Chapter 2

Multivariate Distributions

2.1.2

$$P(A_5) = \frac{7}{8} - \frac{4}{8} - \frac{3}{8} + \frac{2}{8} = \frac{2}{8}.$$

2.1.5

$$\begin{aligned}\int_0^\infty \int_0^\infty \left[2g(\sqrt{x_1^2+x_2^2})/\pi\sqrt{x_1^2+x_2^2}\right] dx_1 dx_2 &= \int_0^\infty \int_0^{\pi/2} [2g(\rho)/\pi\rho]\rho \, d\theta d\rho \\ &= \int_0^\infty g(\rho)\, d\rho = 1.\end{aligned}$$

2.1.6

$$\begin{aligned}G(z) &= P(X+Y \le z) = \int_0^z \int_0^{z-x} e^{-x-y}\, dy dx \\ &= \int_0^z [1-e^{-(z-x)}]e^{-x}\, dx = 1 - e^{-z} - ze^{-z}. \\ g(z) &= G'(z) = \begin{cases} ze^{-z} & 0 < z < \infty \\ 0 & \text{elsewhere.}\end{cases}\end{aligned}$$

2.1.7

$$\begin{aligned}G(z) &= P(XY \le z) = 1 - \int_z^1 \int_{z/x}^1 dy dx \\ &= 1 - \int_z^1 \left(1 - \frac{z}{x}\right) dx = z - z\log z \\ g(z) &= G'(z) = \begin{cases} -\log z & 0 < z < 1 \\ 0 & \text{elsewhere.}\end{cases}\end{aligned}$$

Why is $-\log z > 0$?

2.1.8

$$f(x,y) = \begin{cases} \frac{\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}}{\binom{52}{13}} & x \geq 0, y \geq 0, x+y \leq 13, x \text{ and } y \text{ integers} \\ 0 & \text{elsewhere.} \end{cases}$$

2.1.10

$$P(X_1 + X_2 \leq 1) = 15 \int_0^{1/2} x_1^2 \left[\int_{x_1}^{1-x_1} x_2 \, dx_2 \right] dx_1.$$

2.1.14

$$\begin{aligned} E\left[e^{t_1X_1+t_2X_2}\right] &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{t_1 i + t_2 j}\left(\frac{1}{2}\right)^{i+j} \\ &= \sum_{i=1}^{\infty}\left(e^{t_1}\frac{1}{2}\right)^{i}\sum_{j=1}^{\infty}\left(e^{t_1}\frac{1}{2}\right)^{j} \\ &= \left[\frac{1}{1-2^{-1}e^{t_1}} - 1\right]\left[\frac{1}{1-2^{-1}e^{t_2}} - 1\right], \end{aligned}$$

provided $t_i < \log 2$, $i = 1, 2$.

2.2.1

$$p(y_1, y_2) = \begin{cases} \left(\frac{2}{3}\right)^{y_2}\left(\frac{1}{3}\right)^{2-y_2} & (y_1,y_2) = (0,0),(-1,1),(1,1),(0,2) \\ 0 & \text{elsewhere.} \end{cases}$$

2.2.2

$$p(y_1, y_2) = \begin{cases} y_1/36 & y_1 = y_2, 2y_2, 3y_2; y_2 = 1,2,3 \\ 0 & \text{elsewhere.} \end{cases}$$

y_1	1	2	3	4	6	9
$p(y_1)$	1/36	4/36	6/36	4/36	12/36	9/36

2.2.4 The inverse transformation is given by $x_1 = y_1y_2$ and $x_2 = y_2$ with Jacobian $J = y_2$. By noting what the boundaries of the space $\mathcal{S}(X_1, X_2)$ map into, it follows that the space $\mathcal{T}(Y_1, Y_2) = \{(y_1, y_2) : 0 < y_i < 1, i = 1, 2\}$. The pdf of (Y_1, Y_2) is $f_{Y_1,Y_2}(y_1, y_2) = 8y_1y_2^3$.

2.2.5 The inverse transformation is $x_1 = y_1 - y_2$ and $x_2 = y_2$ with Jacobian $J = 1$. The space of (Y_1, Y_2) is $\mathcal{T} = \{(y_1, y_2) : -\infty < y_i < \infty, i = 1, 2\}$. Thus the joint pdf of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2),$$

which leads to formula (2.2.1).

