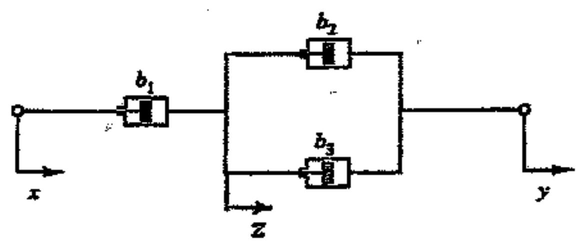
CHAPTER 3

B-3-1. Since the same force transmits the shaft, we have

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) + b_3(\dot{y} - \dot{z}) \tag{1}$$

where displacement z is defined in the figure below.



In terms of the equivalent viscous friction coefficient, the force f is given by

$$f = b_{eq}(\dot{y} - \dot{x}) \tag{2}$$

From Equation (1) we have

$$b_1\dot{z} + b_2\dot{z} + b_3\dot{z} = b_1\dot{x} + b_2\dot{y} + b_3\dot{y}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2 + b_3}\left[b_1\dot{x} + (b_2 + b_3)\dot{y}\right] \tag{3}$$

By substituting Equation (3) into Equation (1), we have

$$f = b_1(\dot{z} - \dot{x}) = b_1\left\{\frac{1}{b_1 + b_2 + b_3}\left[b_1\dot{x} + (b_2 + b_3)\dot{y}\right] - \dot{x}\right\}$$

$$= b_1\frac{b_2 + b_3}{b_1 + b_2 + b_3}(\dot{y} - \dot{x}) \tag{4}$$

Hence, by comparing Equations (2) and (4), we obtain

$$b_{eq} = b_1\frac{b_2 + b_3}{b_1 + b_2 + b_3} = \frac{1}{\frac{1}{b_2 + b_3} + \frac{1}{b_1}}$$

B-3-2.

(a)

$$m\ddot{x} + kx = u$$

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + k}$$

(b) Define the displacement of a point between springs k_1 and k_2 as y. Then the equations of motion for this system become

$$m\ddot{x} + k_2(x-y) = u$$
$$k_1 y = k_2(x-y)$$

From the second equation, we have

$$k_1 y + k_2 y = k_2 x$$

or

$$y = \frac{k_2}{k_1+k_2}x$$

Then

$$m\ddot{x} + \frac{k_1 k_2}{k_1+k_2}x = u$$

or

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + \frac{k_1 k_2}{k_1+k_2}}$$

B-3-3.

$$m_1\ddot{y}_1 + b_1(\dot{y}_1 - \dot{y}_2) + k_1 y_1 = u_1$$
$$m_2\ddot{y}_2 + b_1(\dot{y}_2 - \dot{y}_1) + k_2 y_2 = u_2$$

Define

$$x_1 = y_1, \quad x_2 = \dot{y}_1, \quad x_3 = y_2, \quad x_4 = \dot{y}_2$$

Then

$$m_1\dot{x}_2 + b_1(x_2 - x_4) + k_1 x_1 = u_1$$
$$m_2\dot{x}_4 + b_1(x_4 - x_2) + k_2 x_3 = u_2$$

Hence

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{1}{m_1}\left[b_1(x_2 - x_4) + k_1 x_1\right] + \frac{1}{m_1}u_1$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -\frac{1}{m_2}\left[b_1(x_4 - x_2) + k_2 x_3\right] + \frac{1}{m_2}u_2$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b_1}{m_1} & 0 & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b_1}{m_2} & -\frac{k_2}{m_2} & -\frac{b_1}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

B-3-4.

