CHAPTER 3

Section 3.2

1. (b) a set is LD if it is not LI, so it can't be both. NO.

2. (b) $\{x^2, x^2+x, x^2+x+1, x-1\}$. $(x^2+x)-(x^2) = x$
$= \frac{1}{2}[(x^2+x+1)-(x^2)+(x-1)]$

i.e., $1(x^2+x)-\frac{1}{2}(x^2+x+1)-\frac{1}{2}(x^2)-\frac{1}{2}(x-1)=0$.

(g) $6(0)+0(x)+0(x^3)=0$, where $6,0,0$ are not all zero.

(h) $6(x)-3(2x)+0(x^2)=0$, where $6,-3,0$ are not all zero.

3. (b) Use Theorem 3.2.2:

$$W[e^{a_1x},\dots,e^{a_nx}] = \begin{vmatrix} e^{a_1x} & e^{a_2x} & \cdots & e^{a_nx} \\ a_1e^{a_1x} & a_2e^{a_2x} & \cdots & a_ne^{a_nx} \\ \vdots & \vdots & & \vdots \\ a_1^{n-1}e^{a_1x} & a_2^{n-1}e^{a_2x} & \cdots & a_n^{n-1}e^{a_nx} \end{vmatrix}.$$

By property D7 in Section 10.4, this

$$= e^{a_1x}e^{a_2x}\cdots e^{a_nx} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}.$$

The latter determinant is a Vandermonde determinant (see Exercise 17, Section 10.4) so if the a_j's are distinct then that determinant, and hence W (since the e^{a_jx} factors are nonzero for all x), is nonzero. It follows from Theorem 3.2.2 that if the a_j's are distinct then $\{e^{a_1x},\dots,e^{a_nx}\}$ is LI. Surely, if the a_j's are not distinct then the set is LD. For suppose $a_1=a_3$, for instance. Then $4e^{a_1x}+0e^{a_2x}-4e^{a_3x}+0e^{a_4x}+\cdots+0e^{a_nx}=0$ with the coefficients $4,0,-4,0,\dots,0$ not all 0.

(c)
$$W[1,1+x,1+x^2] = \begin{vmatrix} 1 & 1+x & 1+x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = \text{etc} = 2 \neq 0$$
so (Theorem 3.2.2) LI

(e)
$$W[\sin x, \cos x, \sinh x] = \begin{vmatrix} \sin x & \cos x & \sinh x \\ \cos x & -\sin x & \cosh x \\ -\sin x & -\cos x & \sinh x \end{vmatrix} = \text{etc} = -2\sinh x,$$
which is not identically 0 on any interval. Hence (Theorem 3.2.2), LI.

(f)
$W[x,x^2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$, which is not identically 0 on any interval. Hence (Theorem 3.2.2), LI. Since there are only two functions in the set, it is simpler to use Theorem 3.2.4: neither is a scalar multiple of the other; hence, they are LI.

(g) LI by Theorem 3.2.4.

4. (b)
$$W[\sin 2x, \cos 2x] = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0$$
so (Thm. 3.2.3) LI.

(c) (As in (b), we'll omit the straightforward verification that the functions are indeed solutions of the ODE.) $W = \begin{vmatrix} e^x & xe^x & e^{4x} \\ e^x & e^x+xe^x & 4e^{4x} \\ e^x & 2e^x+xe^x & 16e^{4x} \end{vmatrix} = (e^x)(e^x)(e^{4x}) \begin{vmatrix} 1 & x & 1 \\ 1 & 1+x & 4 \\ 1 & 2+x & 16 \end{vmatrix}$,

by property D7 (Section 10.4), $= e^{6x}(9) \neq 0$ so (Thm. 3.2.3) LI. Of course, we don't <u>need</u> property D7, we could simply use (B5c) in Appendix B.

5. (a) $W'(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' = (y_1 y_2' - y_1' y_2)' = y_1 y_2'' + \cancel{y_1' y_2'} - \cancel{y_1' y_2'} - y_1'' y_2$

$= y_1 y_2'' - y_1'' y_2$

$= y_1(-p_1 y_2' - \cancel{p_2 y_2}) - (-p_1 y_1' - \cancel{p_2 y_1}) y_2$ since $y_1'' + p_1 y_1' + p_2 y_1 = 0$ and $y_2'' + p_1 y_2' + p_2 y_2 = 0$

$= p_1(y_1' y_2 - y_1 y_2')$

$= -p_1 \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -p_1 W(x)$

Then, (9) in Section 2.2 gives $W(x) = W(\xi)\, e^{-\int_\xi^x p_1(t)dt}$. ✓

6. (a) If they are LD then $a_1 u_1(x) + a_2 u_2(x) = 0$ (on I) with a_1, a_2 not both 0. Thus a_1 and/or a_2 are nonzero. Let $a_2 \neq 0$, say. Then we can divide by a_2 and obtain $u_2(x) = -\frac{a_1}{a_2} u_1(x)$, so u_1 is expressible as a multiple of u_2. Conversely, suppose one (say u_1) can be expressed as a multiple of the other: $u_1 = \alpha u_2$. Then $1 u_1(x) - \alpha u_2(x) = 0$ where not both coefficients are zero (since the first is 1); hence u_1, u_2 are LD.

(b) Let $u_2(x) = 0$, say. Then surely $0u_1(x) + 5u_2(x) + 0u_3(x) + \cdots + 0u_n(x) = 0$ with the coefficients $0, 5, 0, \ldots, 0$ not all 0. Hence, the set is LD.

(c) $a_1 u_1(x) + \cdots + a_n u_n(x) = b_1 u_1(x) + \cdots + b_n u_n(x)$ gives $(a_1 - b_1) u_1(x) + \cdots + (a_n - b_n) u_n(x) = 0$. Since $u_1, \ldots, u_n$ are LI, it follows that $a_1 - b_1 = 0, \ldots, a_n - b_n = 0$; i.e., $a_1 = b_1, \ldots, a_n = b_n$.

7. No, it does <u>not</u> follow. For ex., 1 and x are LI (Thm 3.2.4), 1 and $1+2x$ are LI, and x and $1+2x$ are LI, yet $\{1, x, 1+2x\}$ is LD since $1(1) - 2(x) + 1(1+2x) = 0$.

8. No, because the theorem does not apply since its conditions are not met. Specifically, $p_1(x) = -4/x$ and $p_2(x) = 6/x^2$ are not continuous on any interval containing the point $x=0$.

Section 3.3

1. (b) $e^x - e^{2x}$ and e^x are solutions (as is easily verified by substitution) and they are LI (one is not a multiple of the other), so $C_1(e^x - e^{2x}) + C_2 e^x$ <u>is</u> a general solution.

(c) $e^{-x} + e^{2x}$ is a solution, but we need <u>two</u> LI solutions for a general solution.

(e) No, we need <u>three</u> LI solutions.

(f) Yes. (g) No (h) Yes (i) Yes

2. (b) e^{3x} and $\cosh 3x$ <u>are</u> solutions, they are LI, and there are two of them. Hence $\{e^{3x}, \cosh 3x\}$ is a basis for $y''-9y=0$.
   (c) No, because $\sinh 3x$ and $2\cosh 3x$ are not solutions of the ODE.
   (e) Yes, they are 3 LI solutions so they constitute a basis. (f) Yes
3. (c) On $0<x<\infty$? Yes. On $-\infty<x<0$? Yes.
4. (b) No; neither e^x nor e^{-x} is a solution of the ODE
   (d) $x+x\ln|x|$ and $x-x\ln|x|$ are LI solutions of the ODE on any interval not containing the origin—such as $-\infty<x<0$, $0<x<\infty$, and $6<x<10$.
5. (b) It is not, because it contains only 6 LI solutions; e.g., the $\sinh x$ is a linear combination of the e^x and the e^{-x} and the $\cosh 2x$ is a linear combination of the e^{2x} and the e^{-2x}.
6. Yes, $y(x)=3$ is a solution. No contradiction; when we say that Thm 3.3.2 does not hold for nonlinear or nonhomogeneous we are saying that if $y_1(x)$ and $y_2(x)$ are solutions of a nonlinear " " equation (the ODE in this exercise is nonlinear) ... then $C_1y_1(x)+C_2y_2(x)$ is not <u>necessarily</u> a solution too—it <u>could</u> be, by coincidence, as in this case.
8. (b) The answer is $y(x)=-1-2x^2-\frac{2}{3}x^4-\frac{4}{45}x^6+\cdots$, as can be checked using these Maple commands:

```
Order := 8;
dsolve({diff(y(x),x,x)-4*y(x)=0, y(0)=-1, D(y)(0)=0},
       y(x), type=series);
```

   (c) $y(x)=2-5x+\frac{13}{2}x^2-\frac{35}{6}x^3+\frac{97}{24}x^4-\frac{55}{24}x^5+\cdots$
   (e) $y(x)=2-3x-\frac{1}{6}x^4+\frac{3}{20}x^5+\cdots$
9. (b) The ODE is of the type (5a) and the conditions are initial conditions like (5b). $p_1(x)=2$ and $p_2(x)=3$ are continuous for all x so, by Thm 3.3.1, the problem admits a unique solution on $-\infty<x<\infty$.
   (f) $p_1(x)=x/\sin x$ is continuous on $-\pi<x<\pi$ (containing the initial point $x=2$) as are $p_2(x)=p_3(x)=p_4(x)=0$, so, by Thm 3.3.1, the problem admits a unique solution on that interval.
11. (c) $y(x)=C_1\cos x+C_2\sin x$, $\quad y(1)=1=C_1\cos 1+C_2\sin 1$
   $\quad y(2)=2=C_1\cos 2+C_2\sin 2$
   has a unique solution for C_1, C_2 because $\begin{vmatrix}\cos 1 & \sin 1\\ \cos 2 & \sin 2\end{vmatrix}=\cos 1\sin 2-\sin 1\cos 2=\sin(2-1)$ $=\sin 1\neq 0$. Namely, $C_1=-0.920$, $C_2=1.779$. Thus, the boundary-value problem has the unique solution $y(x)=-0.920\cos x+1.779\sin x$.
13. Surely (10) implies (13.1a) (by choosing $\alpha=\beta=1$) and (13.1b) (by choosing $\beta=0$), but we also need to show that (13.1a,b) imply (10), which we do next: $L[\alpha u+\beta v]=L[\alpha u]+L[\beta v]$ (by 13.1a) $=\alpha L[u]+\beta L[v]$ (by 13.1b).
14. If (11) holds for k, then $L[\alpha_1u_1+\cdots+\alpha_{k+1}u_{k+1}]=L[1(\alpha_1u_1+\cdots+\alpha_ku_{k+1})+\alpha_{k+1}u_{k+1}]$ $=1\,L[\alpha_1u_1+\cdots+\alpha_ku_k]+\alpha_{k+1}L[u_{k+1}]$ by (11) with $k=2$. Further, from (11) this $=\alpha_1L[u_1]+\cdots+\alpha_kL[u_k]+\alpha_{k+1}L[u_{k+1}]$, so (11) holds for $k+1$. Hence, $P(k)$ holds for <u>all</u> $k\geq 1$.

Section 3.4

4. (b) $y(x) = A + Be^{x}$

(c) $y(x) = A + Be^{-x}$, $y(0)=3$ and $y'(0)=0$ give $A+B=3$, $-B=0$ so $B=0$, $A=3$, $y(x)=3$.

