Chapter 2

Discrete Distributions

2.1 Random Variables of the Discrete Type

2.1-2 **(a)**

$$f(x) = \begin{cases} 0.6, & x = 1, \\ 0.3, & x = 5, \\ 0.1, & x = 10, \end{cases}$$

(b)

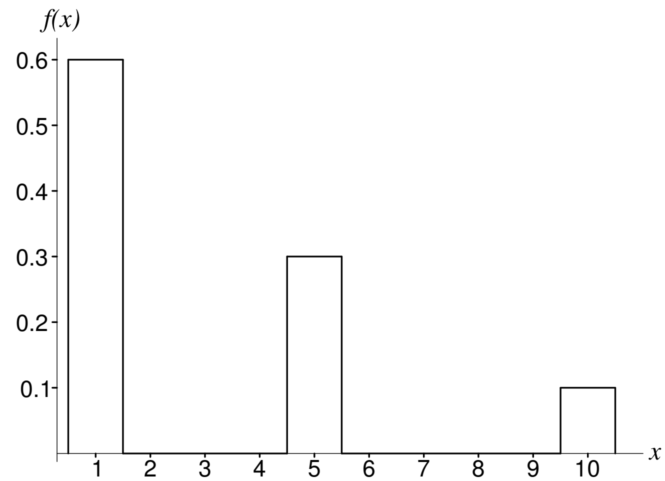

Figure 2.1–2: A probability histogram

2.1-4 **(a)** $f(x) = \dfrac{1}{10}, \qquad x = 0, 1, 2, \cdots, 9;$

(b)

$\mathcal{N}(\{0\})/150 = 11/150 = 0.073;$ $\mathcal{N}(\{5\})/150 = 13/150 = 0.087;$
$\mathcal{N}(\{1\})/150 = 14/150 = 0.093;$ $\mathcal{N}(\{6\})/150 = 22/150 = 0.147;$
$\mathcal{N}(\{2\})/150 = 13/150 = 0.087;$ $\mathcal{N}(\{7\})/150 = 16/150 = 0.107;$
$\mathcal{N}(\{3\})/150 = 12/150 = 0.080;$ $\mathcal{N}(\{8\})/150 = 18/150 = 0.120;$
$\mathcal{N}(\{4\})/150 = 16/150 = 0.107;$ $\mathcal{N}(\{9\})/150 = 15/150 = 0.100.$

(c)

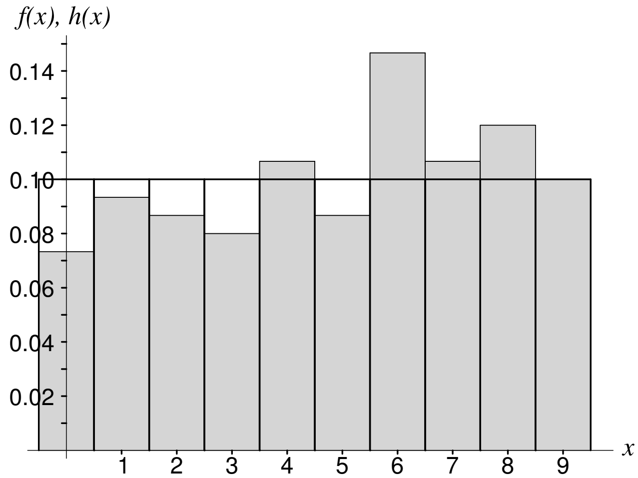

Figure 2.1–4: Michigan daily lottery digits

2.1-6 **(a)** $f(x) = \dfrac{6 - |\,7 - x\,|}{36}, \ x = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$

(b)

