Chapter 2 – Sequences

Section 2.1

1. $\dfrac{1}{\sqrt{n+1}} < 0.02 \Leftrightarrow 1 < (0.02)\sqrt{n+1} \Leftrightarrow n > 2499$. Thus, if $n^* = 2{,}500$, then for any $n \ge n^*$ the given inequality will be true.

2. **(a)** We prove that $\lim_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$ be given. We need to find $n^* \in N$ so that $|a_n - 0| < \varepsilon$ for all $n \ge n^*$. But, $|a_n - 0| = \dfrac{1}{2n-3}$ if $n \ge 2$, and $\dfrac{1}{2n-3} < \varepsilon$ if $n > \dfrac{1}{2\varepsilon} + \dfrac{3}{2}$. Thus, if $n^* > \max\left\{2, \dfrac{1}{2\varepsilon} + \dfrac{3}{2}\right\}$, then $|a_n| < \varepsilon$ for all $n \ge n^*$. It is also alright to write that $|a_n - 0| = \dfrac{1}{2n-3} < \dfrac{1}{n}$ if $n > 3$. Thus, if $n > 3$ and $n > \dfrac{1}{\varepsilon}$, then $|a_n - 0| < \varepsilon$. So, choose $n^* > \max\left\{3, \dfrac{1}{\varepsilon}\right\}$.

(b) We prove that $\lim_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$ be given. We need to find $n^* \in N$ so that $|a_n - 0| < \varepsilon$ for all $n \ge n^*$. But, $|a_n| = \dfrac{n}{n^2 - 2}$ if $n \ge 2$, and $\dfrac{n}{n^2-2} \le \dfrac{n}{\frac{1}{2}n^2} = \dfrac{2}{n}$ if $n \ge 2$, and $\dfrac{2}{n} < \varepsilon$ if $n > \dfrac{2}{\varepsilon}$. Thus, if $n^* > \max\left\{2, \dfrac{2}{\varepsilon}\right\}$, then $|a_n| < \varepsilon$ for all $n \ge n^*$.

(c) We prove that $\lim_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$ be given. We need to find $n^* \in N$ so that $|a_n - 0| < \varepsilon$ for all $n \ge n^*$. But, $|a_n| = \dfrac{1}{n^p} < \varepsilon$ if $n > \sqrt[p]{\dfrac{1}{\varepsilon}}$. Thus, if $n^* > \sqrt[p]{\dfrac{1}{\varepsilon}}$, then $|a_n| < \varepsilon$ for all $n \ge n^*$.

(d) We prove that $\lim_{n\to\infty} a_n = \dfrac{1}{2}$. Let $\varepsilon > 0$ be given. We need to find $n^* \in N$ so that $\left|a_n - \dfrac{1}{2}\right| < \varepsilon$ for all $n \ge n^*$. But, $\left|a_n - \dfrac{1}{2}\right| = \dfrac{\sqrt{n}}{4n + 2\sqrt{n}} < \dfrac{\sqrt{n}}{4n} = \dfrac{1}{4\sqrt{n}} < \varepsilon$ if $n > \dfrac{1}{16\varepsilon^2}$. Thus, if $n^* > \dfrac{1}{16\varepsilon^2}$, then $\left|a_n - \dfrac{1}{2}\right| < \varepsilon$.

(f) Suppose $\{a_n\}$ converges to A. Then, there exists $n^* \in N$ such that for a particular $\varepsilon > 0$, say, 1, we have $\left|(-1)^n - A\right| < 1$ for all $n \ge n^*$. Now, if $n \ge n^*$ is even, then we have $|1 - A| < 1$, which implies that $A > 0$. If $n \ge n^*$ is odd, then we have $|-1 - A| < 1$, which implies $A < 0$. Due to Theorem 2.1.9 this is a contradiction.

(g) Since $a_n = \sqrt{n+1} - \sqrt{n} = \dfrac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$, we will prove that $\lim_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$ be given. Since $\sqrt{n+1} > \sqrt{n}$, we have that $|a_n - 0| = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} < \dfrac{1}{2\sqrt{n}} < \varepsilon$ if $n > \dfrac{1}{4\varepsilon^2}$. Thus, if $n^* > \dfrac{1}{4\varepsilon^2}$, then $|a_n - 0| < \varepsilon$ for all $n \ge n^*$.

(h) We will prove that $\{a_n\}$ does not converge by using Definition 2.1.6. *Case 1.* Suppose that $A \ge 0$ is any arbitrary real number, $\dfrac{1}{2}$ is a particular $\varepsilon > 0$, and n^* is an arbitrary natural number. We will show that $|a_m - A| \ge \dfrac{1}{2}$ for some $m > n^*$. To this end, let m be odd and write $|a_m - A| = \left|(-1)\left(\dfrac{m}{m+1}\right) - A\right| \ge$

$\frac{m}{m+1} \geq \frac{1}{2}$. *Case 2.* Suppose that $A < 0$ and proceed in a similar fashion.

(k) We will prove that $\{a_n\}$ does not converge by using Definition 2.1.6. *Case 1.* Suppose that $A \geq \frac{3}{4}$ is any arbitrary real number, $\frac{1}{4}$ is a particular $\varepsilon > 0$, and n^* is an arbitrary natural number. We will show that $|a_m - A| \geq \frac{1}{4}$ for some $m > n^*$. To this end, let m be any even natural number. Then, $|a_m - A| \geq \left|\frac{1}{m} - A\right| \geq \left|\frac{1}{m} - \frac{3}{4}\right| \geq \frac{1}{4}$. *Case 2.* Suppose $A < \frac{3}{4}$ and proceed in a similar fashion.

3. **(a)** Using Example 1.3.3, we have $a_n = \frac{1+2+\cdots+n}{n^2} = \frac{\frac{n(n+1)}{2}}{n^2} = \frac{n+1}{2n}$, which we will prove tends to $\frac{1}{2}$. To this end, let $\varepsilon > 0$ be given. Since, $\left|a_n - \frac{1}{2}\right| = \frac{1}{2n} < \varepsilon$ if $n > \frac{1}{2\varepsilon}$. Thus, if $n^* > \frac{1}{2\varepsilon}$, then $\left|a_n - \frac{1}{2}\right| < \varepsilon$ for all $n \geq n^*$.

