

**Solution Manual for
SIGNALS AND SYSTEMS
USING MATLAB**

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Chapter 0

From the Ground Up

0.1 Basic Problems

0.1 Let $z = 8 + j3$ and $v = 9 - j2$,

(a) Find

$$(i) \operatorname{Re}(z) + \operatorname{Im}(v), \quad (ii) |z + v|, \quad (iii) |zv|, \quad (iv) \angle z + \angle v, \quad (v) |v/z|, \quad (vi) \angle(v/z)$$

(b) Find the trigonometric and polar forms of

$$(i) z + v, \quad (ii) zv, \quad (iii) z^* \quad (iv) zz^*, \quad (v) z - v$$

Answers: (a) $\operatorname{Re}(z) + \operatorname{Im}(v) = 6$; $|v/z| = \sqrt{85}/\sqrt{73}$; (b) $zz^* = |z|^2 = 73$.

Solution

(a) i. $\operatorname{Re}(z) + \operatorname{Im}(v) = 8 - 2 = 6$

ii. $|z + v| = |17 + j1| = \sqrt{17^2 + 1}$

iii. $|zv| = |72 - j16 + j27 + 6| = |78 + j11| = \sqrt{78^2 + 11^2}$

iv. $\angle z + \angle v = \tan^{-1}(3/8) - \tan^{-1}(2/9)$

v. $|v/z| = |v|/|z| = \sqrt{85}/\sqrt{73}$

vi. $\angle(v/z) = -\tan^{-1}(2/9) - \tan^{-1}(3/8)$

(b) i. $z + v = 17 + j = \sqrt{17^2 + 1}e^{j \tan^{-1}(1/17)}$

ii. $zv = 78 + j11 = \sqrt{78^2 + 11^2}e^{j \tan^{-1}(11/78)}$

iii. $z^* = 8 - j3 = \sqrt{64 + 9}(e^{-j \tan^{-1}(3/8)})^* = \sqrt{73}e^{j \tan^{-1}(3/8)}$

iv. $zz^* = |z|^2 = 73$

v. $z - v = -1 + j5 = \sqrt{1 + 25}e^{-j \tan^{-1}(5)}$

0.2 Use Euler's identity to

(a) show that

$$(i) \cos(\theta - \pi/2) = \sin(\theta), \quad (ii) -\sin(\theta - \pi/2) = \cos(\theta), \quad (iii) \cos(\theta) = \sin(\theta + \pi/2).$$

(b) to find

$$(i) \int_0^1 \cos(2\pi t) \sin(2\pi t) dt, \quad (ii) \int_0^1 \cos^2(2\pi t) dt.$$

Answers: (b) 0 and 1/2.

Solution

(a) We have

$$i. \cos(\theta - \pi/2) = 0.5(e^{j(\theta-\pi/2)} + e^{-j(\theta-\pi/2)}) = -j0.5(e^{j\theta} - e^{-j\theta}) = \sin(\theta)$$

$$ii. -\sin(\theta - \pi/2) = 0.5j(e^{j(\theta-\pi/2)} - e^{-j(\theta-\pi/2)}) = 0.5j(-j)(e^{j\theta} + e^{-j\theta}) = \cos(\theta)$$

$$iii. \sin(\theta + \pi/2) = (je^{j\theta} + je^{-j\theta})/(2j) = \cos(\theta)$$

(b) i. $\cos(2\pi t) \sin(2\pi t) = (1/4j)(e^{j4\pi t} - e^{-j4\pi t})$ so that

$$\int_0^1 \cos(2\pi t) \sin(2\pi t) dt = \frac{1}{4j} \frac{e^{j4\pi t}}{4\pi j} \Big|_0^1 + \frac{1}{4j} \frac{e^{-j4\pi t}}{4\pi j} \Big|_0^1 = 0 + 0 = 0$$

ii. We have

$$\cos^2(2\pi t) = \frac{1}{4}(e^{j4\pi t} + 2 + e^{-j4\pi t}) = \frac{1}{2}(1 + \cos(4\pi t))$$

so that its integral is 1/2 since the integral of $\cos(4\pi t)$ is over two of its periods and it is zero.

0.3 Use Euler's identity to

(a) show the identities

$$(i) \quad \cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$(ii) \quad \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta),$$

(b) find an expression for $\cos(\alpha) \cos(\beta)$, and for $\sin(\alpha) \sin(\beta)$.

Answers: $e^{j\alpha} e^{j\beta} = \cos(\alpha + \beta) + j \sin(\alpha + \beta) = [\cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)] + j[\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)]$.

Solution

(a) Using Euler's identity the product

$$\begin{aligned} e^{j\alpha} e^{j\beta} &= (\cos(\alpha) + j \sin(\alpha))(\cos(\beta) + j \sin(\beta)) \\ &= [\cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)] + j[\sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)] \end{aligned}$$

while

$$e^{j(\alpha+\beta)} = \cos(\alpha + \beta) + j \sin(\alpha + \beta)$$

so that equating the real and imaginary parts of the above two equations we get the desired trigonometric identities.