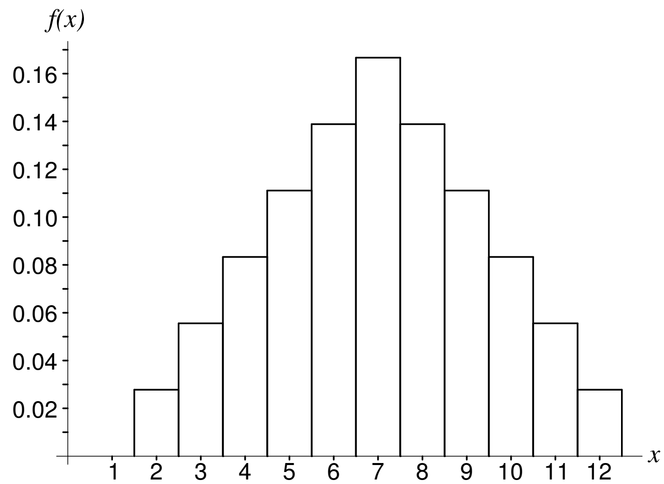

Figure 2.1–6: Probability histogram for the sum of a pair of dice

2.1-8 **(a)** The space of W is $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

$P(W = 0) = P(X = 0, Y = 0) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$, assuming independence.

$P(W = 1) = P(X = 0, Y = 1) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$,

$P(W = 2) = P(X = 2, Y = 0) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$,

$P(W = 3) = P(X = 2, Y = 1) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$,

$P(W = 4) = P(X = 0, Y = 4) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$,

$P(W = 5) = P(X = 0, Y = 5) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$,

$P(W = 6) = P(X = 2, Y = 4) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$,

$P(W = 7) = P(X = 2, Y = 5) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

That is, $f(w) = P(W = w) = \frac{1}{8}, \quad w \in S$.

(b)

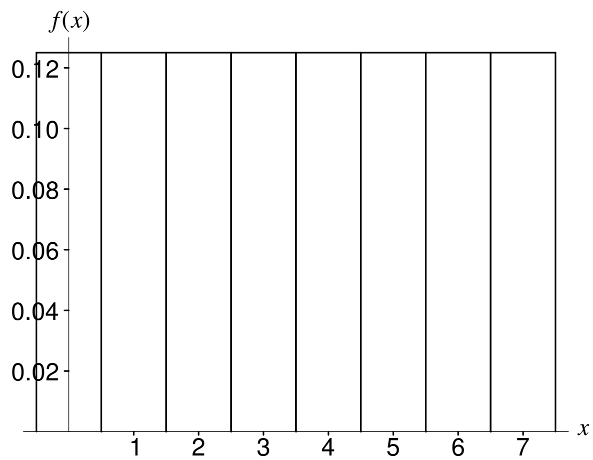

Figure 2.1–8: Probability histogram of sum of two special dice

2.1-10 **(a)** $\dfrac{\binom{3}{1}\binom{47}{9}}{\binom{50}{10}} = \dfrac{39}{98}$;

(b) $\displaystyle\sum_{x=0}^{1} \frac{\binom{3}{x}\binom{47}{10-x}}{\binom{50}{10}} = \frac{221}{245}$.

2.1-12 $$\begin{aligned} P(X \ge 4 \,|\, X \ge 1) &= \frac{P(X \ge 4)}{P(X \ge 1)} = \frac{1 - P(X \le 3)}{1 - P(X = 0)} \\ &= \frac{1 - [1 - 1/2 + 1/2 - 1/3 + 1/3 - 1/4 + 1/4 - 1/5]}{1 - [1 - 1/2]} = \frac{2}{5}. \end{aligned}$$

2.1-14 $$P(X \ge 1) = 1 - P(X = 0) = 1 - \frac{\binom{3}{0}\binom{17}{5}}{\binom{20}{5}} = 1 - \frac{91}{228} = \frac{137}{228} = 0.60.$$

2.1-16 **(a)** $P(2, 1, 6, 10)$ means that 2 is in position 1 so 1 cannot be selected. Thus

$$P(2, 1, 6, 10) = \frac{\binom{1}{0}\binom{1}{1}\binom{8}{5}}{\binom{10}{6}} = \frac{56}{210} = \frac{4}{15};$$

(b) $$P(i, r, k, n) = \frac{\binom{i-1}{r-1}\binom{1}{1}\binom{n-i}{k-r}}{\binom{n}{k}}.$$