$$J\ddot{\theta} = T$$

where

$$T = -2k\theta a^2 - mgl\sin\theta$$

$$J = ml^2$$

For small θ,

$$ml^2\ddot{\theta} = -2ka^2\theta - mgl\theta$$

or

$$\ddot{\theta} + \left(\frac{2ka^2}{ml^2} + \frac{g}{l}\right)\theta = 0$$

B-3-5. Note that

$$x_G = x + l\sin\theta, \quad y_G = l\cos\theta$$

For x direction:

$$M\ddot{x} + m\ddot{x}_G = u$$

or

$$M\ddot{x} + m\frac{d^2}{dt^2}(x + l\sin\theta) = u$$

Since

$$\frac{d^2}{dt^2}\sin\theta = (-\sin\theta)\dot{\theta}^2 + (\cos\theta)\ddot{\theta}$$

we have

$$(M+m)\ddot{x} - ml(\sin\theta)\dot{\theta}^2 + ml(\cos\theta)\ddot{\theta} = u$$

For small θ and small $\dot{\theta}^2$, we have

$$(M+m)\ddot{x} + ml\ddot{\theta} = u \tag{1}$$

<u>For rotational motion:</u>

$$J\ddot{\theta} = mgl\sin\theta - m\ddot{x}l\cos\theta$$

where

$$J = I + ml^2 \,, \qquad I = m\frac{l^2}{3}$$

Thus,

$$(I+ml^2)\ddot{\theta} = mgl\sin\theta - m\ddot{x}l\cos\theta$$

For small θ, we have

$$(I+ml^2)\ddot{\theta} = mgl\theta - ml\ddot{x} \tag{2}$$

From Equation (2),

$$\ddot{x} = g\theta - \frac{I+ml^2}{ml}\ddot{\theta}$$

Substituting this last equation into Equation (1), we obtain

$$(M+m)\left(g\theta - \frac{I+ml^2}{ml}\ddot{\theta}\right) + ml\ddot{\theta} = u$$

or

$$(M+m)g\theta - \frac{(M+m)I+Mml^2}{ml}\ddot{\theta} = u$$

Thus,

$$\ddot{\theta} = \frac{ml(M+m)g}{(M+m)I+Mml^2}\theta - \frac{ml}{(M+m)I+Mml^2}u \tag{3}$$

Also, from Equation (1) we have

$$(M+m)\ddot{x} + ml\frac{mgl\theta - ml\ddot{x}}{I+ml^2} = u$$

or

$$[MI + m(I+Ml^2)]\ddot{x} + m^2l^2g\theta = u(I+ml^2)$$

from which we get

$$\ddot{x} = -\frac{m^2 \ell^2 g}{MI + m(I + M\ell^2)}\theta + \frac{I + m\ell^2}{MI + m(I + M\ell^2)} u \tag{4}$$

Equations (3) and (4) describe the system dynamics in terms of differential equations.

By taking the Laplace transform of Equation (3), we obtain

$$\left[s^2 - \frac{m\ell(M+m)g}{(M+m)I + Mm\ell^2}\right]\Theta(s) = -\frac{m\ell}{(M+m)I + Mm\ell^2} U(s)$$

or

$$\left\{\left[MI + m(I + M\ell^2)\right]s^2 - m\ell(M+m)g\right\}\Theta(s) = -m\ell\, U(s)$$

Hence

$$\frac{\Theta(s)}{U(s)} = -\frac{m\ell}{[MI + m(I + M\ell^2)]s^2 - m\ell(M+m)g} \tag{5}$$

By taking the Laplace transform of Equation (4), we get

$$s^2 X(s) = -\frac{m^2\ell^2 g}{MI + m(I + M\ell^2)}\Theta(s) + \frac{I + m\ell^2}{MI + m(I + M\ell^2)} U(s)$$

Hence

$$\frac{s^2 X(s)}{U(s)} = -\frac{m^2\ell^2 g}{MI + m(I + M\ell^2)}\frac{\Theta(s)}{U(s)} + \frac{I + m\ell^2}{MI + m(I + M\ell^2)}$$

or

$$\frac{X(s)}{U(s)} = \frac{m^2\ell^2 g}{[MI + m(I + m\ell^2)]s^2}\,\frac{m\ell}{[MI + m(I + m\ell^2)]s^2 - m\ell(M+m)g}$$

$$+ \frac{I + m\ell^2}{[MI + m(I + M\ell^2)]s^2} \tag{6}$$

Equations (5) and (6) define the inverted pendulum control system in terms of transfer functions.

