CHAPTER 2

SECTION 2.1

2.1.1 Option 1 accumulated value is $50,000(1+i)^{24}$.

Option 2 annuity payment is K, where $K = \dfrac{50,000}{a_{\overline{24}|.1}} = 5564.99$

Then $5564.99 s_{\overline{24}|.05} = 50,000(1+i)^{24}$ so that $i = 6.9\%$.

2.1.2 Suppose the annual effective rate is i. Then $900 s_{\overline{10}|i} = 1000 a_{\overline{\infty}|i}$

so that $900\left[\dfrac{(1+i)^{10}-1}{i}\right] = 1000\left[\dfrac{1}{i}\right]$, and then $(1+i)^{10} = \dfrac{19}{9}$.

We then have $K s_{\overline{5}|i} \times (1+i)^5 = 1000 a_{\overline{\infty}|i}$ so that

$$K = \frac{1000}{(1+i)^{10} - (1+i)^5} = 1519.42.$$

2.1.3
$$s_{\overline{n+k}|i} = \left[(1+i)^{n+k-1} + (1+i)^{n+k-2} + \cdots + (1+i)^{k+1} + (1+i)^k\right]$$
$$+\left[(1+i)^{k-1} + (1+i)^{k-2} + \cdots + (1+i) + 1\right]$$
$$= (1+i)^k\left[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)^1 + 1\right] + s_{\overline{k}|i}$$
$$= (1+i)^k \cdot s_{\overline{n}|i} + s_{\overline{k}|i}$$

2.1.4
$$1000 s_{\overline{300}|.01} = Y a_{\overline{300}|.01} \longrightarrow Y = \frac{1000 s_{\overline{300}|.01}}{a_{\overline{300}|.01}}$$
$$= 1000(1.01)^{300} = 19,788.47$$

2.1.5 (i) $100s_{\overline{7}|.0075} = 715.95$

(ii) $100s_{\overline{19}|.0075} = 2033.87$

(iii) $100\left[s_{\overline{19}|.0075}\cdot(1.00875)^9(1.01)^4 + s_{\overline{9}|.00875}\cdot(1.01)^4 + s_{\overline{4}|.01}\right]$
$= 3665.12$

(iv) $3665.12(.01) = 36.65$

2.1.6 $98s_{\overline{n}|}(1+i)^{2n} + 196s_{\overline{2n}|} = 8000,\ (1+i)^n = 2$

$\rightarrow 196\left[2s_{\overline{n}|} + s_{\overline{2n}|}\right] = 8000 \rightarrow 2s_{\overline{n}|} + s_{\overline{2n}|} = 40.82$

$\rightarrow \frac{2(1+i)^n-2}{i} + \frac{(1+i)^{2n}-1}{i} = 40.82$

$\rightarrow \frac{2}{i} + \frac{3}{i} = 40.82 \rightarrow i = .1225.$

2.1.7 (a) $10(1.05)^{30}\cdot s_{\overline{10}|.05} + 20(1.05)^{20}\cdot s_{\overline{10}|.05}$
$+ 30(1.05)^{10}\cdot s_{\overline{10}|.05} + 40s_{\overline{10}|.05} = 2328.82$

(b) $10\left[s_{\overline{40}|} - s_{\overline{30}|} + 2(s_{\overline{30}|} - s_{\overline{20}|}) + 3(s_{\overline{20}|} - s_{\overline{10}|}) + 4s_{\overline{10}|}\right]$
$= 0\left[s_{\overline{10}|.05} + s_{\overline{20}|.05} + s_{\overline{30}|.05} + s_{\overline{40}|.05}\right]$

2.1.8 $\sum_{t=1}^{10} s_{\overline{t}|.10} = \sum_{t=1}^{10}\frac{(1.10)^t-1}{.10} = 10\left[\ddot{s}_{\overline{10}|.10} - 10\right] = 11S - 100$

2.1.9 $I_t = i\cdot s_{\overline{t-1}|i} = (1+i)^{t-1} - 1 \rightarrow \sum_{t=1}^{n} I_t$
$= \sum_{t=1}^{n}\left[(1+i)^{t-1} - 1\right] = s_{\overline{n}|i} - n$

Total interest = total accumulated value − total deposit

2.1.10 After n years Smith's AV is $\left[80s_{\overline{10}|.06} + 200\right](1.06)^{n-10}$ and Brown's AV is $40s_{\overline{n-10}|.06} + P$. Thus, if $n = 15$, then $P = 14.53$, $n = 20 \rightarrow P = 17.19$, $n = 25 \rightarrow 20.75$.

2.1.11 (a) $\frac{s_{\overline{2n}|i}}{s_{\overline{n}|i}} = \frac{(1+i)^{2n}-1}{(1+i)^n-1} = (1+i)^n + 1 = \frac{210}{70} = 3$

$\rightarrow (1+i)^n = 2,\ 70 = s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$

$\rightarrow i = \frac{1}{70} = .014286,\ s_{\overline{3n}|i} = s_{\overline{n}|i} + (1+i)^n s_{\overline{2n}|i} = 490$

(b) $\frac{s_{\overline{3n}|i}}{s_{\overline{n}|i}} = \frac{(1+i)^{3n}-1}{(1+i)^n-1} = (1+i)^{2n} + (1+i)^n + 1 = \frac{X}{Y}$

$\rightarrow$ quadratic equation in $z = (1+i)^n : z^2 + z + \frac{Y-X}{Y} = 0$

$\rightarrow (1+i)^n = z = \frac{-1\pm\sqrt{1-\frac{4(Y-X)}{Y}}}{2}$ (discard negative root)

$\rightarrow v^n = \frac{1}{(1+i)^n} = \frac{2}{-1+\sqrt{1-\frac{4(Y-X)}{Y}}}$

(c) $s_{\overline{n}|i} = (1+i)^2\cdot s_{\overline{n-2}|i} + (1+i) + 1$

$\rightarrow 36.34(1+i)^2 + (1+i) - 47.99 = 0$

$\rightarrow 1+i = 1.1355$, or -1.1630 (discard negative root)

2.1.12 $AV = s_{\overline{n}|.11} + (1.11)^n s_{\overline{m}|.07} = 128 + (1.11)^n(34)$

Since $s_{\overline{n}|.11} = \frac{(1.11)^n-1}{.11} = 128$, it follows that $(1.11)^n = 15.08$
$\rightarrow AV = 640.72$

2.1.13 We accumulate the payments to the beginning of the 6^{th} year (time 5) and then accumulate them for another 5 years.

$20s_{\overline{6}|.1}(1.1)^5 + Xs_{\overline{3}|.1}(1.1)^5 = 200(1.04)^{10} \rightarrow X = 8.92$

2.1.14 An investment of amount 1 is equal to the present value of the return of principal in n years plus the present value of the interest generated over the n years.

2.1.15 2825.49

2.1.16 The equivalent effective annual rate of interest is $i = (1.04)^2 - 1 = .0816$. The balance on January 1, 2010 is

$$100{,}000(1.04)^{20} + 5000s_{\overline{10}|i} - 12{,}000s_{\overline{10}|i}(1.04) = 109{,}926.$$

2.1.17 Annuity (a) has present value $55a_{\overline{20}|i}$.

The present value of annuity (b) can be formulated as $30a_{\overline{10}|i} + 60v^{10}a_{\overline{10}|i} + 90v^{20}a_{\overline{10}|i}$. Note that annuity (a) can also be written as $55a_{\overline{20}|i} = 55a_{\overline{10}|i} + 55v^{10}a_{\overline{10}|i}$. Both annuities have the same present value X, so that

$$55a_{\overline{10}|i} + 55v^{10}a_{\overline{10}|i} = 30a_{\overline{10}|i} + 60v^{10}a_{\overline{10}|i} + 90v^{20}a_{\overline{10}|i}.$$

After canceling the factor $a_{\overline{10}|i}$ the equation becomes

$$55 + 55v^{10} = 30 + 60v^{10} + 90v^{20}.$$

With $v^{10} = y$, this becomes the quadratic equation

$$90y^2 + 5y - 25 = 0,$$

or equivalently $18y^2 + y - 5 = 0$. The roots are $y = .50, \ -.556$. We ignore the negative root for $v^{10} = y$. Therefore, $v^{10} = .50$ so that $v = (.50)^{.1}$, and then $i = .0718$. Finally, $X = 55a_{\overline{20}|.0718} = 575$.

2.1.18 Option 1 corresponds to a single deposit earning ordinary compound interest (compounded annually), and the accumulated value at the end of 24 years is $10{,}000(1+i)^{24}$.

Under Option 2, the 10,000 purchases an annuity-immediate at 10% paying K per year, so that $10{,}000 = Ka_{\overline{24}|.1}$ (the purchase price of 10,000 is the present value of the annuity-immediate being purchased). Solving for K results in

$$K = \frac{10{,}000}{a_{\overline{24}|.1}} = \frac{10{,}000(.1)}{1-v_{.1}^{24}} = \frac{1000}{.898474} = 1{,}113.$$

Under Option 2, the payments of 1,113 will be received at the end of each year for 24 years (it is implicitly understood that with an annuity-immediate the payments begin one period after the annuity is purchased – this is referred to as the "end" of the year). If, as the payments are received, they are deposited into an account earning interest at effective annual interest rate 5%, then the accumulated value of the account at the end of 24 years is

$$1113s_{\overline{24}|.05} = (1.113)\left[\frac{(1.05)^{24}-1}{.05}\right] = (1113)(44.502) = 49{,}531.$$

Since Option 1 results in the same accumulated value, we have $10{,}000(1+i)^{24} = 49{,}531$, from which it follows that $i = .0689$.

