^2 a^2 \left[-\frac{2}{A^3}\left(\frac{A}{a} - 1\right) + \frac{1}{A^2}\left(\frac{1}{a}\right)\right] = 0$$

Multiply by A^3 gives:

$$2\left(\frac{A}{a} - 1\right) = \frac{A}{a} \quad \text{or} \quad \frac{A}{a} = 2$$

Since areas are proportional to the squares of diameters, $(p_2 - p_1)$ is a maximum when the diameter of the exit pipe is $\sqrt{2}$ times the diameter of the inlet pipe.

<u>Energy dissipation rate</u> From energy balance:

$$\Delta\left(\frac{u^2}{2}\right) + \frac{\Delta p}{\rho} + \underbrace{g\Delta z + w}_{\text{zero}} + \mathcal{F} = 0$$

$$\mathcal{F} = \frac{u^2}{2} - \frac{v^2}{2} + \frac{p_1 - p_2}{\rho} = \frac{u^2}{2} - \frac{v^2}{2} + \rho v^2\left(1 - \frac{A}{a}\right)$$

$$= \frac{v^2}{2}\left(\frac{A^2}{a^2} - 1 + 2 - 2\frac{A}{a}\right) = \frac{v^2}{2}\left(\frac{A}{a} - 1\right)^2$$

$$= \frac{u^2}{2}\frac{a^2}{A^2}\left(\frac{A}{a} - 1\right)^2 \text{ (per unit mass flowing)}$$

<u>Hence the overall rate of energy dissipation is</u>

$$m\mathcal{F} = \rho a u \frac{u^2}{2}\frac{a^2}{A^2}\left(\frac{A}{a} - 1\right)^2$$

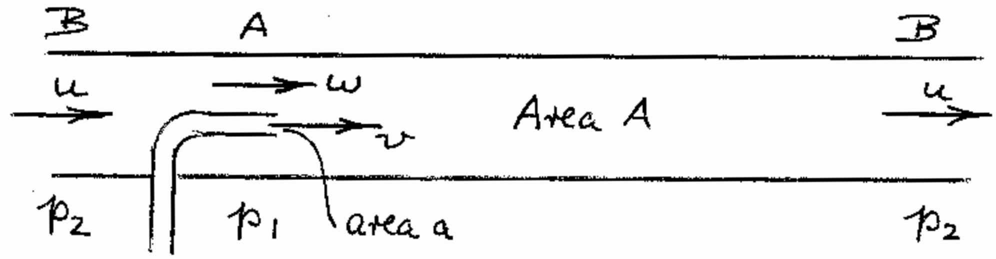
$$= \frac{\rho a u^3}{2}\left(\frac{1}{4}\right)(2-1)^2$$

$$= \frac{\rho a u^3}{8}$$

2.42

Manifold in Reactor Base

"Unwind" the circular path and treat as linear.



Continuity $B \to A$ $\quad uA = w(A-a) \quad$ or $\quad w = \dfrac{uA}{A-a}$

Bernoulli $B \to A$ $\quad \dfrac{u^2}{2} + \dfrac{p_2}{\rho} = \dfrac{w^2}{2} + \dfrac{p_1}{\rho}$

or $\quad p_1 - p_2 = \dfrac{\rho}{2}(u^2 - w^2)$

Momentum $A - B$ $\quad \underbrace{(p_1 - p_2)A}_{\text{Pressure}} + \underbrace{uA\rho}_{m}(\underset{\text{inlet}}{w} - \underset{\text{exit}}{u}) + \underset{\text{jet}}{\rho a v^2} = 0$

Substitute for $p_1 - p_2$, cancel ρ, and divide by $A/2$:

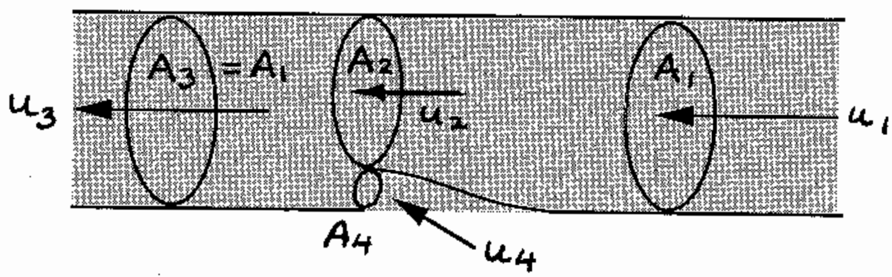
$$\underbrace{(u^2 - w^2) + 2u(w-u)}_{(u-w)(\underbrace{u + w - 2u}_{w-u})} + \frac{2av^2}{A} = 0$$

or $\quad \dfrac{2v^2 a}{A} = (w-u)^2$

$$v\sqrt{\frac{2a}{A}} = w - u = \frac{uA}{A-a} - u = \frac{ua}{A-a}$$

Hence $\quad u = v(A-a)\sqrt{\dfrac{2}{aA}}$

Leaking Ventilation Duct



Areas: $A_3 = A_1, \quad \frac{A_4}{A_1} = 0.1, \quad \frac{A_2}{A_1} = 0.9 \qquad (1)$

