Solution to Problems in Chapter 2, Section 2.10

2.1. $$Q = \int \mathbf{v} \bullet \mathbf{n} dA = \int_{y=0}^{3}\int_{x=0}^{2}\left(\frac{3}{\sqrt{2}}x + \frac{6}{\sqrt{2}}y\right)dxdy = \int_{y=0}^{3}\left(\frac{3}{2\sqrt{2}}x^2 + \frac{6}{2\sqrt{2}}yx\right)\Bigg|_{x=0}^{2} dy$$

$$Q = \int_{y=0}^{3}\left(\frac{6}{\sqrt{2}} + \frac{12}{\sqrt{2}}y\right)dy = \left(\frac{6}{\sqrt{2}}y + \frac{6}{\sqrt{2}}y^2\right)\Bigg|_{y=0}^{3} = \frac{72}{\sqrt{2}}$$

$Q = 50.91\ cm^3 s^{-1}$

2.2. $|\boldsymbol{n}| = 1 = \sqrt{a^2 + a^2 + a^2} = \sqrt{3}a$

Rearranging, $a = 1/\sqrt{3}$

2.3. $$\nabla \bullet (\rho \mathbf{v}\mathbf{v}) = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \bullet (\rho \mathbf{v}\mathbf{v}) = \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \bullet \left(\rho \mathbf{e}_x \mathrm{v}_x \mathbf{v} + \rho \mathbf{e}_y \mathrm{v}_y \mathbf{v} + \rho \mathbf{e}_z \mathrm{v}_z \mathbf{v}\right)$$

$$= \frac{\partial}{\partial x}(\rho \mathrm{v}_x \mathbf{v}) + \frac{\partial}{\partial y}(\rho \mathrm{v}_y \mathbf{v}) + \frac{\partial}{\partial z}(\rho \mathrm{v}_z \mathbf{v})$$

Differentiating term by term,

$$\nabla \bullet (\rho \mathbf{v}\mathbf{v}) = \mathbf{v}\left(\frac{\partial}{\partial x}(\rho \mathrm{v}_x) + \frac{\partial}{\partial y}(\rho \mathrm{v}_y) + \frac{\partial}{\partial z}(\rho \mathrm{v}_z)\right) + \rho \mathrm{v}_x \frac{\partial}{\partial x}(\mathbf{v}) + \rho \mathrm{v}_y \frac{\partial}{\partial y}(\mathbf{v}) + \rho \mathrm{v}_z \frac{\partial}{\partial z}(\mathbf{v})$$

$$\nabla \bullet (\rho \mathbf{v}\mathbf{v}) = \mathbf{v}\nabla \bullet (\rho \mathbf{v}) + \rho \mathbf{v} \bullet \nabla \mathbf{v}$$

2.4. (a) For a two-dimensional steady flow, the acceleration is:

$$\mathbf{a} = \mathrm{v}_x \frac{\partial \mathbf{v}}{\partial \mathrm{x}} + \mathrm{v}_y \frac{\partial \mathbf{v}}{\partial \mathrm{y}}$$

For $\mathbf{v} = U_o(x^2 - y^2 + x)\mathbf{e}_x - U_o(2xy + y)\mathbf{e}_y$,

$$\frac{\partial \mathbf{v}}{\partial \mathrm{x}} = U_o(2x+1)\mathbf{e}_x - U_o 2y\mathbf{e}_y \qquad \frac{\partial \mathbf{v}}{\partial \mathrm{y}} = U_o(-2y)\mathbf{e}_x - U_o(2x+1)\mathbf{e}_y$$

$$\mathbf{a} = U_o^2\left(x^2 - y^2 + x\right)\left((2x+1)\mathbf{e}_x - 2y\mathbf{e}_y\right) - U_o^2\left(2xy + y\right)\left(-2y\mathbf{e}_x - (2x+1)\mathbf{e}_y\right)$$

Collecting terms:

$$\mathbf{a} = U_o^2\left[\left(x^2 - y^2 + x\right)(2x+1) + (2xy + y)2y\right]\mathbf{e}_x - U_o^2\left[\left(x^2 - y^2 + x\right)2y - (2xy + y)(2x+1)\right]\mathbf{e}_y$$

$$\mathbf{a} = U_o^2\left[2x^3 + 3x^2 - 2xy^2 - y^2 + x + 4xy^2 + 2y^2\right]\mathbf{e}_x - U_o^2\left[2yx^2 - 2y^3 + 2xy - 4x^2y + 2xy + 2xy + y\right]\mathbf{e}_y$$

$$\mathbf{a} = U_o^2\left[2x^3 + 3x^2 + x + 2xy^2 + y^2\right]\mathbf{e}_x - U_o^2\left[-2yx^2 - 2y^3 + 6xy + y\right]\mathbf{e}_y$$

$$\mathbf{a} = U_o^2\left[\left(2x^2 + 2y^2 + 3x + 1\right)x + y^2\right]\mathbf{e}_x + U_o^2\left(2x^2 + 2y^2 - 6x - 1\right)y\mathbf{e}_y$$

At y = 1 and x = 0 $\quad \mathbf{a} = (2)^2\left(\mathbf{e}_x + \mathbf{e}_y\right) = 4\mathbf{e}_x + 4\mathbf{e}_y$

At y = 1 and x = 2

$$\mathbf{a} = (2)^2\left[(8+2+6+1)2+1\right]\mathbf{e}_x + (2)^2(8+2-12-1)\mathbf{e}_y = 140\mathbf{e}_x - 12\mathbf{e}_y$$

(b) From equation 2.2.6

$$Q = \int \mathbf{v}\bullet\mathbf{n}dA = \int \mathrm{v}_x dydz$$

since $\mathbf{n} = \mathbf{e}_x$.

$$Q = \int_{y=0}^{5}\int_{z=0}^{3} U_o\left(x^2 - y^2 + x\right)\Big|_{x=5} dydz = 3U_o \int_{y=0}^{5}\left(30 - y^2\right)dy = 3U_o\left(30y - \frac{y^3}{3}\right) = 6\left(150 - \frac{125}{3}\right) = 650 \text{ m}^3\text{s}^{-1}$$

2.5. (a) $a_x = \boldsymbol{e}_x\boldsymbol{a} = \mathrm{v}_x\dfrac{\partial \mathrm{v}_x}{\partial x} = U_0\left(1-\dfrac{x}{L}\right)^{-2}\dfrac{\partial}{\partial x}\left[U_0\left(1-\dfrac{x}{L}\right)^{-2}\right]$

$$\frac{\partial}{\partial x}\left[\left(1-\frac{x}{L}\right)^{-2}\right] = \frac{2}{L}\left(1-\frac{x}{L}\right)^{-3}$$

$$a_x = \boldsymbol{e}_x\boldsymbol{a} = \mathrm{v}_x\frac{\partial \mathrm{v}_x}{\partial x} = U_0^{\,2}\left(1-\frac{x}{L}\right)^{-2}\frac{\partial}{\partial x}\left[\left(1-\frac{x}{L}\right)^{-2}\right] = \frac{2U_0^{\,2}}{L}\left(1-\frac{x}{L}\right)^{-5}$$

For values given:

$$a_x = \frac{50 \text{ m}^2/\text{s}^2}{2 \text{ m}}(1-0.5)^{-5} = \left(25 \text{ m/s}^2\right)/(1/32) = 800 \text{ m/s}^2$$

(b) (1) The “no slip” boundary condition is not satisfied.
(2) At x = L, the acceleration is undefined!