2.1.19 The phrase "at the end of each year" indicates an annuity immediate.

$$Xa_{\overline{n}|} = X\left(\frac{1-v^n}{i}\right) = 493, \ 3Xa_{\overline{2n}|} = 3X\left(\frac{1-v^{2n}}{i}\right) = 2748.$$

Using the factorization $1 - v^{2n} = (1-v^n)(1+v^n)$, we have

$$\frac{3Xa_{\overline{2n}|}}{Xa_{\overline{n}|}} = \frac{3(1-v^{2n})/i}{(1-v^n)/i} = 3(1+v^n) = \frac{2748}{493} = 5.574 \rightarrow v^n = .858.$$

This idea has arisen in exam questions a number of times over the years. A similar factorization could be applied if $s_{\overline{n}|}$ and $s_{\overline{2n}|}$ were given. A more involved situation arises if $a_{\overline{n}|}$ and $a_{\overline{3n}|}$ are given. In that case, we use the factorization

$$1 - v^{3n} = (1-v^n)(1+v^n+v^{2n}).$$

2.1.20 $10{,}000 = K \cdot a_{\overline{10}|.03} + 200v^5 \cdot a_{\overline{5}|.03} \rightarrow K = 1079.68$

2.1.21 $a_{\overline{n}|} = \frac{1-v^n}{i} = \frac{1}{i} - v^n \cdot \frac{1}{i} = a_{\overline{\infty}|} - v^n \cdot a_{\overline{\infty}|}$

2.1.22 (b) $v^k a_{\overline{n}|i} = v^k[v + v^2 + \cdots + v^n]$

$$= v^{k+1} + v^{k+2} + \cdots + v^{k+n}$$

$$= [v + v^2 + \cdots + v^k + v^{k+1} + v^{k+2} + \cdots + v^{k+n}] - [v + v^2 + \cdots + v^k]$$

$$= a_{\overline{n+k}|} - a_{\overline{k}|}$$

2.1.23 330.80

$$PV = 10v^{40}\left[s_{\overline{10}|.05} + s_{\overline{20}|.05} + s_{\overline{30}|.05} + s_{\overline{40}|.05}\right]$$

$$= 10\left[v^{30}a_{\overline{10}|.05} + v^{20}a_{\overline{20}|.05} + v^{10}a_{\overline{30}|.05} + a_{\overline{40}|.05}\right]$$

$$= 10\left[a_{\overline{40}|} - a_{\overline{10}|} + a_{\overline{40}|} - a_{\overline{20}|} + a_{\overline{40}|} - a_{\overline{30}|} + a_{\overline{40}|}\right]$$

2.1.24 $Y = s_{\overline{k}|j} + (1+j)^k s_{\overline{n}|i}, \quad X = a_{\overline{n}|i} + v_i^n \cdot a_{\overline{k}|}$

$$\rightarrow X(1+i)^n \cdot (1+j)^k$$

$$= a_{\overline{n}|i} \cdot (1+i)^n \cdot (1+j)^k + v_i^n \cdot a_{\overline{k}|j} \cdot (1+i)^n \cdot (1+j)^k$$

$$= (1+j)^k s_{\overline{n}|i} + s_{\overline{k}|j}$$

2.1.25 $\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-v^n} = \frac{i - iv^n + iv^n}{1-v^n} = \frac{i(1-v^n)}{1-v^n} + \frac{iv^n}{1-v^n}$

$$= i + \frac{i}{(1+i)^n - 1} = i + \frac{1}{s_{\overline{n}|i}}$$

2.1.26 $a_{\overline{n}|i} = v^n a_{\overline{2n}|i} \rightarrow \frac{v^n a_{\overline{2n}|i}}{a_{\overline{n}|i}} = \frac{v^n(1-v^{2n})}{1-v^n}$

$$= v^n(1+v^n) = v^n + v^{2n} = 1$$

$\rightarrow v^{2n} + v^n - 1 = 0 \rightarrow v^n = \frac{-1 \pm \sqrt{1+4}}{2} = .6180$ or -1.6180.

(discard negative root)

2.1.27 (a) $\ddot{a}_{\overline{n}|i} = \frac{1-v^n}{d} = \frac{1-v^n}{i/(1+i)} = (1+i) \cdot \frac{1-v^n}{i} = (1+i) \cdot a_{\overline{n}|i}$

$$= a_{\overline{n}|i} + i \cdot a_{\overline{n}|i} = a_{\overline{n}|i} + i \cdot \frac{1-v^n}{i} = a_{\overline{n}|i} + 1 - v^n$$

$$= 1 + (v + v^2 + \cdots + v^{n-1} + v^n) - v^n$$

$$= 1 + (v + v^2 + \cdots + v^{n-1}) = 1 + a_{\overline{n-1}|i}$$

(b) $\ddot{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d} = \frac{(1+i)^n - 1}{i/(1+i)} = (1+i) \cdot \frac{(1+i)^n - 1}{i} = (1+i) \cdot s_{\overline{n}|i}$

$$= s_{\overline{n}|i} + i \cdot s_{\overline{n}|i} = s_{\overline{n}|i} + i \cdot \frac{(1+i)^n - 1}{i} = s_{\overline{n}|i} + (1+i)^n - 1$$

$$= \left[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} + (1+i)^n\right] - 1$$

$$= s_{\overline{n+1}|i} - 1$$

2.1.28 At rate j per month, $5000 = 117.38a_{\overline{12n}|j} \rightarrow a_{\overline{12n}|j} = 42.5967$,
$10{,}000 = 113.40\ddot{s}_{\overline{12n}|j} \rightarrow \ddot{s}_{\overline{12n}|j} = 88.1834$.
Using the identities $\ddot{s}_{\overline{12n}|j} = s_{\overline{12n}|j} - 1 + (1+j)^{12n}$ and
$s_{\overline{12n}|j} = (1+j)^{12n} \cdot a_{\overline{12n}|j}$, we have $(1+j)^{12n} = 2.045646$.
Then $v^{12n} = .488843$, so that $j = \frac{1-v^{12n}}{a_{\overline{12n}|j}} = .012$.
Then $i = (1+j)^{12} - 1 = .1539$.

2.1.29 (a) $v_j + v_j \cdot v_i + v_j^2 \cdot v_i + v_j^2 \cdot v_i^2 + v_j^3 \cdot v_i^2 + v_j^3 \cdot v_i^3 + \cdots$

$$= v_j[1+v_i][1+v_j \cdot v_i + (v_j \cdot v_i)^2 + \cdots]$$
$$= v_j[1+v_i] \cdot \frac{1}{1-v_j \cdot v_i}$$

(b) (i) $[v + v^3 + v^5 + \cdots] + 2[v^2 + v^4 + v^6 + \cdots]$

$$= [v+2v^2][1+v^2+v^4+\cdots] = \frac{v+2v^2}{1-v^2}$$

(ii) $\frac{1+2v}{1-v^2}$

2.1.30 (a) $X = \frac{L}{2a_{\overline{10|}} - a_{\overline{5|}}},\ Y = \frac{L}{a_{\overline{10|}}},\ Z = \frac{L}{a_{\overline{10|}}+a_{\overline{5|}}}$

$\rightarrow \frac{1}{X} = \frac{2a_{\overline{10|}} - a_{\overline{5|}}}{L},\ \frac{3}{2Y} = \frac{\frac{3}{2}a_{\overline{10|}}}{L},\ \frac{1}{Z} = \frac{a_{\overline{10|}} + a_{\overline{5|}}}{L}.$

Since $a_{\overline{10|}} < 2a_{\overline{5|}}$ it follows that

$2a_{\overline{10|}} - a_{\overline{5|}} < \frac{3}{2}a_{\overline{10|}} < a_{\overline{10|}} + a_{\overline{5|}}$, so that $\frac{1}{X} < \frac{3}{2Y} < \frac{1}{Z}$.

(b) $P_1 = \frac{Li}{1-v_i^n}$ and $P_2 = \frac{L \cdot 2i}{1-v_{2i}^n}$. Since $v_i^n > v_{2i}^n$ it follows that

$$\frac{P_2}{2P_1} = \frac{1-v_i^n}{1-v_{2i}^n} < 1.$$

2.1.31 $10(X+Y) = 10{,}233 \rightarrow X+Y = 1023.3$

$5000 = Y(v+v^3+v^5+\cdots+v^{19}) + X(v^2+v^4+v^6+\cdots+v^{20})$

$\rightarrow 5000 = [vY+v^2X][1+v^2+v^4+\cdots+v^{18}]$

$$= [vY+v^2X]\left[\frac{1-v^{20}}{1-v^2}\right] = 4.720263X + 5.097884Y$$

Solving the two equations for X and Y results in $X = 573.76$ and $Y = 449.54$.