Continuity $A_1 u_1 = A_2 u_2 \quad \text{or} \quad u_2 = \frac{A_1}{A_2} u_1 \qquad (2)$

$$A_1 u_1 + A_4 u_4 = A_3 u_3 = A_1 u_3 \qquad (3)$$

Hence $u_3 = u_1 + \frac{A_4}{A_1} u_4 = u_1 + 0.1\, u_4 \qquad (4)$

Bernoulli 1 → 2 $\quad \frac{u_1^2}{2} + \frac{p_1}{\rho} = \frac{u_2^2}{2} + \frac{p_2}{\rho} \qquad (5)$

Hence $\frac{p_1 - p_2}{\rho} = \frac{u_2^2}{2} - \frac{u_1^2}{2} = \frac{u_1^2}{2}\left(\frac{A_1^2}{A_2^2} - 1\right) \qquad (6)$

Momentum (2) $\longrightarrow$ (3) (to the left)

$$p_2 \underbrace{(A_2 + A_4)}_{A_1} - p_3 \underset{A_1}{A_3} + \underbrace{\rho A_2 u_2^2}_{\rho A_1 u_1 u_2}$$

$$+ \rho A_4 u_4^2 - \rho \underset{A_1}{A_3} u_3^2 = 0 \qquad (7)$$

Hence

$$\frac{p_2 - p_3}{\rho} = u_3^2 - u_1 u_2 - \frac{A_4}{A_1} u_4^2 = 0 \quad (8)$$

Adding (6) + (8)

$$\frac{p_1 - p_3}{\rho} = \frac{u_1^2}{2}\left(\frac{A_1^2}{A_2^2} - 1\right) + (u_1 + 0.1 u_4)^2 - u_1 \frac{A_1}{A_2} u_1 - \frac{A_4}{A_1} u_4^2$$

Inserting area ratios from (1) and putting $u_1 = 3 \frac{m}{s}$:

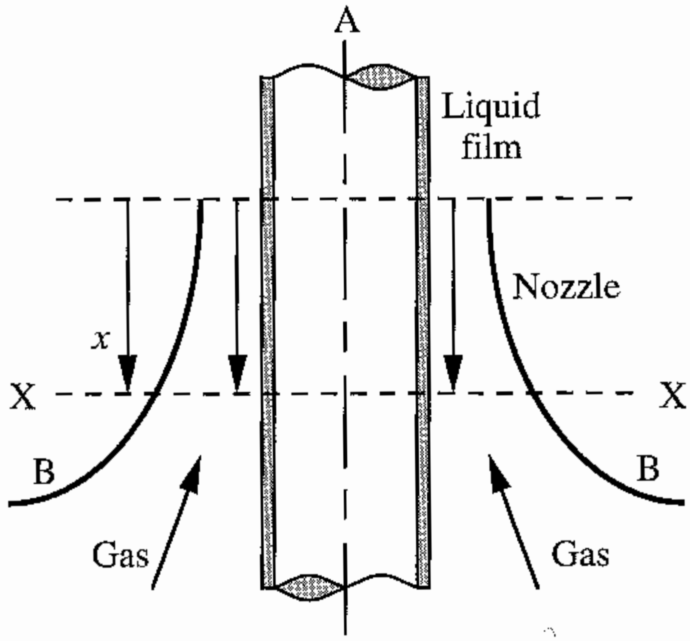
$$0.09 u_4^2 - 0.6 u_4 - 0.0557 + \frac{p_1 - p_3}{\rho} = 0$$

Thus for a specified $(p_1 - p_3)/\rho$, u_4 can be calculated and the inward leakage rate from

$$Q_4 = u_4 A_4$$

Considerations of friction would require a knowledge of the wall shear stress, as in Chapter 3.