2.2 Mathematical Expectation

2.2-2 $$E(X) = (-1)\left(\frac{4}{9}\right) + (0)\left(\frac{1}{9}\right) + (1)\left(\frac{4}{9}\right) = 0;$$

$$E(X^2) = (-1)^2\left(\frac{4}{9}\right) + (0)^2\left(\frac{1}{9}\right) + (1)^2\left(\frac{4}{9}\right) = \frac{8}{9};$$

$$E(3X^2 - 2X + 4) = 3\left(\frac{8}{9}\right) - 2(0) + 4 = \frac{20}{3}.$$

2.2-4 $$\begin{aligned} 1 &= \sum_{x=0}^{6} f(x) = \frac{9}{10} + c\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) \\ c &= \frac{2}{49}; \\ E(\text{Payment}) &= \frac{2}{49}\left(1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{5} + 5 \cdot \frac{1}{6}\right) = \frac{71}{490} \text{ units.} \end{aligned}$$

2.2-6 Note that $\displaystyle\sum_{x=1}^{\infty} \frac{6}{\pi^2 x^2} = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{6}{\pi^2}\frac{\pi^2}{6} = 1$, so this is a pdf

$$E(X) = \sum_{x=1}^{\infty} x \, \frac{6}{\pi^2 x^2} = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x} = +\infty$$

and it is well known that the sum of this harmonic series is not finite.

2.2-8 $E(|X - c|) = \dfrac{1}{7} \displaystyle\sum_{x \in S} |x - c|$, where $S = \{1, 2, 3, 5, 15, 25, 50\}$.

When $c = 5$,

$$E(|X - 5|) = \frac{1}{7}\left[(5 - 1) + (5 - 2) + (5 - 3) + (5 - 5) + (15 - 5) + (25 - 5) + (50 - 5)\right].$$

If c is either increased or decreased by 1, this expectation is increased by 1/7. Thus $c = 5$, the median, minimizes this expectation while $b = E(X) = \mu$, the mean, minimizes $E[(X-b)^2]$. You could also let $h(c) = E(\,|\,X - c\,|\,)$ and show that $h'(c) = 0$ when $c = 5$.

2.2-10 $(1)\cdot\dfrac{15}{36} + (-1)\cdot\dfrac{21}{36} = \dfrac{-6}{36} = \dfrac{-1}{6};$

$(1)\cdot\dfrac{15}{36} + (-1)\cdot\dfrac{21}{36} = \dfrac{-6}{36} = \dfrac{-1}{6};$

$(4)\cdot\dfrac{6}{36} + (-1)\cdot\dfrac{30}{36} = \dfrac{-6}{36} = \dfrac{-1}{6}.$

2.2-12 **(a)** The average class size is $\dfrac{(16)(25)+(3)(100)+(1)(300)}{20} = 50;$

(b)

$$f(x) = \begin{cases} 0.4, & x = 25, \\ 0.3, & x = 100, \\ 0.3, & x = 300, \end{cases}$$

(c) $E(X) = 25(0.4) + 100(0.3) + 300(0.3) = 130.$

2.3 Special Mathematical Expectations

2.3-2 **(a)**

$$\begin{aligned}
\mu &= E(X) \\
&= \sum_{x=1}^{3} x\,\frac{3!}{x!\,(3-x)!}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x} \\
&= 3\left(\frac{1}{4}\right)\sum_{k=0}^{2}\frac{2!}{k!\,(2-k)!}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{2-k} \\
&= 3\left(\frac{1}{4}\right)\left(\frac{1}{4}+\frac{3}{4}\right)^2 = \frac{3}{4}; \\
E[X(X-1)] &= \sum_{x=2}^{3} x(x-1)\,\frac{3!}{x!\,(3-x)!}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x} \\
&= 2(3)\left(\frac{1}{4}\right)^2\frac{3}{4} + 6\left(\frac{1}{4}\right)^3 \\
&= 6\left(\frac{1}{4}\right)^2 = 2\left(\frac{1}{4}\right)\left(\frac{3}{4}\right); \\
\sigma^2 &= E[X(X-1)] + E(X) - \mu^2 \\
&= (2)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2 \\
&= (2)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right);
\end{aligned}$$