2.1.32 $11{,}000 = 367.21\ [a_{\overline{37|}.01} - v^k] = 11{,}309.89 - 367.21v^k$

$\rightarrow v_{.01}^k = .8439 \rightarrow k = \frac{\ln(.8439)}{\ln(v_{.01})} = 17$

2.1.33 (a) $v^t \cdot s_{\overline{n|}} = \frac{(1+i)^n - 1}{i} = \frac{(1+i)^{n-t} - v^t}{i}$

$$= \frac{(1+i)^{n-t} - 1 + 1 - v^t}{i} = a_{\overline{t|}} + s_{\overline{n-t|}}$$

for $t > n$, $v^t \cdot s_{\overline{n|}} = v^{t-n} \cdot v^n \cdot s_{\overline{n|}} = v^{t-n} \cdot a_{\overline{n|}}$

SECTION 2.2

2.2.1 (a) $X = \frac{50{,}000}{a_{\overline{300|}j}} = 447.24$, where $j = (1.05)^{1/6} - 1$ is the one-month effective rate of interest.

(b) $(X+100)a_{\overline{n|}j} = 50{,}000 \rightarrow a_{\overline{n|}j} = 91.366987$

$$\rightarrow n = \frac{\ln[1-91.366987j]}{\ln[v_j]} = 168.5$$

(or using a calculator function). Then the 168[th] payment of $X+100 = 547.24$ occurs on December 31, 2023, and the amount of the additional final payment will be Y where

$$547.24a_{\overline{168|}j} + Y \cdot v_j^{169} = 50{,}000 \rightarrow Y = 290.30.$$

2.2.2 Derek's accumulated value should be

$$1200\ddot{s}_{\overline{25|}.06} = 1200(1.06) \cdot \frac{(1.06)^{25}-1}{.06} = 69{,}787.66.$$

Anne's accumulated value should be

$$1200s_{\overline{25|}.06} = 1200 \cdot \frac{(1.06)^{25}-1}{.06} = 65{,}837.41.$$

Ira's accumulated value should be $100\ddot{s}_{\overline{300|}j}$, where

$j = (1.06)^{1/12} - 1 = .00486755$ is the equivalent one-month compound interest rate. Then

$$100\ddot{s}_{\overline{300|}j} = 100(1.00486755) \cdot \frac{(1.00486755)^{300}-1}{.00486755} = 67{,}958.10.$$

2.2.3 Quarterly interest rate is j, where $(1.07)^{1/4} - 1 = .01706$.

$450s_{\overline{40|}j}(1.07)^5 = Y\ddot{a}_{\overline{4|}} \rightarrow Y = 9872.$

2.2.4 We denote the 4-year rate of interest by j. Then the accumulated value at the end of 40 years is $X = 100\ddot{s}_{\overline{10}|j}$ (10 4-year periods, with valuation one full 4-year period after the 10^{th} deposit). The accumulated value at the end of 20 years is $100\ddot{s}_{\overline{5}|j}$. We are given that $100\ddot{s}_{\overline{10}|j} = 5\times 100\ddot{s}_{\overline{5}|j}$.

This is equivalent to $\frac{(1+j)^{10}-1}{d_j} = 5\times\frac{(1+j)^5-1}{d_j}$, where $d_j = \frac{j}{1+j}$ is the 4-year discount rate equivalent to the 4-year interest rate j. Factoring the left hand side of the equation, we get

$$\frac{[(1+j)^5-1]\cdot[(1+j)^5+1]}{d_j} = 5\times\frac{(1+j)^5-1}{d_j},$$

from which it follows that $(1+j)^5+1 = 5$, and then $(1+j)^5 = 4$ and $j = .3195$, and $d_j = \frac{.3195}{1.3195} = .2421$. Then

$$X = 100\ddot{s}_{\overline{10}|j} = 100\cdot\frac{(1+j)^{10}-1}{d_j} = 100\cdot\frac{16-1}{.2421} = 6195.$$

2.2.5 Let j = 6-month interest rate, and d_j = 6-month discount rate.

Then $\ddot{a}_{\overline{\infty}|j} = \frac{1}{d_j} = \frac{1+j}{j} = 20 \rightarrow j = \frac{1}{19}$.

Let k = 2-year rate of interest, and d_k = 2-year discount rate.

$X\ddot{a}_{\overline{\infty}|k} = 20 \rightarrow X = 20d_k = 20\cdot\frac{k}{1+k} = 20\left[\frac{(1+j)^4-1}{(1+j)^4}\right]$

(since $(1+j)^4 = 1+k$). Therefore $X = 20\left[\frac{\left(1+\frac{1}{19}\right)^4-1}{\left(1+\frac{1}{19}\right)^4}\right] = 3.71$.

2.2.6 Let P be the monthly payment Sally receives from Tim. Since Sally's yield over the 5 years is 3.725% every 6 months, the value of her accumulated deposits at the end of 5 years must be $10{,}000(1.03725)^{10} = Ps_{\overline{60}|.005}$ (the deposits accumulate at ½% per month). Solving for P results in $P = 206.62$. Let k be the monthly rate on Tim's loan. Then $10{,}000 = 206.62\cdot a_{\overline{60}|k}$. Using the calculator unknown interest function we get $k = .0073$, so that the nominal annual interest rate on Tim's loan is $12k = .088$.

2.2.7 $10{,}000(1.05)^n \geq \frac{2000}{.05} = 40{,}000 \rightarrow (1.05)^n \geq 4$

$$\rightarrow n \geq \frac{\ln(4)}{\ln(1.05)} = 28.4.$$

At time 28 the accumulated value is $10{,}000(1.05)^{28} = 39{,}201.29$ and at time 29 the accumulated value is $10{,}000(1.05)^{29} = 41{,}161.36$. Since 40,000 is the target value of the fund, a reduced scholarship of 1161.36 can be awarded at time 29 (September 1, 1999), while still allowing for the full payment of 2000 in perpetuity from time 30 (September 1, 2000) on.

2.2.8 Monthly effective interest is at rate $j = .0075$, and effective annual interest is at rate $i = (1.0075)^{12}-1 = .09380690$. After n complete years the accumulated value is $100\ddot{s}_{\overline{12n}|j} + 1000s_{\overline{n}|i}$. In order for this to exceed 100,000, we must have (using $d_j = \frac{j}{1+j} = .007444$),

$$\ddot{s}_{\overline{12n}|j} + 10s_{\overline{n}|i} = \frac{(1+i)^n-1}{d_j} + 10\cdot\frac{(1+i)^n-1}{i}$$

$$= 240.9353(1+i)^n - 240.9353 \geq 1000.$$

Thus, $(1+i)^n \geq 5.15 \rightarrow n \geq \frac{\ln(5.15)}{\ln(1+i)} = 18.3$.

$n = 18 \rightarrow \ddot{s}_{\overline{216}|j} + 10s_{\overline{18}|i} = 969.2$, and

$n = 19 \rightarrow \ddot{s}_{\overline{228}|j} + 10s_{\overline{19}|i} = 1078.2$.

The account exceeds 100,000 sometime between January 1 and December 31, 2013. The balance on April 1 after the deposit is 99,521, and the balance on April 30 just after interest is 100,268.

2.2.9 $100s_{\overline{n}|.04} = 200a_{\overline{2n}|.04} \rightarrow v^{2n}+v^n-.50 = 0 \rightarrow v^n = .366025$ (ignore negative root) $\rightarrow n = 25.6 \rightarrow 26$ deposits.

2.2.10 $500s_{\overline{n}|.05} \geq 1000s_{\overline{10}|.05}(1.05)^{n-10} \rightarrow a_{\overline{n}|.05} \geq 2a_{\overline{10}|.05} \rightarrow n \geq 30.32$
On January 1, 2015, Account A has a balance of 33,373 and Account B has a balance of 33,219. On January 1, 2016 the balances are 35,042 and 35,380.

2.2.11 $1000 = 100 \cdot \left[a_{\overline{4}|.035} + v_{.035}^4 \cdot a_{\overline{8}|i}\right] \rightarrow a_{\overline{8}|i} = 7.260287.$
Using the unknown interest calculator function we get $i = 2.208\%$.

2.2.12 For the insurer, after expenses, the profit is
$3368.72\left[(.80)(1.125)^{25} + (.90)\cdot \ddot{s}_{\overline{24}|.125}\right] - 250{,}000 = 234{,}829.$

The rate of return earned by the policyholder is i where
$3368.72\ddot{s}_{\overline{25}|i} = 250{,}000 \rightarrow i = .076.$

2.2.13 $12{,}000 = 592.15a_{\overline{24}|j} = 426.64a_{\overline{36}|j}$

$\rightarrow \frac{592.15}{426.64}[1+v^{12}] = 1 + v^{12} + v^{24} \rightarrow v^{12} = .846321$

$\rightarrow j = .0140 \rightarrow i^{(12)} = .1680.$
$12{,}000 = K \cdot a_{\overline{48}|j} \rightarrow K = 345.02.$

2.2.14 Three-year rate of interest is $j = (1+i)^3 - 1$. PV of perpetuity starting in 6 years (two 3-year periods) is $v_j \cdot \frac{10}{j} = \frac{10}{(1+j)j} = 32$ $\rightarrow 32j^2 + 32j - 10 = 0 \rightarrow j = .25$ or -1.25 (we ignore negative root). Therefore $(1+i)^3 = 1.25$. Let the 4-month interest rate be k. Then $(1+k)^3 = 1+i = (1.25)^{1/3}$. PV of perpetuity-immediate of 1 every 4 months is $X = \frac{1}{k} = \frac{1}{(1.25)^{1/9} - 1} = 39.84.$

2.2.15 The 2-month effective rate of interest is j.

(a) $25a_{\overline{36}|j} = 150a^{(6)}_{\overline{6}|.06} = 755.83$, where $j = (1.06)^{1/6} - 1$

(b) $25v_j^4\, a_{\overline{36}|j} = 50v_{.02}^2 a^{(2)}_{\overline{18}|.02} = 724.08$, where $j = (1.02)^{1/2} - 1$

(c) $25(1+j)s_{\overline{36}|j} = 1092.02$, where $j = (.97)^{-1/3} - 1$

(d) $25(1+j)^6 s_{\overline{36}|j} = 1144.57$, where $j = e^{.01} - 1$

2.2.16 The series is the same as a perpetuity-immediate of 1 per month plus a perpetuity-immediate of 1 per year. At monthly rate j the present value of the monthly perpetuity is $\frac{1}{j}$, and the present value of the annual perpetuity is $\frac{1}{i}$, where $i = (1+j)^{12} - 1 = j \cdot s_{\overline{12}|j}$ is the equivalent effective annual rate of interest.