4. ($\Rightarrow$) Let $\varepsilon > 0$ be given. We need to show that $\lim_{n\to\infty} |a_n| = 0$, that is, we need to find n^* so that $\left||a_n| - 0\right| < \varepsilon$ for all $n \geq n^*$. Since $\lim_{n\to\infty} a_n = 0$, there exists n_1 such that $|a_n - 0| < \boxed{\varepsilon}$ for all $n \geq n_1$. But, if $n^* = n_1$, we have $\left||a_n| - 0\right| = |a_n| < \varepsilon$ for all $n \geq n^*$. Proof of the converse is similar.

5. Let $\varepsilon > 0$ be given. We wish to show that $\lim_{n\to\infty} |a_n| = |A|$, that is, we need to find n^* so that $\left||a_n| - |A|\right| < \varepsilon$ for all $n \geq n^*$. Since $\lim_{n\to\infty} a_n = A$, there exists n_1 such that $|a_n - A| < \boxed{\varepsilon}$ for all $n \geq n_1$. But, if $n^* = n_1$, using Corollary 1.8.6, we have $\left||a_n| - |A|\right| \leq |a_n - A| < \varepsilon$ for all $n \geq n^*$.

The converse is false. Choose $a_n = (-1)^n$.

6. ($\Rightarrow$) Let $\varepsilon > 0$ be given. We need to show that $\lim_{n\to\infty} (a_n - A) = 0$, that is, we need to find n^* so that $|(a_n - A) - 0| < \varepsilon$ for all $n \geq n^*$. Since $\lim_{n\to\infty} a_n = A$, there exists n_1 such that $|a_n - A| < \boxed{\varepsilon}$ for all $n \geq n_1$. But, if $n^* = n_1$, we have $|(a_n - A) - 0| = |a_n - A| < \varepsilon$ for all $n \geq n^*$. Proof of the converse is similar.

8. Let $\varepsilon > 0$ be given. We need to show that $\lim_{n\to\infty} a_n = A$, that is, we need to find n^* so that $|a_n - A| < \varepsilon$ for all $n \geq n^*$. Since $\lim_{n\to\infty} b_n = 0$, there exists n_1 such that $|b_n - 0| < \boxed{\frac{\varepsilon}{k+1}}$ for all $n \geq n_1$. But, if $n^* = n_1$, we have $|a_n - A| \leq k|b_n| < k \cdot \frac{\varepsilon}{k+1} < \varepsilon$ for all $n \geq n^*$. We chose $\frac{\varepsilon}{k+1}$ instead of $\frac{\varepsilon}{k}$ in the case $k = 0$.

9. Use Exercise 7.

10. We will prove that $\{b_n\}$ converges to A. Let $\varepsilon > 0$ be given. We need to find n^* so that $|b_n - A| < \varepsilon$ for all $n \geq n^*$. Since $\lim_{n\to\infty} a_n = A$, there exists n_1 such that $|a_n - A| < \boxed{\frac{\varepsilon}{2}}$ for all $n \geq n_1$. But, if $n^* = n_1$, then we have $|b_n - A| = \left|\frac{a_n + a_{n+1}}{2} - A\right| \leq \left|\frac{a_n}{2} - A\right| + \left|\frac{a_{n+1}}{2} - A\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

12. Suppose that $\lim_{n\to\infty} a_n = A \neq 0$. By Theorem 2.1.12 there exists n^* such that $\frac{1}{2}|A| \leq |a_n|$ for all $n \geq n^*$, which in turn implies that $\left|\frac{1}{a_n}\right| < \frac{2}{|A|}$. Thus, if $M = \max\left\{\frac{2}{|A|}, \frac{1}{|a_1|}, \frac{1}{|a_2|}, \cdots, \frac{1}{|a_{n^*-1}|}\right\}$ then, $\left|\frac{1}{a_n}\right| \le M$ for all $n \in N$.

13. Choose any $t \in (0,1)$ and fix it. By Remark 2.1.8, part (g), there exists n^* such that $|a_n - A| < \boxed{(1-t)A}$ for all $n \ge n^*$. But, $|a_n| = |a_n - A + A| \ge |A| - |a_n - A| > |A| - (1-t)|A| = t|A|$. If $t = 0$, then the conclusion $|a_n| \ge t|A|$ becomes $|a_n| \ge 0$, which is certainly true. If $t = 1$, then the conclusion $|a_n| \ge t|A|$ becomes $|a_n| \ge |A|$, which need not hold. For example, pick $a_n = 1 - \frac{1}{n}$.

14. We will show that if $c > 0$, then $\lim_{n\to\infty} \sqrt[n]{c} = 1$.

Case 1. If $0 < c < 1$, then $\sqrt[n]{c} < 1$. So, there exists $d_n > 0$ such that $\sqrt[n]{c} = \frac{1}{1+d_n}$. By binomial theorem we have $(1+d_n)^n = 1 + nd_n + \frac{n(n-1)}{2}(d_n)^2 + \cdots + (d_n)^n > nd_n$, and thus, $c = \left(\frac{1}{1+d_n}\right)^n = \frac{1}{(1+d_n)^n} < \frac{1}{nd_n}$ from which it follows that $0 < cnd_n < 1$, that is, $0 < d_n < \frac{1}{cn}$ for all $n \in N$. Thus, $0 < 1 - \sqrt[n]{c} = 1 - \frac{1}{1+d_n} = \frac{d_n}{1+d_n} < d_n < \frac{1}{cn}$ for all $n \in N$. Thus, $\left|\sqrt[n]{c} - 1\right| < \frac{1}{cn}$, and so by Exercise 8, $\lim_{n\to\infty} \sqrt[n]{c} = 1$ if $0 < c < 1$.

Case 2. If $c = 1$, then $\sqrt[n]{c} = 1$ for each value of n. Thus, by Exercise 6 we have $\lim_{n\to\infty} \sqrt[n]{c} = \lim_{n\to\infty} 1 = 1$.