2.3.2

$$
\begin{array}{ll}
(a) & c_1 \int_0^{x_2} x_1/x_2^2\,dx_1 = \frac{c_1}{2} = 1 \Rightarrow c_1 = 2 \text{ and } c_2 = 5. \\
(b) & 10x_1x_2^2, 0 < x_1 < x_2 < 1; \text{zero elsewhere} \\
(c) & \int_{1/4}^{1/2} 2x_1/(5/8)^2\,dx = \frac{64}{25}\left(\frac{1}{4} - \frac{1}{16}\right) = \frac{12}{25}. \\
(d) & \int_{1/4}^{1/2}\int_{x_1}^{1} 10x_1x_2^2\,dx_2dx_1 = \int_{1/4}^{1/2} \frac{10}{3}x_1(1-x_1^3)\,dx_1 = \frac{135}{512}.
\end{array}
$$

2.3.3

$$
\begin{aligned}
f_2(x_2) &= \int_0^{x_2} 21x_1^2x_2^3\,dx_1 = 7x_2^6, \quad 0 < x_2 < 1. \\
f_{1|2}(x_1|x_2) &= 21x_1^2x_2^3/7x_2^6 = 3x_1^2/x_2^3, \quad 0 < x_1 < x_2. \\
E(X_1|x_2) &= \int_0^{x_2} x_1(3x_1^2/x_2^3)\,dx_1 = \frac{3}{4}x_2. \\
G(y) &= P\left(\frac{3}{4}X_2 \le y\right) = \int_0^{4y/3} 7x_2^6\,dx_2 = \left(\frac{4y}{3}\right)^7, \quad 0 < y < \frac{3}{4} \\
g(y) &= \begin{cases} 7\left(\frac{4}{3}\right)^7 y^6 & 0 < y < \frac{3}{4} \\ 0 & \text{elsewhere.} \end{cases} \\
E(Y) &= \frac{7}{8}\frac{3}{4} = \frac{21}{32}. \\
\text{Var}(Y) &= \frac{7}{1024}. \\
E(X_1) &= \frac{21}{32}. \\
\text{Var}(X_1) &= \frac{553}{15360} > \frac{7}{1024}.
\end{aligned}
$$

2.3.8 The marginal pdf of X is

$$
f_X(x) = 2\int_x^\infty e^{-x}e^{-y}\,dy = 2e^{-2x}, \quad 0 < x < \infty.
$$

Hence, the conditional pdf of Y given $X = x$ is

$$
f_{Y|X}(y|x) = \frac{2e^{-x}e^{-y}}{2e^{-2x}} = e^{-(y-x)}, \quad 0 < x < y < \infty,
$$

with conditional mean

$$
E(Y|X = x) = \int_x^\infty ye^{-(y-x)}\,dy = x + 1, \quad x > 0.
$$

2.3.9 For Part (c):

$$\binom{13}{x_2}\binom{13}{x_3}\binom{13}{2-x_2-x_3}\Big/\binom{39}{2}, \quad \text{where integers } x_2, x_3 \geq 0 \text{ and } x_2 + x_3 \leq 2.$$

2.3.11

$$(a) \qquad f_1(x_1)f_{2|1}(x_2|x_1) = 1 \cdot \frac{1}{x_1}, \quad 0 < x_2 < x_1 < 1.$$

$$(b) \qquad \int_{1/2}^{1}\int_{1-x_1}^{x_1} \frac{1}{x_1}\, dx_2 dx_1 = \int_{1/2}^{1} \frac{2x_1 - 1}{x_1}\, dx_1 = 2(1/2) + \log(1/2) = 1 - \log 2.$$

2.3.12

$$(b) \qquad \int_2^\infty e^{-x}\, dx \Big/ \int_1^\infty e^{-x}\, dx = e^{-2}/e^{-1} = e^{-1}.$$

2.4.1 For Part (c):

$$\text{cov} = (0)(0)(1/3) + (1)(1)(1/3) + (2)(0)(1/3) - (1)(1/3) = 0.$$

Thus $\rho = 0$ and yet X and Y are dependent.

2.4.3

$$\rho^2 = (1/2)(1/2) = 1/4 \Rightarrow \rho = 1/2.$$

2.4.7

$$h(v) = \text{var}(X) + 2v\text{cov}(X,Y) + v^2\text{var}(Y) \geq 0,$$

for all v. Hence, the discriminant of this quadratic must satisfy $b^2 - 4ac \leq 0$ which yields

$$[2\text{cov}(X,Y)]^2 - 4\text{var}(X)\text{var}(Y) \leq 0.$$

Equivalently,

$$\rho^2 = [\text{cov}(X,Y)]^2/\text{var}(X)\text{var}(Y) \leq 1.$$

2.4.11 Let $Y = (X_1 - \mu_1) + (X_2 - \mu_2)$. Then the mean of Y is 0 and its variance is

$$\text{Var}(Y) = \text{Var}(X_1 + X_2) = \sigma^2 + \sigma^2 + 2\rho\sigma^2 = 2\sigma^2(1-\rho).$$

Use Chebyshev's inequality to obtain the result.