2.2.17 $\int_0^u \delta_y\, dy = \int_0^u \left[p + \frac{se^{-st}}{e^{-st}+r}\right] dt = pu - \ln\left[\frac{e^{-su}+r}{1+r}\right]$

$\rightarrow e^{-\int_0^u \delta_y\, dy} = \left[\frac{e^{-su}+r}{1+r}\right]\cdot e^{-pu}$

$\rightarrow \bar{a}_{\overline{n}|} = \int_0^n \left[\frac{e^{-su}+r}{1+r}\right]\cdot e^{-pu}\, du = \frac{r(1-e^{-pn})}{(1+r)p} + \frac{(1-e^{-(p+s)n})}{(1+r)(p+s)}$

2.2.18 $L = K' \cdot a_{\overline{n/2}|i} \rightarrow$

$$K' = \frac{L}{a_{\overline{n/2}|i}} = \frac{Lj}{1-v^{n/2}} = \frac{Lj}{1-v^n}[1+v^{n/2}]$$
$$= \frac{L}{a_{\overline{n}|i}}[1+v^{n/2}]$$
$$= K[1+v^{n/2}] \leq 2K$$

2.2.19 (a) The 6-month effective interest rate is

$$j = (1.01)^3 - 1 = .030301.$$

$$500 s_{\overline{n}|j} = 10{,}000 \rightarrow n = \frac{\left[\ln\left[1+\frac{10{,}000j}{500}\right]\right]}{\ln(1+j)} = 15.87.$$

With the 15^{th} deposit (January 1, 2006, the balance is $500 s_{\overline{15}|j} = 9320.00$. With interest (1% every 2 months) on February 28 the balance is $9320.00(1.01) = 9413.20$, on April 30 it is 9507.33, and on June 30 it is 9602.41. On July 1, 2006, the deposit of 500 brings the balance to 10,102.41.

(b) The 6-month effective interest rate is

$j = (1.04)^{1/2} - 1 = .01980390.$

$n = 17.01$. With the 17^{th} deposit the balance is 9989.75. $9989.75[1+(.04)t] = 10{,}000 \rightarrow t = .025652$ years, or 9.4 days. Close the account on January 11, 2007.

2.2.20 (a) $s_{\overline{20}|.03}(1.04)^n + s_{\overline{n}|.04} \geq 100 \rightarrow 51.870375(1.04)^n \geq 125$

$\rightarrow n \geq 22.4 \quad (23)$

(b) $s_{\overline{n}|.03}(1.04)^n + s_{\overline{n}|.04} \geq 100.$

Trial and error:

$n = 21 \rightarrow 97.316; \qquad n = 22 \rightarrow 106.618.$

2.2.21 $n \cdot \ln(1.0075) = .008333n - \frac{(.008333)^2 \cdot n^2}{2} \rightarrow n = 24.8$

$n \cdot \ln(1.0075) = .008333n - \frac{(.008333)^2 \cdot n^2}{2} + \frac{(.008333)^3 \cdot n^3}{3}$

$\rightarrow n = 29.7$ or 300.6 (300.6 is an unrealistic answer)

2.2.22 $B = A + (1+i)^n \rightarrow (1+i)^n = B - A$

$\rightarrow A = \frac{(1+i)^n - 1}{i} = \frac{B-A-1}{i} \rightarrow i = \frac{B-A-1}{A}$

2.2.23 (a) Follows from the Intermediate Value Theorem of calculus.

(b) (i) $\lim_{i\to\infty} s_{\overline{n}|i} = \infty$, (ii) $\lim_{i\to -1} s_{\overline{n}|i} = 0$, and (iii) $s_{\overline{n}|i}$ is an increasing function of i. If $J, n > 0$ and $M > 0$, then the equation has a unique solution for i.

2.2.24 $(1+i)A + (B-A-I)\bar{s}_{\overline{1}|i} = B \rightarrow (1+i)A + (B-A-I)\left(1+\frac{i}{2}\right) \doteq B$

$\rightarrow i \doteq \frac{2I}{A+B-I}$

2.2.25 (a) $s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{\left[(1+j)^m\right]^n - 1}{(1+j)^m - 1}$

$\ddot{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{1 - v_i} = \frac{\left[(1+j)^m\right]^n - 1}{1 - v_j^m}$

$\ddot{a}_{\overline{n}|i} = \ddot{s}_{\overline{n}|i} \cdot v_j^{mn} = \frac{1 - v_j^{mn}}{1 - v_j^m}$

(b) $s_{\overline{n}|i} = \frac{\left[(1+j)^m\right]^n - 1}{(1+j)^m - 1} = \frac{\frac{\left[(1+j)^m\right]^n - 1}{j}}{\frac{(1+j)^m - 1}{j}} = \frac{s_{\overline{mn}|j}}{s_{\overline{m}|j}}$

(c) $1+i = e^{\delta} \rightarrow s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = \frac{e^{n\delta} - 1}{e^{\delta} - 1}$

(d) $a_{\overline{\infty}|i} = \frac{1}{i} = \frac{1}{(1+j)^m - 1} = \frac{a_{\overline{\infty}|j}}{s_{\overline{m}|j}}$

$\ddot{a}_{\overline{\infty}|i} = \frac{1}{d} = \frac{1}{1 - v_j^m} = \frac{a_{\overline{\infty}|j}}{a_{\overline{m}|j}}$

2.2.26 $1+i = (1+j)^m \quad 1+j = (1+i)^{1/m} \quad v_i = v_j^m \quad v_j = v_i^{1/m}$

(a) $\frac{1}{m}\cdot s_{\overline{nm}|j} = \frac{1}{m}\cdot\frac{(1+j)^{nm}-1}{j} = \frac{1}{m}\cdot\frac{(1+i)^n-1}{(1+i)^{1/m}-1}$

2.2.27 (a) $\lim_{m\to\infty} s_{\overline{n}|i}^{(m)} = \lim_{m\to\infty}\frac{(1+i)^n-1}{i^{(m)}} = \frac{(1+i)^n-1}{\lim_{m\to\infty} i^{(m)}} = \frac{(1+i)^n-1}{i^{(\infty)}}$

(b) Since $d < d^{(m)} < \delta < i^{(m)} < i$, it follows that

$$a_{\overline{n}|i} < a_{\overline{n}|i}^{(m)} < \bar{a}_{\overline{n}|i} < \ddot{a}_{\overline{n}|i}^{(m)} < \ddot{a}_{\overline{n}|i}.$$

(c) $\bar{s}_{\overline{n}|i} = \int_0^1[1+i(1-t)]\,dt = 1+\frac{i}{2}$

SECTION 2.3

2.3.1 The annuity is paid monthly, and the interest rate is quoted as .5% per month, but the geometric increase in the payments occurs once per year. In order to use the geometric payment annuity present value formula $K\left[\frac{1-\left(\frac{1+r}{1+j}\right)^n}{j-r}\right]$, the payment period, interest period and geometric growth period must coincide. In a situation such as this, where those periods do not coincide, it is necessary to conform to the geometric growth period, which, in this case, is one year with $r=.05$. The equivalent interest rate per year is the effective annual rate $i=(1.005)^{12}-1=.06168$. Since the payments are at the ends of successive months, for each year we must find a single payment at the end of each year that is equivalent to the monthly payments for that year. For the first year, the single payment at the end of the year that is equivalent in value to the 12 monthly payments during the first year is $2000s_{\overline{12}|.005}=24,671=K$.

The monthly payments in the second year are each 2000(1.05), so that the single payment at the end of the second year that is equivalent in value to the 12 monthly payments during the second year is $2000(1.05)s_{\overline{12}|.005}=24,677(1.05)=K(1.05)$. In a similar way, the single payments at the ends of the successive years that are equivalent in value to the monthly payments during those year are K, $K(1.05)$, $K(1.05)^2,\ldots,K(1.05)^{19}$ (the 20^{th} year would have had 19 years of growth in the payment amount). Now, we have interest period, (equivalent) payment period and geometric growth period all being 1 year, so that the present value of the annuity, valued one year before the first equivalent annual payment, is

$$K\left[\frac{1-\left(\frac{(1+r)}{1+i}\right)^n}{i-r}\right] = (24,671)\left[\frac{1-\left(\frac{1.05}{1.06168}\right)^{20}}{.06168-.05}\right] = 419,242.$$

This answer is based on some roundoff. If exact calculator values are used, the answer is 419,253.