(b) We have

$$\begin{aligned} \cos(\alpha) \cos(\beta) &= 0.5(e^{j\alpha} + e^{-j\alpha}) 0.5(e^{j\beta} + e^{-j\beta}) \\ &= 0.25(e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)}) + 0.25(e^{j(\alpha-\beta)} + e^{-j(\alpha-\beta)}) \\ &= 0.5 \cos(\alpha + \beta) + 0.5 \cos(\alpha - \beta) \end{aligned}$$

Now,

$$\begin{aligned} \sin(\alpha) \sin(\beta) &= \cos(\alpha - \pi/2) \cos(\beta - \pi/2) \\ &= 0.5 \cos(\alpha - \pi/2 + \beta - \pi/2) + 0.5 \cos(\alpha - \pi/2 - \beta + \pi/2) \\ &= 0.5 \cos(\alpha + \beta - \pi) + 0.5 \cos(\alpha - \beta) \\ &= -0.5 \cos(\alpha + \beta) + 0.5 \cos(\alpha - \beta) \end{aligned}$$

0.4 Consider the calculation of roots of an equation $z^N = \alpha$ where $N \geq 1$ is an integer and $\alpha = |\alpha|e^{j\phi}$ a nonzero complex number.

- (a) First verify that there are exactly N roots for this equation and that they are given by $z_k = re^{j\theta_k}$ where $r = |\alpha|^{1/N}$ and $\theta_k = (\phi + 2\pi k)/N$ for $k = 0, 1, \dots, N - 1$.
- (b) Use the above result to find the roots of the following equations

$$(i) z^2 = 1, \quad (ii) z^2 = -1, \quad (iii) z^3 = 1, \quad (iv) z^3 = -1.$$

and plot them in a polar plane (i.e., indicating their magnitude and phase). Explain how the roots are distributed in the polar plane.

Answers: Roots of $z^3 = -1 = 1e^{j\pi}$ are $z_k = 1e^{j(\pi+2\pi k)/3}$, $k = 0, 1, 2$, equally spaced around circle of radius r .

Solution

(a) Replacing $z_k = |\alpha|^{1/N} e^{j(\phi+2\pi k)/N}$ in z^N we get $z_k^N = |\alpha| e^{j(\phi+2\pi k)} = |\alpha| e^{j\phi} = \alpha$ for any value of $k = 0, \dots, N - 1$.

(b) Applying the above result we have:

- For $z^2 = 1 = 1e^{j2\pi}$ the roots are $z_k = 1e^{j(2\pi+2\pi k)/2}$, $k = 0, 1$. When $k = 0$, $z_0 = e^{j\pi} = -1$ and $z_1 = e^{j2\pi} = 1$.
- When $z^2 = -1 = 1e^{j\pi}$ the roots are $z_k = 1e^{j(\pi+2\pi k)/2}$, $k = 0, 1$. When $k = 0$, $z_0 = e^{j\pi/2} = j$, and $z_1 = e^{j3\pi/2} = -j$.
- For $z^3 = 1 = 1e^{j2\pi}$ the roots are $z_k = 1e^{j(2\pi+2\pi k)/3}$, $k = 0, 1, 2$. When $k = 0$, $z_0 = e^{j2\pi/3}$; for $k = 1$, $z_1 = e^{j4\pi/3} = e^{-j2\pi/3} = z_0^*$; and for $k = 2$, $z_2 = 1e^{j(2\pi)} = 1$.
- When $z^3 = -1 = 1e^{j\pi}$ the roots are $z_k = 1e^{j(\pi+2\pi k)/3}$, $k = 0, 1, 2$. When $k = 0$, $z_0 = e^{j\pi/3}$; for $k = 1$, $z_1 = e^{j\pi} = -1$; and for $k = 2$, $z_2 = 1e^{j(5\pi)/3} = 1e^{j(-\pi)/3} = z_0^*$

(c) Notice that the roots are equally spaced around a circle of radius r and that the complex roots appear as pairs of complex conjugate roots.

0.5 Consider a function of $z = 1 + j1$, $w = e^z$

- (a) Find (i) $\log(w)$, (ii) $\mathcal{R}e(w)$, (iii) $\mathcal{I}m(w)$
- (b) What is $w + w^*$, where w^* is the complex conjugate of w ?
- (c) Determine $|w|$, $\angle w$ and $|\log(w)|^2$?
- (d) Express $\cos(1)$ in terms of w using Euler's identity.

Answers: $\log(w) = z$; $w + w^* = 2\mathcal{R}e[w] = 2e \cos(1)$.

Solution

(a) If $w = e^z$ then

$$\log(w) = z = 1 + j1$$

given that the \log and e functions are the inverse of each other.
The real and imaginary of w are

$$w = e^z = e^1 e^{j1} = \underbrace{e \cos(1)}_{\text{real part}} + j \underbrace{e \sin(1)}_{\text{imaginary part}}$$

(b) The imaginary parts are cancelled and the real parts added twice in

$$w + w^* = 2\mathcal{R}e[w] = 2e \cos(1)$$

(c) Replacing z

$$w = e^z = e^1 e^{j1}$$

so that $|w| = e$ and $\angle w = 1$.