2.6. (a) Using the definition of the volumetric flow rate, Q

$$Q = \int \mathbf{v}\bullet\mathbf{n}dA = \int_0^{2\pi}\int_0^{R_i} \mathrm{v}_z r dr d\theta$$

The cross-sectional area element in cylindrical coordinates is rdrdθ. Since the velocity does not vary with angular position, substitution for v_z and integration in the angular direction yields:

$$Q = \int_0^{2\pi}\int_0^{R_i} \mathrm{v}_{\max}\left(1-\frac{r^2}{R^2}\right) r dr d\theta = 2\pi \mathrm{v}_{\max}\int_0^{R_i}\left(1-\frac{r^2}{R_i^{\,2}}\right) r dr$$

R_i is used to denote the local radius within the stenosis. Integrating in the radial direction yields:

$$Q = 2\pi \mathrm{v}_{\max}\int_0^{R_i}\left(1-\frac{r^2}{R_i^{\,2}}\right) r dr = 2\pi \mathrm{v}_{\max}\left(\frac{r^2}{2} - \frac{r^4}{4R_i^{\,2}}\right)_{r=0}^{R} = \frac{\pi R_i^{\,2}}{2}\mathrm{v}_{\max}$$

Solving for v_{max}: $$v_{max} = \frac{2Q}{\pi R_i^2} = \frac{2Q}{\pi R_0^2\left[1-0.5\left(1-4\left(\frac{z}{L}\right)^2\right)^{1/2}\right]^2}$$

Outside the stenosis, $R_i = R_0$ and:

$$v_{max} = \frac{2Q}{\pi R_0^2}$$

(b) At z = 0, the velocity in the stensosis is

$$v_{max} = \frac{2Q}{\pi R_i^2} = \frac{2Q}{\pi R_0^2[0.5]^2} = \frac{8Q}{\pi R_0^2}$$

$$R_i = R_0\left[1-0.5\left(1-4\left(\frac{z}{L}\right)^2\right)^{1/2}\right] = 0.5R_0$$

The shear stress in the stenosis is:

$$\tau_{rz}\Big|_{stenosis} = \mu\frac{\partial v_r}{\partial z} = \mu\frac{\partial}{\partial r}\left[v_{max}\left(1-\frac{r^2}{R_i(z=0)^2}\right)\right]_{r=R_i} = -\frac{2\mu R_i(z=0)v_{max}}{R_i(z=0)^2} = -\frac{32\mu Q}{\pi R_0^3}$$

Outside the stenosis the shear stress is:

$$\tau_{rz} = \mu\frac{\partial}{\partial r}\left[v_{max}\left(1-\frac{r^2}{R_0^2}\right)\right]_{r=R_0} = -\frac{2\mu v_{max}}{R_0} = -\frac{4\mu Q}{\pi R_0^3}$$

2.7. Evaluating Equation (2.7.30) for y = -h/2 yields:

$$\tau_w = \tau_{yx}(y=-h/2) = \frac{\Delta p}{L}\frac{h}{2} \qquad \text{(S2.7.1)}$$

From Equations (2.7.23) and (2.7.26),

$$\frac{\Delta p}{L} = \frac{8\mu v_{max}}{h^2} = \frac{12\mu Q}{wh^3} \qquad \text{(S2.7.2)}$$

Replacing $\Delta p/L$ in Equation (S2.7.1) with the expression in Equation (S2.7.2) yields

$$\tau_w = \frac{6\mu Q}{wh^2}$$

Solving for h: $h = \sqrt{\frac{6\mu Q}{w\tau_w}}$

Inserting the values provided for Q, w, μ and τ_w yields h = 0.051 cm.

2.8. (a) $\Delta p = \rho g h = (1\ \text{g cm}^{-3})(980\ \text{cm s}^{-2})(2.5 \times 10^{-4}\ \text{cm}) = 0.245\ \text{dyne cm}^{-2}$

(b) Rearranging equation (2.4.16) we have

$$T_c = \frac{\Delta p}{2\left(\frac{1}{R_p} - \frac{1}{R_c}\right)}$$

T_c = 1.838 x 10^{-5} dyne cm^{-1}

2.9. (a) To find the radius use Equation (2.4.16) and treat the pipet radius as the capillary radius R_c = R_{cap}.

$$\Delta p = 2T_c\left(\frac{1}{R_{cap}} - \frac{1}{R_c}\right)$$

For R_c = 6.5 μm
T_c = 0.06 mN/m= 6 x 10^{-5} N/m(1 x 10^{-6} m/μm) = 6 x 10^{-11} N/μm

Δp = 0.2 mm Hg
Since 1.0133×10^5 N m^{-2} = 760 mm Hg
0.2 mm Hg = 26.7 N m^{-2}(1 m/10^6 μm)^2 = 2.67 x 10^{-11} N μm^{-2}

Solving for R_{cap}

$$\frac{1}{R_{pcap}} = \frac{1}{R_c} + \frac{\Delta p}{2T_c}$$

$$R_{cap} = \frac{1}{\frac{1}{R_c} + \frac{\Delta p}{2T_c}}$$

$$R_{cap} = \frac{1}{\frac{1}{R_c} + \frac{\Delta p}{2T_c}} = \frac{1}{\frac{1}{6.5} + \frac{2.67}{2(6)}} = 2.66\ \mu m$$

While this result satisfies the law of Laplace, we need to assess whether the surface area is no greater than the maximum surface area of the cell, 1.4 times the surface area of a spherical cell, or 743.3 μm^2. The factor of 1.4 accounts for the excess surface area. Ideally, a larger cell entering a smaller capillary with look like a cylinder with hemispheres on each end. The cylinder will have length l and radius equal to the capillary. The hemispheres will have a radius equal to the capillary radius. The volume must remain constant, so

$$V = \frac{4}{3}\pi R_c^3 + \pi R_c^2 L$$

Solving for the length,

$$L = \frac{V - \frac{4}{3}\pi R_c^{\,3}}{\pi R_c^{\,2}} = \frac{\frac{4}{3}\pi\left(R^3 - R_c^{\,3}\right)}{\pi R_c^{\,2}} = \frac{4}{3}\frac{\left(6.5^3 - 2.66^3\right)}{2.66^2} = 48.2\ \mu\text{m}$$

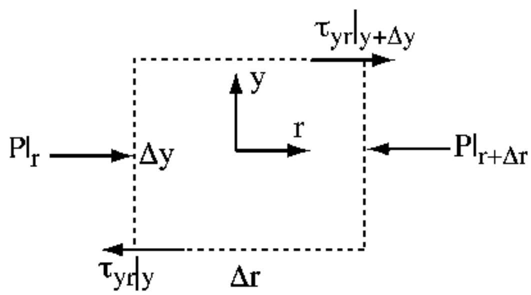
The resulting surface area is $SA = 4\pi R_c^{\,2} + 2\pi R_c L = \pi\left(4 * 2.66^2 + 2 * 48.2 * 2.66\right) = 894.6\ \mu\text{m}^2$
This is larger than the surface area 530.9 μm^2 or 1.4 times the surface area 743.3 μm^2.

To find the radius and length, one could iteratively solve for L and surface area of use the fzero function in MATLAB. After several iterations, the result approaches a radius of 3.3 µm and L = 29.2 µm.

If the cell had no excess area, then the cell would have no capacity to enter a capillary smaller than itself!

(b) Whether or not excess area is not considered, a cell with a radius of 3.0 µm can enter the capillary.

2.10. A momentum balance is applied on a differential volume element, 2πrΔrΔy, as shown in the figure below.



$$p\big|_r 2\pi r\Delta y - p\big|_{r+\Delta r} 2\pi(r+\Delta r)\Delta y + \tau_{yr}\big|_{y+\Delta y} 2\pi r\Delta r - \tau_{yr}\big|_y 2\pi r\Delta r = 0 \qquad \text{(S2.10.1)}$$

Divide each term by 2πrΔrΔy and take the limit as Δr and Δy go to zero results in the following expression:

$$\frac{1}{r}\frac{d(rp)}{dr} = \frac{d\tau_{yr}}{dy} \qquad \text{(S2.10.2)}$$

Note that if the gap distance h is much smaller than the radial distance, then curvature is not significant. Each side is equal to a constant C_1. Solving for the shear stress, $\tau_{yr} = C_1 y + C_2$. Substituting Newton's law of viscosity and integrating yields:

$$\text{v}_r = \frac{C_1 y^2}{2\mu} + \frac{C_2}{\mu} y + C_3 \qquad \text{(S2.10.3)}$$

Applying the boundary conditions that $\text{v}_r = 0$ at $y = \pm h/2$,

$$0 = \frac{C_1 h^2}{8\mu} + \frac{C_2}{\mu}\frac{h}{2} + C_3 \qquad \text{(S2.10.4a)}$$

$$0 = \frac{C_1 h^2}{8} - \frac{C_2}{\mu}\frac{h}{2} + C_3 \qquad \text{(S2.10.4b)}$$

Adding Equations (S2.10.4a) and (S2.10.4b) and solving for C_3,

$$C_3 = -\frac{C_1 h^2}{8} \qquad \text{(S2.10.5)}$$

Inserting Equation (S2.10.5) into Equation (S2.10.4a) yields $C_2 = 0$. Thus the velocity is:

$$\mathrm{v}_r = \frac{C_1}{\mu}\left(\frac{y^2}{2} - \frac{h^2}{8}\right) \qquad \text{(S2.10.6)}$$