Next, we shall obtain a state-space representation of the system. Define state variables by

$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

$$x_3 = x$$

$$x_4 = \dot{x}$$

$$y_1 = \theta = x_1$$

$$y_2 = x = x_3$$

Then, from the definition of state variables and Equations (3) and (4), the state equation and output equation can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{m\ell(M+m)g}{MI+m(I+M\ell^2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{m^2\ell^2 g}{MI+m(I+M\ell^2)} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{m\ell}{MI+m(I+M\ell^2)} \\ 0 \\ \dfrac{I+m\ell^2}{MI+m(I+M\ell^2)} \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

B-3-6. The equations for the system are

$$m_1 \ddot{x}_1 = -k_1 x_1 - b_1 \dot{x}_1 - k_3 (x_1 - x_2) + u$$

$$m_2 \ddot{x}_2 = -k_2 x_2 - b_2 \dot{x}_2 - k_3 (x_2 - x_1)$$

Rewriting, we have

$$m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_1 x_1 + k_3 x_1 = k_3 x_2 + u$$

$$m_2 \ddot{x}_2 + b_2 \dot{x}_2 + k_2 x_2 + k_3 x_2 = k_3 x_1$$

Assuming the zero initial condition and taking the Laplace transforms of these two equations, we obtsin

$$(m_1 s^2 + b_1 s + k_1 + k_3) X_1(s) = k_3 X_2(s) + U(s) \qquad (1)$$

$$(m_2 s^2 + b_2 s + k_2 + k_3) X_2(s) = k_3 X_1(s) \qquad (2)$$

By eliminating $X_2(s)$ from Equations (1) and (2), we get

$$(m_1 s^2 + b_1 s + k_1 + k_3) X_1(s) = \frac{k_3^2}{m_2 s^2 + b_2 s + k_2 + k_3} X_1(s) + U(s)$$

Hence

$$\frac{X_1(s)}{U(s)} = \frac{m_2 s^2 + b_2 s + k_2 + k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}$$

From Equation (2), we obtain

$$\frac{X_2(s)}{X_1(s)} = \frac{k_3}{m_2 s^2 + b_2 s + k_2 + k_3}$$

Hence

$$\frac{X_2(s)}{U(s)} = \frac{X_2(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)} = \frac{k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}$$

B-3-7. The equations for the given circuit are as followw:

$$R_1 i_1 + L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = e_i$$

$$R_2 i_2 + \frac{1}{C}\int i_2\,dt + L\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) = 0$$

$$\frac{1}{C}\int i_2\,dt = e_o$$

Taking the Laplace transforms of these three equations, assuming zero initial conditions, gives

$$R_1 I_1(s) + L\left[s I_1(s) - s I_2(s)\right] = E_i(s) \qquad (1)$$

$$R_2 I_2(s) + \frac{1}{Cs} I_2(s) + L\left[s I_2(s) - s I_1(s)\right] = 0 \qquad (2)$$

$$\frac{1}{Cs} I_2(s) = E_o(s) \qquad (3)$$

From Equation (2) we obtain

$$\left(R_2 + \frac{1}{Cs} + Ls\right) I_2(s) = Ls\, I_1(s)$$

or

$$I_2(s) = \frac{LCs^2}{LCs^2 + R_2 Cs + 1} I_1(s) \qquad (4)$$

Substituting Equation (4) into Equation (1), we get

$$\left(R_1 + Ls - Ls\,\frac{LCs^2}{LCs^2 + R_2 Cs + 1}\right) I_1(s) = E_i(s)$$

or

$$\frac{LC(R_1 + R_2)s^2 + (R_1 R_2 C + L)s + R_1}{LCs^2 + R_2 Cs + 1} I_1(s) = E_i(s) \qquad (5)$$