Using the result in (a)

$$|\log(w)|^2 = |z|^2 = 2$$

(d) According to Euler's equation

$$\cos(1) = 0.5(e^j + e^{-j}) = 0.5 \left(\frac{w}{e} + \frac{w^*}{e} \right)$$

which can be verified using $w + w^*$ obtained above.

0.6 A phasor can be thought of as a vector, representing a complex number, rotating around the polar plane at a certain frequency in radians/second. The projection of such a vector onto the real axis gives a cosine with a certain amplitude and phase. This problem will show the algebra of phasors which would help you with some of the trigonometric identities that are hard to remember.

- (a) When you plot $y(t) = A \sin(\Omega_0 t)$ you notice that it is a cosine $x(t) = A \cos(\Omega_0 t)$ shifted in time, i.e.,

$$y(t) = A \sin(\Omega_0 t) = A \cos(\Omega_0(t - \Delta_t)) = x(t - \Delta_t)$$

how much is this shift Δ_t ? Better yet, what is $\Delta_\theta = \Omega_0 \Delta_t$ or the shift in phase? One thus only need to consider cosine functions with different phase shifts instead of sines and cosines.

- (b) From above, the phasor that generates $x(t) = A \cos(\Omega_0 t)$ is Ae^{j0} so that $x(t) = \mathcal{R}e[Ae^{j0} e^{j\Omega_0 t}]$. The phasor corresponding to the sine $y(t)$ should then be $Ae^{-j\pi/2}$. Obtain an expression for $y(t)$ similar to the one for $x(t)$ in terms of this phasor.
- (c) From the above results, give the phasors corresponding to $-x(t) = -A \cos(\Omega_0 t)$ and $-y(t) = -A \sin(\Omega_0 t)$. Plot the phasors that generate \cos , \sin , $-\cos$ and $-\sin$ for a given frequency. Do you see now how these functions are connected? How many radians do you need to shift in positive or negative direction to get a sine from a cosine, etc.
- (d) Suppose then you have the sum of two sinusoids, for instance $z(t) = x(t) + y(t) = A \cos(\Omega_0 t) + A \sin(\Omega_0 t)$, adding the corresponding phasors for $x(t)$ and $y(t)$ at some time, e.g., $t = 0$, which is just a sum of two vectors, you should get a vector and the corresponding phasor. For $x(t)$, $y(t)$, obtain their corresponding phasors and then obtain from them the phasor corresponding to $z(t) = x(t) + y(t)$.
- (e) Find the phasors corresponding to

$$(i) 4 \cos(2t + \pi/3), (ii) -4 \sin(2t + \pi/3), (iii) 4 \cos(2t + \pi/3) - 4 \sin(2t + \pi/3)$$

Answers: $\sin(\Omega_0 t) = \cos(\Omega_0(t - T_0/4)) = \cos(\Omega_0 t - \pi/2)$ since $\Omega_0 = 2\pi/T_0$; $z(t) = \sqrt{2}A \cos(\Omega_0 t - \pi/4)$; (e) (i) $4e^{j\pi/3}$; (iii) $4\sqrt{2}e^{j7\pi/12}$.

Solution

- (a) Shifting to the right a cosine by a fourth of its period we get a sinusoid, thus

$$\sin(\Omega_0 t) = \cos(\Omega_0(t - T_0/4)) = \cos(\Omega_0 t - \Omega_0 T_0/4) = \cos(\Omega_0 t - \pi/2)$$

since $\Omega_0 = 2\pi/T_0$ or $\Omega_0 T_0 = 2\pi$.

- (b) The phasor that generates a sine is $Ae^{-j\pi/2}$ since

$$y(t) = \mathcal{R}e[Ae^{-j\pi/2} e^{j\Omega_0 t}] = \mathcal{R}e[Ae^{j(\Omega_0 t - \pi/2)}] = A \cos(\Omega_0 t - \pi/2)$$

which equals $A \sin(\Omega_0 t)$.

- (c) The phasors corresponding to $-x(t) = -A \cos(\Omega_0 t) = A \cos(\Omega_0 t + \pi)$ is $Ae^{j\pi}$. For

$$-y(t) = -A \sin(\Omega_0 t) = -A \cos(\Omega_0 t - \pi/2) = A \cos(\Omega_0 t - \pi/2 + \pi) = A \cos(\Omega_0 t + \pi/2)$$

the phasor is $Ae^{j\pi/2}$. Thus, relating any sinusoid to the corresponding cosine, the magnitude and angle of this cosine gives the magnitude and phase of the phasor that generates the given sinusoid.