The volumetric flow rate is:

$$Q = \int \mathbf{v} \bullet \mathbf{n} dA = \int_{y=-h/2}^{h/2} 2\pi r v_r dy = \frac{2\pi r C_1}{\mu} \int_{y=-h/2}^{h/2} \left(\frac{y^2}{2} - \frac{h^2}{8}\right) dy \qquad \text{(S2.10.7)}$$

$$Q = \frac{2\pi r C_1}{\mu}\left(\frac{y^3}{6} - \frac{h^2 y}{8}\right)\Bigg|_{y=-h/2}^{h/2} = \frac{2\pi r C_1 h^3}{\mu}\left(\frac{1}{24} - \frac{1}{8}\right) = -\frac{\pi r C_1 h^3}{6\mu} \qquad \text{(S2.10.8)}$$

Solving for C_1 and inserting into equation (S2.10.6)

$$\mathrm{v}_r = -\frac{6Q}{\pi r h^3}\left(\frac{y^2}{2} - \frac{h^2}{8}\right) \qquad \text{(S2.10.9)}$$

The shear stress can thus be written as;

$$\tau_w = \tau_{yr}\Big|_{y=-h/2} = \mu \frac{dv_r}{dr}\Bigg|_{y=-h/2} = -\frac{6Q}{\pi r h^3} y\Bigg|_{y=-h/2} = \frac{3\mu Q}{\pi r h^2} \qquad \text{(S2.10.10)}$$

2.11. Flow rate per fiber, Q_f = Q/250 = 0.8 mL/60 s = 0.01333 mL/s

Average velocity per fiber: $\langle v_f \rangle = Q_f/\pi R_f^{\,2}$ = (0.01333 mL/s)/(3.14159*(0.01 cm)^2)
$\langle v_f \rangle$ = 42 cm/s

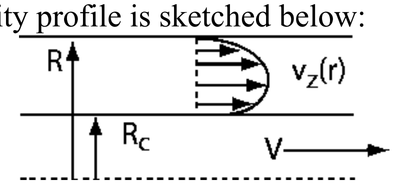
Re = $\rho \langle v_f \rangle D_f/\mu$ = 1.05*42*0.02/0.03 = 29.7.

L_e = 0.058DRe = 0.058*(0.02 cm)(29.7) = 0.034 cm $\ll$ L = 30 cm.

2.12. (a) The momentum balance is the same as that used for the case of pressure-driven flow in a cylindrical tube in Section 2.7.3.

$$\frac{dp}{dz} = \frac{1}{r}\frac{d(r\tau_{rz})}{dr} \qquad \text{(S2.12.1)}$$

(b) The velocity profile is sketched below:



Integrating the momentum balance and substituting Newton's law of viscosity,

$$\tau_{rz} = -\frac{\Delta p}{2L} r + \frac{C_1}{r} = \mu \frac{d\mathrm{v}_z}{dr} \qquad \text{(S2.12.2)}$$

Note that the shear stress and shear rate are a maximum at $r = \sqrt{\frac{2C_1}{\Delta p / L}}$. Assuming that C_1 is greater than zero, then r will have a maximum in the fluid.

(c) Integrating Equation (S2.12.2) yields:

$$v_z = -\frac{\Delta p}{4\mu L} r^2 + \frac{C_1}{\mu} \ln(r) + C_2 \qquad \text{(S2.12.3)}$$

Applying the boundary conditions

$$V = -\frac{\Delta p}{4\mu L} R_C{}^2 + \frac{C_1}{\mu} \ln(R_C) + C_2 \qquad \text{(S2.12.4a)}$$

$$0 = -\frac{\Delta p}{4\mu L} R^2 + \frac{C_1}{\mu} \ln(R) + C_2 \qquad \text{(S2.12.4b)}$$

Subtracting

$$V = -\frac{\Delta p}{4\mu L}\left(R_C{}^2 - R^2\right) + \frac{C_1}{\mu} \ln\left(\frac{R_C}{R}\right) \qquad \text{(S2.12.5)}$$

Solving for C_1:

$$C_1 = \frac{\mu V}{\ln\left(\frac{R_C}{R}\right)} + \frac{\Delta p}{4L}\frac{\left(R_C{}^2 - R^2\right)}{\ln\left(\frac{R_C}{R}\right)} \qquad \text{(S2.12.6a)}$$

Using this result to find C_2

$$C_2 = \frac{\Delta p}{4\mu L} R^2 - \left(\frac{V}{\ln\left(\frac{R}{R_C}\right)} + \frac{\Delta p}{4\mu L}\frac{\left(R_C{}^2 - R^2\right)}{\ln\left(\frac{R}{R_C}\right)}\right)\ln(R) \qquad \text{(S2.12.6b)}$$

The resulting expression for the velocity profile is

$$v_z = \frac{\Delta p R^2}{4\mu L}\left(1 - \frac{r^2}{R^2}\right) + \left(V + \frac{\Delta p}{4\mu L}\left(R_C{}^2 - R^2\right)\right)\frac{\ln\left(\frac{r}{R}\right)}{\ln\left(\frac{R}{R_C}\right)} \qquad \text{(S2.12.7)}$$

(d) The shear stress is:

$$\tau_{zr} = \mu\frac{\mathrm{d}v_z}{\mathrm{d}r} = -\frac{r\Delta p}{2L} + \left(\frac{\mu V + \frac{\Delta p}{4L}\left(R_C{}^2 - R^2\right)}{\ln\left(\frac{R}{R_C}\right)}\right)\left(\frac{1}{r}\right) \qquad \text{(S2.12.8)}$$

(e) At r = R, the shear stress is:

$$\tau_{zr} = \mu \frac{\mathrm{dv}_z}{\mathrm{dr}} = -\frac{R\Delta p}{2L} + \left(\frac{\mu V + \frac{\Delta p}{4L}\left(R_C{}^2 - R^2\right)}{\ln\left(\frac{R}{R_C}\right)} \right)\left(\frac{1}{R}\right) \qquad \text{(S2.12.9)}$$

For the values provided

$$\tau_{zr}\Big|_{r=R} = -\frac{(0.17\text{ cm})\left(100\text{ dyne/cm}^3\right)}{2}$$

$$+\left((0.03\text{ g/cm/s}) + \left(25\text{ dyne/cm}^3\right)\left((0.15\text{ cm})^2 - (0.17\text{ cm})^2\right)\right)\frac{\left(\frac{1}{(0.17\text{ cm})}\right)}{\ln\left(\frac{0.17}{0.15}\right)}$$

$$\tau_{zr}\Big|_{r=R} = -16.0\text{ dyne/cm}^2$$

This compares with a shear stress of -8.5 dyne/cm^2 in the absence of the catheter.

2.13. For a Newtonian fluid

$$\mathrm{v}_z = \frac{\Delta p R^2}{4\mu L}\left(1 - \frac{r^2}{R^2}\right) = \mathrm{v}_{max}\left(1 - \frac{r^2}{R^2}\right) = 2\langle \mathrm{v}\rangle\left(1 - \frac{r^2}{R^2}\right) \qquad \dot{\gamma}_r = -\frac{dv_z}{dr}\Big|_{r=R} = \frac{4Q}{\pi R^3} \qquad \text{(S2.7.1a,b)}$$

$$\overline{U} = \frac{Q}{2\pi R^3} \qquad \text{(S2.7.2)}$$

By comparing equations (S2.7.1b) and (S2.7.2), we find that,

$$\frac{Q}{\pi R^3} = \frac{\dot{\gamma}_r}{4} = 2\overline{U}$$

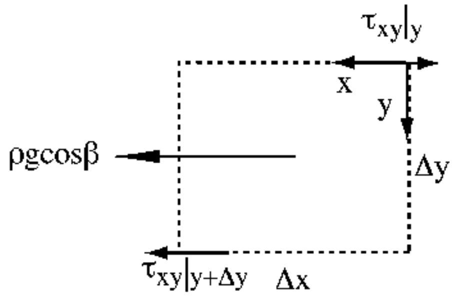
As a result, $\dot{\gamma}_r = 8\overline{U}$.

For a power law fluid, the average velocity and shear rate equal (equations 2.7.52 and 2.7.55):

$$\dot{\gamma} = -\frac{dv_z}{dr}\Big|_{r=R} = (3 + 1/n)\frac{Q}{\pi R^3} = 2(3 + 1/n)\overline{U}$$

The constant relating shear rate and the reduced velocity equals 2(3n+1)/n. If n = 1, for a Newtonian fluid, then 2(3n+1)/n = 8.