Case 3. If $c > 1$, then by Exercise 32 of Section 1.9, we have $\sqrt[n]{c} > 1$. Therefore, $\sqrt[n]{c} = 1 + b_n$ with $b_n > 0$. Thus, we have $c = (1+b_n)^n = 1 + nb_n + \frac{n(n-1)}{2}(b_n)^2 + \cdots + (b_n)^n > nb_n$. Therefore, $c > nb_n$ and consequently, $b_n < \frac{c}{n}$. Thus, we have $\left|\sqrt[n]{c} - 1\right| = |b_n| = b_n < \frac{c}{n}$ for all $n \in N$. By Exercise 8, $\lim_{n\to\infty} \sqrt[n]{c} = 1$ if $c > 1$.

15. We will show that if $c > 0$, then $\lim_{n\to\infty} \sqrt[n]{n} = 1$. If $n > 1$, then by Exercise 32 of Section 1.9, we have $\sqrt[n]{n} > 1$. Therefore, $\sqrt[n]{n} = 1 + b_n$ with $b_n > 0$. Thus, we have $n = (1+b_n)^n = 1 + nb_n + \frac{n(n-1)}{2}(b_n)^2 + \cdots + (b_n)^n > 1 + \frac{n(n-1)}{2}(b_n)^2$. Thus, if $n > 1$, we have $n - 1 > \frac{n(n-1)}{2}(b_n)^2$, which gives $2 > n(b_n)^2$, or equivalently, $b_n < \frac{\sqrt{2}}{\sqrt{n}}$. Thus, $\left|\sqrt[n]{n} - 1\right| = b_n < \sqrt{2} \cdot \frac{1}{\sqrt{n}}$. Since, by Exercise 2(c), $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, Exercise 8 gives the desired conclusion.

16. We will show that $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$. Note that, as verified in Exercise 4(g) of Section 1.3, by the binomial theorem, we have $2^n = (1+1)^n = 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{3!} + \cdots + 1 > \frac{n(n-1)(n-2)}{6}$ for all $n \in N$. Thus, if $n \ge 4$ (why?) we have that $\frac{n^2}{2^n} < \frac{6n^2}{n(n-1)(n-2)} = \frac{6n}{n^2 - 3n + 2} < \frac{6n}{n^2 - 3n} = \frac{6}{n-3}$. Hence, $\left|\frac{n^2}{2^n} - 0\right| < \frac{6}{n-3}$, and by Exercise 8, the desired result follows.

17. We will prove that $\lim_{n\to\infty} nr^n = 0$ if $|r| < 1$. If $r = 0$, then $nr^n = 0$ for each n. Therefore, $\lim_{n\to\infty} nr^n =$

$\lim_{n\to\infty} 0 = 0$. If $|r| < 1$, there exists $b > 0$ such that $|r| = \frac{1}{1+b}$. But, $nr^n = (1+b)^n = 1 + nb + \frac{n(n-1)}{2}b^2 + \cdots$ $+ b^n > \frac{n(n-1)}{2}b^2$, for all $n \in N$. Therefore, $\left|nr^n - 0\right| = nr^n = n \cdot \frac{1}{(1+b)^n} < \frac{2}{b^2} \cdot \frac{1}{n-1}$. Now apply Exercise 8.

18. By Examples 1.3.4 and 1.4.5 we have $a + ar + ar^2 + \cdots + ar^n = a\left(1 + r + r^2 + \cdots + r^n\right) = a\left(\frac{r^{n+1} - 1}{r-1}\right) =$ $\frac{a}{1-r} - \frac{a}{1-r} \cdot r^{n+1}$. To show that $\{a_n\}$ converges to $\frac{a}{1-r}$, we can write $\left|a + ar + ar^2 + \cdots + ar^n - \frac{a}{1-r}\right| =$ $\frac{|a|}{1-r} \cdot |r|^{n+1}$. Thus, using Theorem 2.1.13 and by Exercise 8 we have that $\{a_n\}$ converges to $\frac{a}{1-r}$.

19. **(a)** Set $a = 1$ and $r = \frac{1}{2}$ in Exercise 18 to get $\lim_{n\to\infty} a_n = 2$.

(b) We write $1.\overline{9} = 1 + 0.9 + 0.09 + 0.009 + \cdots = 1 + \lim_{n\to\infty}\left(\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n}\right) = 2$.

20. **(a)** No, not if $a_n = n$.

(b) No, not if $a_n = \frac{1}{n}$.

(c) No, not if $a_n = 0$, or $a_n = n^n$, or ... .

21. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty}\left(a_n - a_{n-2}\right) = 0$, there exists n_1 such that $|a_n - a_{n-2}| < \boxed{\varepsilon}$ for all $n \ge n_1$. Next, observe that for any $n > n_1$ we can write, $a_n - a_{n-1} = \left(a_n - a_{n-2}\right) - \left(a_{n-1} - a_{n-3}\right) + \left(a_{n-2} - a_{n-3}\right) -$ $\cdots \pm \left(a_{n_1+1} - a_{n_1-1}\right) \mp \left(a_{n_1} - a_{n_1-1}\right)$. Thus, if $n > n_1$, we have, $|a_n - a_{n-1}| \le (n - n_1)\varepsilon + |a_{n_1} - a_{n_1-1}|$, and hence, $\lim_{n\to\infty} \frac{a_n - a_{n-1}}{n} = 0$.

Section 2.2

1. Let $\varepsilon > 0$ be given. We need to find n^* such that $\left|\frac{1}{b_n} - \frac{1}{B}\right| < \varepsilon$ for all $n \ge n^*$. Since B and b_n are not 0, by Theorem 2.1.12, there exists n_1 such that $|b_n| > \frac{|B|}{2}$ if $n \ge n_1$. Since, $\lim_{n\to\infty} b_n = B$, there exists n_2 such that $|b_n - B| < \boxed{\frac{B^2\varepsilon}{2}}$ for all $n \ge n_2$. If $n^* = \max\{n_1, n_2\}$, then for all $n \ge n^*$ we have $\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|B - b_n|}{|b_n B|} < \frac{|b_n - B|}{\frac{|B|}{2} \cdot |B|} = \frac{2|b_n - B|}{B^2} < \varepsilon$.