2.5.2 X_1 and X_2 are dependent because $0 < x_1 < x_2 < \infty$ is not a product space.

2.5.4 Because X_1 and X_2 are independent, the probability equals

$$\left[\int_0^{1/3} 2x_1\, dx_1\right]\left[\int_0^{1/3} 2(1-x_2)\, dx_2\right] = (1/3)^2[1 - (2/3)^2] = 5/81.$$

2.5.7 The marginal pdf of X_1 is given by

$$f_{X_1}(x_1) = \int_{-2-\sqrt{1-(x_1-1)^2}}^{-2+\sqrt{1-(x_1-1)^2}} \frac{1}{\pi}\, dx_2 = \frac{2}{\pi}\sqrt{1-(x_1-1)^2}, \quad 0 < x < 2.$$

The random variables X_1 and X_2 are not independent.

2.5.8 X and Y are dependent because $0 < y < x < 1$ is not a product space.

$$E(X|y) = \int_y^1 x[2x/(1-y^2)]\, dx = \frac{2(1-y^2)}{3(1-y^2)}.$$

2.5.9

$$\begin{aligned} P(X+Y \le 60) &= P(X \le 10) + \int_{10}^{20}\int_{40}^{60-x} \frac{1}{300}\, dy\, dx \\ &= \frac{1}{3} + \int_{10}^{20} (20-x)/300\, dx = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}. \end{aligned}$$

2.5.12

$$\begin{aligned} P(|X_1 - X_2| = 1) &= P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0) \\ &= P(X_1 = 0)P(X_2 = 1) + P(X_1 = 1)P(X_2 = 0) = \frac{1}{3}. \end{aligned}$$

2.6.1 For Part (g):

$$E(X|y,z) = \int_0^1 x\frac{3(x+y+z)/2}{3((1/2)+y+z)/2}\, dx = \frac{(1/3)+(y/2)+(z/2)}{(1/2)+y+z}.$$

2.6.3

$$\begin{aligned} G(y) &= 1 - P(y < X_i, i = 1,2,3,4) = 1 - [(1-y)^3]^4 = 1 - (1-y)^{12} \\ g(y) &= G'(y) = \begin{cases} 12(1-y)^{11} & 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

2.6.6 Multiply both members of $E[X_1 - \mu_1 | x_2, x_3] = b_2(x_2 - \mu_2) + b_3(x_3 - \mu_3)$ by the joint pdf of X_2 and X_3 and denote the result by (1). Multiply both members of (1) by $(x_2 - \mu_2)$ and integrate (or sum) on x_2 and x_3. This gives (2), $\rho_{12}\sigma_1\sigma_2 = b_2\sigma_2^2 + 3\rho_{23}\sigma_1\sigma_2$. Return to (1) and multiply each member by $(x_3 - \mu_3)$ and integrate (or sum) on x_2 and x_3. This yields (3) $\rho_{13}\sigma_1\sigma_3 = b_2\rho_{23}\sigma_2\sigma_3 + b_3\sigma_3^2$. Solve (2) and (3) for b_2 and b_3.

2.6.9

$$\begin{aligned} (a) \quad & \int_0^\infty \int_{x_1}^\infty e^{-x_1-x_2}\, dx_2\, dx_1 \Big/ \int_0^\infty \int_{x_1/2}^\infty e^{-x_1-x_2}\, dx_2\, dx_1 \\ & + \int_0^\infty e^{-2x_1}\, dx_1 \Big/ \int_0^\infty e^{-3x_1/2}\, dx_1 = \frac{1}{2}\frac{2}{3} = \frac{3}{4}. \end{aligned}$$

2.7.1

$$x_1 = y_1y_2y_3,\ x_2 = y_2y_3 - y_1y_2y_3,\ x_3 = y_3 - y_2y_3.$$

with $J = y_2y_3^2$, and $0 < y_1 < 1, 0 < y_2 < 1, 0 < y_3 < \infty$. This yields

$$g(y_1, y_2, y_3) = y_2y_3^2e^{-y_3} = (1)(2y_2)(y_3^2e^{-y_3}/2) = g_1(y_1)g_2(y_2)g_3(y_3).$$

2.7.2

$$x_1 = \sqrt{y},\ x_2 = -\sqrt{y} \text{ and } J_i = \tfrac{1}{2\sqrt{y}},\ i = 1.2.$$

This yields

$$g(y) = \frac{1}{2}\left(\frac{1}{2\sqrt{y}}\right) + \frac{1}{2}\left(\frac{1}{2\sqrt{y}}\right) = \frac{1}{2\sqrt{y}}, \quad 0 < y < 1.$$