2.14. A schematic of a volume element is shown below.



The momentum balance is:

$$\rho g \cos\beta \Delta x \Delta y \Delta z + \left(\tau_{xy}\Big|_{y+\Delta y} - \tau_{xy}\Big|_{y} \right) \Delta x \Delta z = 0 \tag{S2.14.1}$$

where the z direction is normal to the x-y plane. Dividing each term by the volume element ΔxΔyΔz and taking the limit as Δy goes to zero yields:

$$\frac{d\tau_{xy}}{dy} = -\rho g \cos\beta \tag{S2.14.2}$$

Note that at the air-liquid interface the shear stress is zero. This is because the viscosity of the gas is much less than the viscosity of the liquid. As a result the velocity gradient at y = 0 is zero. Integrating the momentum balance and applying the boundary condition at y =0 yields:

$$\tau_{zx} = -\rho g y \cos\beta \tag{S2.14.3}$$

For a Bingham plastic, the velocity gradient is zero for a shear stress below the yield stress, $\tau_o = -\rho g \delta \cos\beta$. For the angles given, the difference in yield stress is only 6% so the manufacturer's claim is exaggerated.

2.15. For fluid 1, since the resulting velocity is linearly related to the original velocity, the fluid is Newtonian.

For fluid 2, there is no linear or power dependence between the velocities suggesting that the fluid is a Bingham plastic. Applying Equation (2.7.12b) to the base case and cases (2a) and (2b)

$$V_1 = \frac{(\tau_1 - \tau_0)H}{\mu_0} \quad \text{or} \quad \tau_1 = \frac{\mu_0 V_1}{H} + \tau_0 \tag{S2.15.1a,b}$$

$$3V_1 = \frac{(2\tau_1 - \tau_0)H}{\mu_0} \tag{S2.15.2}$$

$$7V_1 = \frac{(4\tau_1 - \tau_0)H}{\mu_0} \tag{S2.15.3}$$

Substituting Equation (S2.15.1b) for τ_1 into equation (S2.15.2)

$$3V_1 = \frac{\left(2\left(\frac{\mu_0 V_1}{H} + \tau_0 \right) - \tau_0 \right) H}{\mu_0} = 2V_1 + \frac{\tau_0}{\mu_0} H \tag{S2.15.4}$$

Solving for V_1, $$V_1 = \frac{\tau_0}{\mu_0} H \tag{S.2.15.5}$$

Substituting equations (S2.15.5) into equation (S2.15.1a), yields.

$$\tau_o = 0.5\tau_1 \tag{S.2.15.6}$$

Inserting Equations (S2.15.5) and (S2.15.6) into equation (S2.9.3) verifies that equation (S2.15.6) is the correct relation.

For fluid 3, the resulting velocity is proportional to the square of the original velocity. The fluid is a power law fluid. For a power law fluid, equation (2.7.7) becomes

$$\tau_{yx} = m\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy} = m\left(\frac{dv_x}{dy}\right)^n \quad \text{(S2.15.7)}$$

Rearranging yields an expression for the velocity gradient

$$\frac{dv_x}{dy} = \left(\frac{\tau_{yx}}{m}\right)^{1/n} \quad \text{(S2.15.8)}$$

Integrating, and evaluating the boundary condition that $v_x = 0$ at $y = 0$.

$$v_x = \left(\frac{\tau_{yx}}{m}\right)^{1/n} y \quad \text{(S2.15.9)}$$

Evaluating the velocity at y = H for the conditions given,

$$4V_1 = \left(\frac{2\tau_1}{m}\right)^{1/n} H \quad \text{(S2.15.10a)}$$

$$16V_1 = \left(\frac{4\tau_1}{m}\right)^{1/n} H \quad \text{(S2.15.10b)}$$

Dividing equation (S2.15.10b) by equation (S2.15.10a) yields:

$$4 = (2)^{1/n} \quad \text{(S2.15.11)}$$

Solving, yields n = 0.5.

2.16. For a Bingham plastic the momentum balance is unchanged from equation (2.7.57)

$$\frac{1}{r^2}\frac{d(r^2\tau_{r\theta})}{dr} = 0$$

which after integration yields $r^2\tau_{r\theta} = C_1$ If the shear stress is less than τ_o then the shear rate is zero. Thus, $\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = 0$ or $v_\theta = C_2 r$ where C_2 is a constant. The inner cylinder is not moving. Although the shear stress is lower on the outer cylinder, the only way the boundary condition at r = εR can be satisfied is for $v_\theta = 0$. Thus when $\tau_{r\theta} < \tau_o$, $v_\theta = 0$.

For $\tau_{r\theta}$ greater than τ_o, $\tau_{r\theta} = \tau_0 + \mu_0\dot{\gamma}_0 = \tau_0 + \mu_0 \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{C_1}{r^2}$

Rearranging $\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{C_1}{\mu_o r^3} - \frac{\tau_o}{\mu_o r}$. Integrating we have:

$$v_\theta = -\frac{C_1}{2\mu_o r} - \frac{\tau_o r}{\mu_o} \ln r + C_2 r$$

Applying the boundary condition at r = R and r = εR we have $\tau_{r\theta} > \tau_0$,

$$v_\theta = -\frac{\tau_o}{\mu_o} r \ln(r/\varepsilon R) + \left(\frac{r}{\varepsilon R} - \frac{\varepsilon R}{r}\right)\left[\frac{\Omega \varepsilon R}{1-\varepsilon^2} - \frac{\tau_o}{\mu_o}\frac{\varepsilon R \ln \varepsilon}{1-\varepsilon^2}\right]$$

The yield stress τ_o can be determined from the torque required to begin rotation of the outer cylinder, $T = 2\pi R^2 L \tau_o$. Once rotation begins, the viscosity can be determined by relating the torque to the shear stress at r = R (Equation (2.7.69)). For a Bingham plastic the result is:

$$T = 2\pi R^2 L\tau_{r\theta}\Big|_{r=R} = \frac{4\pi L\varepsilon^2 R^2}{1-\varepsilon^2}\left[\mu_o\Omega - \tau_o \ln\varepsilon\right]$$

A plot of the torque versus the rotational speed Ω will be linear with a slope proportional to μ_o and an intercept proportional to τ_o.

2.17. For a power law fluid the shear stress is related to the shear rate as:

$$\tau_{r\theta} = m\left|r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right|^{n-1}\left(r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right) \qquad \text{(S2.17.1)}$$

Since v_q increases with r, the derivative is positive, the shear stress can be written as;

$$\tau_{r\theta} = m\left(r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right)^n = \frac{C_1}{r^2} \qquad \text{(S2.17.2)}$$

Rearranging equation (S2.11.2) yields

$$\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{1}{r}\left(\frac{C_1}{mr^2}\right)^{1/n} \qquad \text{(S2.17.3)}$$

Integrating and expressing the results in terms of v_θ,

$$v_\theta = \left(\frac{C_1}{m}\right)^{1/n} r^{\frac{n-2}{n}} + rC_2 \qquad \text{(S2.17.4)}$$

Applying the boundary conditions at r = εR and r = R, the velocity profile is:

$$v_\theta = \frac{\Omega\varepsilon R}{1-\varepsilon^{2/n}}\left[\frac{r}{\varepsilon R} - \left(\frac{\varepsilon R}{r}\right)^{2/n-1}\right] \qquad \text{(S2.17.5)}$$

For n = 1 (Newtonian fluid) this result is equal to equation (2.7.67).
The shear stress is:

$$\tau_{r\theta} = m\left(\frac{\Omega}{1-\varepsilon^{2/n}}\right)^n\left(\frac{2}{n}\right)^n\left(\frac{\varepsilon R}{r}\right)^2 \qquad \text{(S2.17.6)}$$

Finally, the torque is:

$$T = 2\pi R^2 L\tau_{r\theta}\Big|_{r=R} = 2\pi R^2 L\left[m\varepsilon^2\left(\frac{\Omega}{1-\varepsilon^{2/n}}\right)^n\left(\frac{2}{n}\right)^n\right] \qquad \text{(S2.17.7)}$$

To find m and n, take the logarithm of the left and right hand sides of equation (S2.17.7),

$$log\left(\frac{T}{2\pi R^2 L\varepsilon^2}\right) = log(m) + n\,log\left[\left(\frac{\Omega\varepsilon R}{1-\varepsilon^{2/n}}\right)\left(\frac{2}{n}\right)\right] = log(m) + n\,log\left(\frac{2}{n}\right) + n\,log\left(\frac{\Omega\varepsilon R}{1-\varepsilon^{2/n}}\right)$$

A plot of the log of the torque/$2\pi R^2 L\varepsilon$ versus $\log(\Omega\varepsilon R/(1-\varepsilon^{2/n}))$has an intercept equal to the log(m)+nlog(2/n) and a slope equal to n.