2. Let $\varepsilon > 0$ be given. We need to find n^* such that $|ca_n - cA| < \varepsilon$ for all $n \ge n^*$. Since $\lim_{n\to\infty} a_n = A$, there exists n_1 such that $|a_n - A| < \boxed{\frac{\varepsilon}{|c|+1}}$ for all $n \ge n_1$. (We use $|c| + 1$ to avoid 0 in the denominator.) Choose $n^* = n_1$. Then, for all $n \ge n^*$ we have $|ca_n - cA| = |c||a_n - A| < |c| \cdot \frac{\varepsilon}{|c|+1} < \varepsilon$.

3. Let $\varepsilon > 0$ be given. We need to find n^* such that $\left|(a_n)^p - A^p\right| < \varepsilon$ for all $n \ge n^*$. Since $\{a_n\}$ converges, by Theorem 2.1.11, it is bounded. Therefore, there exists $M > 0$ such that $|a_n| \le M$ for all n. Also, since $\lim_{n\to\infty} a_n = A$, there exists n_1 such that $|a_n - A| < \boxed{\varepsilon\left(M^{p-1} + M^{p-2}A + \cdots + A^{p-1}\right)^{-1}}$. If $n^* = n_1$, then for all $n \ge n^*$ we have $\left|(a_n)^p - A^p\right| = |a_n - A|\left|(a_n)^{p-1} + (a_n)^{p-2}A + \cdots + A^{p-1}\right| \le$ $|a_n - A|\left(M^{p-1} + M^{p-2}A + \cdots + A^{p-1}\right) < \varepsilon$. Hence, the result follows. Observe that Theorem 2.2.1, part (b), could have been used $p-1$ times to prove this result.

4. Observe that for any two real numbers a and b we have $ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right]$. Since $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ by Theorem 2.2.1 we have that $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$, and thus, $\lim_{n\to\infty}(a_n \pm b_n)^2 = (A \pm B)^2$. Hence, $\lim_{n\to\infty}\left[(a_n + b_n)^2 - (a_n - b_n)^2\right] = (A+B)^2 - (A-B)^2 = 4AB$. Exercise 2 completes the proof.

5. Let $\varepsilon > 0$ be given. We need to find n^* such that $|a_n b_n - 0| < \varepsilon$ for all $n \ge n^*$. Since $\{b_n\}$ is bounded, there exists $M > 0$ such that $|b_n| \le M$ for all n. Since $\lim_{n\to\infty} a_n = 0$ there exists n_1 such that $|a_n - 0| < \boxed{\frac{\varepsilon}{M}}$ for all $n \ge n_1$. If $n^* = n_1$, then for all $n \ge n^*$ we have that $|a_n b_n - 0| = |a_n||b_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon$. Hence, $\lim_{n\to\infty} a_n b_n = 0$.

6. By Exercise 6 from Section 2.1, $\lim_{n\to\infty} a_n = A \Leftrightarrow \lim_{n\to\infty}(a_n - A) = 0$ and $\lim_{n\to\infty} b_n = B \Leftrightarrow \lim_{n\to\infty}(b_n - B) = 0$ and thus bounded by say, M. Since $a_n b_n - AB = (a_n - A)(b_n - B) + A(b_n - B) + B(a_n - A)$, employing Theorem 2.2,7 we see that $\lim_{n\to\infty}(a_n - A)(b_n - B) = 0$. (Observe that we cannot split this limit into two since that is what we are trying to prove.) Also, $A \lim_{n\to\infty}(b_n - B) = A \cdot 0 = 0$ and $B \lim_{n\to\infty}(a_n - A) = B \cdot 0 = 0$. Thus, $\lim_{n\to\infty}(a_n b_n - AB) = \lim_{n\to\infty}(a_n - A)(b_n - B) + A \lim_{n\to\infty}(b_n - B) + B \lim_{n\to\infty}(a_n - A) = 0$. Hence, by Exercise 6 from Section 2.1, $\lim_{n\to\infty} a_n b_n = AB$.

7. We prove that $\lim_{n\to\infty}\sqrt{a_n} = \sqrt{A}$, where $A = \lim_{n\to\infty} a_n$. Let $\varepsilon > 0$ be given. We need to find n^* such that $\left|\sqrt{a_n} - \sqrt{A}\right| < \varepsilon$ for all $n \ge n^*$. *Case 1.* Suppose $A = 0$. Since $\lim_{n\to\infty} a_n = A = 0$, there exists n_1 such that $|a_n - 0| = a_n < \boxed{\varepsilon^2}$ for all $n \ge n_1$. If $n^* = n_1$, then for all $n \ge n^*$ we have $\left|\sqrt{a_n} - \sqrt{A}\right| = \left|\sqrt{a_n} - 0\right| = \sqrt{a_n} <$ $\sqrt{\varepsilon^2} = |\varepsilon| = \varepsilon$.

Case 2. Suppose $A \ne 0$. Since $\lim_{n\to\infty} a_n = A$, there exists n_2 such that $|a_n - A| < \boxed{\sqrt{A}\,\varepsilon}$ for all $n \ge n_2$. If $n^* = n_2$, then for all $n \ge n^*$ we rationalize and write $\left|\sqrt{a_n} - \sqrt{A}\right| = \frac{|a_n - A|}{\sqrt{a_n} + \sqrt{A}} \le \frac{|a_n - A|}{\sqrt{A}} < \frac{\sqrt{A}\varepsilon}{\sqrt{A}} = \varepsilon$. Hence, $\lim_{n\to\infty}\sqrt{a_n} = \sqrt{A}$.

To prove that $\lim_{n\to\infty}\sqrt[3]{a_n} = \sqrt[3]{A}$, follow a similar procedure.

8. **(a)** Let $\varepsilon > 0$ be given. We will show that $A - B < \varepsilon$ and employ Exercise 13 from Section 1.8. Since $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, there exist n_1 and n_2 such that $|a_n - A| < \boxed{\frac{\varepsilon}{2}}$ for all $n \ge n_1$, and

$|b_n - B| < \boxed{\frac{\varepsilon}{2}}$ for all $n \ge n_2$. Now, if we choose $n^* = \max\{n_1, n_2\}$, then for all $n \ge n^*$ we have $A - a_n < \frac{\varepsilon}{2}$ and $b_n - B < \frac{\varepsilon}{2}$. Therefore, for all $n \ge n^*$, since $a_n \le b_n$, we have $A - B = (A - a_n) + (b_n - B) + (a_n - b_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0 = \varepsilon$. Thus, by Exercise 13 from Section 1.8 we have $A - B \le 0$. Hence, $A \le B$.