(n) $y = e^{\lambda x} \to \lambda^4 - 1 = 0$, $\lambda^4 = 1$, $\lambda^2 = \pm 1$, $\lambda = \pm 1, \pm i$ so $y(x) = Ae^{x} + Be^{-x} + Ce^{ix} + De^{-ix}$
or $y(x) = E\cosh x + F\sinh x + G\cos x + H\sin x$, for example.

(o) $y = e^{\lambda x} \to \lambda^4 - 2\lambda^2 - 3 = 0$, $\lambda^2 = (2 \pm \sqrt{4+12})/2 = 1 \pm 2 = 3, -1$; $\lambda = \pm\sqrt{3}$ and $\pm i$
so $y(x) = Ae^{\sqrt{3}x} + Be^{-\sqrt{3}x} + C\cos x + D\sin x$.

5. (e) dsolve({diff(y(x),x,x) - 4*diff(y(x),x) - 5*y(x) = 0, y(1)=1, D(y)(1)=0},
y(x)); gives $y(x) = \frac{5}{6}\frac{e^{-x}}{e^{-1}} + \frac{1}{6}\frac{e^{5x}}{e^{5}}$

(n) dsolve(diff(y(x),x,x,x,x) - y(x) = 0, y(x)); gives
$y(x) = C_1 e^{x} + C_2 \cos x + C_3 \sin x + C_4 e^{-x}$

6. (b) $y(x) = (A+Bx)e^{-3x}$, $\left.\begin{array}{l} y(1) = e = (A+B)e^{-3} \\ y'(1) = -2 = -(3A+2B)e^{-3} \end{array}\right\} \Rightarrow \begin{array}{l} A = 2e^{3}(1-e), \\ B = e^{3}(3e-2) \end{array}$
so $y(x) = [2(1-e) + (3e-2)x]e^{-3(x-1)}$

(c) $y(x) = A + Bx + Cx^2$, $y(0) = 3 = A$, $y'(0) = -5 = B$, $y''(0) = 1 = 2C$,
so $y(x) = 3 - 5x + \frac{1}{2}x^2$

8. (b) $(\lambda - 2i)(\lambda + 2i) = \lambda^2 + 4$, so the ODE is $y'' + 4y = 0$. $y(x) = Ae^{i2x} + Be^{-i2x}$
or $C\cos 2x + D\sin 2x$.

(c) $(\lambda-(4-2i))(\lambda-(4+2i)) = \lambda^2 - 8\lambda + 20$, so the ODE is $y'' - 8y' + 20y = 0$
with general solution $y(x) = Ae^{(4-2i)x} + Be^{(4+2i)x} = e^{4x}(C\cos 2x + D\sin 2x)$

(f) $(\lambda-1)^2(\lambda+2) = \lambda^3 - 3\lambda + 2$, so the ODE is $y''' - 3y' + 2y = 0$
with general solution $y(x) = (A+Bx)e^{x} + Ce^{-2x}$.

9. (b) $\lambda^2 - 3i\lambda - 2 = 0$ gives $\lambda = (3i \pm \sqrt{-9+8})/2 = i, 2i$ so $y(x) = Ae^{ix} + Be^{i2x}$

(c) $\lambda^2 + i\lambda - 1 = 0$ gives $\lambda = (-i \pm \sqrt{-1+4})/2 = (-i \pm \sqrt{3})/2$ so $y(x) = e^{-ix/2}(Ae^{\sqrt{3}x/2} + Be^{-\sqrt{3}x})$

10. Remember that, in Maple, $i = \sqrt{-1}$ is written as I.

11. (a) $(D-\lambda_1)\underbrace{(D-\lambda_2)y}_{u} = 0$. $u' - \lambda_1 u = 0$ gives $u_1 = Ae^{\lambda_1 x}$. Then $(D-\lambda_2)y = u$ becomes
$y' - \lambda_2 y = Ae^{\lambda_1 x}$ which, being first-order linear, gives
$y(x) = e^{-\int -\lambda_2 dx}\left(\int e^{\int -\lambda_2 dx} Ae^{\lambda_1 x}dx + B\right) = e^{\lambda_2 x}\left(\int Ae^{(\lambda_1-\lambda_2)x}dx + B\right)$
$= e^{\lambda_2 x}\left(\frac{A}{\lambda_1-\lambda_2}e^{(\lambda_1-\lambda_2)x} + B\right) = Ce^{\lambda_1 x} + Be^{\lambda_2 x}$ (B, C arbitrary constants)

12. (b) $\lambda \approx -2.52, -0.239 \pm 0.858i$. Each $\operatorname{Re}\lambda < 0$, so stable. The Maple command used was fsolve (x^3 + 3*x^2 + 2*x + 2 = 0, x, complex);

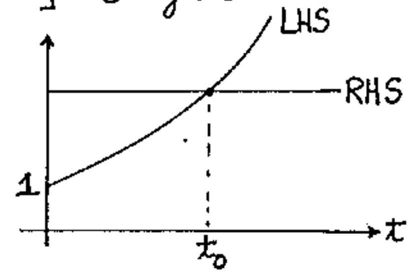
(e) $\lambda \approx -0.793 \pm 0.458i, -0.297 \pm 2.236i, +0.590 \pm 0.597i$ hence unstable, because of the +0.590. This result is in accord with Theorem 3.4.4 because the polynomial in λ has mixed signs.

Section 3.5

1. (b) $3\cos 6t - 4\sin 6t = E\sin(\omega t + \phi)$, $E = \sqrt{3^2+4^2} = 5$, $\phi = \tan^{-1}\left(\frac{3}{-4}\right) = -0.6435$ rad

5. It is striking that the frequency $\omega = \sqrt{k/m}$ is fixed; i.e., it is independent of x_0 (and x_0'). While true for the linear oscillator $m\ddot{x} + kx = 0$, it is not true for nonlinear oscillators, as we will see in Chapter 7.

6. (a) The form $x(t) = e^{-\alpha t}(Ae^{\sqrt{\Gamma}t} + Be^{-\sqrt{\Gamma}t})$ will be convenient, where α is $c/2m$ and $\sqrt{\Gamma}$ is $\sqrt{\alpha^2 - \omega^2}$; A, B are, of course, dictated by the initial conditions. Then $x'(t) = e^{-\alpha t}[-\alpha A e^{\sqrt{\Gamma}t} - \alpha B e^{-\sqrt{\Gamma}t} + A\sqrt{\Gamma}e^{\sqrt{\Gamma}t} - B\sqrt{\Gamma}e^{-\sqrt{\Gamma}t}] = 0$ gives $e^{2\sqrt{\Gamma}t} = \frac{B}{A}\frac{\sqrt{\Gamma}+\alpha}{\sqrt{\Gamma}-\alpha}$. The graphs of the LHS,



and RHS are sketched at the right. Since the LHS is a monotone function of t and the RHS is a constant, we have exactly one flatspot (at t_0) if the initial conditions are such that RHS > 1 and none if RHS < 1. The foregoing is for the overdamped case. For the critically damped case $x(t) = (A + Bt)e^{-\alpha t}$ and $x'(t) = (-\alpha A + B - \alpha B t)e^{-\alpha t} = 0$ gives "t_0" $= (B - \alpha A)/(\alpha B)$. If the latter is negative then there are no flat spots on $0 \le t < \infty$, and if it is positive then there is one flat spot on $0 \le t < \infty$.

(b) Let $m = k = 1$ and $c = c_{cr} = \sqrt{4mk} = 2$. Then $\alpha = c/2m = 2/2 = 1$ so $x(t) = (A+Bt)e^{-t}$. $t_0 = (B - \alpha A)/(\alpha B) = (B - A)/B$. If $B = 1$ and $A = 2$ then $t_0 < 0$ so there are no flatspots; in this case $x(0) = x_0 = 2$ and $x'(0) = x_0' = -1$. (c) If instead we let $B = 1$ and $A = -1$, then $t_0 = 2 > 0$ so there is one flatspot; in this case $x(0) = x_0 = -1$ and $x'(0) = x_0' = 2$. Of course these choices are by no means unique.

NOTE that this is a "design" question — how to design the physical system (i.e., how to choose m, c, k, x_0, x_0') so as to achieve a certain behavior.

7. (a) $x(t) = e^{-\alpha t}(A\cos\sqrt{\Gamma}t + B\sin\sqrt{\Gamma}t)$, where α is $c/2m$ and $\sqrt{\Gamma}$ is $\sqrt{\omega^2 - (c/2m)^2}$. $x'(t) = 0$ gives $\tan\sqrt{\Gamma}t = (\sqrt{\Gamma}B - \alpha A)/(\alpha B + \sqrt{\Gamma}A) \equiv *$, say. The latter has roots $\sqrt{\Gamma}t = \sqrt{\Gamma}t_0 + n\pi$ (where $t_0 = \tan^{-1}\frac{*}{\sqrt{\Gamma}}$ in $-\frac{\pi}{2} < t_0 < \frac{\pi}{2}$). But successive flat spots are max, min, max, ..., so to consider successive maxima change the $n\pi$ to $2n\pi$ and write $\sqrt{\Gamma}t = \sqrt{\Gamma}t_0 + 2n\pi$. Then, if x_n and x_{n+1} are successive maxima of $x(t)$,

$$r = \frac{x_n}{x_{n+1}} = \frac{\exp[-\alpha(t_0 + 2n\pi/\sqrt{\Gamma})][A\cos(\sqrt{\Gamma}t_0 + 2n\pi) + B\sin(\sqrt{\Gamma}t_0 + 2n\pi)]}{\exp[-\alpha(t_0 + 2(n+1)\pi/\sqrt{\Gamma})][A\cos(\sqrt{\Gamma}t_0 + 2(n+1)\pi) + B\sin(\sqrt{\Gamma}t_0 + 2(n+1)\pi)]}$$

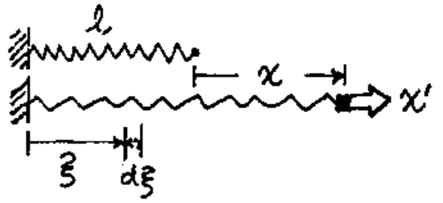
$$= \exp(+2\pi\alpha/\sqrt{\Gamma})$$ is a constant (i.e., doesn't change with n)

(b) logarithmic decrement $\delta = \ln r = \ln\exp\left(\frac{2\pi\alpha}{\sqrt{\Gamma}}\right) = \frac{2\pi\alpha}{\sqrt{\Gamma}} = \frac{2\pi c/(2m)}{\sqrt{\omega^2 - (c/2m)^2}}$

8. If $\epsilon \ll 1$, then $\theta'' + \epsilon\theta' + \frac{g}{L}\theta = 0$ is underdamped and its solution is given by (12) with $m \to 1$, $c \to \epsilon$, $k \to g/L$:

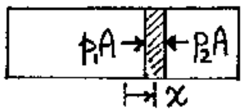
$$\theta(t) = e^{-\epsilon t/2}\left[A\cos\sqrt{\tfrac{g}{L} - (\tfrac{\epsilon}{2})^2}\,t + B\sin\sqrt{\tfrac{g}{L} - (\tfrac{\epsilon}{2})^2}\,t\right]$$

The oscillation frequency, $\sqrt{(g/L) - (\epsilon/2)^2}$, is a constant, even as the magnitude damps out due to the $\exp(-\epsilon t/2)$ factor.