Packed-Column Flooding



Continuity

$$Q = uA = u(a + bx^2)$$

$$u = \frac{Q}{a + bx^2}$$

Bernoulli

$$\frac{u^2}{2} + \frac{p}{\rho_G} = c \quad \text{(constant)}$$

(ignore slight elevation effect, since u is large, ρ is small)

$$\frac{1}{2}\frac{Q^2}{(a+bx^2)^2} + \frac{p}{\rho_G} = c$$

$$\frac{dp}{dx} = -\frac{\rho_G Q^2}{2}\,\frac{-2(2bx)}{(a+bx^2)^3} = \frac{2b\rho_G Q^2 x}{(a+bx^2)^3}$$

Pressure gradient is a maximum if

$$\frac{d}{dx}\left(\frac{dp}{dx}\right) = 2b\rho_G Q^2 \, \frac{(a+bx^2)^3 - x(3\times 2bx)(a+bx^2)^2}{(a+bx^2)^6} = 0$$

Hence

$$(a+bx^2)^3 = 6bx^2(a+bx^2)^2$$

$$a + bx^2 = 6bx^2 \qquad \text{or} \quad x = \sqrt{\frac{a}{5b}}$$

Maximum pressure gradient is then

$$\left(\frac{dp}{dx}\right)_{max} = \frac{2b\rho_G Q^2 x}{\underbrace{(a+bx^2)}_{6bx^2}{}^3} = \frac{2b\rho_G Q^2\sqrt{a/5b}}{\frac{216}{125}a^3}$$

$$\frac{6a}{5} \rightarrow 6bx^2$$

$$\left(\frac{dp}{dx}\right)_{max} = \frac{125\, b\rho_G Q^2}{108a^3}\sqrt{\frac{a}{5b}}$$

At onset of flooding, this pressure gradient (note that p increases in the downwards direction) just suffices to overcome the usual hydrostatic gradient in the liquid, causing it to back up:

$$\left(\frac{dp}{dx}\right)_L = \rho_L g = \frac{125\, b\rho_G Q^2}{108a^3}\sqrt{\frac{a}{5b}}$$

Hence

$$Q^2 = \frac{108\, g\rho_L a^3}{125\rho_G b}\sqrt{\frac{5b}{a}} \qquad \text{at onset of flooding}$$

2.45

Interpretation of Pressure Head

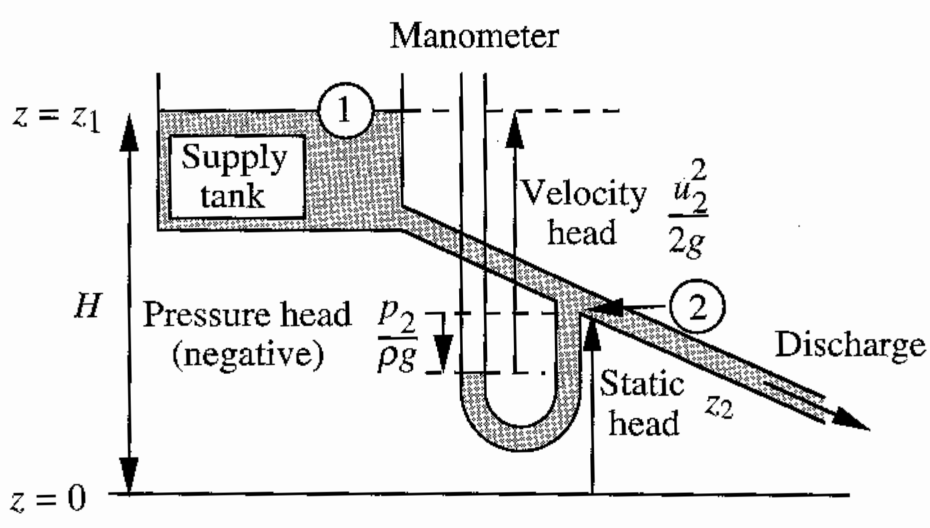
Without friction, the sum of the three heads remains constant