From Equations (3) and (4), we have

From Equations (5) and (6), we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{Ls}{LC(R_1+R_2)s^2+(R_1R_2C+L)s+R_1}$$

B-3-8. Equations for the circuit are

$$\frac{1}{C_1}\int(i_1-i_2)\,dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int(i_2-i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0$$

$$\frac{1}{C_2}\int i_2\,dt = e_o$$

The Laplace transforms of these three equations, with zero initial conditions, are

$$\frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right] + R_1 I_1(s) = E_i(s) \tag{1}$$

$$\frac{1}{C_1 s}\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0 \tag{2}$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s) \tag{3}$$

Equation (1) can be rewritten as

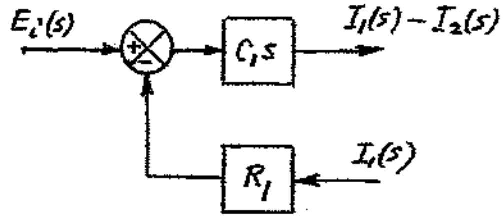
$$C_1 s\left[E_i(s) - R_1 I_1(s)\right] = I_1(s) - I_2(s) \tag{4}$$

Equation (4) gives the block diagram shown in Figure (a). Equation (2) can be modified to

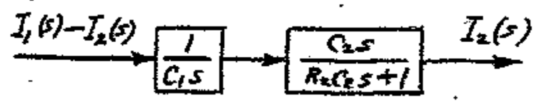
$$I_2(s) = \frac{C_2 s}{R_2 C_2 s + 1}\,\frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right] \tag{5}$$

Equation (5) yields the block diagram shown in Figure (b). Also, Equation (3) gives the block diagram shown in Figure (c). Combining the block diagrams of Figures (a), (b), and (c), we obtain Figure (d). This block diagram can be succesively modified as shown in Figures (e) through (h). In this way, we can obtain the transfer function $E_o(s)/E_i(s)$ of the given circuit..

(a)


$E_i(s)$
$C_1 s$
$I_1(s) - I_2(s)$
R_1
$I_1(s)$

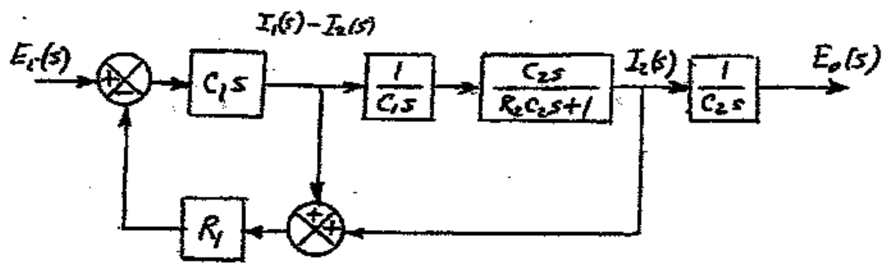
(b)


$I_1(s) - I_2(s)$
$\frac{1}{C_1 s}$
$\frac{C_2 s}{R_2 C_2 s + 1}$
$I_2(s)$

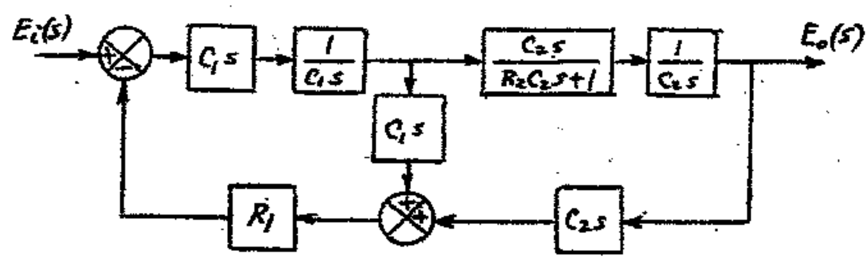
(c)