2.3.2 (i) $1000\left[(1.01)^{29}+(.99)(1.01)^{28}+\cdots+(.99)^{29}\right]$

$$= 1000(1.01)^{29}\left[\frac{1-\left(\frac{.99}{1.01}\right)^{30}}{1-\frac{.99}{1.01}}\right] = 30,407$$

(ii) 59,704

(iii) 151,906

2.3.3 $k\% = .01k$ in decimal form. The present value of the perpetuity-immediate is $30a_{\overline{\infty}|.01k}=\frac{30}{.01k}$. The 10-year annuity has geometrically increasing payments, with $r=.01k$, and the valuation rate for present value is $i=.01k$. Since $i=r$, the present value of the geometrically increasing annuity is

$$Knv=(53)(10)\left(\frac{1}{1+.01k}\right)=\frac{530}{1+.01k}.$$

We are told that Jeff and Jason each use the same amount to purchase their annuities, and therefore $\frac{30}{.01k}=\frac{530}{1+.01k}$. Solving for k results in $k=6(\%)$.

2.3.4
$$PV = 10v + 10v^2 + 10v^3 + 10v^4 + 10v^5 + 10(1+.01K)v^6 + 10(1+.01K)^2 v^7 + \cdots$$
$$= 10a_{\overline{4}|.092} + 10v^4\left[v + (1+.01K)v^2 + (1+.01K)^2 v^3 + \cdots\right]$$
$$= 10a_{\overline{4}|.092} + 10v^4 \frac{1}{i-.01K} = 167.50 \rightarrow K = 4.$$

2.3.5 In order to use the geometric payment annuity formula, the payment period, interest period and geometric growth period must all coincide. In this case the payments are monthly and the geometric growth (inflation) is annual. We deal with this situation by determining a single annual payment at the end of each year which is equal to the accumulated value of the 12 monthly payments for that year. Suppose that first year's monthly payment is R. Then a single payment at the end of the year that is equivalent to the 12 month-end payments is $Rs_{\overline{12}|j}$, where j is the monthly interest rate that is found from the equation $(1+j)^{12} = 1.06$. Therefore, the equivalent annual payment for the first year is $Rs_{\overline{12}|j} = 12.3265R$.

In the second year the monthly payments are $1.032R$, so the single payment at the end of the second year that is equivalent to the monthly payments in the second year is

$$1.032Rs_{\overline{12}|j} = (1.032)(12.3265R).$$

In the same way we can see that the monthly payment in the third year is equivalent to a single year end payment of

$$(1.032)^2 Rs_{\overline{12}|j} = (1.032)^2(12.3265R).$$

This pattern continues to the 20^{th} year, when the monthly payment is equivalent to a single year end payment of

$$(1.032)^{19} Rs_{\overline{12}|j} = (1.032)^{19}(12.3265R).$$

The present value of the equivalent annual payments is

$$12.3265R \cdot \left[\frac{1-\left(\frac{1.032}{1.06}\right)^{20}}{.06-.032}\right].$$

We are told that the buyout package (present value) has a value of 100,000.

Therefore $12.3265R \cdot \left[\frac{1-\left(\frac{1.032}{1.06}\right)^{20}}{.06-.032}\right] = 100{,}000$, from which we get $R = 548$.

2.3.6 (a) $(1+r)^n \ddot{s}_{\overline{n}|j} = (1+r)^n \frac{\left(\frac{1+i}{1+r}\right)^n - 1}{1+\frac{1+r}{1-i}} = \frac{(1+i)^n - (1+r)^n}{1-\frac{1+r}{1-i}}$

$$AV = (1+i)^n \cdot PV = \frac{(1+i)^n - (1+r)^n}{1-\frac{1+r}{1+i}}$$

(b) Use $\frac{1}{1+i} = \frac{1}{1+r} v_j$

2.3.7
$$100{,}000 = 2000(v + v^2 + v^3)\left[1 + v^3(1+r) + v^6(1+r) + \cdots\right]$$
$$= \frac{2000a_{\overline{3}|.045}}{1-v^3(1+r)} \rightarrow r = .0784$$

2.3.8 (a) Final salary is $18{,}000(1.04)^{36} = 73{,}871$. Total career salary is $18{,}000\left[1 + (1.04) + (1.04)^2 + \cdots + (1.04)^{36}\right] = 18{,}000s_{\overline{37}|.04} = 1{,}470{,}640$, so career average annual salary is 39,747. Pension is $(.70)(39{,}747) = 27{,}823 = (.377)(73{,}871)$.

(b) $(37)(.025)(39{,}747) = 36{,}766$

(c) Average salary in final 10 years is
$(.10)(18{,}000)\left[(1.04)^{27}+(1.04)^{28}+\cdots+(1.04)^{36}\right] = 62{,}312$
Pension is $(.025)(37)(62{,}312)=57{,}639$.

(d) Accumulated amount after 37 years is

$$(.06)(18{,}000)\left[(1.06)^{36}+(1.04)(1.06)^{35}+\cdots+(1.04)^{36}\right](1.06)^{1/2}$$

$$=(.06)(18{,}000)(1.06)^{36}\left[\frac{1-\left(\frac{1.04}{1.06}\right)^{37}}{1-\frac{1.04}{1.06}}\right](1.06)^{1/2}=242{,}845.$$

Then $242{,}845 = X\cdot\ddot{a}_{\overline{20}|.06} \rightarrow X = 19{,}974$.

2.3.9 The total payout over 20 years is

$$2000\times12\times\left[1+1.03+(1.03)^2+\cdots+(1.03)^{19}\right] = 24{,}000\times\frac{(1.03)^{20}-1}{1.03-1} = 644{,}889.$$

Note that in the 20th year, there will have been 19 annual inflationary increases since the first year. We formulate the present value in a way that is similar to that in Example 2.18 and in Exercise 2.3.5 above. The value at the end of each year of that year's payments is $P\cdot s_{\overline{12}|j}$, where $j = (1+i)^{1/12}-1$ is the equivalent monthly rate of interest and P is the monthly payment. The monthly payments are 2000 in the first year, $2000(1.03)$ in the second year, $2000(1.03)^2$ in the third year, ..., $2000(1.03)^{19}$ in the 20^{th} year. Now using the equivalent annual payment at the end of each year, the present value is

$$2000\cdot s_{\overline{12}|j}\times\left[v_i+(1.03)v_i^2+(1.03)^2v_i^3+\cdots+(1.03)^{19}v_i^{20}\right] = 2000\cdot s_{\overline{12}|j}\times\frac{1-\left(\frac{1.03}{1+i}\right)^{20}}{i-.03}.$$

We set this equal to the given present value of 346,851 and solve for i. This requires a numerical solution. MS EXCEL Solver gives a solution of $i=.0640$.

2.3.10 $X = Z\left[7+.05(Is)_{\overline{6}|.06}\right] = 8.1615Z$,

$$Y = Z\left[14+.025(Is)_{\overline{13}|.03}\right] = 16.5719Z \rightarrow \frac{Y}{X} = \frac{16.57}{8.16} = 2.03.$$

2.3.11 Sandy's annuity has present value
$90a_{\overline{\infty}|i}+10(Ia)_{\overline{\infty}|i} = \frac{90}{i}+10\left(\frac{1}{i}+\frac{1}{i^2}\right)$.
Danny's annuity has present value $180\ddot{a}_{\overline{\infty}|i} = \frac{180}{d}$.
We are told that $\frac{90}{i}+10\left(\frac{1}{i}+\frac{1}{i^2}\right) = \frac{180(1+i)}{i}$.
We solve the quadratic equation $18i^2+8i-1 = 0$ which results in $i=.102$ (ignore the negative root $-.346$).

2.3.12 With monthly rate j, $X=2(Ia)_{\overline{60}|j}$.

We are given 3-month rate

$.0225 \rightarrow (1+j)^3 = 1.0225 \rightarrow j=.007444$.

$$X = 2\cdot\frac{\ddot{a}_{\overline{60}|.00744}-60v_{.007444}^{60}}{.007444} = 2729.$$

2.3.13 The progression of fund X and deposits to fund Y are described in the following timeline.

Fund X *earns* interest at rate 6%.

	Amount in Fund X	Deposit to Fund Y	Fund Y Interest
0	1000		
1	900	100	+60
2	800	100	+54
3	700	100	+48
4	600	100	+42
5	500	100	+36
6	400	100	+30
7	300	100	+24
8	200	100	+18
9	100	100	+12
10	0	100	+6

The deposits into Fund Y consist of a combination of level deposits of 100 each for 10 years, along with decreasing deposits. The accumulated value in Fund Y is

$$100s_{\overline{10}|.09}+6(Ds)_{\overline{10}|.09} = 1519.30+565.38 = 2085.$$

2.3.14 The timeline of the payments is

	0	1	2	3	4	…	n	$n+1$	$n+2$
Pmt.			1	2	3	…	$n-1$	n	n

The schedule of payments can be written as the combination of two series

Time	1	2	3	4	…	n	$n+1$	$n+2$	
	n	n	n	n	…	n	n	n	…
	$-n$	$-(n-1)$	$-(n-2)$	$-(n-3)$	…	-1			

In other words, the perpetuity can be written as a level perpetuity immediate of n per year <u>minus</u> a decreasing n-year annuity-immediate whose payments start at n and decrease by 1 per year.