(b) Choose $a_n = 0$, $b_n = \frac{1}{n}$, and $n_1 = 1$.

9. Proof is by contradiction. Suppose that $A < 0$. Then, since $\lim_{n\to\infty} a_n = A$, there exists n_2 such that for all $n \ge n_2$ we have $|a_n - A| < \boxed{\frac{|A|}{2}}$. Therefore, by Exercise 14(a) from Section 1.8, we have $-\frac{|A|}{2} < a_n - A < \frac{|A|}{2}$, which is equivalent to $A - \frac{|A|}{2} < a_n < A + \frac{|A|}{2}$. But, $A + \frac{|A|}{2}$ is negative, which implies that $a_n < 0$ for all $n \ge n_2$. This is a contradiction to the hypothesis.

Another way to prove $A \ge 0$ is to apply Exercise 8(a) with $b_n = 0$ and the reverse inequality.

10. Suppose $\{a_n\}$ converges to A and to B. By Theorem 2.2.1, part (f), with $a_n = b_n$, we have that $A \le B$. Similarly, $B \le A$. Hence, $A = B$.

11. (a) $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{1}{1} = 1.$

(b) $\lim_{n\to\infty} r^{\frac{n+1}{2}} = \lim_{n\to\infty} \sqrt{r}\sqrt{r^n} = \sqrt{r} \lim_{n\to\infty} \sqrt{r^n} = \sqrt{r}\cdot\sqrt{0} = 0.$

(c) Since $n < 2^n$, which can be proven by induction, we have $0 < \frac{1}{2^n} < \frac{1}{n}$. But, $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus, by the sandwich theorem, $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

(d) Since $\lim_{n\to\infty} \frac{|r|}{n} = 0$, there exists $n_1 \in N$ such that $\frac{|r|}{n} < \frac{1}{2}$ if $n \ge n_1$. Now, since first we are looking for $\lim_{n\to\infty} \frac{|r^n|}{n!}$, assume that $n > n_1$ and write $0 \le \frac{|r^n|}{n!} = \frac{|r|}{1}\cdot\frac{|r|}{2}\cdot\frac{|r|}{3}\cdot\ldots\cdot\frac{|r|}{n_1}\cdot\frac{|r|}{n_1+1}\cdot\ldots\cdot\frac{|r|}{n} < \left(\frac{|r|}{1}\cdot\frac{|r|}{2}\cdot\frac{|r|}{3}\cdot\ldots\cdot\frac{|r|}{n_1}\right)\frac{1}{2}\cdot\ldots\cdot\frac{1}{2} = M\cdot\frac{1}{2}\cdot\ldots\cdot\frac{1}{2} = M\left(\frac{1}{2}\right)^{n-n_1}$, which tends to 0 as n goes to ∞. Now apply the sandwich theorem and Theorem 2.1.14..

(e) Use Exercise 2(c) of Section 2.1 and the sandwich theorem.

(f) Rationalizing, we write $0 < \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}$. Now apply the sandwich theorem.

(g) If we rationalize the numerator, we obtain $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$, which tends to $\frac{1}{2}$ as n goes to infinity.

(h) We rationalize the numerator, to obtain $a_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{n}$. So, by the sandwich theorem the limit is 0.

(i) Since $\sqrt[n]{n} < \sqrt[n]{n+\sqrt{n}} < \sqrt[n]{n+n} = \sqrt[n]{2n} = \sqrt[n]{2}\sqrt[n]{n}$, and $\lim_{n\to\infty} \sqrt[n]{2} = 1 = \lim_{n\to\infty} \sqrt[n]{n}$, by the sandwich theorem, the limit is 1.

(j) Since $\sqrt[n]{2^{n+1}} = 2\sqrt[n]{2}$ and $\lim_{n\to\infty} \sqrt[n]{2} = 1$, then limit of the desired expression is 2.

(k) For $n > 3$ we can write $0 < \frac{n^2}{n!} = \frac{n \cdot n}{n(n-1)(n-2)\cdots(2)(1)} < \frac{n}{(n-1)(n-2)} < \frac{n}{n^2-3n} = \frac{1}{n-3}$. Now apply the sandwich theorem.

(l) $\lim_{n\to\infty} \frac{1}{n}\sin\frac{1}{n} = \left(\lim_{n\to\infty} \frac{1}{n}\right)\cdot\left(\lim_{n\to\infty} \sin\frac{1}{n}\right) = 0\cdot 0 = 0$.

(m) If $m = 2n$, then we write, $\lim_{n\to\infty} n\sin\frac{1}{2n} = \lim_{m\to\infty}\left(\frac{1}{2}m\right)\sin\frac{1}{m} = \frac{1}{2}\lim_{m\to\infty} m\sin\frac{1}{m} = \frac{1}{2}\cdot 1 = \frac{1}{2}$.

12. Even though $\{a_n\}$ converges to 0, $\{a_nb_n\}$ need not converge. For example, choose $a_n = \frac{1}{n}$ and $b_n = n^2$. Then $\{a_nb_n\}$ diverges to $+\infty$. But if $b_n = kn,\ k \in \Re$, then $\{a_nb_n\}$ converges to k. However, if $\{b_n\}$ is bounded, whether convergent or not, $\{a_nb_n\}$ will converge to 0. This requires a proof.

13. Not true. Choose $a_n = \frac{1}{n}$ and $b_n = n$. Observe that if in addition we were to assume that $\{a_n\}$ converges to a nonzero value A, then the statement is true because then we can apply Theorem 2.2.1, part (c), and write $b_n = \frac{a_nb_n}{a_n}$. Since $\{b_n\}$ is a quotient of 2 converging sequences and all the conditions of Theorem 2.2.1, part (c), are satisfied, we can conclude that $\{b_n\}$ must converge.

14. To prove that $\lim_{n\to\infty} \sqrt[n]{c} = 1$, for $0 < c < 1$, write $0 < \left|\sqrt[n]{c} - 1\right| < \frac{1}{cn}$, and for $c > 1$, write $0 < \sqrt[n]{c} - 1 < \frac{c}{n}$. To prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$, write $0 < \sqrt[n]{n} - 1 < \sqrt{\frac{2}{n}}$, which tends to 0. To prove that $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$, write $0 < \frac{n^2}{2^n} < \frac{6}{n-3}$. To prove that $\lim_{n\to\infty} nr^n = 0$, write $0 < nr^n < \frac{2}{b^2}\cdot\frac{1}{n-1}$, b a real constant.