(b)
$$\begin{aligned}\mu &= E(X)\\ &= \sum_{x=1}^{4} x\,\frac{4!}{x!\,(4-x)!}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}\\ &= 4\left(\frac{1}{2}\right)\sum_{k=0}^{3}\frac{3!}{k!\,(3-k)!}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{3-k}\\ &= 4\left(\frac{1}{2}\right)\left(\frac{1}{2}+\frac{1}{2}\right)^3 = 2;\end{aligned}$$

$$\begin{aligned}E[X(X-1)] &= \sum_{x=2}^{4} x(x-1)\,\frac{4!}{x!\,(4-x)!}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}\\ &= 2(6)\left(\frac{1}{2}\right)^4 + (6)(4)\left(\frac{1}{2}\right)^4 + (12)\left(\frac{1}{2}\right)^4\\ &= 48\left(\frac{1}{2}\right)^4 = 12\left(\frac{1}{2}\right)^2;\\ \sigma^2 &= (12)\left(\frac{1}{2}\right)^2 + \frac{4}{2} - \left(\frac{4}{2}\right)^2 = 1.\end{aligned}$$

2.3-4 $E[(X-\mu)/\sigma] = (1/\sigma)[E(X)-\mu] = (1/\sigma)(\mu-\mu) = 0;$

$E\{[(X-\mu)/\sigma]^2\} = (1/\sigma^2)E[(X-\mu)^2] = (1/\sigma^2)(\sigma^2) = 1.$

2.3-6 $f(1) = \dfrac{3}{8},\ f(2) = \dfrac{2}{8},\ f(3) = \dfrac{3}{8}$

$\mu = 1\cdot\dfrac{3}{8} + 2\cdot\dfrac{2}{8} + 3\cdot\dfrac{3}{8} = 2,$

$\sigma^2 = 1^2\cdot\dfrac{3}{8} + 2^2\cdot\dfrac{2}{8} + 3^2\cdot\dfrac{3}{8} - 2^2 = \dfrac{3}{4}.$

2.3-8
$$\begin{aligned}E(X) &= \sum_{x=1}^{4} x\cdot\frac{2x-1}{16}\\ &= \frac{50}{16} = 3.125;\\ E(X^2) &= \sum_{x=1}^{4} x^2\cdot\frac{2x-1}{16}\\ &= \frac{85}{8};\\ \mathrm{Var}(X) &= \frac{85}{8} - \left(\frac{25}{8}\right)^2 = \frac{55}{64} = 0.8594;\\ \sigma &= \frac{\sqrt{55}}{8} = 0.9270.\end{aligned}$$

2.3-10 We have $N = N_1 + N_2$. Thus

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)f(x) \\ &= \frac{\displaystyle\sum_{x=2}^{n} x(x-1)\,\frac{N_1!}{x!\,(N_1-x)!}\cdot\frac{N_2!}{(n-x)!\,(N_2-n+x)!}}{\dbinom{N}{n}} \\ &= N_1(N_1-1)\,\frac{\displaystyle\sum_{x=2}^{n} \frac{(N_1-2)!}{(x-2)!\,(N_1-x)!}\cdot\frac{N_2!}{(n-x)!\,(N_2-n+x)!}}{\dbinom{N}{n}}. \end{aligned}$$

In the summation, let $k = x-2$, and in the denominator, note that

$$\binom{N}{n} = \frac{N!}{n!(N-n)!} = \frac{N(N-1)}{n(n-1)}\binom{N-2}{n-2}.$$

Thus

$$\begin{aligned} E[X(X-1)] &= \frac{N_1(N_1-1)}{\dfrac{N(N-1)}{n(n-1)}} \sum_{k=0}^{n-2} \frac{\dbinom{N_1-2}{k}\dbinom{N_2}{n-2-k}}{\dbinom{N-2}{n-2}} \\ &= \frac{N_1(N_1-1)(n)(n-1)}{N(N-1)}. \end{aligned}$$