2.18. Letting $t_0 = 0$, the shear rate function is:

$$\dot{\gamma}_x(t) = \begin{cases} \gamma_0/\varepsilon, & \text{for} \quad -\varepsilon < t < 0 \\ 0, & \text{for} \quad t < -\varepsilon \text{ or } t > 0 \end{cases}$$

Thus, equation (2.5.15a) becomes: $\tau_{yx} = \int_{-\varepsilon}^{0} G(t-t')\frac{\gamma_0}{\varepsilon}dt'$

Rearranging the above equation, we have: $\tau_{yx} = \gamma_0 \frac{\int_{-\varepsilon}^{0} G(t-t')dt'}{\varepsilon}$

(b) According to L'Hopital's rule, the limit of this expression as ε goes to 0 is:

$$\lim_{\varepsilon \to 0} \tau_{yx} = \gamma_0 \frac{\frac{d}{d\varepsilon}\left(\int_{-\varepsilon}^{0} G(t-t')dt'\right)\Big|_{\varepsilon=0}}{1}$$

Applying Leibnitz's rule when differentiating the integral, we have:

$$\lim_{\varepsilon \to 0} \tau_{yx} = \gamma_0 G(t-\varepsilon)\Big|_{\varepsilon=0} = \gamma_0 G(t)$$

2.19. Since the apparent viscosity depends upon the shear rate, the fluid is not Newtonian. For a power law fluid, $\eta_{app} = \mathrm{m}\dot{\gamma}^{\mathrm{n-1}}$. Taking the logarithm of each side yields.

$$\ln\left(\eta_{app}\right) = \ln(\mathrm{m}) + (\mathrm{n}-1)\ln(\dot{\gamma})$$

The data are plotted in Figure S2.19.1. From the slope n= 0.499 ≈ 0.50 so the cytoplasm is shear thinning. The value of m is 147.4 Pa s. If the results are presented in terms of the base 10 log

$$\log\left(\eta_{app}\right) = \log(\mathrm{m}) + (\mathrm{n}-1)\log(\dot{\gamma})$$

The regression line is $\log\left(\eta_{app}\right) = 2.169 + 0.499\log(\dot{\gamma})$

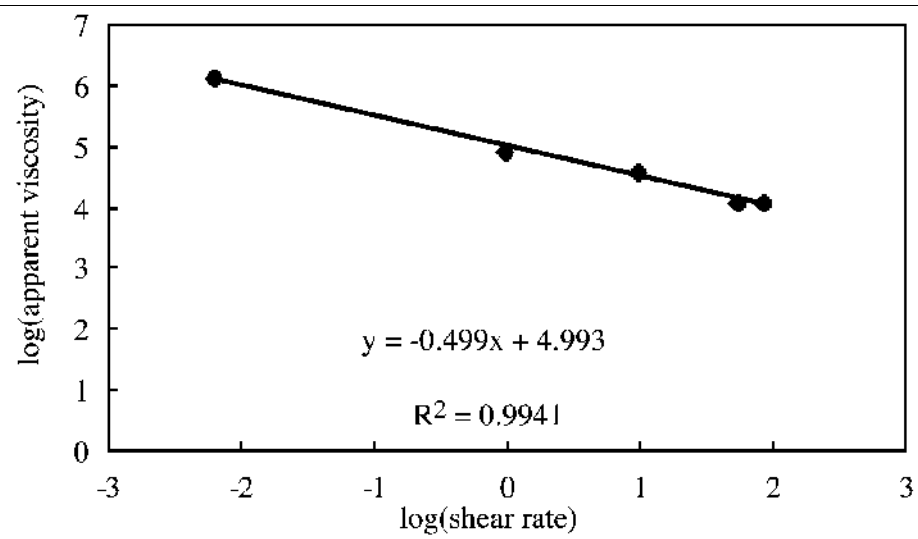


Figure S2.19.1

2.20. A plot of the shear stress versus shear rate revealed that while a straight line gives a good fit, there is some curvature to the data suggesting a shear thinning fluid. A log-log plot of shear stress versus shear rate yields n = 0.855 (Figure S2.20.1). So the fluid does exhibit some shear thinning behavior. The apparent viscosity is $\eta_{app} = 1.42\dot{\gamma}^{-0.145}$

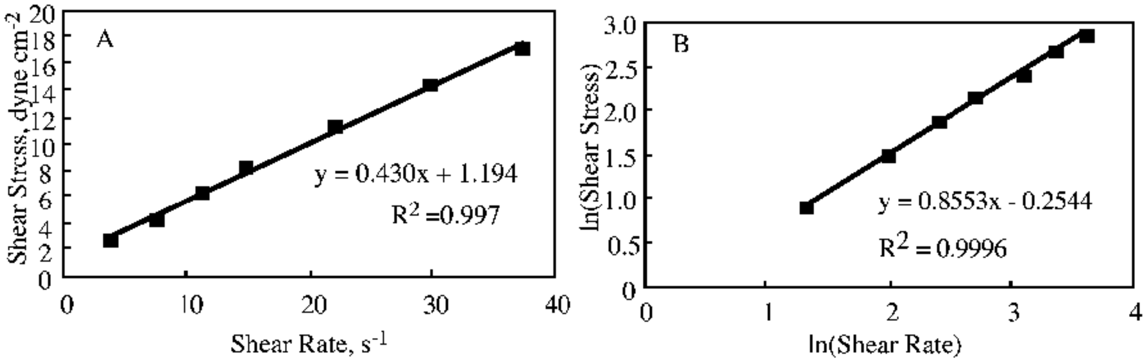


Figure S2.20.1

After the enzyme is added the apparent viscosity decreases and is much less sensitive to shear rate, as determined by the following regression of data (Figure S2.20.2) $\eta_{app} = 0.01\dot{\gamma}^{-0.0225}$. The enzyme functions by clipping the hyaluronic acid chains decreasing their length. As a result the hyaluronic acid offers much less resistance to flow.

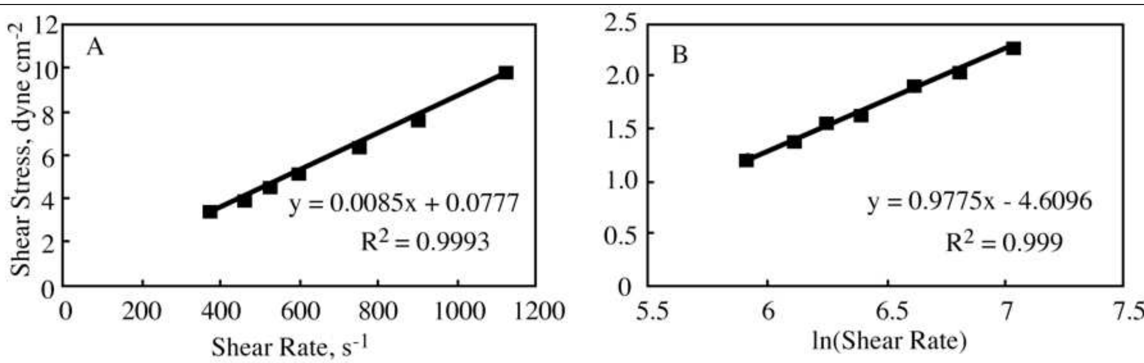


Figure S2.20.2

2.21. (a) Since the momentum balance is independent of the type of fluid, begin with Equation 2.7.36 with $C_2 = 0$.

$$\tau_{rz} = -\frac{\Delta p}{2L} r$$

The shear stress is greatest at r = R. If $\tau_{rz}(r = R) < \tau_0$, v_z = constant for $0 \leq r \leq R$. Since the velocity at r = R is zero, $v_z = 0$. Thus, the following criterion must be met for the fluid to flow.