9. (b) 

$$\text{KE in spring} = \int_{\xi=0}^{\xi=l+x} \frac{1}{2}\left(\frac{d\xi}{l+x} m_s\right)\left[\frac{\xi}{l+x}x'\right]^2$$
$$= \frac{1}{2}\frac{m_s x'^2}{(l+x)^3}\frac{(l+x)^3}{3} = \frac{1}{6}m_s x'^2$$

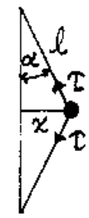
Including this spring KE gives (9.2), and d/dt of (9.2) gives (9.3).

10. (a) 

Newton's 2nd law → $mx'' = (p_1 - p_2)A$.
Boyle's law → $p_2(L-x)A = p_1(L+x)A = p_0 LA$
gives $p_1 = \frac{p_0 L}{L+x}$, $p_2 = \frac{p_0 L}{L-x}$

so $mx'' + p_0 LA\left(\frac{1}{L-x} - \frac{1}{L+x}\right) = 0$,
$mx'' + \frac{2p_0 ALx}{L^2 - x^2} = 0$

(b) Nonlinear due to the $x/(L^2-x^2)$ term.
(c) Taylor series: $x/(L^2-x^2) = \frac{x}{L^2}\frac{1}{1-\left(\frac{x}{L}\right)^2} = \frac{x}{L^2}\left(1 + \frac{x^2}{L^2} + \frac{x^4}{L^4} + \cdots\right) \sim \frac{x}{L^2}$
gives the linearized version
for small x (i.e., for $|x/L| \ll 1$): $mx'' + 2\frac{p_0 A}{L}x = 0$.
(d) freq $= \sqrt{\frac{2p_0 A}{mL}}\ \frac{\text{rad}}{\text{sec}}\ \frac{1\,\text{cycle}}{2\pi\,\text{rad}} = \frac{1}{2\pi}\sqrt{\frac{2p_0 A}{mL}}\ \frac{\text{cycles}}{\text{sec}}$.
(e) Yes



11. (a) $mx'' = -2\tau\sin\alpha$ (see sketch at right) $= -2\tau(l)x/l$
so $mx'' + 2\frac{\tau(\sqrt{l_0^2+x^2})}{\sqrt{l_0^2+x^2}}x = 0$

(b) Nonlinear because $\tau(\sqrt{l_0^2+x^2})x/\sqrt{l_0^2+x^2}$ is not a linear function of x.
(c) $\tau[l(x)] = \tau[l(0)] + \frac{d\tau}{dx}\Big|_{x=0}x + \frac{1}{2!}\frac{d^2\tau}{dx^2}\Big|_{x=0}x^2 +$ etc.

$\hookrightarrow \frac{d\tau}{dx} = \frac{d\tau}{dl}\frac{dl}{dx} = \frac{d\tau}{dl}\frac{1}{2}\frac{2x}{l}$, $\frac{d^2\tau}{dx^2} = \frac{d\tau}{dl}\frac{1}{l} + \frac{x}{l}\frac{d^2\tau}{dl^2}\frac{1}{2}\frac{2x}{l}$
so $\frac{d\tau}{dx}\Big|_x = 0$ and $\frac{d^2\tau}{dx^2}\Big|_{x=0} = \tau'(l_0)/l_0 + 0$

Thus, $\tau[l(x)] = \tau(l_0) + 0x + \frac{1}{2!}\frac{\tau'(l_0)}{l_0}x^2 + \cdots$.
It might be clearer to proceed, instead, like this:
$\tau[l(x)] = \tau(\sqrt{l_0^2+x^2}) = \tau\{l_0[1+(\frac{x}{l_0})^2]^{1/2}\} = \tau\{l_0(1+\frac{1}{2}\frac{x^2}{l_0^2} - \frac{1}{8}\frac{x^4}{l_0^4}+\cdots)\}$
$= \tau[l_0 + \underbrace{(\frac{1}{2}\frac{x^2}{l_0}+\cdots)}_{\text{Call this } Z}] = \tau(l_0+Z) = \tau(l_0) + \tau'(l_0)Z + \frac{1}{2!}\tau''(l_0)Z^2 + \cdots$
$= \tau(l_0) + \tau'(l_0)(\frac{1}{2}\frac{x^2}{l_0}+\cdots) + \frac{1}{2!}\tau''(l_0)(\frac{1}{2}\frac{x^2}{l_0}+\cdots)^2 + \cdots$
Rearranging (formally) in ascending powers of x gives
$\tau[l(x)] = \tau(l_0) + \tau'(l_0)\frac{x^2}{2l_0}$ + terms of order $x^4, x^6, \ldots$.

Since we want the Taylor series of $\tau[l(x)]/l(x)$ we also need to expand the $1/l(x)$ factor and then multiply its series into the series for $\tau[l(x)]$.

$$\frac{1}{\ell(x)} = (\ell_0^2+x^2)^{-1/2} = \frac{1}{\ell_0}\left[1+\left(\frac{x}{\ell_0}\right)^2\right]^{-1/2} = \frac{1}{\ell_0}\left[1-\frac{1}{2}\frac{x^2}{\ell_0^2}+\cdots\right], \text{ so}$$

$$\frac{\tau[\ell(x)]}{\ell(x)} = \left[\tau(\ell_0)+\tau'(\ell_0)\frac{x^2}{2\ell_0}+\cdots\right]\frac{1}{\ell_0}\left(1-\frac{1}{2}\frac{x^2}{\ell_0^2}+\cdots\right) = \frac{\tau(\ell_0)}{\ell_0} + \left[\tau'(\ell_0)\frac{1}{2\ell_0^2}-\frac{\tau(\ell_0)}{2\ell_0^3}\right]x^2+\cdots$$

(d) Linearizing (i.e., keeping terms through x to the first power), $\frac{\tau[\ell(x)]}{\ell(x)}x \sim \frac{\tau(\ell_0)}{\ell_0}x$ so the linearized ODE is $mx'' + \underbrace{\left(2\frac{\tau(\ell_0)}{\ell_0}\right)}_{\text{"}k_{\text{equiv.}}\text{"}}x = 0$

Frequency $= \sqrt{k_{eq.}/m}\ \frac{\text{rad}}{\text{sec}} = \sqrt{\frac{2\tau_0}{\ell_0 m}}\ \frac{\text{rad}}{\text{sec}}\ \frac{1\text{ cycle}}{2\pi\text{ rad}} = \frac{1}{2\pi}\sqrt{\frac{2\tau_0}{m\ell_0}}$ cycles/sec

12. $\sum$ Vertical forces $=0$ gives $N_1+N_2=mg$
$\sum$ Moments about left-hand cylinder gives $N_2L = mg(x+\frac{L}{2})$
so $N_2 = mg(\frac{1}{2}+\frac{x}{L})$, $N_1 = mg(\frac{1}{2}-\frac{x}{L})$

Then, $mx'' = \sum$ Horizontal forces
$= \mu N_1 - \mu N_2 = \mu mg(\frac{1}{2}-\frac{x}{L}) - \mu mg(\frac{1}{2}+\frac{x}{L})$

or, $mx'' + \frac{2mg\mu}{L}x = 0$.

Frequency $= \sqrt{2mg\mu/mL} = \sqrt{2g\mu/L}$ rad/sec

13. (a) Let potential energy (due to gravity) $=0$ when m is in the position shown.

so $PE = mg(L\cos\theta\sin\alpha)$
$KE = \frac{1}{2}m(L\theta')^2$
so $PE+KE = mgL\cos\theta\sin\alpha + \frac{1}{2}m(L\theta')^2 = \text{const}$.
d/dt gives $-mgL\sin\theta\,\theta'\sin\alpha + \frac{1}{2}mL^2 2\theta''\theta' = 0$
$\theta'' + \frac{g}{L}\sin\alpha\sin\theta = 0$

(b) Linearized, $\theta'' + \frac{g\sin\alpha}{L}\theta = 0$ so freq. $= \sqrt{\frac{g\sin\alpha}{L}}\ \frac{\text{rad}}{\text{sec}} = \frac{1}{2\pi}\sqrt{\frac{g\sin\alpha}{L}}\ \frac{\text{cycles}}{\text{sec}}$

Section 3.6

1.(b) $y=x^\lambda$ gives $\lambda-1=0$ so $\lambda=1$, $y=Ax$, $y(2)=5=2A$ so $A=5/2$ and $y(x)=5x/2$ $(-\infty<x<\infty)$

(c) $\lambda^2-\lambda+\lambda=0$, $\lambda=0,0$, $y(x)=(A+B\ln|x|)x^0 = A+B\ln|x| = \begin{cases} A+B\ln x & \text{for } 0<x<\infty \\ A+B\ln(-x) & \text{for } -\infty<x<0 \end{cases}$

(e) $\lambda^2-\lambda+\lambda-9=0$, $\lambda=\pm3$, $y=Ax^3+Bx^{-3}$. $y(2)=1=8A+B/8$ and $y'(2)=2=12A-3B/16$
so $y(x)=\frac{7}{48}x^3-\frac{4}{3}x^{-3}$ on $0<x<\infty$

(f) $\lambda^2-\lambda+\lambda+1=0$, $\lambda=\pm i$, $y=A\cos(\ln x)+B\sin(\ln x)$.
$y(1)=1=A$, $y'(1)=0=B$, so $y(x)=\cos(\ln x)$ on $0<x<\infty$.

(h) $\lambda=2,-1$, $y=Ax^2+B/x$. $y(-5)=3=25A-B/5$, $y'(-5)=0=-10A-B/25$; $A=1/25$, $B=-10$,
so $y(x)=x^2/25-10/x$ on $-\infty<x<0$

(m) $\lambda(\lambda-1)(\lambda-2)-2\lambda=0$, $\lambda=0,0,3$; $y(x)=A+B\ln|x|+Cx^3$. $y(1)=2=A+C$, $y'(1)=0=B+3C$.
$y''(1)=0=-B+6C$ gives $A=2$, $B=C=0$, $y(x)=2$ on $-\infty<x<\infty$.

(o) $\lambda^2-\lambda+\lambda-K^2=0$, $\lambda=\pm K$, $y(x)=A|x|^K+B|x|^{-K} = \begin{cases} Ax^K+Bx^{-K} & \text{on } 0<x<\infty \\ Ax^K+B(-x)^{-K} & \text{on } -\infty<x<0 \end{cases}$

(g) $\lambda(\lambda-1)(\lambda-2)+2\lambda-2=0$, $\lambda=1, 1\pm i$; $y(x)=Ax+x[B\cos(\ln|x|)+C\sin(\ln|x|)]$ on $0<x<\infty$ or on $-\infty<x<0$.

2. (m) dsolve({x^2*diff(y(x),x,x,x)-2*diff(y(x),x)=0, y(1)=2, D(y)(1)=0, D(D(y))(1)=0}, y(x)); gives the solution $y(x)=2$. Note the D(y)(1) and D(D(y))(1) designations for $y'(1)$ and $y''(1)$.

6. Recall that two functions are LI if and only if one is not a constant multiple of the other. Thus, the two solutions in (33) are LI if and only if $\star = \int Y(x)^{-2} e^{-\int a_1(x)dx}\,dx \neq$ constant. Well, $\frac{d}{dx}\int Y(x)^{-2}e^{-\int a_1 dx}\,dx = e^{-\int a_1 dx}/Y^2(x)=0$ is impossible (since the exponential function is nowhere 0 and $Y^2(x)\neq\infty$) so $\star \neq$ constant and the solutions are LI.