- (d) If $z(t) = x(t) + y(t) = A \cos(\Omega_0 t) + A \sin(\Omega_0 t)$, the phasor corresponding to $z(t)$ is the sum of the phasors Ae^{j0} , corresponding to $A \cos(\Omega_0 t)$, with the phasor $Ae^{-j\pi/2}$, corresponding to $A \sin(\Omega_0 t)$, which gives $\sqrt{2}Ae^{-j\pi/4}$ (equivalently the sum of a vector with length A and angle 0 with another vector of length A and angle $-\pi/2$). We have that

$$z(t) = \mathcal{R}e \left[\sqrt{2}Ae^{-j\pi/4} e^{j\Omega_0 t} \right] = \sqrt{2}A \cos(\Omega_0 t - \pi/4)$$

- (e) i. Phasor $4e^{j\pi/3}$
 ii. $-4 \sin(2t + \pi/3) = 4 \cos(2t + \pi/3 + \pi/2)$ with phasor $4e^{j5\pi/6}$
 iii. We have

$$\begin{aligned} 4 \cos(2t + \pi/3) - 4 \sin(2t + \pi/3) &= \mathcal{R}e[(4e^{j\pi/3} + 4e^{j(\pi/2+\pi/3)})e^{j2t}] \\ &= \mathcal{R}e[4e^{j\pi/3} \underbrace{(1 + e^{j\pi/2})}_{\sqrt{2}e^{j\pi/4}} e^{j2t}] \\ &= \mathcal{R}e[4\sqrt{2}e^{j7\pi/12} e^{j2t}] \end{aligned}$$

so that the phasor is $4\sqrt{2}e^{j7\pi/12}$

0.7 To get an idea of the number of bits generated and processed by a digital system consider the following applications:

- (a) A compact disc (CD) is capable of storing 75 minutes of “CD quality” stereo (left and right channels are recorded) music. Calculate the number of bits that are stored in the CD as raw data.
Hint: find out what ‘CD quality’ means in the binary representation of each sample.
- (b) Find out what the vocoder in your cell phone is used for. To attaining “telephone quality” voice you use a sampling rate of 10,000 samples/sec, and that each sample is represented by 8 bits. Calculate the number of bits that your cell-phone has to process every second that you talk. Why would you then need a vocoder?
- (c) Find out whether text messaging is cheaper or more expensive than voice. Explain how the text messaging works.
- (d) Find out how an audio CD and an audio DVD compare. Find out why it is said that a vinyl long-play record reproduces sounds much better. Are we going backwards with digital technology in music recording? Explain.
- (e) To understand why video streaming in the internet is many times of low quality, consider the amount of data that needs to be processed by a video compressor every second. Assume the size of a video frame, in pixels, is 352×240 , and that an acceptable quality for the image is obtained by allocating 8 bits/pixel and to avoid jerking effects we use 60 frames/second.
 - How many pixels need to be processed every second?
 - How many bits would be available for transmission every second?
 - The above is raw data, compression changes the whole picture (literally), find out what some of the compression methods are.

Answers: (a) About 6.4 Gbs; vocoder (short for voice encoder) reduces number of transmitted bits while keeping voice recognizable.

Solution

(a) Assuming a maximum frequency of 22.05 kHz for the acoustic signal, the numbers of bytes (8 bits per byte) for two channels (stereo) and a 75 minutes recording is greater or equal to: $2 \times 22,050 \text{ samples/channel/second} \times 2 \text{ bytes/sample} \times 2 \text{ channels} \times 75 \text{ minutes} \times 60 \text{ seconds/minute} = 7.938 \times 10^8 \text{ bytes}$. Multiplying by 8 we get the number of bits. CD quality means that the signal is sampled at 44.1 kHz and each sample is represented by 16 bits or 2 bytes.

(b) The raw data would consist of $8 \text{ (bits/sample)} \times 10,000 \text{ (samples/sec)} = 80,000 \text{ bits/sec}$. The vocoder is part of a larger unit called a digital signal processor chip set. It uses various procedures to reduce the number of bits that are transmitted while still keeping your voice recognizable. When there is silence it does not transmit, letting another signal use the channel during pauses.

(c) Texting between cell phones is possible by sending short messages (160 characters) using the short message services (SMS). Whenever your cell-phone communicates with the cell phone tower there is an exchange of messages over the control channel for localization, and call setup. This channel provides a pathway for SMS messages by sending packets of data. Except for the cost of storing messages, the procedure is rather inexpensive and convenient to users.

(d) For CD audio the sampling rate is 44.1 kHz with 16 bits/sample. For DVD audio the sampling rate is 192 kHz with 24 bits/sample. The sampling process requires getting rid of high frequencies in the signal, also each sample is only approximated by the binary representation, so analog recording could sound better in some cases.

(e) The number of pixels processed every second is: 352×240 pixels/frame $\times 60$ frames/sec. The number of bits available for transmission every second is obtained by multiplying the above answer by 8 bits/pixel. There many compression methods JPEG, MPEG, etc.

0.8 The geometric series

$$S = \sum_{n=0}^{N-1} \alpha^n$$

will be used quite frequently in the next chapters so let us look at some of its properties:

- (a) Suppose $\alpha = 1$ what is S equal to?
 (b) Suppose $\alpha \neq 1$ show that

$$S = \frac{1 - \alpha^N}{1 - \alpha}$$

Verify that $(1 - \alpha)S = (1 - \alpha^N)$. Why do you need the constraint that $\alpha \neq 1$? Would this sum exist if $\alpha > 1$? Explain.