The present value of the perpetuity can be formulated as the combination of the two present values.

$$PV = na_{\overline{\infty}|.105}-(Da)_{\overline{n}|.105} = n\left(\frac{1}{.105}\right)-\frac{n-a_{\overline{n}|}}{.105} = \frac{a_{\overline{n}|}}{.105} = 77.1.$$

Therefore $a_{\overline{n}|.105} = 8.0955 \rightarrow n = 19.$

2.3.15 $12{,}000 = 395a_{\overline{36}|.01}+X\left[a_{\overline{12}|.01}-v^{24}a_{\overline{12}|.01}\right] \rightarrow X = 44.98$

2.3.16 $A+nB, A+(n-1)B, A+(n-2)B,\ldots,A+2B, A+B$

The series of payments is $100, 97, 94,\ldots, 31, 28 \equiv 25+(25)(3)$, $25+(24)(3),\ 25+(23)(3),\ldots,\ 25+(2)(3), 25+3$. the PV is $25a_{\overline{25}|}+3(Da)_{\overline{25}|}$.

2.3.17 If the deposits had been made at the 8% rate then the accumulated value at the end of 20 years would be $300\ddot{s}_{\overline{20}|.08}=14{,}827$. The actual investment accumulates to

$$300(20)+300i(Is)_{\overline{20}|i/2} = 6000+300i\left[\frac{\ddot{s}_{\overline{20}|i/2}-20}{i/2}\right] = 6000+600\left[\ddot{s}_{\overline{20}|i/2}-20\right],$$

and we set this equal to 14,827. Therefore

$$6000+600\left[\ddot{s}_{\overline{20}|i/2}-20\right] = 14{,}827 \rightarrow \ddot{s}_{\overline{20}|i/2} = 34.71 \rightarrow \frac{i}{2}=.05.$$

2.3.18 PV of Annuity 2 = 2×(PV of Annuity 1)

$\rightarrow 11a_{\overline{\infty}|} - (Da)_{\overline{10}|} = 2(Da)_{\overline{10}|}$

$\rightarrow 3(Da)_{\overline{10}|} = 11a_{\overline{\infty}|} \rightarrow 3\left[\frac{10-a_{\overline{10}|}}{i}\right] = \frac{11}{i}$

$\rightarrow 3a_{\overline{10}|} = 19 \rightarrow a_{\overline{10}|} = \frac{19}{3} \rightarrow i = .093 \rightarrow (Da)_{\overline{10}|} = 39.4.$

2.3.19 If j is the monthly effective interest rate, i the effective annual interest rate and r the annual inflation rate used for valuation purposes, then the present value of the perpetuity-immediate is $\frac{400s_{\overline{12}|j}}{i-r}$.

(i) PV before deindexing = 168,620, PV after deindexing = 84,310

(ii) PV before deindexing = 56,207, PV after deindexing = 42,155

(iii) PV before deindexing = 166,497, PV after deindexing = 83,249

(iv) PV before deindexing = 164,354, PV after deindexing = 82,177

2.3.20 (a) $500{,}000 = 1000 \cdot nv \rightarrow n = 505$

(b) Balance just after t^{th} withdrawal is

$1000(1.01)^t\left[v + (1.01)v^2 + \cdots + (1.01)^{504-t}v^{505-t}\right]$

$= 1000(1.01)^t(505-t)v = f(t).$

$f'(t) = 1000v(1.01)^t\left[(505-t)\cdot\ln(1.01) - 1\right] = 0$

$\rightarrow t = 505 - \frac{1}{\ln(1.01)} = 404.5$

At $t = 404$ the balance is

$1000(1.01)^{404}(505-404)v = 5{,}569{,}741$, and at $t = 405$ the balance is $1000(1.01)^{405}(505-405)v = 5{,}569{,}741$

2.3.21 (a) $100{,}000 = 6250a_{\overline{20}|} + 750(Ia)_{\overline{20}|}$; $i = .1014$

(b) $100{,}000 = 7000\left[\frac{1-\left(\frac{1.10}{1+i}\right)^{20}}{i-.10}\right]$; $i = .1266$

2.3.22 (a) $X = (1+i)^{n-1} + 2(1+i)^{n-2} + 3(1+i)^{n-3} + \cdots + (n-1)(1+i) + n$

$(1+i)X = (1+i)^n + 2(1+i)^{n-1} + 3(1+i)^{n-2}$

$+ \cdots + (n-1)(1+i)^2 + n(1+i)$

$\rightarrow iX = (1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) - n = \ddot{s}_{\overline{n}|i} - n$

(b) $s_{\overline{n+1}|i}$ is $n+1$ payments of 1 each plus interest on an increasing total deposit.

2.3.23 (a) $(Ia)_{\overline{n}|} + (Da)_{\overline{n}|}$

$= (v + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1} + nv^n)$

$+ \left(nv + (n-1)v^2 + (n-2)v^3 + \cdots + 2v^{n-1} + v^n\right)$

$= (n+1)v + (n+1)v^2 + (n+1)v^3 + \cdots + (n+1)v^n$

$= (n+1)a_{\overline{n}|}$

(b) $\sum_{k=0}^{n-1} {}_{k|}a_{\overline{n-k}|} = \sum_{k=0}^{n-1} v^k \cdot \frac{1-v^{n-k}}{i}$

$= \sum_{k=0}^{n-1} \frac{v^k - v^n}{i}$

$= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$

$= (Ia)_{\overline{n}|}$

2.3.24 $(Ia)_{\overline{\infty|}} = \lim_{n\to\infty} (Ia)_{\overline{n|}} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n|}} - n\cdot v^n}{i}$

$$= \frac{\ddot{a}_{\overline{\infty|}}}{i} - \lim_{n\to\infty} \frac{n\cdot v^n}{i}$$

$$= \frac{1}{d\cdot i} - 0$$

The increasing perpetuity immediate can be looked at as a combination of a level perpetuity immediate of 1 per year, and each year another perpetuity immediate of 1 per year starts up, so that the annual payment grows by 1 every year forever.

2.3.25 (a) (i) $\frac{d}{di}(v + v^2 + v^3 + \cdots + v^n)$

$$= \frac{d}{di}\left[(1+i)^{-1} + (1+i)^{-2} + (1+i)^{-3} + \cdots + (1+i)^{-n}\right]$$

$$= -(1+i)^{-2} - 2(1+i)^{-3} - 3(1+i)^{-4} - \cdots - n(1+i)^{-n-1}$$

$$= -v(Ia)_{\overline{n|}}$$

(b) $\frac{d}{dn}\int_0^n v^t\, dt = v^n$

2.3.26 Using the chain rule with $K = \frac{100{,}000}{a_{\overline{300|}j}}$

$$\rightarrow \frac{d}{di^{(2)}} K = \frac{100{,}000 v_j (Ia)_{\overline{300|}j}}{\left[a_{\overline{300|}j}\right]^2} \cdot \frac{d}{di^{(2)}} j, \text{ and}$$

$$\frac{d}{di^{(2)}} j = \frac{d}{di^{(2)}}\left[\left(1+\frac{i^{(2)}}{2}\right)^{1/6} - 1\right] = \frac{1}{12}\left(1+\frac{i^{(2)}}{2}\right)^{-5/6}.$$

The numerical values of the derivative are

$i^{(2)} = .21 \rightarrow \frac{d}{di^{(2)}} K = 7459.13$ (or 74.59 per 1% increase in $i^{(2)}$)

$i^{(2)} = .13 \rightarrow \frac{d}{di^{(2)}} K = 7101.66$ (or 71.02 per 1% increase in $i^{(2)}$).

2.3.27 Balance December 31, 2004 is $500{,}000(1+i)$; withdrawal January 1, 2005 is $\frac{500{,}000(1+i)}{19}$, leaving a balance of $\frac{18}{19}\cdot 500{,}000(1+i)$. Balance December 31, 2005 is $\frac{18}{19}\cdot 500{,}000(1+i)^2$; withdrawal January 1, 2006 is $\frac{18}{19}\cdot\frac{500{,}000(1+i)^2}{18} = \frac{500{,}000(1+i)^2}{19}$, leaving a balance of $\frac{17}{19}\cdot 500{,}000(1+i)^2$. Withdrawal on January 1, $2004+t$ is $\frac{500{,}000(1+i)^t}{19}$.

2.3.28 (a) $100{,}000 = 1000 a_{\overline{n|}.0075}$

$\rightarrow n = 185.5$

$\rightarrow 100{,}000 = 1000 a_{\overline{185|}.0075} + Xv^{186}$

$\rightarrow X = 532.46$

(b) $100{,}000 = 990 a_{\overline{n|}.0075} + 10(Ia)_{\overline{n|}.0075}$

$\rightarrow$ (by trial and error) $n = 99$

$\rightarrow 100{,}000 = 990 a_{\overline{99|}.0075} + 10(Ia)_{\overline{99|}.0075} + Xv^{100}$

$\rightarrow X = 761.19$

(c) $100{,}000 = \frac{1000}{1.01} a_{\overline{n|}j}$,

where $v_j = \frac{1.01}{1.0075}$

$\rightarrow n = 90$

$\rightarrow 100{,}000 = \frac{1000}{1.01} a_{\overline{90|}j} + Xv^{91}_{.0075}$

$\rightarrow X = 93.85$

(d) Total withdrawn: (a)185,532, (b) 148,271, (c)144,957. The more rapidly the payments increase, the more quickly the account is exhausted and the smaller the total withdrawn.