2.3-12 **(a)** $f(x) = \left(\dfrac{364}{365}\right)^{x-1}\left(\dfrac{1}{365}\right), \qquad x = 1, 2, 3, \ldots,$

(b)

$$\begin{aligned} \mu &= \frac{1}{\dfrac{1}{365}} = 365, \\ \sigma^2 &= \frac{\dfrac{364}{365}}{\left(\dfrac{1}{365}\right)^2} = 132{,}860, \\ \sigma &= 364.500; \end{aligned}$$

(c)

$$\begin{aligned} P(X > 400) &= \left(\frac{364}{365}\right)^{400} = 0.3337, \\ P(X < 300) &= 1 - \left(\frac{364}{365}\right)^{299} = 0.5597. \end{aligned}$$

2.3-14 $P(X \geq 100) = P(X > 99) = (0.99)^{99} = 0.3697.$

2.3-16 **(a)** $f(x) = (1/2)^{x-1}, \quad x = 2, 3, 4, \ldots;$

$$\textbf{(b)}\quad M(t) = E[e^{tx}] = \sum_{x=2}^{\infty} e^{tx}(1/2)^{x-1} = 2\sum_{x=2}^{\infty}(e^t/2)^x = \frac{2(e^t/2)^2}{1-e^t/2} = \frac{e^{2t}}{2-e^t}, \qquad t < \ln 2;$$

$$\textbf{(c)}\quad M'(t) = \frac{4e^{2t}-e^{3t}}{(2-e^t)^2}$$

$$\mu = M'(0) = 3;$$

$$M''(t) = \frac{(2-e^t)^2(8e^{2t}-3e^{3t}) - (4e^{2t}-e^{3t})2*(2-e^t)(-e^t)}{(2-e^t)^4}$$

$$\sigma^2 = M''(0) - \mu^2 = 11 - 9 = 2;$$

(d) **(i)** $P(X \le 3) = 3/4$; **(ii)** $P(X \ge 5) = 1/8$; **(iii)** $P(X = 3) = 1/4$.

2.3-18 $$P(X > k+j \,|\, X > k) = \frac{P(X > k+j)}{P(X > k)} = \frac{q^{k+j}}{q^k} = q^j = P(X > j).$$

2.4 The Binomial Distribution

2.4-2 $f(-1) = \dfrac{11}{18}, \qquad f(1) = \dfrac{7}{18};$

$$\mu = (-1)\frac{11}{18} + (1)\frac{7}{18} = -\frac{4}{18};$$

$$\sigma^2 = \left(-1+\frac{4}{18}\right)^2\left(\frac{11}{18}\right) + \left(1+\frac{4}{18}\right)^2\left(\frac{7}{18}\right) = \frac{77}{81}.$$

2.4-4 **(a)** X is $b(7,\ 0.15)$;

(b) **(i)** $P(X \ge 2) = 1 - P(X \le 1) = 1 - 0.7166 = 0.2834;$

(ii) $P(X = 1) = P(X \le 1) - P(X \le 0) = 0.7166 - 0.3206 = 0.3960;$

(iii) $P(X \le 3) = 0.9879.$

2.4-6 **(a)** X is $b(15, 0.75)$; $\qquad 15 - X$ is $b(15, 0.25)$;

(b) $P(X \ge 10) = P(15 - X \le 5) = 0.8516;$

(c) $P(X \le 10) = P(15 - X \ge 5) = 1 - P(15 - X \le 4) = 1 - 0.6865 = 0.3135;$

(d) $P(X = 10) = P(X \ge 10) - P(X \ge 11)$
$= P(15 - X \le 5) - P(15 - X \le 4) = 0.8516 - 0.6865 = 0.1651;$

(e) $\mu = (15)(0.75) = 11.25, \quad \sigma^2 = (15)(0.75)(0.25) = 2.8125; \quad \sigma = \sqrt{2.8125} = 1.67705.$

2.4-8 **(a)** $1 - 0.01^4 = 0.99999999$; **(b)** $0.99^4 = 0.960596$.