- (c) Suppose now that $N = \infty$, under what conditions will S exist? if it does, what would S be equal to? Explain.
 (d) Suppose again that $N = \infty$ in the definition of S . The derivative of S with respect to α is

$$S_1 = \frac{dS}{d\alpha} = \sum_{n=0}^{\infty} n\alpha^{n-1}$$

obtain a rational expression to find S_1 .

Answers: $S = N$ when $\alpha = 1$, $S = (1 - \alpha^N)/(1 - \alpha)$ when $\alpha \neq 1$.

Solution

- (a) If $\alpha = 1$ then

$$S = \sum_{n=0}^{N-1} 1 = \underbrace{1 + 1 + \cdots + 1}_{N \text{ times}} = N$$

- (b) The expression

$$\begin{aligned} S(1 - \alpha) &= S - \alpha S \\ &= (1 + \alpha + \cdots + \alpha^{N-1}) - (\alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N) \\ &= 1 - \alpha^N \end{aligned}$$

as the intermediate terms cancel. So that

$$S = \frac{1 - \alpha^N}{1 - \alpha}, \quad \alpha \neq 1$$

Since we do not want the denominator $1 - \alpha$ to be zero, the above requires that $\alpha \neq 1$. If $\alpha = 1$ the sum was found in (a). As a finite sum, it exists for any finite values of α .

Putting (a) and (b) together we have

$$S = \begin{cases} (1 - \alpha^N)/(1 - \alpha) & \alpha \neq 1 \\ N & \alpha = 1 \end{cases}$$

- (c) If N is infinite, the sum is of infinite length and we need to impose the condition that $|\alpha| < 1$ so that α^n decays as $n \rightarrow \infty$. In that case, the term $\alpha^N \rightarrow 0$ as $N \rightarrow \infty$, and the sum is

$$S = \frac{1}{1 - \alpha} \quad |\alpha| < 1$$

If $|\alpha| \geq 1$ this sum does not exist, i.e., it becomes infinite.

(d) The derivative becomes

$$S_1 = \frac{dS}{d\alpha} = \sum_{n=0}^{\infty} n\alpha^{n-1} = \frac{1}{(1-\alpha)^2}.$$

0.2 Problems using MATLAB

0.9 Derivative and finite difference — Let $y(t) = dx(t)/dt$, where $x(t) = 4 \cos(2\pi t)$, $-\infty < t < \infty$. Find $y(t)$ analytically and determine a value of T_s for which $\Delta[x(nT_s)]/T_s = y(nT_s)$ (consider as possible values $T_s = 0.01$ and $T_s = 0.1$). Use the MATLAB function *diff* or create your own to compute the finite difference. Plot the finite difference in the range $[0, 1]$ and compare it with the actual derivative $y(t)$ in that range. Explain your results for the given values of T_s .

Answers: $y(t) = -8\pi \sin(2\pi t)$ has same sampling period as $x(t)$, $T_s \leq 0.5$; $T_s = 0.01$ gives better results.

Solution

The derivative is

$$y(t) = \frac{dx(t)}{dt} = -8\pi \sin(2\pi t)$$

which has the same frequency as $x(t)$, thus the sampling period should be like in the previous problem, $T_s \leq 0.5$.

```
% Pr. 0.9
clear all
% actual derivative
Tss=0.0001;t1=0:Tss:3;
y=-8*pi*sin(2*pi*t1);
figure(2)
% forward difference
Ts=0.01;t=[0:Ts:3];N=length(t);
subplot(211)
xa=4*cos(2*pi*t); % sampled signal
der1_x=forwarddiff(xa,Ts,t,y,t1);

clear der1_x
% forward difference
Ts=0.1;t=[0:Ts:3];N=length(t);
subplot(212)
xa=4*cos(2*pi*t); % sampled signal
der1_x=forwarddiff(xa,Ts,t,y,t1);
```

The function *forwarddiff* computes and plots the forward difference and the actual derivative.

```
function der=forwarddiff(xa,Ts,t,y,t1)
% % forward difference
% % xa: sampled signal using Ts
% % y: actual derivative defined in t
N=length(t);n=0:N-2;
der=diff(xa)/Ts;
stem(n*Ts,der,'filled');grid;xlabel('t, nT_s')
hold on
plot(t1,y,'r'); legend('forward difference','derivative')
hold off
```

For $T_s = 0.1$ the finite difference looks like the actual derivative but shifted, while for $T_s = 0.01$ it does not.