15. Due to partial fraction decomposition we have $a_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$. Thus, $s_n = a_1 + a_2 + \cdots + a_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$, which tends to $\frac{1}{2}$ as n goes to infinity.

16. Due to partial fraction decomposition we have $a_n \equiv \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Thus, $b_n = a_1 + a_2 + \cdots + a_n = 1 - \frac{1}{n+1}$, which tends to 1 as n goes to infinity.

17. Write $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-2}} - \frac{1}{2^{n-1}}\right) = 1 + 1 - \frac{1}{2^{n-1}}$, which tends to 2 as $n \to \infty$.

18. **(a)** Yes. Choose $a_n = \sqrt{n}$.

(b) Suppose $\{a_n\}$ is bounded. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, the product must tend to 0, (see Exercise 5). This contradicts the fact that $L \neq 0$.

19. To prove $\{a_n\}$ converges to -1, we show that $\lim_{n\to\infty} (a_n + 1) = 0$ and then apply Exercise 6 from Section 2.1. To this end, we write $\lim_{n\to\infty} (a_n + 1) = \lim_{n\to\infty} \left(\frac{b_n + 1}{b_n - 1} + 1\right) = \lim_{n\to\infty} \frac{2b_n}{b_n - 1} = \frac{0}{0-1} = 0$.

21. If $\alpha = \beta$, $\alpha \geq 0$, then $a_n = \sqrt[n]{\alpha^n + \alpha^n} = \alpha\sqrt[n]{2} \to \alpha \cdot 1 = \alpha = \beta$, as $n \to \infty$, by Exercise 14 in Section 2.1. If $0 \leq \alpha < \beta$, then $a_n = \sqrt[n]{\alpha^n + \beta^n} = \beta\sqrt[n]{\left(\frac{\alpha}{\beta}\right)^n + 1} \to \beta \cdot 1 = \beta$.

Section 2.3

1. Let $M > 0$ be given. We want to find $n^* \in N$ so that $b_n > M$ for all $n \geq n^*$. Since $\lim_{n\to\infty} a_n = +\infty$, there exists $n_1 \in N$ such that $a_n > \boxed{M}$ for all $n \geq n_1$. Now choose $n^* = n_1$. Then, for all $n \geq n^*$ we have $b_n \geq a_n > M$.

2. (a) Let $M > 0$ be given. We want to find n^* so that $a_n + b_n > M$ for all $n \geq n^*$. Since $\{a_n\}$ diverges to $+\infty$, there exists n_1 such that for all $n \geq n_1$ we have $a_n > \boxed{M - K}$ where K is a lower bound of $\{b_n\}$. Now choose $n^* = n_1$. Then for all $n \geq n^*$ we have $a_n + b_n > (M - K) + K = M$. Note that if $\{b_n\}$ is not bounded below, then the result is false. For example, choose $a_n = n$ and $b_n = -n$.

(b) Let $M > 0$ be given. We want to find n^* so that $a_n b_n > M$ for all $n \geq n^*$. Since $\{a_n\}$ diverges to $+\infty$, there exists n_1 such that for all $n \geq n_1$ we have $a_n > \boxed{\frac{M}{K}}$ where $K > 0$ is a lower bound of $\{b_n\}$. Now choose $n^* = n_1$. Then for all $n \geq n^*$ we have $a_n b_n > \frac{M}{K} \cdot K = M$. Note that if $\{b_n\}$ is not bounded below, then Theorem 2.3.3, part (b), is not true. For example, choose $a_n = n$ and $b_n = -n$. Then, $\{a_n b_n\}$ diverges to $-\infty$. If $\{b_n\}$ is bounded below but not by a positive constant, the result is still not true. For example, choose $a_n = n$ and $b_n = -1$. Then, $\{a_n b_n\}$ diverges to $-\infty$. Or, choose $a_n = n$ and $b_n = \frac{c}{n}$, $c \in \Re^+$ so that $\{b_n\}$ is bounded below by 0. Then $\{a_n b_n\}$ converges to c. Also, see the answer to Exercise 6. If $b_n = 0$, then $\{a_n b_n\}$ converges to 0.

(c) Let $M > 0$ be given. If c is a positive constant, we want to find n^* so that $ca_n > M$ for all $n \geq n^*$. Since $\{a_n\}$ diverges to $+\infty$, there exists n_1 such that $a_n > \boxed{\frac{M}{c}}$ for all $n \geq n_1$. Now choose $n^* = n_1$. Then, for all $n \geq n^*$ we have $ca_n > c \cdot \frac{M}{c} = M$. If c is a negative constant, we want to find n^* so that $ca_n < -M$ for all $n \geq n^*$. Since $\{a_n\}$ diverges to $+\infty$, there exists n_1 such that $a_n > \boxed{-\frac{M}{c}}$ for all $n \geq n_1$. Now choose $n^* = n_1$. Then, for all $n \geq n^*$ we have $ca_n < c \cdot \frac{M}{-c} = -M$.

3. (a) Since for $n > 2$ we can write $\frac{n^2 + 1}{n - 2} > \frac{n^2}{n - 2} > \frac{n^2}{n} = n$, which tends to $+\infty$ as n goes to infinity, by the

comparison theorem, we have $\lim_{n\to\infty} a_n = +\infty$.

(b) Since $n^3 - n + 1 > \frac{n^3}{2}$, and $2n + 4 \le 4n$ for $n \ge 2$, we conclude that whenever $n \ge 2$, we have $\frac{n^3 - n + 1}{2n + 4} > \frac{n^3}{4n} = \frac{1}{4}n^2$. By the comparison theorem, the sequence $\{a_n\}$ diverges to $+\infty$.

(c) Show $\{-a_n\}$ tends to $+\infty$.

4. **(a)** $a_n = (-1)^n$.

(b) $a_n = 0$ for n even and $a_n = -n$ for n odd.

(c) $a_n = 0$ for n even and $a_n = n$ for n odd.