7. (a) $x^2y''-xy'-3y=0$. $x=e^t$, $dx/dt=e^t$, $dt/dx=e^{-t}$

$dy/dx = dY/dt\; dt/dx = e^{-t}\,dY/dt$

$d^2y/dx^2 = \frac{d}{dt}\left(e^{-t}\frac{dY}{dt}\right)\frac{dt}{dx} = \left(-e^{-t}\frac{dY}{dt}+e^{-t}\frac{d^2Y}{dt^2}\right)e^{-t}$

so $e^{2t}\left(-e^{-t}\frac{dY}{dt}+e^{-t}\frac{d^2Y}{dt^2}\right)e^{-t} - e^t\left(e^{-t}\frac{dY}{dt}\right)-3Y=0$,

$d^2Y/dt^2 - 2dY/dt - 3Y=0$, $Y(t)=Ae^{-t}+Be^{3t}$. But $x=e^t \to t=\ln x$,

so $y(x)=Ae^{-\ln x}+Be^{3\ln x} = A/x+Bx^3$.

8. (a) We saw in 7(a) that $xDy=DY$. Then $xD(xDy)=D^2Y$ gives

$x^2D^2y+\underbrace{xDy}_{DY}=D^2Y$ or $x^2D^2y = D^2Y-DY=D(D-1)Y$.

Next, $xD(x^2D^2y) = DD(D-1)Y$

$x^3D^3y + \underbrace{2x^2D^2y}_{D(D-1)Y} = D^2(D-1)Y$

so $x^3D^3y = D^2(D-1)Y - 2D(D-1)Y$

$= D(D-1)(D-2)Y$,

and so on.

9. (b) $\Phi = A+B\ln r$, $\Phi'(r_1)=0=B/r_1 \Rightarrow B=0$ so $\Phi(r)=A$. Then $\Phi(r_2)=\Phi_2=A$, so $\Phi(r)=\Phi_2$

10. (b) $u=A+B/r$, $u'(r_1)=3=-B/r_1^2 \Rightarrow B=-3r_1^2$ so $u(r)=A-3r_1^2/r$. Then $u(r_2)=0=A-3r_1^2/r_2$ gives $A=3r_1^2/r_2$ so $u(r)=3r_1^2\left(\frac{1}{r_2}-\frac{1}{r}\right)$

11. (b) Seek $y(x)=A(x)x$. $y'=A+A'x$, $y''=A'+A'+A''x$ so

$x(2A'+A''x)+x(\cancel{A}+A'x)-\cancel{Ax}=0$, $x^2A''+(2x+x^2)A'=0$ or, with $A'=p$,

$\frac{dp}{p}+\left(\frac{2}{x}+1\right)dx=0$, $\ln p+2\ln x+x=B$, $p=A'=e^{B-x-2\ln x}=C\,e^{-x}/x^2$

so $A(x)=C\int e^{-x}x^{-2}dx$. Thus, $y(x)=Ax+Cx\int e^{-x}\,dx/x^2$.

12. (b) dsolve(x*diff(y(x),x,x)+x*diff(y(x),x)-y(x)=0, y(x)); gives

$y(x)=C_1x+C_2(-e^{-x}+\mathrm{Ei}(1,x)x)$.

Is this equivalent to our solution in (11b)? ?Ei gives us the Maple definition of the exponential integral function as

$$\mathrm{Ei}(n,x)=\int_1^\infty \frac{e^{-xt}}{t^n}\,dt$$

Integrating by parts, $y(x) = Ax + Cx\int e^{-x}\frac{dx}{x^2} = C_1x + C_2x\int_x^\infty \frac{e^{-\xi}}{\xi^2}d\xi$ $\left(\begin{smallmatrix} u=e^{-\xi} \\ dv=d\xi/\xi^2\end{smallmatrix}\right)$

$= C_1x + C_2x\left[-e^{-\xi}\frac{1}{\xi}\Big|_x^\infty - \int_x^\infty\left(-\frac{1}{\xi}\right)(-e^{-\xi}d\xi)\right] \overset{\text{Let } \xi = xt}{=} C_1x + C_2x\left[\frac{e^{-x}}{x} - \int_1^\infty \frac{e^{-xt}}{xt}x\,dt\right]$

$= C_1x + C_2\left(e^{-x} - x\int_1^\infty e^{-xt}\,dt/t\right) = C_1x + C_2\left(e^{-x} - x\,\mathrm{Ei}(1,x)\right)$, which agrees with the Maple solution. ✓

13. (48) says $y''+a_1y'+a_2y = y''-(a+b)y'+(ab-b')y$. Since this identity is to hold for all (twice-differentiable) functions, we can let $y=1$ and x, in turn. These give $0a_1+a_2 = ab-b'$ and $a_1 + \cancel{xa_2} = -(a+b)+(ab\cancel{-b'})x$, so $a_2 = ab-b'$, $a_1 = -(a+b)$.

15. $(D+x)\underbrace{(D-x)y}_{u}=0 \to \frac{du}{dx}+xu=0$, $\frac{du}{u}=-x\,dx$, $u=Ae^{-x^2/2}$, $y' \underbrace{-x}_{p(x)} y = \underbrace{Ae^{-x^2/2}}_{q(x)}$

$y(x) = e^{-\int -x\,dx}\left[\int e^{\int -x\,dx}Ae^{-x^2/2}dx + B\right] = Be^{+x^2/2} + Ae^{x^2/2}\int e^{-x^2}dx$, which (with A and B interchanged) is the same as (57).

16. If a,b are constants then (50a,b) become

$$a' = a^2+a_1a+a_2, \quad b' = -b^2-a_1b-a_2,$$

both of which are satisfied if a and b are constants, namely, solutions of $\lambda^2+a_1\lambda+a_2=0$, say λ_1,λ_2. Then $(D-\lambda_1)\underbrace{(D-\lambda_2)y}_{u}=0$. $(D-\lambda_1)u=0$ gives

$u(x)=Ae^{\lambda_1x}$. Then $(D-\lambda_2)y = Ae^{\lambda_1 x}$, or $y'-\lambda_2y=Ae^{\lambda_1x}$, so

$y(x)=e^{-\int-\lambda_2dx}\left[\int e^{\int-\lambda_2dx}Ae^{\lambda_1x}dx+B\right] = e^{\lambda_2x}\left(A\int e^{(\lambda_1-\lambda_2)x}dx+B\right)$

$= e^{\lambda_2x}\frac{A}{\lambda_1-\lambda_2}e^{(\lambda_1-\lambda_2)x}+Be^{\lambda_2x} = Ce^{\lambda_1x}+Be^{\lambda_2x}$ if $\lambda_1\ne\lambda_2$. If $\lambda_1=\lambda_2=\lambda$,

then the foregoing gives $y(x)=e^{\lambda x}\left(A\int e^{0x}dx+B\right) = (Ax+B)e^{\lambda x}$. These results are the same as obtained by the elementary methods given in Section 3.4.

17. (b) $x^2y''+xy'+9y=0$, so $a_1(x)=1/x$, $a_2(x)=9/x^2$. Then (50a,b) give

$a' = a^2+\frac1x a+\left(\frac{9}{x^2}+\frac{1}{x^2}\right)$ and $b'=-b^2-\frac1x b-\frac{9}{x^2}$. Try $a=\alpha/x$ and $b=\beta/x$.

Then $-\frac{\alpha}{x^2}=\frac{\alpha^2}{x^2}+\frac{\alpha}{x^2}+\frac{10}{x^2}$ and $-\cancel{\frac{\beta}{x^2}} = -\frac{\beta^2}{x^2}-\cancel{\frac{\beta}{x^2}}-\frac{9}{x^2}$, so $\alpha=-1\pm3i$, $\beta=\pm3i$.

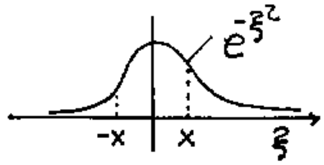
Choose $a(x)=\frac{-1+3i}{x}$ and $b(x)=-\frac{3i}{x}$, say. Then the factored ODE is $\left(D-\frac{-1+3i}{x}\right)\underbrace{\left(D+\frac{3i}{x}\right)y}_{u}=0$. $u'-\frac{-1+3i}{x}u=0$ gives $u=Ax^{3i-1}$. Then

$\left(D+\frac{3i}{x}\right)y=u$ becomes $y'+\frac{3i}{x}y=Ax^{3i-1}$, so $y(x)=e^{-\int\frac{3i}{x}dx}\left(\int e^{\int\frac{3i}{x}dx}Ax^{3i-1}dx+B\right)$

$=x^{-3i}\left(\int x^{3i}Ax^{3i-1}dx+B\right) = x^{-3i}\left(\frac{A}{6i}x^{6i}+B\right) = Cx^{3i}+Bx^{-3i}$, which is the same result as is obtained by seeking $y(x)=x^\lambda$.

18. $\mathrm{erf}(-x)=\frac{2}{\sqrt\pi}\int_0^{-x}e^{-\xi^2}d\xi = -\frac{2}{\sqrt\pi}\int_{-x}^0 e^{-\xi^2}d\xi$

$=-\frac{2}{\sqrt\pi}\int_0^x e^{-\xi^2}d\xi = -\mathrm{erf}(x)$ because



the graph of the integrand is symmetric about $x=0$.

19. (a) $\ln x^a = \int_1^{x^a} \frac{dt}{t} \overset{t=u^a}{=} \int_1^{x} \frac{a u^{a-1}}{u^a}\,du = a\int_1^x \frac{du}{u} = a \ln x$

(b) $\ln xy = \int_1^{xy} \frac{dt}{t} = \int_1^x \frac{dt}{t} + \boxed{\int_x^{xy} \frac{dt}{t}}$ ← Let $t = x\tau$

$= \ln x + \int_1^y \frac{x\,d\tau}{x\tau} = \ln x + \ln y$

Section 3.7

1. (b) Yes, $\cos x \sinh 2x \to \{\cos x \sinh 2x, \sin x \sinh 2x, \cos x \cosh 2x, \sin x \cosh 2x\}$

(c) No, $\ln x \to \ln x, 1/x, 1/x^2, 1/x^3, \ldots$ without end.