$I_2(s)$
$\frac{1}{C_2 s}$
$E_o(s)$

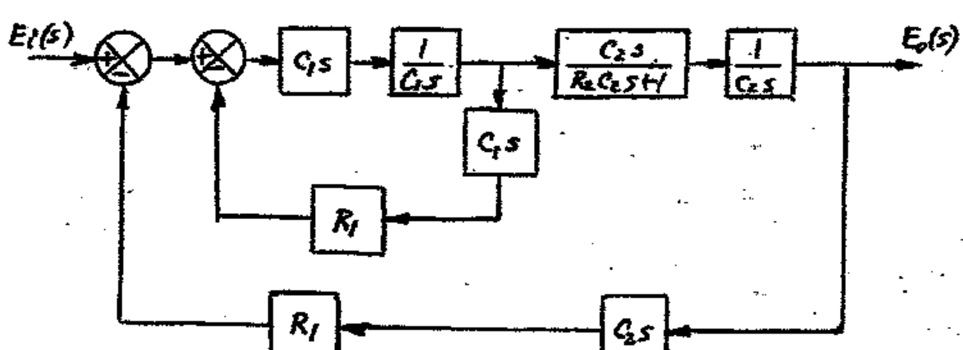
(d)


$E_i(s)$
$C_1 s$
$I_1(s) - I_2(s)$
$\frac{1}{C_1 s}$
$\frac{C_2 s}{R_2 C_2 s + 1}$
$I_2(s)$
$\frac{1}{C_2 s}$
$E_o(s)$
R_1

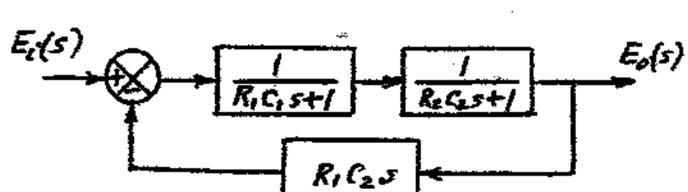
(e)


$E_i(s)$
$C_1 s$
$\frac{1}{C_1 s}$
$\frac{C_2 s}{R_2 C_2 s + 1}$
$\frac{1}{C_2 s}$
$E_o(s)$
$C_1 s$
R_1
$C_2 s$

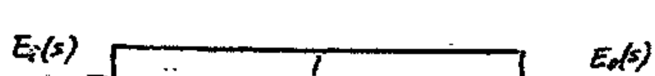
(f)


$E_i(s)$
$C_1 s$
$\frac{1}{C_1 s}$
$\frac{C_2 s}{R_2 C_2 s + 1}$
$\frac{1}{C_2 s}$
$E_o(s)$
$C_1 s$
R_1
R_1
$C_2 s$

(g)

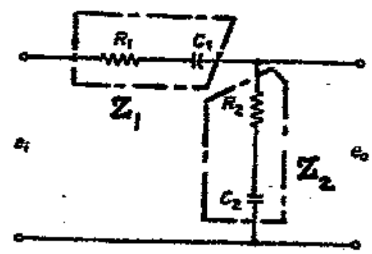

$E_i(s)$
$\frac{1}{R_1 C_1 s + 1}$
$\frac{1}{R_2 C_2 s + 1}$
$E_o(s)$
$R_1 C_2 s$

(h)