$$\left|\tau_{rz}(r = R)\right| = -\frac{\Delta p}{2L} R > \tau_0$$

(b) When $\tau_{rz}(r = R) > \tau_0$, the fluid begins to move. Since $\tau_{rz}(r = 0) = 0$, the shear stress at some point in the fluid, say r_0, equals the yield stress. Thus for $0 \leq r \leq r_0$, the velocity is constant. The location of r_0 is determined by solving for the yield stress

$$r_0 = \frac{2L\tau_0}{\Delta p}$$

In the region $0 \le r \le r_0$, $\tau_{rz} < 0$ and the constitutive relation is:

$$\tau_{rz} = -\frac{\Delta p}{2L} r = -\tau_0 + \mu_0 \frac{d\mathrm{v}_z}{dr}$$

Rearranging and integrating once, we have:

$$\mathrm{v}_z = \frac{\tau_0}{\mu_0} r - \frac{\Delta p}{4\mu_0 L} r^2 + C_3$$

Applying the boundary condition at r = R that $\mathrm{v}_z = 0$, the C_3 equals:

$$C_3 = -\frac{\tau_0}{\mu_0} R + \frac{\Delta p}{4\mu_0 L} R^2$$

Replacing C_3 yields the final expression for $\mathrm{v}_z(r)$.

$$\mathrm{v}_z = \frac{\Delta p R^2}{4\mu_0 L}\left(1 - \frac{r^2}{R^2}\right) - \frac{\tau_0 R}{\mu_0}\left(1 - \frac{r}{R}\right) \qquad r_0 \le r \le R$$

$$\mathrm{v}_z = \frac{\Delta p R^2}{4\mu_0 L}\left(1 - \frac{r_0^{\,2}}{R^2}\right) - \frac{\tau_0 R}{\mu_0}\left(1 - \frac{r_0}{R}\right) \qquad 0 \le r \le r_0$$

The volumetric flow rate is:

$$Q = \int_{r=0}^{R}\int_{\theta=0}^{2\pi} \mathrm{v}_z r dr d\theta = 2\pi\left[\int_{r=0}^{r_0} \mathrm{v}_z r dr + \int_{r=r_0}^{R} \mathrm{v}_z r dr\right] = 2\pi\left[\mathrm{v}_z\left(r = r_0\right)\frac{r_0^{\,2}}{2} + \int_{r=r_0}^{R} \mathrm{v}_z r dr\right]$$

$$Q = 2\pi\left[\left(\frac{\Delta p R^2}{4\mu_0 L}\left(1 - \frac{r_0^{\,2}}{R^2}\right) - \frac{\tau_0 R}{\mu_0}\left(1 - \frac{r_0}{R}\right)\right)\frac{r_0^{\,2}}{2} + \int_{r=r_0}^{R}\left(\frac{\Delta p R^2}{4\mu_0 L}\left(1 - \frac{r^2}{R^2}\right) - \frac{\tau_0 R}{\mu_0}\left(1 - \frac{r}{R}\right)\right) r dr\right]$$

Replacing τ_0 with r_0.

$$Q = 2\pi\frac{\Delta p R^2}{4\mu_0 L}\left[\left(\left(1 - \frac{r_0^{\,2}}{R^2}\right) - \frac{2r_0}{R}\left(1 - \frac{r_0}{R}\right)\right)\frac{r_0^{\,2}}{2} + \int_{r=r_0}^{R}\left(\left(1 - \frac{r^2}{R^2}\right) - \frac{2r_0}{R}\left(1 - \frac{r}{R}\right)\right) r dr\right]$$

Integrating:

$$Q = 2\pi\frac{\Delta p R^2}{4\mu_0 L}\left[\left(\left(1 - \frac{r_0^{\,2}}{R^2}\right) - \frac{2r_0}{R}\left(1 - \frac{r_0}{R}\right)\right)\frac{r_0^{\,2}}{2} + \left(\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right) - \frac{2r_0}{R}\left(\frac{r^2}{2} - \frac{r^3}{3R}\right)\right)\Bigg|_{r=r_0}^{R}\right]$$

Evaluating the limits.

$$Q = 2\pi\frac{\Delta p R^2}{4\mu_0 L}\left[\left(\frac{r_0^{\,2}}{2} - \frac{r_0^{\,4}}{2R^2}\right) - \frac{2r_0}{R}\left(\frac{r_0^{\,2}}{2} - \frac{r_0^{\,3}}{2R}\right) + \left(\frac{R^2}{2} - \frac{r_0^{\,2}}{2} - \frac{R^2}{4} + \frac{r_0^{\,4}}{4R^2}\right) - \frac{2r_0}{R}\left(\frac{R^2}{2} - \frac{r_0^{\,2}}{2} - \frac{R^2}{3} + \frac{r_0^{\,3}}{2R}\right)\right]$$

Collecting terms

$$Q = 2\pi\frac{\Delta p R^2}{4\mu_0 L}\left[\left(\frac{r_0^{\,2}}{2} - \frac{r_0^{\,4}}{2R^2}\right) - \frac{2r_0}{R}\left(\frac{r_0^{\,2}}{2} - \frac{r_0^{\,3}}{2R}\right) + \left(\frac{R^2}{4} - \frac{r_0^{\,2}}{2} + \frac{r_0^{\,4}}{4R^2}\right) - \frac{2r_0}{R}\left(\frac{R^2}{6} + \frac{r_0^{\,3}}{3R} - \frac{r_0^{\,2}}{2}\right)\right]$$

$$Q = 2\pi\frac{\Delta p R^2}{4\mu_0 L}\left[\left(\frac{R^2}{4} - \frac{r_0^{\,4}}{4R^2}\right) - \frac{2r_0}{R}\left(\frac{R^2}{6} + \frac{r_0^{\,3}}{3R} - \frac{r_0^{\,3}}{2R}\right)\right] = 2\pi\frac{\Delta p R^2}{4\mu_0 L}\left[\left(\frac{R^2}{4} - \frac{r_0^{\,4}}{4R^2}\right) - \frac{2r_0}{R}\left(\frac{R^2}{6} - \frac{r_0^{\,3}}{6R}\right)\right]$$

$$Q = \frac{\pi \Delta p R^4}{8\mu_0 L}\left[\left(1-\frac{r_0^{\ 4}}{R^4}\right)-\frac{4r_0}{3R}\left(1-\frac{r_0^{\ 3}}{R^3}\right)\right]$$

Note that for $r_0 = 0$, the result for a Newtonian fluid is obtained.

2.22. (a) Since δ = R(1-ε)<<R, the magnitude of the radial position does not vary significantly with a differential change in r, equation (2.7.57) can be approximated as:

$$\frac{1}{r^2}\frac{d(r^2\tau_{r\theta})}{dr} \approx \frac{r^2}{r^2}\frac{d\tau_{r\theta}}{dr} = \frac{d\tau_{r\theta}}{dr} = 0 \qquad \text{(S2.22.1)}$$

Since y = r-εR, the momentum balance can be written as:

$$\frac{d\tau_{y\theta}}{dy} = 0 \qquad \text{(S2.22.2)}$$

As a result the shear stress is constant. If curvature can be neglected, equation (2.7.62b) can be written as:

$$\tau_{y\theta} = \mu\frac{dv_y}{dr} \qquad \text{(S2.22.3)}$$

Inserting Equation (S2.22.3) into Equation (S2.22.2), integrating and evaluating the boundary conditions ($y = 0\ v_\theta = 0$, $y = \delta\ v_\theta = \Omega R$) leads to the following expression for the velocity:

$$v_\theta = \frac{\Omega R y}{\delta} \qquad \text{(S2.22.4)}$$

The shear stress is:

$$\tau_{y\theta} = \frac{\mu\Omega R}{\delta} = \frac{\mu\Omega}{1-\varepsilon} \qquad \text{(S2.22.5)}$$

The torque is

$$T = 2\pi R^2 L\tau_{y\theta} = \frac{2\pi R^3 L\mu\Omega}{\delta} = \frac{2\pi R^2 L\mu\Omega}{1-\varepsilon} \qquad \text{(S2.22.6)}$$

(b) Taking the ratio of the shear stress obtained neglecting curvature (equation (S2.22.5) to the exact result (equation (2.7.70)), yields:

$$\frac{\tau_{y\theta}}{\tau_{r\theta}\big|_{r=R}} = \frac{1+\varepsilon}{2\varepsilon^2} \qquad \text{(S2.22.7)}$$

This relation can be used to compute the error induced by neglecting curvature. The error is 0.76% for δ/R equal to 0.005, 1.52% for δ equal to 0.01, and 4.69% for δ equal to 0.03.