2.4-10 **(a)** X is $b(8,\ 0.90)$;

(b) **(i)** $P(X = 8) = P(8 - X = 0) = 0.4305;$

(ii) $P(X \le 6) = P(8 - X \ge 2) = 1 - P(8 - X \le 1) = 1 - 0.8131 = 0.1869;$

(iii) $P(X \ge 6) = P(8 - X \le 2) = 0.9619.$

2.4-12 **(a)**

$$f(x) = \begin{cases} 125/216, & x = -1, \\ 75/216, & x = 1, \\ 15/216, & x = 2, \\ 1/216, & x = 3; \end{cases}$$

(b)

$$\mu = (-1)\cdot\frac{125}{216} + (1)\cdot\frac{75}{216} + (2)\cdot\frac{15}{216} + (3)\cdot\frac{1}{216} = -\frac{17}{216};$$

$$\sigma^2 = E(X^2) - \mu^2 = \frac{269}{216} - \left(-\frac{17}{216}\right)^2 = 1.2392;$$

$$\sigma = 1.11;$$

(c) See Figure 2.4-12.

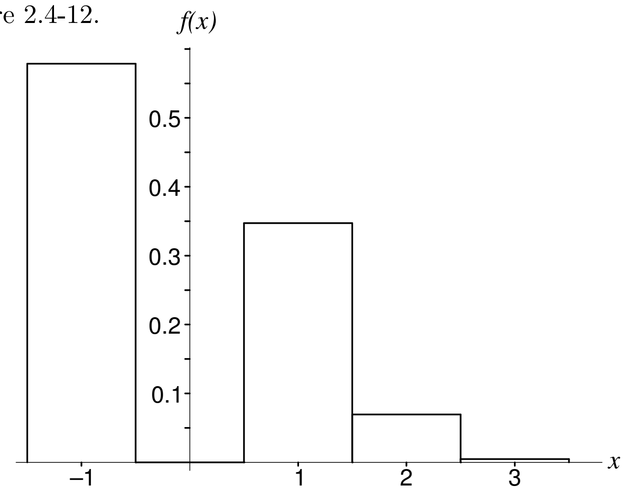

Figure 2.4–12: Losses in chuck-a-luck

2.4-14 Let X equal the number of winning tickets when n tickets are purchased. Then

$$P(X \geq 1) = 1 - P(X = 0)$$
$$= 1 - \left(\frac{9}{10}\right)^n.$$

(a)

$$1 - (0.9)^n = 0.50$$
$$(0.9)^n = 0.50$$
$$n \ln 0.9 = \ln 0.5$$
$$n = \frac{\ln 0.5}{\ln 0.9} = 6.58$$

so $n = 7$.

(b)

$$1 - (0.9)^n = 0.95$$
$$(0.9)^n = 0.05$$
$$n = \frac{\ln 0.05}{\ln 0.09} = 28.43$$

so $n = 29$.

2.4-16 It is given that X is $b(10,\ 0.10)$. We are to find M so that

$P(1000X \le M) \ge 0.99)$ or $P(X \le M/1000) \ge 0.99$. From Appendix Table II,

$P(X \le 4) = 0.9984 > 0.99$. Thus $M/1000 = 4$ or $M = 4000$ dollars.

2.4-18 X is $b(5,\ 0.05)$. The expected number of tests is

$1\,P(X=0) + 6\,P(X>0) = 1\,(0.7738) + 6\,(1-0.7738) = 2.131.$

2.4-20 **(a)** **(i)** $b(5,\ 0.7)$; **(ii)** $\mu = 3.5, \sigma^2 = 1.05$; **(iii)** 0.1607;

(b) **(i)** geometric, $p = 0.3$; **(ii)** $\mu = 10/3, \sigma^2 = 70/9$; **(iii)** 0.51;

(c) **(i)** Bernoulli, $p = 0.55$; **(ii)** $\mu = 0.55, \sigma^2 = 0.2475$; **(iii)** 0.55;

(d) **(ii)** $\mu = 2.1, \sigma^2 = 0.89$; **(iii)** 0.7;

(e) **(i)** discrete uniform on $1, 2, \ldots, 10$; **(ii)** 5.5, 8.25; **(iii)** 0.2.