$E_i(s)$
$E_o(s)$

B-3-9. Impedance Z_1 is

$$Z_1 = R_1 + \frac{1}{C_1 s}$$



Impedance Z_2 is

$$Z_2 = R_2 + \frac{1}{C_2 s}$$

Hence

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2 + \frac{1}{C_2 s}}{R_1 + \frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}}$$

$$= \frac{R_2 C_2 s + 1}{(R_1 C_2 + R_2 C_2) s + 1 + \frac{C_2}{C_1}}$$

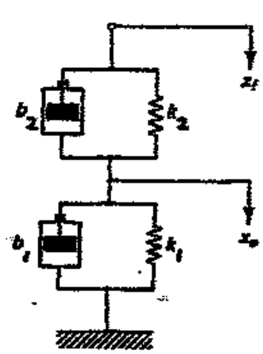
If we change R_1 to b_1, R_2 to b_2, C_1 to $1/k_1$, C_2 to $1/k_2$, then we obtain

$$\frac{R_2 C_2 s + 1}{(R_1 + R_2) C_2 s + 1 + \frac{C_2}{C_1}} = \frac{b_2 \frac{1}{k_2} s + 1}{(b_1 + b_2)\frac{1}{k_2} s + 1 + \frac{k_1}{k_2}}$$

or

$$\frac{X_o(s)}{X_i(s)} = \frac{b_2 s + k_2}{(b_1 + b_2) s + k_2 + k_1} = \frac{b_2 s + k_2}{(b_1 s + k_1) + (b_2 s + k_2)}$$

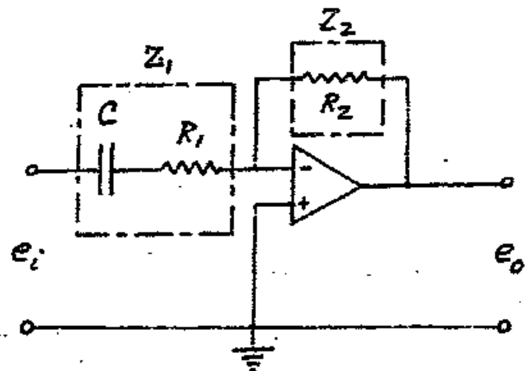
The analogous mechanical system is shown below.



B-3-10.

$$Z_1 = R_1 + \frac{1}{Cs}$$

$$Z_2 = R_2$$



Then

$$E_i(s) = \left(R_1 + \frac{1}{Cs}\right) I(s)$$

$$E_o(s) = -R_2 I(s)$$

Hence

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1 + \frac{1}{Cs}} = -\frac{R_2 Cs}{R_1 Cs + 1}$$

B-3-11. Define the voltage at point A as e_A. Then

$$\frac{E_A(s)}{E_i(s)} = \frac{R_1}{\frac{1}{Cs} + R_1} = \frac{R_1 Cs}{R_1 Cs + 1}$$

Define the voltage at point B as e_B. Then

$$E_B(s) = \frac{R_3}{R_2 + R_3} E_o(s)$$

Noting that

$$\left[E_A(s) - E_B(s)\right] K = E_o(s)$$

and $K \gg 1$, we must have

$$E_A(s) = E_B(s)$$

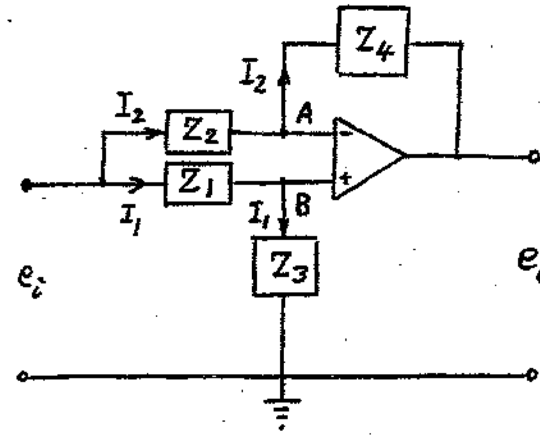
Hence

$$E_A(s) = \frac{R_1 Cs}{R_1 Cs + 1} E_i(s) = E_B(s) = \frac{R_3}{R_2 + R_3} E_o(s)$$

from which we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 + R_3}{R_3} \frac{R_1 Cs}{R_1 Cs + 1} = \frac{\left(1 + \frac{R_2}{R_3}\right) s}{s + \frac{1}{R_1 C}}$$