2. (b) $y' + y = x^4 + 2x$; $y_h = C_1 e^{-x}$; putting $y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$ into the ODE gives
$4Ax^3 + 3Bx^2 + 2Cx + D + Ax^4 + Bx^3 + Cx^2 + Dx + E = x^4 + 2x$.

x^4: $A = 1$
x^3: $4A + B = 0$
x^2: $3B + C = 0$
x: $2C + D = 2$
1: $D + E = 0$

$A = 1, B = -4, C = 12, D = -22, E = 22$
so $y(x) = C_1 e^{-x} + x^4 - 4x^3 + 12x^2 - 22x + 22$

(c) $y' + 2y = 3e^{2x} + 4\sin x$; $y_h = C_1 e^{-2x}$; putting $y_p = \overbrace{Ae^{2x}}^{\text{for } 3e^{2x}} + \overbrace{B\sin x + C\cos x}^{\text{for } 4\sin x}$ into the ODE gives $2Ae^{2x} + B\cos x - C\sin x + 2Ae^{2x} + 2B\sin x + 2C\cos x = 3e^{2x} + 4\sin x$.

e^{2x}: $2A + 2A = 3$
$\sin x$: $-C + 2B = 4$
$\cos x$: $B + 2C = 0$

$A = 3/4, B = 8/5, C = -4/5$
so $y(x) = C_1 e^{-2x} + \frac{3}{4}e^{2x} + \frac{8}{5}\sin x - \frac{4}{5}\cos x$

(h) $y'' + y' = 4xe^x + 3\sin x$. $y_h = C_1 + C_2 e^{-x}$.
$4xe^x \to \{xe^x, e^x\}$, $3\sin x \to \{\sin x, \cos x\}$. No duplication,
so seek $y_p = Axe^x + Be^x + C\sin x + D\cos x$. Putting this in the ODE gives
$Axe^x + Ae^x + Be^x + C\cos x - D\sin x$
$+ Axe^x + Ae^x + Ae^x + Be^x - C\sin x - D\cos x = 4xe^x + 3\sin x$.

xe^x: $2A = 4$
e^x: $3A + 2B = 0$
$\cos x$: $C - D = 0$
$\sin x$: $-D - C = 3$

$A = 2, B = -3, C = -3/2, D = -3/2$
so $y(x) = C_1 + C_2 e^{-x} + 2xe^x - 3e^x - \frac{3}{2}\sin x - \frac{3}{2}\cos x$

(l) $y'' + 2y' = x^2 + 4e^{2x}$. $y_h = C_1 + C_2 e^{-2x}$
$x^2 \to \{x^2, x, 1\}$, $4e^{2x} \to \{e^{2x}\}$ so try $y_p = (Ax^2 + Bx + C) + (De^{2x})$. But the C term duplicates the C_1 term in y_h, so try, instead, $y_p = x(Ax^2 + Bx + C) + (De^{2x})$.
There is no more duplication so we accept $y_p = Ax^3 + Bx^2 + Cx + De^{2x}$ and proceed. Putting the latter into the ODE gives
$2(3Ax^2 + 2Bx + C + 2De^{2x}) + (6Ax + 2B + 4De^{2x}) = x^2 + 4e^{2x}$
x^2: $6A = 1$, x: $4B + 6A = 0$, 1: $2C + 2B = 0$, e^{2x}: $4D + 4D = 4$ gives
$A = 1/6, B = -1/4, C = 1/4, D = 1/2$, so
$y(x) = C_1 + C_2 e^{-2x} + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{2}e^{2x}$

(m) $y''-2y'+y=x^2e^x$. $y_h=(C_1+C_2x)e^x$. $x^2e^x \to \{x^2e^x, xe^x, e^x\}$ so try $y_p = Ax^2e^x+Bxe^x+Ce^x$. But the Bxe^x term duplicates the C_2xe^x term and the Ce^x term duplicates the C_1e^x term, so try $y_p=Ax^3e^x+Bx^2e^x+Cxe^x$. Still, the Cxe^x term duplicates the C_2xe^x term, so try $y_p=Ax^4e^x+Bx^3e^x+Cx^2e^x=(Ax^4+Bx^3+Cx^2)e^x$. Putting this in the ODE gives

$$(Ax^4+Bx^3+Cx^2)e^x-2(4Ax^3+3Bx^2+2Cx+Ax^4+Bx^3+Cx^2)e^x$$
$$+(12Ax^2+6Bx+2C+4Ax^3+3Bx^2+2Cx+4Ax^3+3Bx^2+2Cx+Ax^4+Bx^3+Cx^2)e^x$$
$$=x^2e^x$$

$$\left.\begin{aligned} x^4e^x&: A-2A+A=0\\ x^3e^x&: B-8A-2B+4A+4A+B=0\\ x^2e^x&: C-6B-2C+12A+3B+3B+C=1\\ xe^x&: -4C+6B+2C+2C=0\\ e^x&: 2C=0\end{aligned}\right\}\begin{aligned}&A=1/12,\ B=0,\ C=0,\\ &\text{so } y(x)=(C_1+C_2x)e^x+\tfrac{1}{12}x^4e^x\end{aligned}$$

(p) $y'''-y'=25\cos 2x$. $y_h=C_1+C_2e^x+C_3e^{-x}$. Try $y_p=A\cos 2x+B\sin 2x$.

$(8A\sin 2x-8B\cos 2x)-(-2A\sin 2x+2B\cos 2x)=25\cos 2x$

$$\left.\begin{aligned}\sin 2x&: 8A+2A=0\\ \cos 2x&: -8B-2B=25\end{aligned}\right\}\begin{aligned}&A=0,\ B=-5/2,\\ &\text{so } y(x)=C_1+C_2e^x+C_3e^{-x}-\tfrac{5}{2}\sin 2x\end{aligned}$$

3. (a) dsolve(diff(y(x),x)-3*y(x)= x*exp(2*x)+6, y(x)); gives

$y(x)=C_1e^{3x}-\frac12 xe^{x}-\frac14 e^x-2$

(q) dsolve(diff(y(x),x,x,x)-diff(y(x),x,x)=6*x+2*cosh(x), y(x)); gives quite a messy expression, and the simplify command doesn't help, so let us proceed instead as follows: Integration of

$y'''-y''=6x+2\cosh x$ gives

$y''-y'=3x^2+2\sinh x+A$. Integrating again gives

$y'-y=x^3+2\cosh x+Ax+B$.

Now use general solution of linear first-order equation:

$$\begin{aligned} y(x)&=e^x\left[\int e^{-x}(x^3+e^x+e^{-x}+Ax+B)\,dx+C\right]\\ &=e^x\left[\int(x^3e^{-x}+1+e^{-2x}+Axe^{-x}+Be^{-x})\,dx+C\right]\\ &=e^{-x}\left[(-x^3-3x^2-6x-6)e^{-x}+x-\tfrac12e^{-2x}+A(-x-1)e^{-x}-Be^{-x}+C\right]\end{aligned}$$

4. (b) $y'-y=xe^x+1$. $y_h=Ae^x$ so seek $y_p=A(x)e^x$. Putting that into the ODE gives $A'e^x+Ae^x-Ae^x=xe^x+1$ or $A'=x+e^{-x}$, $A(x)=\frac{x^2}{2}-e^{-x}+C$

so $y(x)=(\frac{x^2}{2}-e^{-x}+C)e^x=Ce^x+\frac{x^2}{2}e^x-1$

(c) $xy'-y=x^3$. $y_h=Ax$ so seek $y_p=A(x)x$. Putting that into the ODE gives $x(A'x+A)-Ax=x^3$, $A'=x$, $A(x)=x^2/2+C$, so $y(x)=(\frac{x^2}{2}+C)x=Cx+x^3/2$.

(h) $y''-2y'+y=6x^2$. $y_h=(A+Bx)e^x$ so seek $y_p=[A(x)+B(x)x]e^x$. That gives

$$\left.\begin{aligned}A'+xB'&=0\\ A'+(1+x)B'&=6x^2e^{-x}\end{aligned}\right\}\to\begin{aligned}&A'=-6x^3e^{-x},\ A(x)=-6e^{-x}(-x^3-3x^2-6x-6)+C\\ &B'=6x^2e^{-x},\ B(x)=6e^{-x}(-x^2-2x-2)+D,\ \text{so}\end{aligned}$$

$$\begin{aligned}y(x)=(A+Bx)e^x&=6(x^3+3x^2+6x+6)+Ce^x-6x(x^2+2x+2)+Dxe^x\\ &=(C+Dx)e^x+6x^2+24x+36\end{aligned}$$

(n) $x^2y''-xy'-3y=4x$. $y_h=Ax^3+Bx^{-1}$ so seek $y_p=A(x)x^3+B(x)x^{-1}$. Obtain

$$\left.\begin{array}{l} x^3A'+x^{-1}B'=0 \\ 3x^4A'-B'=4x \end{array}\right\} \quad \begin{array}{ll} A'=x^{-3}, & A(x)=-x^{-2}/2+C \\ B'=-x, & B(x)=-x^2/2+D, \text{ so} \end{array}$$

$$y(x)=(-x^{-2}/2+C)x^3+(-x^2/2+D)x^{-1}=Cx^3+Dx^{-1}-x$$

6. At most, $\int^x \frac{W_1(\xi)}{W(\xi)}\,d\xi$ and $\int_{\alpha_1}^x \frac{W_1(\xi)}{W(\xi)}\,d\xi$ differ by a constant, and that constant times $y_1(x)$ does not hurt because $y_1(x)$ is a homogeneous solution. Similarly for the other term.

7. Sure it would work. Rather than consider the general case, let us illustrate the effect of this change by re-working problem 4(n), shown at the top of this page.
$x^2y''-xy'-3y=4x$. $y_h=Ax^3+Bx^{-1}$ so seek $y_p=A(x)x^3+B(x)x^{-1}$. Then

$$y_p'=\underbrace{A'x^3+B'x^{-1}}_{\text{Instead of setting this}=0\text{, set it}=6\text{, say.}}+3x^2A-x^{-2}B \quad =6+3x^2A-x^{-2}B$$

$y_p''=3x^2A'+6xA-x^{-2}B'+2x^{-3}B$ and putting these in the ODE gives

$$(3x^4A'+\cancel{6x^3A}-B'+\cancel{2x^{-1}B})-(6x+\cancel{3x^3A}-\cancel{x^{-1}B})-(\cancel{3x^3A}+\cancel{3x^{-1}B})=4x$$

$$\text{so}\ \left.\begin{array}{l} x^3A'+x^{-1}B'=6 \\ 3x^4A'-B'=10x \end{array}\right\} \quad \begin{array}{ll} A'=4x^{-3}, & A(x)=-2x^{-2}+C \\ B'=2x, & B(x)=x^2+D, \text{ so} \end{array}$$

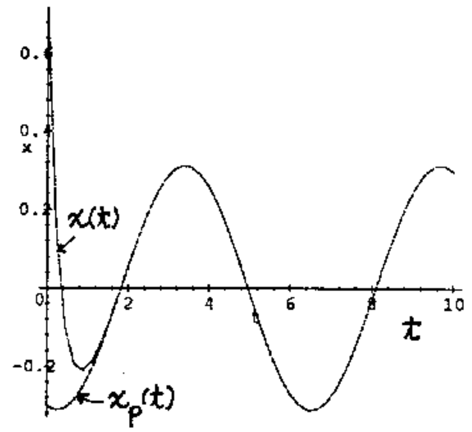
$$y(x)=\left(-\frac{2}{x^2}+C\right)x^3+(x^2+D)\frac{1}{x}=-2x+Cx^3+x+Dx^{-1}=Cx^3+Dx^{-1}-x, \text{ as before.}$$

Section 3.8

6. (a) $mx''+cx'+kx=F_0\cos\Omega t$. Arbitrarily, let $m=1$, $k=32$, $c=c_{cr}=\sqrt{2mk}=8$, $\Omega=1$, $F_0=10$. Then $\omega=\sqrt{k/m}=\sqrt{32}$ and (16) and (19) give the solution as

$$x(t)=e^{-4t}(A+Bt)+\frac{10}{\sqrt{(32-1)^2+8^2}}\cos\Big(t+\underbrace{\tan^{-1}\frac{8}{1-32}}_{\text{the }\tan^{-1}\text{ in the interval }(0,\pi)}\Big)$$

$$=e^{-4t}(A+Bt)+0.3123\cos(t+2.889).$$



Rather than set $x(0)$ and $x'(0)$ and solve for A,B, it is more convenient to do the reverse: set $A=1$, $B=0.5$, say. Then $x(0)=0.6976$ and $x'(0)=-3.578$. To plot, use these maple commands and obtain the plot shown above:

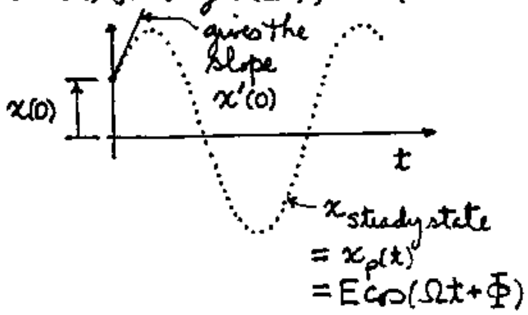
```
> with(plots):
> implicitplot({x=(1+0.5*t)*exp(-4*t)+0.3123*cos(t+2.889),x=0.3123*c
  os(t+2.889)},t=0..10,x=-2..2,numpoints=2000);
```

For the underdamped case shown in Fig. 7 the approach to steady state (shown there as dotted) is oscillatory, but for the critically damped case it is not.