(c) For a power law fluid:

$$\tau_{y\theta} = m\left(\frac{dv_y}{dr}\right)\left|\frac{dv_y}{dr}\right|^{n-1} \qquad \text{(S2.22.8)}$$

The momentum balance indicates that the shear stress is constant and positive.

$$\frac{d\tau_{y\theta}}{dy} = m\left(\frac{dv_y}{dr}\right)^n = 0 \qquad \text{(S2.22.9)}$$

Integrating once

$$\frac{dv_y}{dr} = C \qquad \text{(S2.22.10)}$$

Evaluating the boundary conditions, yields Equation (S2.22.4). Thus, the torque for a power law fluid is:

$$T = 2\pi R^2 L\tau_{y\theta} = 2\pi R^2 Lm\left(\frac{\Omega R}{\delta}\right)^n \qquad \text{(S2.22.11)}$$

Taking the logarithm of both sides of Equation (S2.22.11)

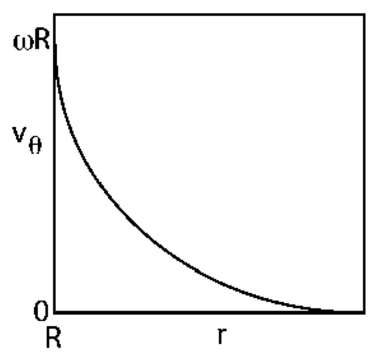
$$\ln(T) = \ln(2\pi R^2 Lm) + n\ln\left(\frac{\Omega R}{\delta}\right) \qquad \text{(S2.22.12)}$$

Thus, a plot of ln(T) versus ln(ΩR/δ) has a slope equal to n and an intercept equal to $\ln(2\pi R^2 Lm)$.

2.23. (a) For low rotational speeds, there is only 1 velocity component, v_θ. This velocity is a function of r only. There is no angular variation in velocity or pressure. Thus there is only one shear stress term, $\tau_{r\theta}$. Applying a momentum balance, Equation (2.7.57):

$$0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\tau_{r\theta}\right)$$

(b)



(c) Integrating once:

$$\tau_{r\theta} = \frac{C_1}{r^2}$$

We need the velocity profile in order to determine the constant. From Equation (2.7.62b), we have.

$$\tau_{r\theta} = \mu r\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{C_1}{r^2}$$

Rearranging $$\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{C_1}{\mu r^3}$$

Integrating $$v_\theta = -\frac{C_1}{2\mu r} + C_2 r$$

The boundary conditions are $r \to \infty$ $v_\theta = 0$

$r = R$ $v_\theta = \omega R$

From the B.C. at $r \to \infty$, $C_2 = 0$. At r = R, $C_1 = -2\mu\, wR^2$.
Thus, the velocity field is:

$$v_\theta = \frac{\omega R^2}{r}$$

(d) $\mathbf{T} = (\mathbf{F}|_{r=R}) \mathrm{x} R\mathbf{e}_r$ The torque and force are determined on the surface of the cylinder. Note that the velocity is constant and the unit normal is in the - r direction. The shear stress can be found by substituting for C_1 in the relation $\tau_{r\theta} = \frac{C_1}{r^2} = -2\mu\omega$

$\mathbf{F}|_{r=R} = -\mathbf{e}_r \bullet (\mathbf{e}_r\mathbf{e}_\theta)\, 2\pi RL\tau_{rq}|_{r=R} = \mathbf{e}_\theta 4\pi RL\mu\omega$.

The torque is $T = 4\pi RL\mu\omega R\mathbf{e}_\theta \mathrm{x} \mathbf{e}_r = \mathbf{e}_z\, 4\pi L\mu w R^2$
The cylinder must exert an equal and opposite torque to remain in motion.

(e) The torque can be measured from the electrical energy needed to keep the motion of the cylinder constant. Then, from a plot of Torque versus $\pi L w R^2$, the viscosity can be found from the slope.

2.24. Using the definitions of the tube and discharge hematocrits provided in the text

$$HCT_T = \frac{2}{R_T^2}\int_0^{R_T} HCT_T(r)rdr = \frac{2}{R_T^2}\int_0^{R_T-\delta} HCT_o rdr = HCT_o\frac{(R_T-\delta)^2}{R_T^2} \quad \text{(S2.24.1)}$$

$$HCT_D = \frac{\int_0^{R_T-\delta} HCT_o v_z(r)rdr}{\int_0^{R_T} v_z(r)rdr} = \frac{HCT_o\int_0^{R_T-\delta}\left(1-\frac{r^2}{R_T^2}\right)rdr}{\int_0^{R_T}\left(1-\frac{r^2}{R_T^2}\right)rdr} = HCT_o\frac{\frac{(R_T-\delta)^2}{2}-\frac{(R_T-\delta)^4}{4R_T^2}}{R_T^2/4}$$

$$HCT_D = HCT_o\left(\frac{2(R_T-\delta)^2}{R_T^2}-\frac{(R_T-\delta)^4}{R_T^4}\right)$$

(a) Since $HCT_F = HCT_D$, the relation between HCT_o and HCT_F is:

$$HCT_o = \frac{HCT_F}{\left(\frac{2(R_T-\delta)^2}{R_T^2}-\frac{(R_T-\delta)^4}{R_T^4}\right)} \quad \text{(S2.24.2)}$$

As expected, $HCT_o > HCT_F$.

(b) Substituting equation (S2.24.2) into equation (S2.24.1) yields:

$$HCT_T = \frac{HCT_F\left(\frac{R_T^2}{(R_T-\delta)^2}\right)}{\left(\frac{2(R_T-\delta)^2}{R_T^2}-\frac{(R_T-\delta)^4}{R_T^4}\right)} = \frac{HCT_F}{2-\frac{(R_T-\delta)^2}{R_T^2}}$$

R_T, μ,m δ/R_T HCT_T/HCT_F

500	0.01	0.9805
400	0.0125	0.9758
250	0.02	0.9619
100	0.05	0.9111
50	0.10	0.8403

2.25. (a) $\tau_{rz}\big|_{r=R_C} = \mu \dfrac{d\text{v}_z}{dr}\bigg|_{r=R} = -\mu \dfrac{d\text{v}_z}{dy}\bigg|_{y=\delta} = -\mu \dfrac{V_C}{\delta}$

(b) A force balance yields:
Pressure X Cross-sectional area of cell = shear stress X area over which stress acts

$$\Delta P \pi R_C^2 = \left(\tau_{rz}\big|_{r=R_C}\right) 2\pi R_C L = \mu \frac{V_C}{\delta} 2\pi R_C L$$

$$V_C = \frac{\Delta p}{2\mu L}\delta R_C = \frac{\Delta p}{2\mu L} R^2 \left(\frac{\delta}{R}\right)\left(1-\frac{\delta}{R}\right)$$

Alternatively, the momentum balance, Equation (2.7.34b), is:

$$\frac{dp}{dz} = \frac{1}{r}\left(\frac{d\left(r\tau_{rz}\right)}{dr}\right)$$

Or after integration and applying the symmetry boundary condition at r = 0.

$$\tau_{rz} = -\frac{\Delta p}{2L} r$$

Evaluating at $r = R_C$ and using the result obtained in part (a) for the shear stress:

$$-\mu \frac{V_C}{\delta} = -\frac{\Delta p}{2L} R_C$$

Rearranging: $$V_C = \frac{\Delta p}{2\mu L}\delta R_C$$

(c) There are two possible ways to approach this. One is to neglect the fluid in the gaps.

$$\langle \text{v} \rangle = \frac{\pi \int_0^{R_C} \text{V}_C r dr}{\pi R^2} = \text{V}_C \frac{R_C}{R} = \text{V}_C \left(1-\frac{\delta}{R}\right)$$

The more general approach is to consider the fluid in the gap.

$$\langle \text{v} \rangle = \frac{\pi \int_0^{R} \text{v}_z r dr}{\pi R^2} = \frac{1}{R^2}\left(\int_0^{R_C} \text{V}_C r dr + \int_{R_C}^{R} \text{v}_z r dr\right) = \text{V}_C\left(1-\frac{\delta}{R}\right) + \text{V}_C \frac{y^3}{3R^3}\bigg|_{R_C}^{R}$$

$$\langle \text{v} \rangle = \text{V}_C\left(1-\frac{\delta}{R}\right) + \frac{\text{V}_C}{3}\left(1-\frac{R_C^3}{R^3}\right) = \text{V}_C\left(1-\frac{\delta}{R}\right) + \frac{\text{V}_C}{3}\left(\frac{\delta^3}{R^3}\right)$$