B-3-12. For the op-amp circuit shown to the right, we have



$$E_A - E_o = Z_4 I_2$$

$$E_B - 0 = Z_3 I_1$$

$$E_A = E_B$$

Hence

$$Z_4 I_2 + E_o = Z_3 I_1$$

or

$$I_2 = \frac{1}{Z_4}(Z_3 I_1 - E_o) \qquad (1)$$

Also,

$$E_i - E_o = (Z_2 + Z_4) I_2 \qquad (2)$$

$$E_i = (Z_1 + Z_3) I_1 \qquad (3)$$

By substituting Equation (1) into Equation (2), we obtain

$$E_i - E_o = (Z_2 + Z_4)\frac{1}{Z_4}(Z_3 I_1 - E_o)$$

By substituting Equation (3) into this last equation, we get

$$(Z_1 + Z_3) I_1 - E_o = \left(\frac{Z_2}{Z_4} + 1\right) Z_3 I_1 - \left(\frac{Z_2}{Z_4} + 1\right) E_o$$

or

$$\left(1 - \frac{Z_2}{Z_4} - 1\right) E_o = \left(Z_1 + Z_3 - \frac{Z_2 Z_3}{Z_4} - Z_3\right) I_1$$

Hence

$$-Z_2 E_o = (Z_1 Z_4 - Z_2 Z_3) I_1 \qquad (4)$$

From Equations (3) and (4), we have

$$\frac{E_o}{E_i} = \frac{\dfrac{Z_2 Z_3 - Z_1 Z_4}{Z_2}}{Z_1 + Z_3} = \frac{Z_2 Z_3 - Z_1 Z_4}{Z_1 Z_2 + Z_2 Z_3}$$

For the current op-amp circuit, we have

$$Z_1 = \frac{1}{Cs}, \quad Z_2 = R_1, \quad Z_3 = R_2, \quad Z_4 = R_1$$

Hence

$$\frac{E_o(s)}{E_i(s)} = \frac{R_1 R_2 - \frac{1}{Cs} R_1}{\frac{1}{Cs} R_1 + R_1 R_2} = \frac{R_2 - \frac{1}{Cs}}{\frac{1}{Cs} + R_2} = \frac{R_2 Cs - 1}{R_2 Cs + 1}$$

B-3-13. Define the current in the armature circuit to be i_a. Then, we have

$$L\frac{di_a}{dt} + Ri_a + K_b\frac{d\theta_m}{dt} = e_i$$

or

$$(Ls+R)I_a(s) + K_b s\,\Theta_m(s) = E_i(s) \tag{1}$$

where K_b is the back emf constant of the motor. We also have

$$J_m\ddot{\theta}_m + T = T_m = Ki_a \tag{2}$$

$$T = \frac{\theta}{\theta_m}T_L = nT_L$$

$$J_L\ddot{\theta} = T_L$$

where K is the motor torque constant and i_a is the armature current. Equation (2) can be rewritten as

$$(J_m + n^2 J_L)\ddot{\theta} = nKi_a$$

or

$$(J_m + n^2 J_L)s^2\Theta(s) = nKI_a(s) \tag{3}$$

By substituting Equation (3) into Equation (1), we obtain

$$(Ls+R)\frac{(J_m+n^2J_L)s^2}{nK}\Theta(s) + K_b s\frac{\Theta(s)}{n} = E_i(s)$$

or

$$\left[(Ls+R)(J_m+n^2J_L)s^2 + KK_b s\right]\Theta(s) = nKE_i(s)$$

Hence

$$\frac{\Theta(s)}{E_i(s)} = \frac{nK}{s\left[(Ls+R)(J_m+n^2J_L)s + KK_b\right]}$$