7. $$\lim_{\Omega\to\omega} x(t)=-\frac{F_0}{m}\lim_{\Omega\to\omega}\frac{\cos\omega t-\cos\Omega t}{\omega^2-\Omega^2}\underset{\text{l'Hôpital}}{=}-\frac{F_0}{m}\lim_{\Omega\to\omega}\frac{t\sin\Omega t}{-2\Omega}=\frac{F_0t}{2m\omega}\sin\omega t$$

9. (a) $x(t) = x_h(t) + E\cos(\Omega t + \Phi)$. We can have $x_h(t) \equiv 0$ by imposing on $x_h(t)$ the initial conditions $x_h(0) = 0$, $x_h'(0) = 0$; these will give $A = B = 0$ in (16). These initial conditions on $x_h(t)$ imply conditions on $x(t)$ through (20), as follows:

$$x(0) = \cancel{x_h(0)} + E\cos\Phi = E\cos\Phi$$
$$x'(0) = \cancel{x_h'(0)} - \Omega E\sin\Phi = -\Omega E\sin\Phi.$$

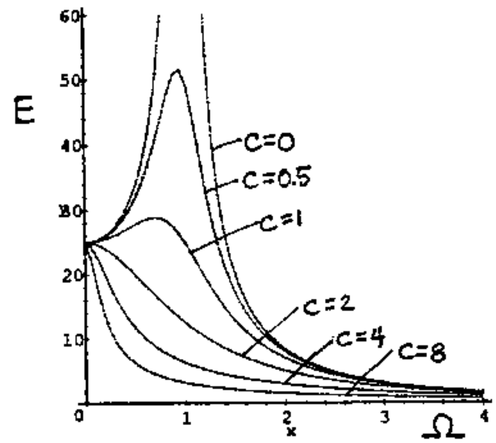


(b) That is, if the steady-state solution is shown as dotted (at the right), then the initial conditions $x(0), x'(0)$ simply start us out along that curve.

10. $mx'' = -k(x - \delta)$ so $mx'' + kx = k\delta_0\cos\Omega t$

11. (a) $m = k = 1$ so $\omega = \sqrt{k/m} = 1$, $F_0 = 25$, $c_{cr} = 2\sqrt{mk} = 2$ so $c = 0, 0.5, 1, 2, 4, 8$.

```
> with(plots):
> implicitplot({y=25/sqrt((1-x^2)^2+0*x^2),y=25/sqrt((1-x^2)^2+0.25*
  x^2),y=25/sqrt((1-x^2)^2+1*x^2),y=25/sqrt((1-x^2)^2+4*x^2),y=25/sq
  rt((1-x^2)^2+16*x^2),y=25/sqrt((1-x^2)^2+64*x^2)},x=0..4,y=0..60,n
  umpoints=4000);
```



12. (a) We want to solve $L[x] = F_0\cos\Omega t$. Consider instead $L[w] = F_0 e^{i\Omega t}$.

Then $L[\operatorname{Re} w + i \operatorname{Im} w] = F_0\cos\Omega t + iF_0\sin\Omega t$

$L[\operatorname{Re} w] + i L[\operatorname{Im} w] =$ " " (by the linearity of L)

Equating real and imaginary parts,

$$L[\operatorname{Re} w] = F_0\cos\Omega t, \quad L[\operatorname{Im} w] = F_0\sin\Omega t$$

so $x(t) = \operatorname{Re} w(t)$.

(d) $w' + 3w = 5e^{i2t}$. $w_p = Ae^{i2t}$, $i2Ae^{i2t} + 3Ae^{i2t} = 5e^{i2t}$ gives $A = 5/(3+2i)$, so $x(t) = \operatorname{Re}\left(\frac{5}{3+2i}e^{i2t}\right) = 5\operatorname{Re}\frac{3-2i}{(3+2i)(3-2i)}(\cos 2t + i\sin 2t)$

$$= \frac{5}{13}(3\cos 2t + 2\sin 2t).$$

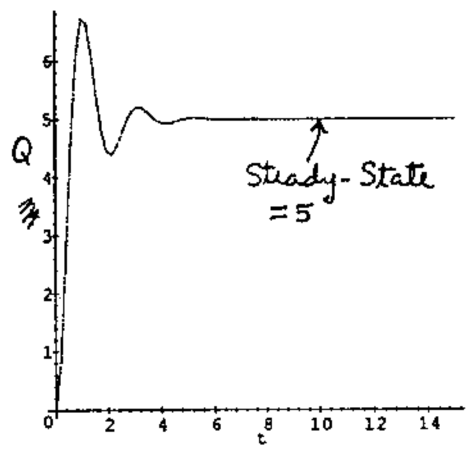
(g) $w'' + 5w' + w = 3e^{i4t}$. $w_p = Ae^{i4t}$, $(-16 + 20i + 1)Ae^{i4t} = 3e^{i4t}$ gives $A = 3/(-15+20i)$, so $x(t) = \operatorname{Im}\left(\frac{3}{-15+20i}\frac{-15-20i}{-15-20i}(\cos 4t + i\sin 4t)\right)$

$$= (-60\cos 4t - 45\sin 4t)/125 = -(12\cos 4t + 9\sin 4t)/25$$

13.(a) $2Q''+4Q'+20Q=100$, $Q(0)=Q'(0)=0$

Obtain $Q(t)=5-e^{-t}(5\cos 3t+\frac{5}{3}\sin 3t)$ by maple dsolve command or by hand. Steady-state solution is $Q(t)\to 5$

```
> with(plots):
> implicitplot(x=5-exp(-t)*(5*cos(3*t)
  +(5/3)*sin(3*t)),t=0..15,x=0..
   10,numpoints=6000);
```



(e) $2Q''+4Q'+20Q=10(1-e^{-t})$, $Q(0)=Q'(0)=0$

The maple dsolve solution is very messy, so let's solve by hand.

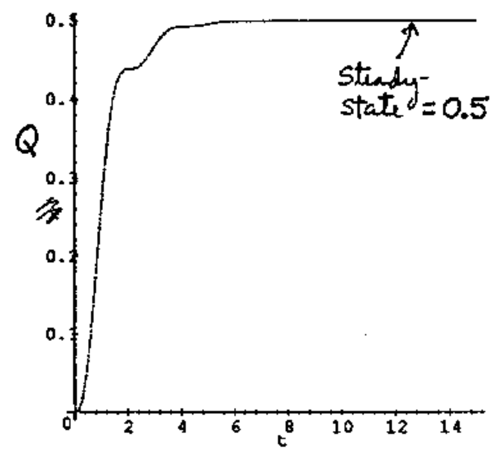
$Q_h=e^{-t}(C_1\cos 3t+C_2\sin 3t)$ and seek $Q_p=A+Be^{-t}$. This gives $A=1/2, B=-5/9$ so $Q(t)=\frac{1}{2}-\frac{5}{9}e^{-t}+e^{-t}(C_1\cos 3t+C_2\sin 3t)$

$Q(0)=0$ and $Q'(0)=0$ give $C_1=1/18$ and $C_2=-1/6$, so

$Q(t)=\frac{1}{2}-\frac{5}{9}e^{-t}+e^{-t}\left(\frac{\cos 3t}{18}-\frac{\sin 3t}{6}\right)$

Steady-state solution is $Q(t)\to 1/2$

```
> implicitplot(x=(1/2)-(5/9)*exp(-t)
  +exp(-t)*((1/18)*cos(3*t)-(1/6)*
    sin(3*t)),t=0..15,
   x=0..10,numpoints=9000);
```



Section 3.9

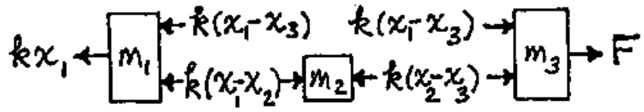
We call your attention especially to Example 8, on the free vibration of a two-mass system. We return to that problem in Section 11.3 and study it there in terms of the matrix eigenvalue problem. It is an important problem, and you may wish to give it added emphasis by discussing it in class, both for Section 3.9 and Section 11.3, and even comparing the two lines of approach to the solution.

3. Assuming $x_1>x_2>x_3>0$:

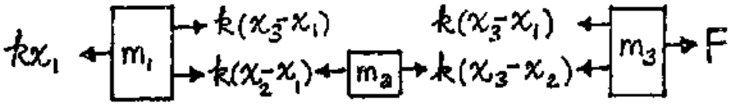
$m_1x_1''=-kx_1-k(x_1-x_3)-k(x_1-x_2)$

$m_2x_2''=k(x_1-x_2)-k(x_2-x_3)$

$m_3x_3''=k(x_1-x_3)+k(x_2-x_3)$

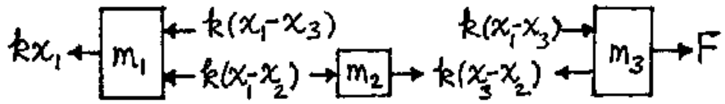


Assuming $x_3>x_2>x_1>0$:



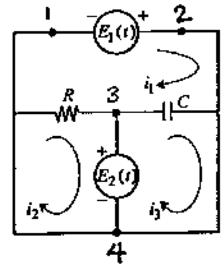
$$m_1x_1''=-kx_1+k(x_2-x_1)+k(x_3-x_1)$$
$$m_2x_2''=-k(x_2-x_1)+k(x_3-x_2)$$
$$m_3x_3''=-k(x_3-x_1)-k(x_3-x_2)$$

Assuming $x_1>x_3>x_2>0$:



$$m_1x_1''=-kx_1-k(x_1-x_3)-k(x_1-x_2)$$
$$m_2x_2''=k(x_1-x_2)+k(x_3-x_2)$$
$$m_3x_3''=-k(x_3-x_2)+k(x_1-x_3)+F$$

The three sets of equations are seen to be identical. Similarly for *any* such assumption, such as $x_3<x_1<0$ and $x_2>0$, and so on.