(d) From parts (b) and (c)

$$\langle \mathrm{v} \rangle = \mathrm{V_C}\left(1-\frac{\delta}{R}\right) = \frac{\Delta p}{2\mu L}R^2\left(\frac{\delta}{R}\right)\left(1-\frac{\delta}{R}\right)^2$$

or

$$\langle \mathrm{v} \rangle = \mathrm{V_C}\left(1-\frac{\delta}{R}\right) = \frac{\Delta p}{2\mu L}R^2\left(\frac{\delta}{R}\right)\left(1-\frac{\delta}{R}\right)\left[\left(1-\frac{\delta}{R}\right)-\frac{1}{3}\left(\frac{\delta^3}{R^3}\right)\right]$$

$$\langle \mathrm{v} \rangle = \frac{R^2}{8\mu_{eff}}\frac{\Delta p}{L}$$

Thus, $\dfrac{\mu_{eff}}{\mu} = \dfrac{1}{4\left(\frac{\delta}{R}\right)\left(1-\frac{\delta}{R}\right)^2}$ (1) or $\dfrac{\mu_{eff}}{\mu} = \dfrac{1}{4\left(\frac{\delta}{R}\right)\left(1-\frac{\delta}{R}\right)^2\left[\left(1-\frac{\delta}{R}\right)-\frac{1}{3}\left(\frac{\delta^3}{R^3}\right)\right]}$ (2)

(e) As shown in the table below, there is a minimum in the viscosity at δ/R between 0.2 and 0.3 and the viscosity increases as δ/R increases.

δ /R	μ_{eff}/μ formula 1	μ_{eff}/μ formula 2
0.1	3.09	3.43
0.2	1.95	2.45
0.3	1.70	2.46
0.4	1.74	3.00

2.26. (a) From Equation (2.7.36), the shear stress of the flow in a cylindrical tube is:

$$\tau_{rz} = -\frac{\Delta pr}{2L} \qquad \text{(S2.26.1)}$$

When $r < r_{c,}\,\tau_{rz} < \tau_0$ and when $r > r_{c,}\,\tau_{rz} > \tau_0$. Therefore,

$$r_c < r < R,\quad \left(\frac{\Delta pr}{2L}\right)^{1/2} = (\tau_0)^{1/2} + (\eta_N)^{1/2}\left(\frac{dv_z}{dr}\right)^{1/2} \qquad \text{(S2.26.2)}$$

$$r < r_c,\quad \frac{d\mathrm{v}_z}{dr} = 0 \qquad \text{(S2.26.3)}$$

Rearrange Equation (S2.26.2), we have,

$$\frac{d\mathrm{v}_z}{dr} = \frac{1}{\eta_N}\left[\left(\frac{\Delta pr}{2L}\right)^{1/2} - (\tau_0)^{1/2}\right]^2 \qquad \text{(S2.26.4)}$$

Integrating Equation (S2.26.4) from r to R:

$$\mathrm{v}_z(R) - \mathrm{v}_z(r) = \frac{1}{\eta_N}\int_r^R \left[\left(\frac{\Delta px}{2L}\right)^{1/2} - (\tau_0)^{1/2}\right]^2 dx$$

$$= \frac{1}{\eta_N}\left[\frac{\Delta p}{4L}x^2 - \frac{4}{3}\left(\frac{\Delta p\tau_0}{2L}\right)x^{3/2} + \tau_0 x\right]_r^R$$

Applying the no slip boundary condition at r = R and evaluating the integral, we have,

$$r_c < r < R, \quad v_z = \frac{\Delta p R^2}{4\eta_N L}\left[\left(1-\frac{r^2}{R^2}\right)-\frac{8}{3}\frac{r_c^{1/2}}{R^{1/2}}\left(1-\frac{r^{3/2}}{R^{3/2}}\right)+\frac{2r_c}{R}\left(1-\frac{r}{R}\right)\right] \qquad \text{(S2.26.5)}$$

Integrating Equation (S2.26.3) from r to r_c,

$$v_z(r_c) - v_z(r) = 0$$

The velocity at r = r_c is obtained from Equation (S2.26.5). For $r < r_c$, the velocity profile is:

$$v_z = \frac{\Delta p R^2}{4\eta_N L}\left[1-\frac{8}{3}\frac{r_c^{1/2}}{R^{1/2}}+\frac{2r_c}{R}-\frac{r_c^2}{3R^2}\right] \qquad \text{(S2.26.6)}$$

(b) Based on Equation (S2.26.1), $\tau_w = -\dfrac{\Delta p R}{2L}$

Combining the above equation and Equation (S2.10.7), we have,

$$\frac{\tau_w}{\tau_o} = \frac{R}{r_c} \qquad \text{(S2.26.7)}$$

(c) Results are plotted in Figure S.2.26.1.

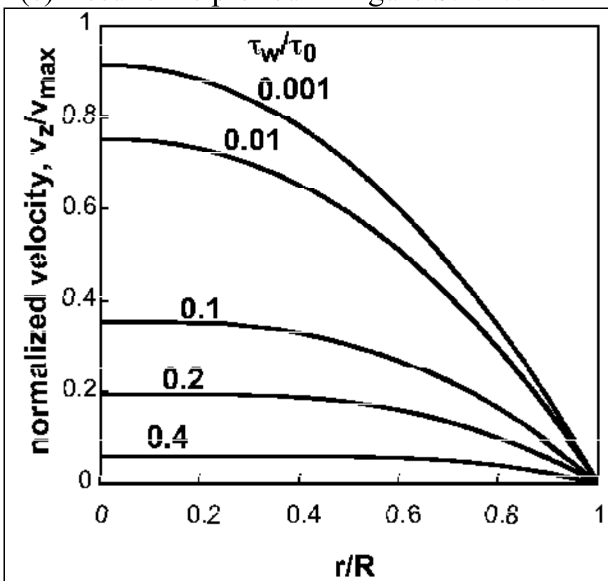


Figure S2.26.1

(d) The wall shear stress (r = R) can be computed from Equation (2.7.36) and is independent of the constitutive equation, $\tau_w = \Delta pR/2L$. The velocity profiles, shear rate and apparent viscosity are dependent on the constitutive equation. For blood, the shear rate at r = R is:

$$\frac{dv_z}{dr} = \frac{\Delta p R^2}{4\eta_N L}\left[-\frac{2}{R}+\frac{8}{3}\frac{r_c^{1/2}}{R^{1/2}}\left(\frac{3}{2}\right)\frac{1}{R}-\frac{2r_c}{R^2}\right] = \frac{\tau_w}{\eta_N}\left[-1+2\left(\frac{\tau_0}{\tau_w}\right)^{1/2}-\frac{\tau_0}{\tau_w}\right]$$

where $\dfrac{r_c}{R} = \dfrac{\tau_0}{\tau_w}$. For τ_w = 15 dyne cm^{-2}, $\tau_0/\tau_w = 0.0013$. The shear rate is 7.2% lower than the value for a Newtonian fluid. Correspondingly, the apparent fluid viscosity at r = R is 7.2% greater than the value for a Newtonian fluid. For $\tau_w = 2$ dyne cm^{-2}, $\tau_0/\tau_w = 0.01$ and the shear rate is 19% lower than the value for a Newtonian fluid and the apparent viscosity at r = R is 19% larger than the value for a Newtonian fluid. For $\tau_w = 0.2$ dyne cm^{-2}, $\tau_0/\tau_w = 0.1$ and the shear rate is 53.3% lower than the value for a Newtonian fluid and the apparent viscosity at r = R is 53.3% larger than the value for a Newtonian fluid.

2.27. The relation between flow and pressure drop is:

$$Q = \frac{\Delta p \pi R^4}{8\mu L}$$

Assuming that flow is proportional to current and pressure drop is proportional to potential difference, this result is analogous to Ohm's Law with a resistance equal to:

$$Resistance = \frac{8\mu L}{\pi R^4}$$

The enzyme treatment decreased the resistance by 14%. Assuming that the enzyme decreased the inner radius of the blood vessel by removing the glycocalyx completely, the change is resistance is due solely to a change in the effective radius of the blood vessel. Thus:

$$0.86 = \frac{R^4}{R^4_{enzyme}}$$

Taking the one-fourth root yields an increase in radius of $R_{enzyme} = 1.038R$. Assuming that the radius after enzyme treat is 14.5 μm, then the glycocalyx thickness is 0.038*14.5 μm = 0.551 μm.