(d) $a_n = (-1)^n n$.

5. Use either a definition or Theorem 2.3.3.

6. **(a)** Use either a definition, or use Theorem 2.1.12 and part (b) of Theorem 2.3.3, or prove by contradiction. So here is a possible sequence of steps one can take: $b_n > \frac{1}{2}B > 0$ eventually, so $a_n b_n > \frac{1}{2}Ba_n \to \infty$. Comparison theorem proves the conclusion.

(b) The sequence $\{a_n b_n\}$ can converge, diverge to $+\infty$, diverge to $-\infty$, or it can oscillate. Examples are: $a_n = n$ and $b_n = \frac{1}{n}$; $a_n = n$ and $b_n = 1$; $a_n = n$ and $b_n = -1$; $a_n = n$ and $b_n = 0$ for n even and $b_n = -1$ for n odd.

7. Since $\{a_n\}$ diverges to infinity, there exists n_1 such that $a_n \ne 0$ for any $n \ge n_1$. Then, for all $n \ge n_1$, we have $b_n = \frac{a_n b_n}{a_n} = \frac{1}{a_n}(a_n b_n)$. Since, $\{a_n\}$ diverges to infinity, by Theorem 2.3.6, $\lim_{n\to\infty} \frac{1}{a_n} = 0$. Since the sequence $\{a_n b_n\}$ converges, by Theorem 2.1.11, it is bounded. Therefore, by Theorem 2.2.7, the sequence $\{b_n\}$ must converge to 0.

8. Write a_n as $a_n = n^p\left(s_p + s_{p-1}n^{-1} + \cdots + s_0 n^{-p}\right) \equiv n^p b_n$. Since $\{b_n\}$ converges to $s_p \ne 0$, it is bounded. Also, $\lim_{n\to\infty} n^p = +\infty$. By Theorem 2.3.3, parts (c) and (d), we have that $\lim_{n\to\infty} a_n = \pm\infty$. It is $+\infty$ if $s_p > 0$ and $-\infty$ if $s_p < 0$.

9. Rewrite a_n as $a_n = n^{p-q} b_n$, where $\{b_n\}$ converges to $\frac{s_p}{t_q}$ by employing theorems in Section 2.1 and 2.2. Since $p > q$, then $\lim_{n\to\infty} n^{p-q} = +\infty$. Therefore, by Theorem 2.3.3, parts (b)–(d), $\lim_{n\to\infty} n^{p-q} b_n = \pm\infty$, depending on the sign of $\frac{s_p}{t_q}$.

10. **(a)** Yes. Let $a_n = n, b_n = (-1)^n n^2$.

(b) Yes. Let $a_n = \frac{1}{n}$ and $b_n = (-1)^n$.

12. **(a)** By Theorems 2.1.11 and 2.1.12, there exists n_1 such that $\boxed{\frac{L}{2}} < \frac{a_n}{b_n} < \boxed{2L}$ for all $n \ge n_1$. Therefore,

$\frac{1}{2}Lb_n < a_n < 2Lb_n$. Using the right-hand inequality, if $\lim_{n\to\infty} a_n = +\infty$, then according to Theorem 2.3.2, we have $\lim_{n\to\infty} 2Lb_n = +\infty$. Thus, by Theorem 2.3.3, part (c), $\lim_{n\to\infty} b_n = +\infty$. In a similar way, using the left-hand inequality, we conclude that $\lim_{n\to\infty} b_n = +\infty$ implies that $\lim_{n\to\infty} a_n = +\infty$. Be careful, students may wish to use a contradiction and assume that $\lim_{n\to\infty} b_n \neq +\infty$ and then try to conclude that $\{b_n\}$ must be bounded. This is of course not true.

(b) Choose, $a_n = 1$ and $b_n = n$. Then, $\lim_{n\to\infty} b_n = +\infty$, which does not imply that $\lim_{n\to\infty} a_n = +\infty$. Next, choose, $a_n = n$ and $b_n = 1$. Then, $\lim_{n\to\infty} a_n = +\infty$, which does not imply that $\lim_{n\to\infty} b_n = +\infty$.

13. (a) By Theorem 2.3.6, the sequence $\left\{\frac{b_n}{a_n}\right\}$ diverges to $+\infty$. Therefore, there exists n_1 such that for any $M > 0$ we have $\frac{b_n}{a_n} > M$, provided $n \geq n_1$. Thus, if $n \geq n_1$, we have $b_n > Ma_n$. If the sequence $\{a_n\}$ diverges to infinity, by Theorem 2.3.3, part (c), we have $\lim_{n\to\infty} Ma_n = +\infty$. Hence, by the comparison theorem, the sequence $\{b_n\}$ diverges to $+\infty$.

(b) Since $b_n > 0$, we can write $a_n = \frac{a_n b_n}{b_n} = \frac{a_n}{b_n} \cdot b_n$. Since $\left\{\frac{a_n}{b_n}\right\}$ converges to 0 and $\{b_n\}$ is bounded, by Theorem 2.2.7 the sequence $\{a_n\}$ converges to 0.

14. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = +\infty$, there exists n_1 such that $a_n > \boxed{M = \frac{(\alpha - \varepsilon\beta)k}{\varepsilon\beta^2}}$. Now choose $n^* = n_1$. Then, for all $n \geq n^*$ we have $\left|\frac{\alpha a_n}{k + \beta a_n} - \frac{\alpha}{\beta}\right| = \frac{\alpha k}{\beta k + \beta^2 a_n} < \frac{\alpha k}{\beta k + \beta^2 M} = \varepsilon$. Hence, the conclusion follows. Argue differently by multiplying the given limit by $\frac{1}{a_n}\Big/\frac{1}{a_n}$.

16. (a) Let $a_n = \frac{b^n}{n!}$. Then, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = b \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$. Thus, by Theorem 2.3.7, part (a), the sequence $\{a_n\}$ converges to 0.

(b) We will show that the sequence $\{a_n\}$ with $a_n = \frac{n^n}{n!}$ diverges to $+\infty$ by proving that $\left\{\frac{1}{a_n}\right\}$ converges to 0 and applying Theorem 2.3.6. To this end, we write $0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n} \leq \frac{1 \cdot n \cdot n \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n} = \frac{1}{n}$, which tends to 0. Therefore, by the sandwich theorem, the sequence $\left\{\frac{1}{a_n}\right\}$ converges to 0, which proves the desired result. Or to prove that $\{a_n\}$ diverges to $+\infty$ directly, use the comparison theorem, since $a_n \geq n$.