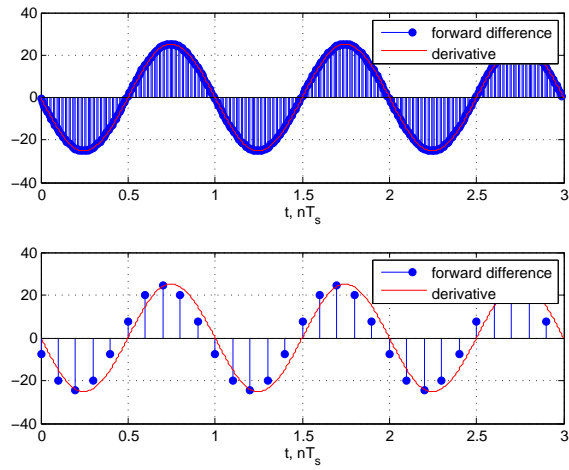


Figure 1: Problem 9: $T_s = 0.01$ sec (top) and $T_s = 0.1$ sec (bottom)

0.10 Backward difference — Another definition for the finite difference is the backward difference:

$$\Delta_1[x(nT_s)] = x(nT_s) - x((n-1)T_s)$$

$(\Delta_1[x(nT_s)]/T_s)$ approximates the derivative of $x(t)$.

- Indicate how this new definition connects with the finite difference defined earlier in this Chapter.
- Solve Problem 9 with MATLAB using this new finite difference and compare your results with the ones obtained there.
- For the value of $T_s = 0.1$, use the average of the two finite differences to approximate the derivative of the analog signal $x(t)$. Compare this result with the previous ones. Provide an expression for calculating this new finite difference directly.

Answers: $\Delta_1[x(n+1)] = x(n+1) - x(n) = \Delta[x(n)]$; $0.5 \{\Delta_1[x(n)] + \Delta[x(n)]\} = 0.5[x(n+1) - x(n-1)]$.

Solution

(a) The backward finite difference (let $T_s = 1$ for simplicity)

$$\Delta_1[x(n)] = x(n) - x(n-1)$$

is connected with the forward finite difference $\Delta[x(n)]$ given in the chapter as follows

$$\Delta_1[x(n+1)] = x(n+1) - x(n) = \Delta[x(n)]$$

That is, $\Delta[x(n)]$ is $\Delta_1[x(n)]$ shifted one sample to the left.

(b) (c) The average of the two finite differences gives

$$0.5 \{\Delta_1[x(n)] + \Delta[x(n)]\} = 0.5[x(n+1) - x(n-1)]$$

which gives a better approximation to the derivative than either of the given finite differences. The following script is used to compute Δ_1 and the average.

```
% Pro 0.10
% compares forward/backward differences
% with new average difference
Ts=0.1;
for k=0:N-2,
    x1=4*cos(2*pi*(k-1)*Ts);
    x2=4*cos(2*pi*k*Ts);
    der_x(k+1)=x2-x1; % backward difference
end
der_x=der_x/Ts;
Tss=0.0001;t1=0:Tss:3;
y=-8*pi*sin(2*pi*t1); % actual derivative
n=0:N-2;
figure(3)
subplot(211)
stem(n*Ts,der_x,'k');grid
hold on
stem(n*Ts,der1_x,'b','filled') % derv1_x forward difference
                                % from Pr. 0.2
hold on
plot(t1,y,'r'); xlabel('t, nT_s')
legend('bck diff','forwd diff','derivative')
```

```

hold off
subplot(212)
stem(n*Ts,0.5*(der_x+der1_x));grid;xlabel('t, nT_s') % average
hold on
plot(t1,y,'r')
hold off
legend('average diff','derivative')

```

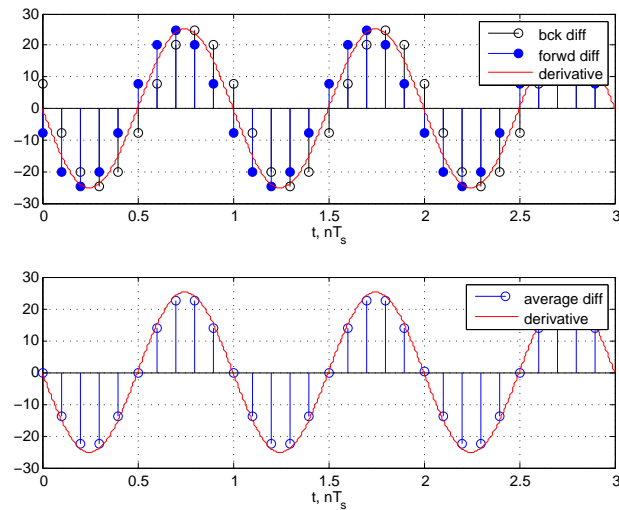


Figure 2: Problem 10: Comparison of different finite differences.

0.11 Sums and Gauss — Three laws in the computation of sums are

$$\text{Distributive: } \sum_k ca_k = c \sum_k a_k$$

$$\text{Associative: } \sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$$

$$\text{Commutative: } \sum_k a_k = \sum_{p(k)} a_{p(k)}$$

for any permutation $p(k)$ of the set of integers k in the summation.

(a) Explain why the above rules make sense when computing sums. To do that consider

$$\sum_k a_k = \sum_{k=0}^2 a_k, \text{ and } \sum_k b_k = \sum_{k=0}^2 b_k.$$

Let c be a constant, and choose any permutation of the values $[0, 1, 2]$ for instance $[2, 1, 0]$ or $[1, 0, 2]$.