2.3.29 $100(Is)_{\overline{12}|}(1+i)^6 = 17{,}177.70$. At $i=.08$, the left-hand side is 16,851.21 and at $i=.085$ it is 17,679.23. By interpolation we get $i \doteq .08197$ (the exact value is .0820).

2.3.30 For each $t<\left(\frac{n+1}{2}\right)$, $tv^t+(n-t+1)v^{n-t+1} < \left(\frac{n+1}{2}\right)v^t+\left(\frac{n+1}{2}\right)v^{n-t+1}$.

This is true since $t<\left(\frac{n+1}{2}\right)$ implies that $n-t+1>t$ so that $\left[\frac{n+1}{2}-t\right]v^{n-t+1} < \left[\frac{n+1}{2}-t\right]v^t$, which is equivalent to $tv^t+(n-t+1)v^{n-t+1} < \left(\frac{n+1}{2}\right)v^t+\left(\frac{n+1}{2}\right)v^{n-t+1}$. Then

$$\begin{aligned}(Ia)_{\overline{n}|} &= v+2v^2+3v^3+\cdots+(n-1)v^{n-1}+nv^n \\ &= [v+nv^n]+\left[2v^2+(n-1)v^{n-1}\right]+\cdots \\ &< \left(\frac{n+1}{2}\right)\left[(v+v^n)+(v^2+v^{n-1})+\cdots\right] \\ &= \left(\frac{n+1}{2}\right)a_{\overline{n}|}.\end{aligned}$$

2.3.31 (a)
$$\begin{aligned}PV = X &= 1+2v^k+3v^{2k}+4v^{3k}+\cdots \\ Xv^k &= v^k+2v^{2k}+3v^{3k}+\cdots \\ \rightarrow X(1-v^k) &= 1+v^k+v^{2k}+v^{3k}+\cdots = \frac{1}{1-v^k} \\ \rightarrow X &= \frac{1}{(1-v^k)^2} = \frac{1}{(ia_{\overline{k}|})^2}\end{aligned}$$

(b)
$$\begin{aligned}Y &= (1+v+v^2+\cdots+v^{k-1})+2(v^k+v^{k+1}+v^{k+2}+\cdots+v^{2k-1}) \\ &\quad +3(v^{2k}+v^{2k+1}+v^{2k+2}+\cdots+v^{3k-1})+\cdots \\ &= \ddot{a}_{\overline{k}|}\left[1+2v^k+3v^{2k}+\cdots\right] = \frac{\ddot{a}_{\overline{k}|}}{(ia_{\overline{k}|})^2} \quad \text{(from part (a))}\end{aligned}$$

2.3.32 Since all t_r's and K_r's are >0, $f(i)$ in Example 2.23 is a decreasing function of i, and $f(0)=\sum_{r=1}^{n}K_r$. Thus, in solving $f(i)=L$, if $f(i)=L>\sum_{r=1}^{n}K_r$ then $i<0$, and if $f(i)=L<\sum_{r=1}^{n}K_r$, then $i>0$.

2.3.33 Let $f(i)=K_1(1+i)^{t_n-t_1}+K_2(1+i)^{t_n-t_2}+\cdots+K_{n-1}(1+i)^{t_n-t_{n-1}}+K_n$. Since $t_n>t_r$ for $r<n$, it follows that $f(i)$ is an increasing function of i. Also, $\lim_{i\to -1} f(i)=0$ and $\lim_{i\to\infty} f(i)=\infty$. It follows that if $L>0$, there is a unique $i>-1$ for which $f(i)=L$.

2.3.34
$$\begin{aligned}PV &= 1+(1+2)v+(1+2+3)v^2+(1+2+3+4)v^3+\cdots \\ &= [1+v+v^2+v^3+\cdots]+2v[1+v+v^2+\cdots] \\ &\quad +3v^2[1+v+v^2+\cdots]+\cdots \\ &= [1+v+v^2+v^3+\cdots][1+2v+3v^2+\cdots] = \ddot{a}_{\overline{\infty}|}(I\ddot{a})_{\overline{\infty}|}\end{aligned}$$

2.3.35
$$\begin{aligned}PV &= v+2v^2+3v^3+\cdots+(n-1)v^{n-1}+nv^n+(n-1)v^{n+1} \\ &\quad +\cdots+2v^{2n-2}+v^{2n-1} \\ &= [v+v^2+v^3+\cdots+v^n]+[v^2+v^3+\cdots+v^{n+1}] \\ &\quad +[v^3+v^4+\cdots+v^{n+1}] \\ &\quad +\cdots+[v^n+v^{n+1}+v^{n+2}+\cdots+v^{2n-1}] \\ &= a_{\overline{n}|}[1+v+v^2+\cdots+v^{n-1}] = a_{\overline{n}|}\ddot{a}_{\overline{n}|}\end{aligned}$$

2.3.36
$$\begin{aligned}PV &= v+2v^2+2v^3+\cdots+(n-1)v^{n-1}+nv^n+nv^{n+1}+nv^{n+2}+\cdots \\ &= (Ia)_{\overline{n}|}+nv^n(v+v^2+v^3+\cdots) = \frac{\ddot{a}_{\overline{n}|}-nv^n}{i}+nv^n\cdot\frac{1}{i}\end{aligned}$$

2.3.37 (a) $\lim_{m\to\infty}(I^{(m)}a)^{(m)}_{\overline{n}|} = \lim_{m\to\infty}\frac{\ddot{a}^{(m)}_{\overline{n}|}-nv^n}{i^{(m)}} = \frac{\ddot{a}_{\overline{n}|}-nv^n}{\delta} = (\overline{I}\,\overline{a})a_{\overline{n}|}$.

(b) $(I\overline{a})a_{\overline{n}|}$ is the present value of an n-year continuously payable annuity for which the rate of payment is 1 per year during the first year, the rate of payment is 2 per year during the second year, etc.

2.3.38 The accumulated value at time t is

$$F_t = F_0\cdot e^{\delta t} + \int_0^t h(s)\cdot e^{\delta(t-s)}\,ds.$$

Then,

$$\frac{d}{dt}F_t = F_0\cdot\delta e^{\delta t} + h(t)\cdot e^{\delta(t-t)} + \int_0^t h(s)\cdot e^{\delta(t-s)}\cdot\delta\,ds = \delta F_t + h(t).$$

2.3.39 (a)
$$\begin{aligned}(\overline{D}\overline{a})_{\overline{n}|} &= \int_0^n (n-t)v^t\,dt \\ &= \int_0^n (n-t)\,d\left(\frac{v^t}{\ln v}\right) \\ &= (n-t)\left(\frac{v^t}{\ln v}\right)\Big|_0^n - \int_0^n\left(-\frac{v^t}{\ln v}\right)dt \\ &= -\frac{n}{\ln v}+\frac{\overline{a}_{\overline{n}|}}{\ln v} \\ &= \frac{\overline{a}_{\overline{n}|}-n}{-\delta} \\ &= \frac{n-\overline{a}_{\overline{n}|}}{\delta}\end{aligned}$$

$$\begin{aligned}(\overline{I}\overline{a})_{\overline{n}|}+(\overline{D}\overline{a})_{\overline{n}|} &= \int_0^n tv^t\,dt + \int_0^n (n-t)v^t\,dt \\ &= \int_0^n nv^t\,dt \\ &= n\overline{a}_{\overline{n}|}\end{aligned}$$

(b)
$$\begin{aligned}\int_0^n t\overline{a}_{\overline{n-t}|}\,dt &= \int_0^n v^t\left(\frac{1-v^{n-t}}{\delta}\right)dt \\ &= \int_0^n \frac{v^t-v^n}{\delta}\,dt = \frac{\overline{a}_{\overline{n}|}-nv^n}{\delta} = (\overline{I}a)_{\overline{n}|}\end{aligned}$$

2.3.40 The present value is

$$\begin{aligned}&Av+(A+B)v^2+(A+2B)v^3+\cdots(A+(n-1)B)v^n \\ &= (A-B)(v+v^2+v^3+\cdots v^n)+Bv+2Bv^2+3Bv^3+\cdots+nBv^n \\ &= (A-B)a_{\overline{n}|}+B(Ia)_{\overline{n}|}\end{aligned}$$

The accumulated value is $(A-B)s_{\overline{n}|}+B(Is)_{\overline{n}|} = As_{\overline{n}|}+B(Is)_{\overline{n-1}|}$

SECTION 2.4

2.4.1 (a) (i) $1000a_{\overline{20}|.12} = 7469.44$

(ii) $\frac{1000s_{\overline{20}|.06}}{1+.12s_{\overline{20}|.06}} = 6794.19$

(iii) $1000v_{.12}^{20}\cdot s_{\overline{20}|.06} = 3813.44$

(b) (i) $7469.44 = v_i^{20}\cdot 1000s_{\overline{20}|.06} \rightarrow v_i^{20} = .203053$

$\rightarrow i = .0830$

(ii) $6749.19 = 1000a_{\overline{20}|i} \rightarrow i = .1356$

(iii) $6749.19 = 1000\cdot v_i^{20}\cdot s_{\overline{20}|.06} \rightarrow v_i^{20} = .18347374$

2.4.2 If $i<j$ then $s_{\overline{n}|j} > s_{\overline{n}|i}$, and $a_{\overline{n}|j} < a_{\overline{n}|i}$ so that

$(1+i')^n = \frac{s_{\overline{n}|j}}{a_{\overline{n}|i}} > \frac{s_{\overline{n}|i}}{a_{\overline{n}|i}} = (1+i)^n$ and

$(1+i')^n = \frac{s_{\overline{n}|j}}{a_{\overline{n}|i}} < \frac{s_{\overline{n}|j}}{a_{\overline{n}|j}} = (1+j)^n$. Thus, $i < i' < j$.