4.(a) Merely for definiteness, suppose $i_1>i_2>i_3>0$. Then

Kirchoff voltage law (loop 1): $E_2-E_1+E_3-E_2+E_1-E_3=0$ or (see p.35)

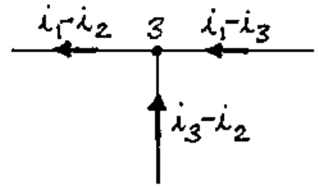
$$E_1(t)-\frac{1}{C}\int(i_1-i_3)dt-R(i_1-i_2)=0 \quad ①$$

Kirch. volt. law (loop 2): $E_4-E_3+E_3-E_4=0$ or

$$-E_2(t)-R(i_2-i_1)=0 \quad ②$$

Kirch. volt. law (loop 3): $E_3-E_4+E_4-E_3=0$ or

$$E_2(t)-\frac{1}{C}\int(i_3-i_1)dt=0 \quad ③$$



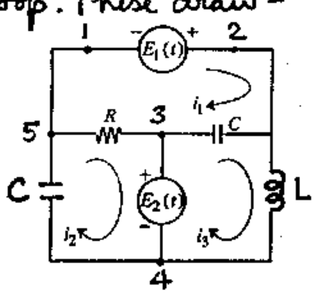
Kirch. current law (junction at 3): By choosing to work with loop currents we do not need to invoke Kirchoff's current law. For ex., suppose we invoke that law at point 3. Then, from the diagram at the right, we have

$$(i_1-i_3)+(i_3-i_2)=(i_1-i_2),$$

but the latter is merely an identity and is therefore automatically satisfied. Taking d/dt of the loop 1,3 equation, to eliminate the integral signs gives this system:

$$\frac{1}{C}(i_1-i_3)+R(i_1'-i_2')=E_1'(t) \quad ④$$
$$R(i_1-i_2)=E_2(t) \quad ⑤$$
$$\frac{1}{C}(i_3-i_1)=E_2'(t) \quad ⑥$$

NOTE: This problem does illustrate the application of Kirchoff's laws but it contains two drawbacks. First, two of the three resulting equations ④-⑥ are algebraic rather than differential. Second, rather than $E_1(t)$ and $E_2(t)$ being arbitrary functions, as I intended, summing equations ①-③ reveals that $E_1(t)$ is necessarily 0, as can also be seen by applying Kirchoff's voltage law to the outer loop. These drawbacks disappear if we include one or more additional elements in the outer loop, for instance as shown at the right. →



In that case, application of Kirchoff's voltage law gives:

loop 1: $E_1-\frac{1}{C}\int(i_1-i_3)dt-R(i_1-i_2)=0$

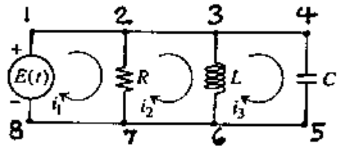
loop 2: $-E_2-\frac{1}{C}\int i_2dt-R(i_2-i_1)=0$

loop 3: $E_2-\frac{1}{C}\int(i_3-i_1)dt-Li_3'=0$

or, taking d/dt of these,
$$\frac{1}{C}(i_1-i_3)+R(i_1'-i_2') = E_1'(t)$$
$$-\frac{1}{C}i_2 - R(i_2'-i_1') = E_2'(t)$$
$$\frac{1}{C}(i_3-i_1) + Li_3'' = E_2'(t).$$

(b) $\left.\begin{aligned} -i_1R-\frac{1}{C}\int i_1\,dt - E &= 0 \\ E - \frac{1}{C}\int i_2\,dt - L\frac{d}{dt}i_2 &= 0 \end{aligned}\right\}$ or, $\begin{aligned} Ri_1' + \frac{1}{C}i_1 &= -E'(t) \\ Li_2'' + \frac{1}{C}i_2 &= E'(t) \end{aligned}$

Observe that these equations are not coupled.

(c) 

Let us apply Kirchoff's voltage law to 3 loops. We could use loops 12781, 13681, 14581 or loops 12781, 23672, 34563. Let us use the former:

12781: $-R(i_1-i_2)+E=0$ $R(i_1-i_2)=E(t)$ ①

13681: $-L\frac{d}{dt}(i_2-i_3)+E=0$ or, $L(i_2'-i_3')=E(t)$ ②

14581: $-\frac{1}{C}\int i_3\,dt + E = 0$ $\frac{1}{C}i_3 = E'(t)$ ③

Equations ①-③ are coupled, but the coupling is trivial; we can solve ③ for i_3, then put that i_3 into ② and solve ② for i_2, then put that i_2 into ① and solve ① for i_1.

5. (b) $(D-1)x+2Dy=0$ ①

$(D+1)x+4Dy=0$ ②

The simplest way to solve by elimination is to subtract twice the first equation from the second, giving $(-D+3)x=0$. Thus, $x=Ae^{3t}$. Putting this in first equation then gives $y'=-\frac{1}{2}(x'-x)=-\frac{1}{2}(3Ae^{3t}-Ae^{3t})=-Ae^{3t}$, $y=-\frac{A}{3}e^{3t}+B$. Or, by Cramer's rule,

$$x=\frac{\begin{vmatrix}0 & 2D\\ 0 & 4D\end{vmatrix}}{\begin{vmatrix}D-1 & 2D\\ D+1 & 4D\end{vmatrix}} = \frac{0}{2D^2-6D},$$ to be understood as $(2D^2-6D)x=0$, so $x=A+Be^{3t}$ ③

$$y=\frac{\begin{vmatrix}D-1 & 0\\ D+1 & 0\end{vmatrix}}{\begin{vmatrix}D-1 & 2D\\ D+1 & 4D\end{vmatrix}} = \frac{0}{2D^2-6D},$$ to be understood as $(2D^2-6D)y=0$, so $y=C+Ee^{3t}$. ④

A, B, C, E are not independent constants. To determine how they are related, put ③ and ④ into ① (the same result is obtained if we put them into ②):
$$(D-1)(A+Be^{3t})+2D(C+Ee^{3t})=0$$
or
$$3Be^{3t}-A-Be^{3t}+6Ee^{3t}=0 \quad \text{or} \quad -A+(2B+6E)e^{3t}=0.$$
Since 1 and e^{3t} are linearly independent, we must have $-A=0$ and $2B+6E=0$ or $A=0$ and $E=-\frac{1}{3}B$, with C remaining arbitrary, so ③ and ④ become
$$x(t)=Be^{3t}, \quad y(t)=C-\frac{B}{3}e^{3t}$$
(B, C arbitrary constants), which is the same result as obtained above.

(c) $\left.\begin{aligned} Dx+(D-1)y &= 5 \\ 2(D+1)x+(D+1)y &= 0\end{aligned}\right\}$ — elimination gives → $[2(D+1)(D-1)-D(D+1)]y = 2(D+1)(5)-(D)(0)$ and $[(D+1)D-2(D-1)(D+1)]x=(D+1)(5)-(D-1)(0)$

or, $(D^2-D-2)x=-5$

$(D^2-D-2)y=10$

Solving these (uncoupled) equations gives $x(t)=\frac{5}{2}+Ae^{-t}+Be^{2t}$

$y(t)=-5+Ce^{-t}+Ee^{2t}$

To determine any relations among A, B, C, E put these solutions into either of the original ODE's, say the first: $Dx+(D-1)y=5$ becomes

$$(-Ae^{-t}+2Be^{2t})+(-Ce^{-t}+2Ee^{2t})-(-5+Ce^{-t}+Ee^{2t})=5$$

or, $(-A-2C)e^{-t}+(2B+2E-E)e^{2t}=0$ so $A=-2C$ and $E=-2B$.

Thus, $x(t)=\frac{5}{2}-2Ce^{-t}+Be^{2t}$

$y(t)=-5+Ce^{-t}-2Be^{2t}$

(e) $\left.\begin{array}{l} Dx+y=\sin t \\ 9x+Dy=4 \end{array}\right\}$ Elimination gives $\begin{array}{l}(D^2-9)x=D(\sin t)-4 = \cos t-4 \\ (D^2-9)y=-9\sin t+D(0) = -9\sin t\end{array}$

with solutions $x(t)=Ae^{3t}+Be^{-3t}-\frac{1}{10}\cos t+\frac{4}{9}$

$y(t)=Ce^{3t}+Ee^{-3t}+\frac{9}{10}\sin t$

To determine any relations among A, B, C, E, put these solutions into either of the original ODE's, say the first: $Dx+y=\sin t$ becomes

$$(3Ae^{3t}-3Be^{-3t}+\tfrac{1}{10}\cancel{\sin t})+(Ce^{3t}+Ee^{-3t}+\tfrac{9}{10}\cancel{\sin t})=\cancel{\sin t}$$

so $3A+C=0$ and $-3B+E=0$ or, $C=-3A$ and $E=3B$. Thus,

$x(t)=Ae^{3t}+Be^{-3t}-\frac{1}{10}\cos t+\frac{4}{9}$

$y(t)=-3Ae^{3t}+3Be^{-3t}+\frac{9}{10}\sin t$

(f) $x(t)=-\frac{8}{3}t^2-\frac{16}{27}-4Ae^{3t}+2Be^{-3t}$

$y(t)=\frac{2}{3}t-\frac{2}{27}-\frac{1}{3}t^2+Ae^{3t}+Be^{-3t}$

(h) $x(t)=Ae^{9t}+4Be^{-t}$, $y(t)=-Ae^{9t}+Be^{-t}$

(i) $x(t)=\frac{52}{49}-\frac{4}{7}t-\frac{4}{3}Ae^{7t}+4Be^{-t}$, $y(t)=\frac{10}{49}-\frac{1}{14}t+Ae^{7t}+Be^{-t}$

(l) $x(t)=A\sin\sqrt{3}t+B\cos\sqrt{3}t+2C+2Et$, $y(t)=2A\sin\sqrt{3}t+2B\cos\sqrt{3}t+C+Et$

(m) $x(t)=\frac{1}{18}t^4-\frac{5}{9}t^2-\frac{8}{27}+A\sin\sqrt{3}t+B\cos\sqrt{3}t+2C+2Et$,

$y(t)=-\frac{11}{18}t^2+\frac{1}{36}t^4+\frac{11}{27}+2A\sin\sqrt{3}t+2B\cos\sqrt{3}t+C+Et$

6.(g) $(2D^2+3)x+(2D+1)y=4e^{3t}-7$

$Dx+(D-2)y=2$

```
deq1:= 2*diff(x(t),t,t)+3*x(t)+2*diff(y(t),t)+y(t)=4*exp(3*t)-7:
deq2:= diff(x(t),t)+diff(y(t),t)-2*y(t)=2:
dsolve({deq1, deq2}, {x(t), y(t)});
```

gives $X(t)=-2+\frac{1}{5}te^{3t}-\frac{3}{25}e^{3t}+Ae^{3t}+(-B+2C)\sin t+(-2B-C)\cos t$

$y(t)=-1-\frac{3}{5}te^{3t}+\frac{9}{25}e^{3t}+(\frac{2}{5}-3A)e^{3t}+B\sin t+C\cos t.$

7. $x_1(t)=G\sin(t+\phi)+H\sin(\sqrt{3}t+\psi)$

$x_2(t)=G\sin(t+\phi)-H\sin(\sqrt{3}t+\psi)$

(a) $x_1(0)=1=G\sin\phi+H\sin\psi$ ①

$x_2(0)=1=G\sin\phi-H\sin\psi$ ②

$x_1'(0)=0=G\cos\phi+\sqrt{3}H\cos\psi$ ③

$x_2'(0)=0=G\cos\phi-\sqrt{3}H\cos\psi$ ④

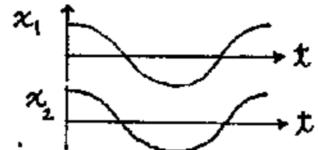
Observe that ①-④ are NOT linear algebraic equations in G, H, ϕ, ψ; they are nonlinear. Nevertheless they are solved easily, as follows:

$$\begin{aligned}
&\text{eqn ①} + \text{eqn ②} \to G\sin\phi = 1 \\
&\text{eqn ①} - \text{eqn ②} \to H\sin\psi = 0 \\
&\text{eqn ②} + \text{eqn ③} \to G\cos\phi = 0 \\
&\text{eqn ②} - \text{eqn ③} \to \sqrt{3}\,H\cos\psi = 0
\end{aligned}
\quad \to \quad
\begin{aligned}
&\phi = \pi/2,\ H = 1 \\
&H = 0,\ \psi \text{ is therefore irrelevant}
\end{aligned}$$