(b) The trick that Gauss played when he was a preschooler can be explained by using the above rules. Suppose you want to find the sum of the integers from 0 to 10,000 (Gauss did it for integers between 0 and 100 but he was then just a little boy, and we can do better!). That is, we want to find S where

$$S = \sum_{k=0}^{10,000} k = 0 + 1 + 2 + \cdots + 10000$$

to do so consider

$$2S = \sum_{k=0}^{10,000} k + \sum_{k=10,000}^0 k$$

and apply the above rules to find then S . Come up with a MATLAB function of your own to do this sum.

(c) Find the sum of an arithmetic progression

$$S_1 = \sum_{k=0}^N (\alpha + \beta k)$$

for constants α and β , using the given three rules.

(d) Find out if MATLAB can do these sums symbolically, i.e., without having numerical values. Use the found symbolic function to calculate the sum in the previous item when $\alpha = \beta = 1$ and $N = 100$.

Answers: $N = 10,000$, $S = N(N + 1)/2$; $S_1 = \alpha(N + 1) + \beta(N(N + 1))/2$.

Solution

(a) The distributive and the associative laws are equivalent to the ones for integrals, indeed

$$\sum_k ca_k = c(\cdots + a_{-1} + a_0 + a_1 + \cdots) = c \sum_k a_k$$

since c does not depend on k . Likewise

$$\sum_k [a_k + b_k] = (\cdots + a_{-1} + b_{-1} + a_0 + b_0 + a_1 + b_1 + \cdots) = \sum_k a_k + \sum_k b_k$$

Finally, when adding a set of numbers the order in which they are added does not change the result. For instance,

$$a_0 + a_1 + a_2 + a_3 = a_0 + a_2 + a_1 + a_3$$

(b) Gauss' trick can be shown in general as follows. Let $S = \sum_{k=0}^N k$ then

$$2S = \sum_{k=0}^N k + \sum_{k=N}^0 k$$

letting $\ell = -k + N$ in the second summation we have

$$2S = \sum_{k=0}^N k + \sum_{\ell=0}^N (N - \ell) = \sum_{k=0}^N (k + N - k) = N \sum_{k=0}^N 1 = N(N + 1)$$

where we let the dummy variables of the two sums be equal. We thus have that for $N = 10^4$

$$S = \frac{N(N + 1)}{2} = \frac{10^4(10^4 + 1)}{2} \approx 0.5 \times 10^8$$

(c) Using the above properties of the sum,

$$\begin{aligned} S_1 &= \sum_{k=0}^N (\alpha + \beta k) = \alpha \sum_{k=0}^N 1 + \beta \sum_{k=0}^N k \\ &= \alpha(N + 1) + \beta \frac{N(N + 1)}{2} \end{aligned}$$

(d) The following script computes numerically and symbolically the various sums.

```
% Pro 0.11
clear all
% numeric
N=100;
S1=[0:1:N];
S2=[N:-1:0];
S=sum(S1+S2)/2
% symbolic
syms S1 N alpha beta k
simple(symsum(alpha+beta*k,0,N))
% computing sum for specific values of alpha, beta and N
subs(symsum(alpha+beta*k,0,N),{alpha,beta,N},{1,1,100})

S = 5050

((2*alpha + N*beta)*(N + 1))/2

5151
```

The answers shown at the bottom.

0.12 Integrals and sums — Suppose you wish to find the area under a signal $x(t)$ using sums. You will need the following result found above

$$\sum_{n=0}^N n = \frac{N(N+1)}{2}$$

(a) Consider first $x(t) = t, 0 \leq t \leq 1$, and zero otherwise. The area under this signal is 0.5. The integral can be approximated from above and below as

$$\sum_{n=1}^{N-1} (nT_s)T_s < \int_0^1 t dt < \sum_{n=1}^N (nT_s)T_s$$

where $NT_s = 1$ (i.e., we divide the interval $[0, 1]$ into N intervals of width T_s). Graphically show for $N = 4$ that the above equation makes sense by showing the right and left bounds as approximations for the area under $x(t)$.

- (b) Let $T_s = 0.001$, use the symbolic function *symsum* to compute the left and right bounds for the above integral. Find the average of these results and compare it with the actual value of the integral.
- (c) Verify the symbolic results by finding the sums on the left and the right of the above inequality using the summation given at the beginning of the problem. What happens when $N \rightarrow \infty$.
- (d) Write a MATLAB script to compute the area under the signal $y(t) = t^2$ from $0 \leq t \leq 1$. Let $T_s = 0.001$. Compare the average of the lower and upper bounds to the value of the integral.