2.4.3 $P_1 = K \cdot a_{\overline{n}|},\ P_2 = \dfrac{K \cdot s_{\overline{n}|j}}{1+i \cdot s_{\overline{n}|j}} = \dfrac{K}{i}\left[1 - \dfrac{1}{1+i \cdot s_{\overline{n}|j}}\right]$

$\dfrac{1}{P_1} = \dfrac{1}{K}\left[\dfrac{1}{s_{\overline{n}|i}} + i\right],\ \dfrac{1}{P_2} = \dfrac{1}{K}\left[\dfrac{1}{s_{\overline{n}|j}} + i\right]$

The result follows from the fact that

$s_{\overline{n}|i} = s_{\overline{n}|j}$ if $i = j$, $s_{\overline{n}|i} > s_{\overline{n}|j}$ if $i > j$, and $s_{\overline{n}|i} < s_{\overline{n}|j}$ if $i < j$.

2.4.4 (a) $(15{,}000 - .10P)s_{\overline{8}|.07} + 10{,}000 = P \ \rightarrow\ P = 80{,}898$

(b) $(15{,}000 - 8500)s_{\overline{8}|.07} + S = 85{,}000 \ \rightarrow\ S = 18{,}311$

2.4.5 $P_2 = \dfrac{s_{\overline{n}|j}}{1+i \cdot s_{\overline{n}|j}} = \dfrac{1}{i}\left[1 - \dfrac{1}{1+i \cdot s_{\overline{n}|j}}\right].$

As j increases, $s_{\overline{n}|j}$ increases, and thus $\dfrac{1}{1+i \cdot s_{\overline{n}|j}}$ decreases, and thus $1 - \dfrac{1}{1+i \cdot s_{\overline{n}|j}}$ increases, showing that $\frac{d}{dj}P_2 > 0$.

2.4.6 If $j < i$ then $(Is)_{\overline{n-1}|j} < (Is)_{\overline{n-1}|i}$, so that

$s_{\overline{n}|i'} = n + i \cdot (Is)_{\overline{n-1}|j} < n + i \cdot (Is)_{\overline{n-1}|i}$

$= n + \ddot{s}_{\overline{n-1}|i} - (n-1) = s_{\overline{n}|i}$

$\rightarrow i' < i$, and $s_{\overline{n}|i'} = n + i \cdot (Is)_{\overline{n-1}|j} > n + j \cdot (Is)_{\overline{n-1}|j} = s_{\overline{n}|j}$

$\rightarrow i' > j$.

2.4.7 $(R - 12{,}250)s_{\overline{18}|.035} = 245{,}000 \ \rightarrow\ R = 22{,}250$

2.4.8 (a) $1000(Ia)_{\overline{20}|.10} = 63{,}920$

(b) $1000(Is)_{\overline{20}|.06} - .10P \cdot s_{\overline{20}|.06} = P \ \rightarrow\ P = 67{,}659$

(c) $P_0(1.10)^t - 1000(Is)_{\overline{t}|.10} = P_t$, until $.10P_{t-1} \le 1000t$

$1000t \cdot s_{\overline{20-t}|.06} + 1000(Is)_{\overline{20-t}|.06} - .10P_t \cdot s_{\overline{20-t}|.06} = P_t$

Solve for t by trial and error.

$t = 8:\ \rightarrow\ P_8 = 86{,}712\ \rightarrow\ P_0 = 61{,}815,\ P_1 = 66{,}997,$
$P_2 = 71{,}698,\ P_3 = 75{,}866,\ P_4 = 79{,}453,$
$P_5 = 82{,}398,\ P_6 = 84{,}638,\ P_7 = 86{,}102$

2.4.9 Machine I: $X(1-d)^{10} = \frac{X}{8} \ \rightarrow\ 1 - d = (.125)^{.1}$

First seven years depreciation equals

$$A - B_7 = X - X(1-d)^7 = X\left[1 - (.125)^{.7}\right] = .7667X.$$

Machine II: First seven years depreciation equals

$$A - B_7 = A - \left[S + \left(\tfrac{1+2+3}{1+2+\cdots+10}\right)(A - S)\right]$$

$$= Y - \left[\tfrac{X}{8} + \left(\tfrac{6}{55}\right)\left(Y - \tfrac{X}{8}\right)\right] = \left(\tfrac{49}{55}\right)\left(Y - \tfrac{X}{8}\right).$$

Setting these equal for machines I and II we get

$.7667X = \left(\frac{49}{55}\right)\left(Y - \frac{X}{8}\right) \ \rightarrow\ .986X = Y.$

2.4.10 (i) $\frac{X-Y}{n} = 1000$

(ii) $\frac{n-3+1}{S_n}(X-Y) = \frac{n-2}{n(n+1)/2}(X-Y) = 800$

(iii) $Y = (.66875)^n X.$

From (i), and (ii) $\frac{2(n-2)}{n+2} = .8 \ \rightarrow\ n = 4$ and $X - Y = 4000$.

From (iii), $X - 4000 = (.66875)^4 X \ \rightarrow\ X = 5000$.

2.4.11 Annual deposits at the end of each year for the 15 years are $.2(20,000)$, $.2(20,000)(.8)$, $.2(20,000)(.8)^2, \ldots, .2(20,000)(.8)^{14}$.

Accumulated value at effective annual 6% is

$$.2(20,000)(1.06)^{14} + .2(20,000(.8)(1.06) \cdot 13$$
$$+ .2(20,000)(.8)(1.06)^{12} + \cdots + .2(20,000)(.8)$$
$$= 4000(1.06)^{14}\left[1 + .8v + (.8)^2v^2 + \cdots + (.8)^{14}v^{14}\right]$$
$$= 4000(1.06)^{14}\left[\frac{1-(.8)^{15}v^{15}}{1-.8v}\right] = 36,329.$$

2.4.12 Under the sum-of-years-digits method, the depreciated value at the end of 4 years in a 10-year depreciation schedule is

$$B_4 = S + \left(\frac{S_{10-4}}{S_{10}}\right)(A-S) = S + \left(\frac{21}{55}\right)(5000-S)$$
$$= 1909.09 + .6182S$$
$$= 2218.$$

We can solve for S, $S = 500$.

Under the straight line method for the remaining 6 years (from time 4 to time 10), the depreciation per year will be

$$\frac{2218-500}{6} = 286.3.$$

CHAPTER 3

SECTION 3.1

3.1.1 (i) $L = 1000a_{\overline{5}|.1} + 500v^5a_{\overline{5}|.1} = 4,967.68$

(ii) $OB_3 = 4967.68(1.1)^3 - 1000s_{\overline{3}|.1} = 3,301.98$

(iii) $I_4 = 3301.98(.1) = 330.20$,
$PR_4 = 1000 - 330.20 = 669.80$

(iv) $OB_8 = 500a_{\overline{2}|.1} = 867.77$

3.1.2 60 monthly payments. Final 20 payments ($41^{st}, 42^{nd}, \ldots$) are

$$1000(.98)^{40}, 1000(.98)^{41}, \ldots, 1000(.98)^{59}.$$

$$OB_{40} = 1000(.98)^{40}v_{.0075} + 1000(.98)^{41}v_{.0075}^2$$
$$+ \cdots + 1000(.98)^{59}v_{.0075}^{20}$$
$$= 1000(.98)^{40}v_{.0075}\left[1 + (.98)v + (.98)^2v^2 + \cdots + (.98)^{19}v^{19}\right]$$
$$= 1000(.98)^{40}v_{.0075} \cdot \frac{1-(.98v)^{20}}{1-.98v} = 6889.$$

3.1.3 $PV = 250a_{\overline{12}|} + 5(Da)_{\overline{12}|} = 2643.84 + 356.16 = 3000$

2.4.11 Annual deposits at the end of each year for the 15 years are $.2(20,000)$, $.2(20,000)(.8)$, $.2(20,000)(.8)^2, \ldots,$ $.2(20,000)(.8)^{14}$.

Accumulated value at effective annual 6% is

$$\begin{aligned} &.2(20,000)(1.06)^{14} + .2(20,000(.8)(1.06)\cdot 13 \\ &\qquad + .2(20,000)(.8)(1.06)^{12} + \cdots + .2(20,000)(.8) \\ &= 4000(1.06)^{14}\left[1 + .8v + (.8)^2 v^2 + \cdots + (.8)^{14} v^{14}\right] \\ &= 4000(1.06)^{14}\left[\frac{1-(.8)^{15} v^{15}}{1-.8v}\right] = 36,329. \end{aligned}$$

2.4.12 Under the sum-of-years-digits method, the depreciated value at the end of 4 years in a 10-year depreciation schedule is

$$\begin{aligned} B_4 &= S + \left(\frac{S_{10-4}}{S_{10}}\right)(A-S) = S + \left(\frac{21}{55}\right)(5000-S) \\ &= 1909.09 + .6182S \\ &= 2218. \end{aligned}$$

We can solve for S, $S = 500$.

Under the straight line method for the remaining 6 years (from time 4 to time 10), the depreciation per year will be

$$\frac{2218-500}{6} = 286.3.$$