Thus, $x_1(t) = \sin(t + \pi/2) = \cos t$ (Recall that $\sin(A+B) = \sin A\cos B + \sin B\cos A$)

$x_2(t) = \sin(t + \pi/2) = \cos t$,

as given by (38). Here, the two masses swing in unison at the low frequency 1, as sketched:



"Low mode" motion

Next, consider the second set of initial conditions:

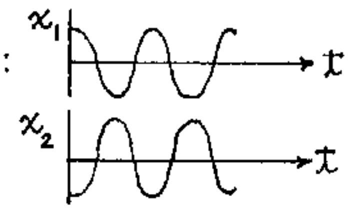
$$\left.\begin{aligned}
x_1(0) &= 1 = G\sin\phi + H\sin\psi \\
x_2(0) &= -1 = G\sin\phi - H\sin\psi \\
x_1'(0) &= 0 = G\cos\phi + \sqrt{3}\,H\cos\psi \\
x_2'(0) &= 0 = G\cos\phi - \sqrt{3}\,H\cos\psi
\end{aligned}\right]$$

Solving these as above gives $G = 0$, ϕ irrelevant, $H = 1$, $\psi = \pi/2$, so

$x_1(t) = \sin(\sqrt{3}t + \pi/2) = \cos\sqrt{3}t$

$x_2(t) = -\sin(\sqrt{3}t + \pi/2) = -\cos\sqrt{3}t$,

as given by (39). Here, the two masses swing in opposition at the high frequency $\sqrt{3}$, as sketched:



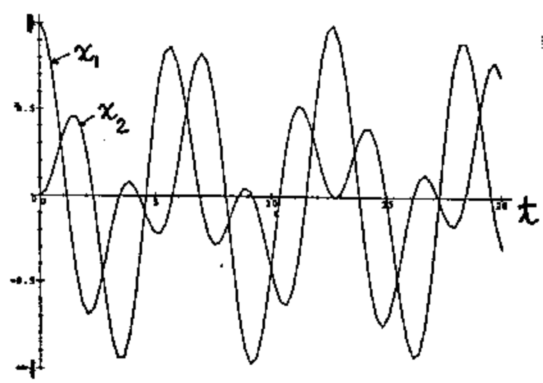
"High mode" motion

(b) This time the initial conditions will excite both modes.

$$\left.\begin{aligned}
x_1(0) &= 1 = G\sin\phi + H\sin\psi \\
x_2(0) &= 0 = G\sin\phi - H\sin\psi \\
x_1'(0) &= 0 = G\cos\phi + \sqrt{3}\,H\cos\psi \\
x_2'(0) &= 0 = G\cos\phi - \sqrt{3}\,H\cos\psi
\end{aligned}\right\} \text{manipulating these as above gives} \longrightarrow
\begin{aligned}
G\sin\phi &= 1/2 \\
H\sin\psi &= 1/2 \\
G\cos\phi &= 0 \\
\sqrt{3}\,H\cos\psi &= 0
\end{aligned}
\quad \text{so } \phi = \psi = \pi/2,\ G = H = 1/2$$

so $x_1(t) = \frac{1}{2}\sin(t + \pi/2) + \frac{1}{2}\sin(\sqrt{3}t + \pi/2) = \frac{1}{2}\cos t + \frac{1}{2}\cos\sqrt{3}t$

$x_2(t) = \frac{1}{2}\sin(t + \pi/2) - \frac{1}{2}\sin(\sqrt{3}t + \pi/2) = \frac{1}{2}\cos t - \frac{1}{2}\cos\sqrt{3}t$



Plot obtained using the Maple command

```
with(plots):
implicitplot({x=0.5*cos(t)
+0.5*cos(sqrt(3)*t), x=
0.5*cos(t)-0.5*cos(sqrt(3)*t)},
t=0..20, x=-2..2,
numpoints=6000);
```

8. (a) $x'+\alpha x=0$ gives $x(t)=Ae^{-\alpha t}$. Then $x+y+z=\gamma$ gives $y=\gamma-x-z=\gamma-Ae^{-\alpha t}-z$ and putting this into $z'=\beta y$ gives the ODE $z'+\beta z=\beta\gamma-\beta Ae^{-\alpha t}$ on z. $z_h=Be^{-\beta t}$, and seeking $z_p=P+Qe^{-\alpha t}$ gives $P=\gamma$ and $Q=-\beta A/(\beta-\alpha)$ so $z(t)=Be^{-\beta t}+\gamma-\frac{\beta A}{\beta-\alpha}e^{-\alpha t}$. Then,

$$\left.\begin{aligned} z(0)&=0=B+\gamma-\beta A/(\beta-\alpha)\\ z'(0)&=0=-\beta B+\alpha\beta A/(\beta-\alpha)\end{aligned}\right\}\text{ gives } A=\gamma,\ B=\alpha\gamma/(\beta-\alpha)$$

so
$$z(t)=\gamma\left[\frac{\alpha e^{-\beta t}-\beta e^{-\alpha t}}{\beta-\alpha}+1\right]$$
$$y(t)=\gamma\left[\frac{\beta e^{-\alpha t}-\alpha e^{-\beta t}}{\beta-\alpha}-e^{-\alpha t}\right]$$
$$x(t)=\gamma e^{-\alpha t}$$

(b) If $\beta=\alpha$ the above expressions for $z(t)$ and $y(t)$ are indeterminate, namely, 0/0. L'Hôpital's rule (as $\beta\to\alpha$) gives $z(t)=\gamma[-\alpha t e^{-\alpha t}-e^{-\alpha t}+1]$

$$y(t)=\gamma[e^{-\alpha t}+\alpha t e^{-\alpha t}-e^{-\alpha t}].$$

Or, of course, we could set $\beta=\alpha$ and re-solve:

$x'+\alpha x=0$ gives $x(t)=Ae^{-\alpha t}$

$z'=\alpha y$

$x+y+z=\gamma$ gives $y=\gamma-Ae^{-\alpha t}-z$

so $z'=\alpha y$ becomes $z'=\alpha(\gamma-Ae^{-\alpha t}-z)$

$$z'+\alpha z=\alpha\gamma-\alpha Ae^{-\alpha t}$$

$z_h=Be^{-\alpha t}$ and this time seek $z_p=P+Qte^{-\alpha t}$. Putting this into ODE gives

$$Qe^{-\alpha t}-\cancel{\alpha Qte^{-\alpha t}}+\alpha P+\cancel{\alpha Qte^{-\alpha t}}=\gamma\alpha-\alpha Ae^{-\alpha t}$$

$e^{-\alpha t}$ terms: $Q=-\alpha A$

Constant terms: $\alpha P=\gamma\alpha$ gives $P=\gamma$

Thus, $z(t)=Be^{-\alpha t}+\gamma-\alpha Ate^{-\alpha t}$

$$\left.\begin{aligned} z(0)&=0=B+\gamma\\ z'(0)&=0=-\alpha B-\alpha A\end{aligned}\right\}\to B=-\gamma,\ A=\gamma,\text{ so } z(t)=-\gamma e^{-\alpha t}+\gamma-\alpha\gamma te^{-\alpha t}$$

$$x(t)=\gamma e^{-\alpha t}$$
$$y(t)=\gamma-x(t)-z(t)=\text{etc.},$$

which agrees with the solution obtained using l'Hôpital's rule.

9. (a) Let $qB/m\equiv\alpha$. Then
$$x''-\alpha y'=0$$
$$\alpha x'+y''=0$$
$$z''=0$$

We might as well integrate these equations once, once, twice, respectively, before proceeding:
$$Dx-\alpha y=E \quad ①$$
$$\alpha x+Dy=G \quad ②$$
$$z=H+It \quad ③$$

Using elimination on the first two of these gives these uncoupled equations

$x''+\alpha^2x=\alpha G$ so $x(t)=J\sin\alpha t+K\cos\alpha t+G/\alpha$ ④

$y''+\alpha^2y=-\alpha E$ so $y(t)=M\sin\alpha t+N\cos\alpha t-E/\alpha$ ⑤

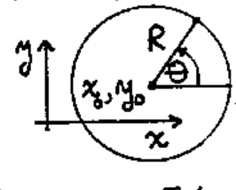
To determine any relations among the integration constants put ④ and ⑤ into

① or ②, say ①: that step gives $\alpha J\cos\alpha t - \alpha K\sin\alpha t - \alpha M\sin\alpha t - \alpha N\cos\alpha t + \cancel{E} = \cancel{E}$
so $N = J$ and $M = -K$. Thus, the general solution is

$$x(t) = J\sin\alpha t + K\cos\alpha t + G/\alpha \qquad ⑥$$
$$y(t) = -K\sin\alpha t + J\cos\alpha t - E/\alpha \qquad ⑦$$
$$z(t) = H + It, \qquad ⑧$$

with the arbitrary integration constants J, K, G, E, H, I. (6 arbitrary independent constants)

(b) For the circle shown at the right, $x = x_0 + R\cos\theta$
$y = y_0 + R\sin\theta$



Comparing these equations with ⑥ and ⑦ we see that we need to have initial conditions such that $J=0$, $G/\alpha = x_0$, $E/\alpha = -y_0$, and $K=R$. That will cause ⑥ and ⑦ to be

$$x(t) = x_0 + R\cos\alpha t = x_0 + R\cos(-\alpha t) \qquad ⑨$$
$$y(t) = y_0 - R\sin\alpha t = y_0 + R\sin(-\alpha t) \qquad ⑩$$

That is, $\theta = -\alpha t$ so the motion is <u>clockwise</u> with angular velocity α. Initial conditions that will result in that motion can be found directly from ⑨ and ⑩: $x(0) = x_0 + R$, $x'(0) = 0$, $y(0) = y_0$, $y'(0) = -\alpha R$.

(c) If $z'(0) \neq 0$ then $z(t) = H + It$ where $I \neq 0$ and in that case the circular x,y motion plus the linear z motion will produce a <u>helix</u>.

NOTE: We showed in (b) that ⑥ and ⑦ <u>can</u> give a circular motion. In fact, the x,y motion is <u>necessarily</u> circular motion at constant angular velocity, for, recalling eqns. (7)–(10) in Section 3.5, ⑥ and ⑦ give

$$x = x_0 + \sqrt{J^2+K^2}\sin(\alpha t + \phi) \quad , \text{ where } \phi = \tan^{-1}(K/J)$$
$$y = y_0 + \sqrt{J^2+K^2}\sin(\alpha t + \psi) \quad , \text{ where } \psi = \tan^{-1}(-J/K)$$

Thus $\tan\phi = K/J$ and $\tan\psi = -J/K$. Hence ψ and ϕ are 90° apart (since the slope $-J/K$ is the negative reciprocal of the slope K/J). If $\psi = \phi + 90^\circ$ then

$$x = x_0 + R\sin(\alpha t + \phi)$$
$$y = y_0 + R\sin(\alpha t + \phi + \pi/2) = y_0 + R\cos(\alpha t + \phi)$$

and if $\psi = \phi - 90^\circ$ then

$$x = x_0 + R\sin(\alpha t + \phi)$$
$$y = y_0 + R\sin(\alpha t + \phi - \pi/2) = y_0 - R\cos(\alpha t + \phi).$$

Either way, the trajectory is a circle, traversed clockwise or counterclockwise) at constant angular velocity $\alpha = qB/m$.