Answer: For $T_s = 1/N$

$$\left[\frac{(N-1)(N-2) + 2(N-1)}{2N^2} \right] \leq \frac{1}{2} \leq \left[\frac{(N-1)(N-2) + 2(N-1)}{2N^2} \right] + \frac{1}{N}$$

Solution

(a) The following figure shows the upper and lower bounds when approximating the integral of t :

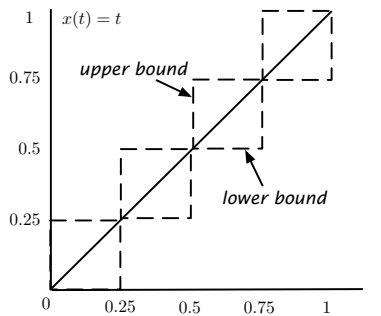


Figure 3: Problem 12: Upper and lower bounds of the integral of t when $N = 4$.

(b) (c) The lower bound for the integral is

$$\begin{aligned} S_\ell &= \sum_{n=1}^{N-1} (nT_s)T_s = T_s^2 \sum_{n=1}^{N-1} n = T_s^2 \sum_{\ell=0}^{N-2} (\ell+1) \\ &= T_s^2 \left[\frac{(N-1)(N-2)}{2} + (N-1) \right] \end{aligned}$$

The definite integral is

$$\int_0^1 t dt = \frac{1}{2}$$

The upper bound is

$$S_u = \sum_{n=1}^N (nT_s)T_s = S_\ell + NT_s^2$$

Letting $NT_s = 1$, or $T_s = 1/N$ we have then that

$$\left[\frac{(N-1)(N-2) + 2(N-1)}{2N^2} \right] \leq \frac{1}{2} \leq \left[\frac{(N-1)(N-2) + 2(N-1)}{2N^2} \right] + \frac{1}{N}$$

for large N the upper and the lower bound tend to $1/2$.

The following script computes the lower and upper bound of the integral of t .

```
% Pr. 0.12
clear all
Ts=0.001;N=1/Ts;
% integral of t from 0 to 1 is 0.5
syms S1 n T k
% lower bound
n=subs(N);T=subs(Ts);
y=simple(symsum(k*T^2,1,n-1));
yy=subs(y)
% upper bound
z=simple(symsum(k*T^2,1,n));
zz=subs(z)
% average
int= 0.5*(yy+zz)
```

giving the following results (the actual integral is $1/2$).

```
yy = 0.4995
zz = 0.5005
int = 0.5000
```

(d) For $y(t) = t^2$, $0 \leq t \leq 1$, the following script computes the upper and the lower bounds and their average:

```
%% integral of t^2 from 0 to 1 is 0.333
% lower bound
y1=simple(symsum(k^2*T^3,1,n-1));
yy1=subs(y1)
% upper bound
z1=simple(symsum(k^2*T^3,1,n));
zz1=subs(z1)
% average
int= 0.5*(yy1+zz1)
```

giving the following results, in this case the value of the definite integral is $1/3$.

```
yy1 = 0.3328  
zz1 = 0.3338  
int = 0.3333
```

0.13 Exponentials — The exponential $x(t) = e^{at}$ for $t \geq 0$ and zero otherwise is a very common continuous-time signal. Likewise, $y(n) = \alpha^n$ for integers $n \geq 0$ and zero otherwise is a very common discrete-time signal. Let us see how they are related. Do the following using MATLAB:

- Let $a = -0.5$, plot $x(t)$
- Let $a = -1$, plot the corresponding signal $x(t)$. Does this signal go to zero faster than the exponential for $a = -0.5$?
- Suppose we sample the signal $x(t)$ using $T_s = 1$ what would be $x(nT_s)$ and how can it be related to $y(n)$, i.e., what is the value of α that would make the two equal?
- Suppose that a current $x(t) = e^{-0.5t}$ for $t \geq 0$ and zero otherwise is applied to a discharged capacitor of capacitance $C = 1$ F at $t = 0$. What would be the voltage in the capacitor at $t = 1$ second?
- How would you obtain an approximate result to the above problem using a computer? Explain.

Answers: $0 < e^{-\alpha t} < e^{-\beta t}$ for $\alpha > \beta \geq 0$; $v_c(1) = 0.79$.

Solution

(a)(b) We have that

$$0 < e^{-\alpha t} < e^{-\beta t}$$

for $\alpha > \beta \geq 0$.

```
% Pr. 0.13
clear all
% compare two exponentials
t=[0:0.001:10];
x=exp(-0.5*t);
x1=exp(-1*t);
figure(6)
plot(t,x,t,x1,'r');
legend('Exponential Signal, a=-0.5','Exponential Signal, a=-1')
grid
axis([0 10 0 1.1]); xlabel('time')
```

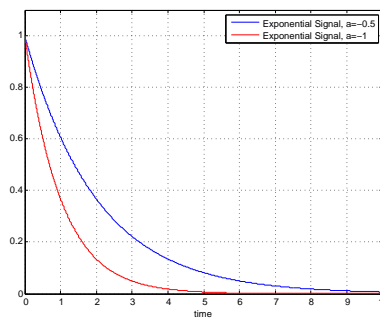


Figure 4: Problem 13: Comparison of exponentials $e^{-0.5t}$ and e^{-t} for $t \geq 0$ and 0 otherwise.

(c) Sampling $x(t) = e^{at}$ using $T_s = 1$, we get

$$x(t)|_{t=n} = e^{an} = \alpha^n$$