2.5 The Negative Binomial Distribution

2.5-2 $\displaystyle \binom{10-1}{5-1}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 = \frac{126}{1024} = \frac{63}{512}.$

2.5-4 Let "being missed" be a success and let X equal the number of trials until the first success. Then $p = 0.01$.

$P(X \le 50) = 1 - 0.99^{50} = 1 - 0.605 = 0.395.$

2.5-6 **(a)**

$$\begin{aligned}
R(t) &= \ln(1-p+pe^t),\\
R'(t) &= \left[\frac{pe^t}{1-p+pe^t}\right]_{t=0} = p,\\
R''(t) &= \left[\frac{(1-p+pe^t)(pe^t)-(pe^t)(pe^t)}{(1-p+pe^t)^2}\right]_{t=0} = p(1-p);
\end{aligned}$$

(b)

$$\begin{aligned}
R(t) &= n\ln(1-p+pe^t),\\
R'(t) &= \left[\frac{npe^t}{1-p+pe^t}\right]_{t=0} = np,\\
R''(t) &= n\left[\frac{(1-p+pe^t)(pe^t)-(pe^t)(pe^t)}{(1-p+pe^t)^2}\right]_{t=0} = np(1-p);
\end{aligned}$$

(c)

$$\begin{aligned}
R(t) &= \ln p + t - \ln[1-(1-p)e^t],\\
R'(t) &= \left[1+\frac{(1-p)e^t}{1-(1-p)e^t}\right]_{t=0} = 1+\frac{1-p}{p} = \frac{1}{p},\\
R''(t) &= \left[(-1)\{1-(1-p)e^t\}^2\{-(1-p)e^t\}\right]_{t=0} = \frac{1-p}{p};
\end{aligned}$$

(d)

$$\begin{aligned}
R(t) &= r\left[\ln p + t - \ln\{1-(1-p)e^t\}\right],\\
R'(t) &= r\left[\frac{1}{1-(1-p)e^t}\right]_{t=0} = \frac{r}{p},\\
R''(t) &= r\left[(-1)\{1-(1-p)e^t\}^{-2}\{-(1-p)e^t\}\right]_{t=0} = \frac{r(1-p)}{p^2}.
\end{aligned}$$

2.5-8 $(0.7)(0.7)(0.3) = 0.147.$

2.5-10 **(a)** Let X equal the number of boxes that must be purchased. Then

$$E(X) = \sum_{x=1}^{12} \frac{1}{(13-x)/12} = \frac{86021}{2310} = 37.2385;$$

(b) $\dfrac{100 \cdot E(X)}{365} \approx 10.2.$

2.6 The Poisson Distribution

2.6-2 $\lambda = \mu = \sigma^2 = 3$ so $P(X = 2) = 0.423 - 0.199 = 0.224.$

2.6-4
$$\begin{aligned} 3\,\frac{\lambda^1 e^{-\lambda}}{1!} &= \frac{\lambda^2 e^{-\lambda}}{2!} \\ e^{-\lambda}\lambda(\lambda - 6) &= 0 \\ \lambda &= 6 \end{aligned}$$

Thus $P(X = 4) = 0.285 - 0.151 = 0.134.$

2.6-6 $\lambda = (1)(50/100) = 0.5$, so $P(X = 0) = e^{-0.5}/0! = 0.607.$

2.6-8 $np = 1000(0.005) = 5;$

(a) $P(X \le 1) \approx 0.040;$

(b) $P(X = 4, 5, 6) = P(X \le 6) - P(X \le 3) \approx 0.762 - 0.265 = 0.497.$

2.6-10 $\sigma = \sqrt{9} = 3,$

$$P(3 < X < 15) = P(X \le 14) - P(X \le 3) = 0.959 - 0.021 = 0.938.$$

2.6-12 Since $E(X) = 0.2$, the expected loss is (0.02)($10,000) = $2,000.