2.7.5 The inverse transformation is $x_1 = \frac{y_1y_3}{1+y_1}$, $x_2 = \frac{y_3}{1+y_1}$, and $x_3 = y_2y_3$, with space $y_i > 0$, $i = 1, 2, 3$. The Jacobian is

$$J = \begin{vmatrix} \frac{y_3}{(1+y_1)^2} & 0 & \frac{y_1}{1+y_1} \\ \frac{-y_3}{(1+y_1)^2} & 0 & \frac{1}{1+y_1} \\ 0 & y_3 & y_2 \end{vmatrix} = \left[\frac{y_3^2}{(1+y_1)^3} + \frac{y_1y_3^2}{(1+y_1)^3}\right] = \frac{y_3^2}{(1+y_1)^2}.$$

2.7.8 Expanding $M(t)$ we get

$$M(t) = \left(\frac{3}{4}\right)^2 e^0 + 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)e^t + \left(\frac{1}{4}\right)^2 e^{2t}.$$

From this, we immediately get the probabilities

$$P(X=0) = \left(\tfrac{3}{4}\right)^2,\ P(X=1) = 2\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) \text{ and } P(X=2) = \left(\tfrac{1}{4}\right)^2.$$

2.8.2 Note that

$$\begin{aligned} \mu_1 &= E(X_i) = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3} \\ E(X_i^2) &= \int_0^1 2x^3\,dx = \frac{2}{4}x^4\Big|_0^1 = \frac{1}{2} \end{aligned}$$

So

$$\sigma^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

Hence,

$$\begin{aligned} E(Y) &= \sum_{i=1}^4 E(X_i) = \frac{8}{3} \\ V(Y) &= \sum_{i=1}^4 V(X_i) = \frac{4}{18}, \end{aligned}$$

where we used the independence of $X_1, \ldots, X_4$ to establish the variance of Y.

2.8.4 By independence

$$\begin{aligned} E(X_1X_2) &= E(X_1)E(X_2) = \mu_1\mu_2 \\ E(X_1^2X_2^2) &= E(X_1^2)E(X_2^2) = (\sigma_1^2+\mu_1^2)(\sigma_2^2+\mu_2^2). \end{aligned}$$

So,

$$V(X_1X_2) = (\sigma_1^2+\mu_1^2)(\sigma_2^2+\mu_2^2) - \mu_1^2\mu_2^2,$$

which simplifies to the answer.

2.8.8 Because in these cases, the correlation coefficient is never influenced by the means, let $\mu_1 = \mu_2 = 0$. Then

$$\begin{aligned} \text{cov}(X,Z) &= E[X(X-Y)] = E(X^2) = \sigma_1^2 \\ \rho &= \sigma_1^2/\sqrt{\sigma_1^2(\sigma_1^2+\sigma_2^2)} = \sigma_1/\sqrt{\sigma_1^2+\sigma_2^2}. \end{aligned}$$

2.8.11

$$\begin{aligned} \text{cov}(W,Z) &= E[(aX+b-a\mu_1-b)(cY+d-c\mu_2-d)] \\ &= acE[(X-\mu_1)(Y-\mu_2)] = ac\rho\sigma_1\sigma_2 \\ \text{correlation coef.} &= \frac{ac\rho\sigma_1\sigma_2}{\sqrt{a^2c^2\sigma_1^2\sigma_2^2}} = \rho. \end{aligned}$$

2.8.13

$$\begin{aligned} \text{cov}(X_1X_2, X_1) &= E[(X_1X_2-\mu_1\mu_2)(X_1-\mu_1)] \\ &= (\mu_1^2+\sigma_1^2)\mu_2 - \mu_1^2\mu_2 - \mu_1^2\mu_2 + \mu_1^2\mu_2 = \sigma_1^2\mu_2. \end{aligned}$$

2.8.15 Without loss of generality, let the means equal zero

$$\begin{aligned} \text{cov}(Y,Z) &= (0.3+0.5+1.0+0.2)\sigma^2 = 2\sigma^2, \\ \text{Answer} &= 2\sigma^2/\sqrt{[1+2(0.3)+1]\sigma^2[1+2(0.2)+1]\sigma^2} = \frac{2}{\sqrt{(2.6)(2.4)}} = 0.801. \end{aligned}$$

2.8.17 Again let $\mu_1 = \mu_2 = 0$.

$$\text{cov}E\{X[Y-\rho(\sigma_2/\sigma_1)X]\} = \rho\sigma_1\sigma_2 - \rho(\sigma_2/\sigma_1)\sigma_1^2 = 0.$$

2.8.18 The function $g(x) = x^2$ is strictly convex. Hence, by Jensen's inequality,

$$(E(S))^2 < E(S^2),$$

which leads to $E(S) < \sigma$.