$$\frac{u_2^2}{2g} + \frac{p_2}{\rho g} + z_2 = H$$

$\frac{u_2^2}{2g}$: Constant, because u_2 is constant from continuity

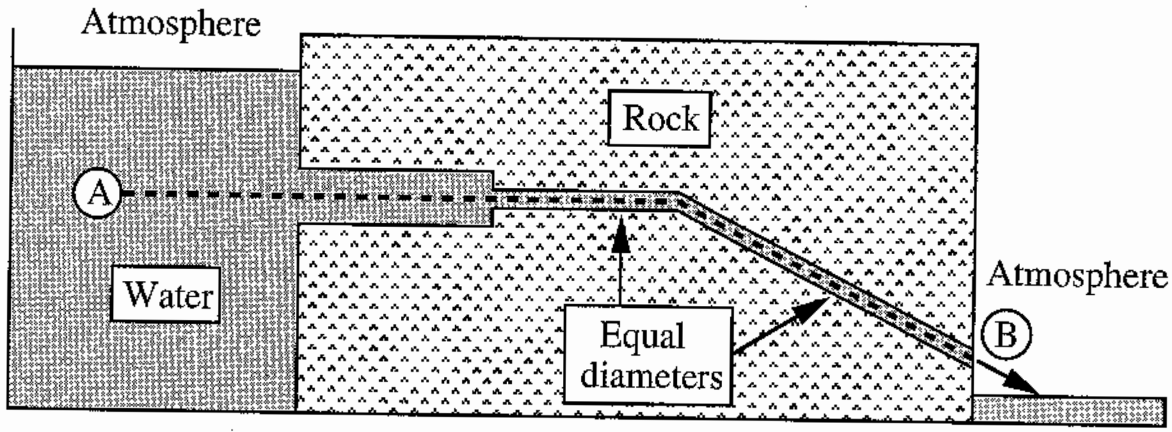
z_2: Decreases from the supply tank to the discharge

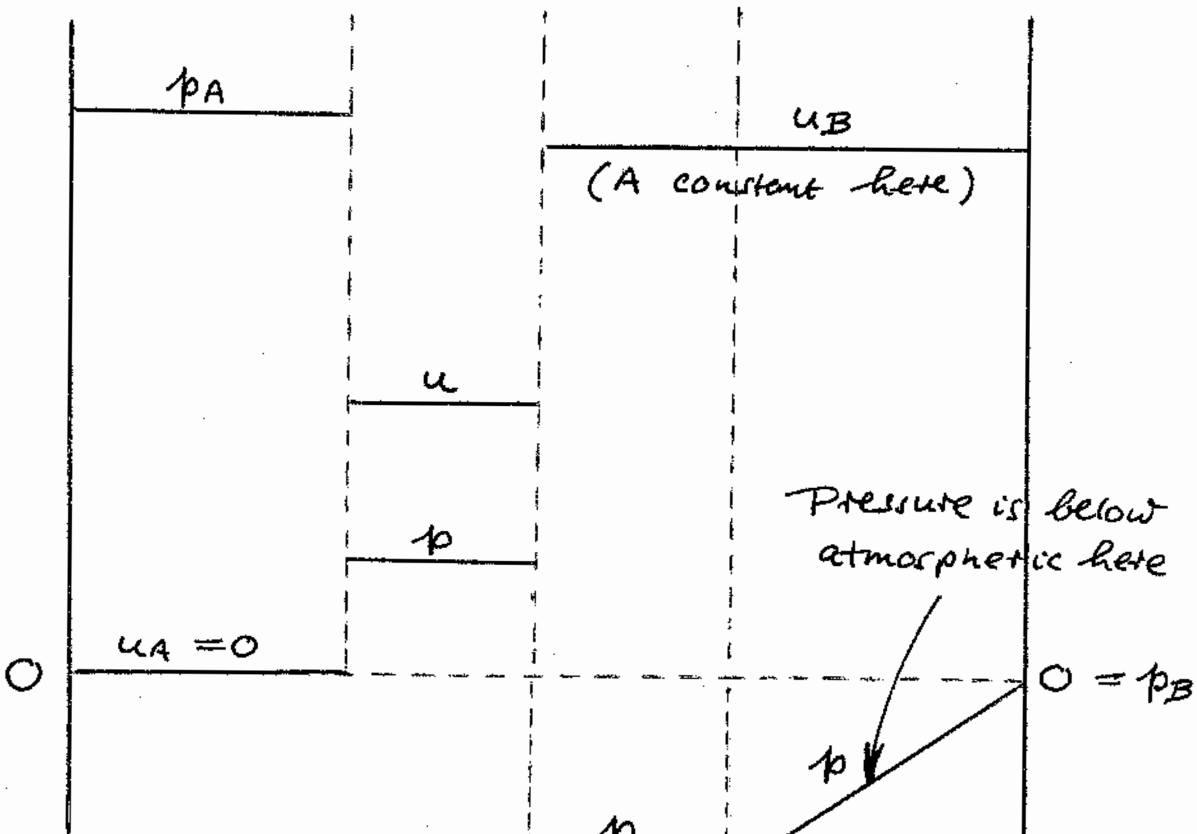
$\frac{p_2}{\rho g}$: Therefore, p_2 must increase as the discharge is approached, where $p_2 = 0$. Hence p_2 must be negative, and Fig 2.6(a) is incorrect as drawn. The correct version follows.



2.46

Velocities and Pressures





The above is based on continuity, $uA = c$, & Bernoulli, $\frac{u^2}{2} + \frac{p}{\rho} + gz = c$