(c) We will prove that $\lim_{n\to\infty} n^k r^n = 0$ using Theorem 2.3.7. Thus, if $a_n = n^k r^n$ we write $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |r| < 1$. Therefore, by Theorem 2.3.7, part (a), the sequence converges to 0.

17. This approach for the second situation is not useful since R represents $\lim_{n\to\infty} a_n$ where $a_n = 2^0 + 2^1 + 2^2 + \cdots + 2^n$. Since this sequence diverges to $+\infty$, $R = +\infty$, and thus the next to the last line involves $\infty - \infty$.

18. **(a)** Since, $\left|\frac{\frac{n^2+3n-3}{n^3}-0}{\frac{2}{n}}\right| = \frac{\left|n^2+3n-3\right|}{2n^2} < \frac{2n^2}{2n^2} = 1$, if $n \geq 2$, by Definition 2.3.9 we have that $\frac{n^2+3n-3}{n^3} = 0 + \mathcal{O}\left(\frac{2}{n}\right)$.

(b) Since, $\left|\frac{\frac{\sin n}{n}-0}{\frac{1}{n}}\right| = \left|\sin n\right| \leq 1$ for all n, by Definition 2.3.9 we have the desired result.

19. Use Definition 2.3.9 to show that both sequences converge to 1 at roughly the same rate.

20. Suppose $\{a_n\}$ converges to a nonzero value. Then a_n can be written as the sum of $b_n = \begin{cases} a_n, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$ and $c_n = \begin{cases} 0, & n \text{ even} \\ a_n, & n \text{ odd.} \end{cases}$ If $\{a_n\}$ converges to 0, then a_n can be written as the sum of $b_n = \begin{cases} a_n - 1, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$ and $c_n = \begin{cases} 1, & n \text{ even} \\ a_n - 1, & n \text{ odd.} \end{cases}$ Note that the second decomposition does not work if the limit is not zero. For example choose $a_n = 2$.

Section 2.4

4. **(a)** Since for $n \geq 1$ we have $2n \geq n+1$, and thus, $n2^{n+1} \geq (n+1)2^n$, which is equivalent to $\frac{n}{2^n} \geq \frac{n+1}{2^{n+1}}$. Hence, $a_n \geq a_{n+1}$, and thus the sequence is decreasing.

(b) If $n \geq 3$, then $2n+1 < n^2$, which gives $n^2+2n+1 < 2n^2$. Therefore, $(n+1)^2 < 2n^2$ which is equivalent to $2^n(n+1)^2 < 2^{n+1}n^2$, which gives that $a_{n+1} < a_n$. So, the sequence is eventually strictly decreasing.

(c) Note that $\frac{a_{n+1}}{a_n} = 3 \cdot \frac{1+3^{2n}}{1+3^{2n+2}} < 3 \cdot \frac{1+3^{2n}}{3^{2n+2}} = \frac{1}{3^{2n+1}} + \frac{1}{3} < 1$, for all n. Since $a_n > 0$, the sequence is strictly decreasing.

(d) Since for all n, $2 > 1$, we have that $2(n+1) > 2n+1$. Thus, $[1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)] \cdot [2(n+1)] > [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)](2n+1)$. Dividing both sides by $2^{n+1}(n+1)!$ we obtain $a_n > a_{n+1}$. Hence, the given sequence is strictly decreasing.

(e) Since $\frac{n+1}{2n+1} < 1$, we multiply both sides of the inequality by $\frac{n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$ to obtain that $a_{n+1} < a_n$. Hence, $\{a_n\}$ is strictly decreasing.

6. **(a)** Since $\frac{a_{n+1}}{a_n} < 1$ and $a_n > 0$, by Exercise 5(a), $\{a_n\}$ converges. (In fact, by Exercise 20 of Section 2.2, $\{a_n\}$ converges to $\frac{1}{2}$.)

(b) $\{a_n\}$ is strictly decreasing and bounded below by 0, thus it converges.

(c) Since $a_n > 0$, by Exercises 4(b) and 5, $\{a_n\}$ converges.

(d) Since $a_{n+1} = a_n + \frac{1}{2^{n+1}}$, $\{a_n\}$ is strictly increasing. By Example 1.3.4, $a_n = \frac{1+\frac{1}{2^{n+1}}}{1-\frac{1}{2}} < 2$. Thus, $\{a_n\}$

is bounded above and hence, converges (to 2, in fact).

(e) Since $a_n > 0$, by Exercises 4(c) and 5, $\{a_n\}$ converges (to 0, in fact).

(f) Since $a_n > 0$, by Exercises 4(e) and 5, $\{a_n\}$ converges.

(g) $\{a_n\}$ converges to 0 and $-\frac{1}{2} \le a_n \le 1$, but it is not monotone.

(h) Note that for all n, $n+1 \ge 2$ is equivalent to $2^n(n!)(n+1) \ge 2 \cdot 2^n(n!)$. Therefore, $a_n \ge a_{n+1}$ and so the sequence is decreasing. Since it is bounded below by 0, it converges.

7. (a) By Exercise 2(t) of Section 1.3 we have that $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \le 2 - \frac{1}{n} < 2$ for all n, and thus $\{a_n\}$ is bounded by 2. Also, $a_{n+1} = a_n + (n+1)^{-2}$, meaning that $\{a_n\}$ is strictly increasing and thus bounded below by $a_1 = 1$. Therefore, $\{a_n\}$ is convergent and $1 \le a_n < 2$ for all n. Thus, by Theorem 2.2.1, part (f), $1 \le A \le 2$. Furthermore, note that $1 + \frac{1}{4} \le a_n < 2$ for all $n \ge 2$. Hence, $\frac{5}{4} \le A \le 2$. The lower bound can be progressively improved.

(b) By Exercise 2(u) of Section 1.3 we have that $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \le \frac{7}{4} - \frac{1}{n} < \frac{7}{4}$ for all $n \ge 2$. Again, since $\{a_n\}$ is strictly increasing, it converges. Since $a_2 = 1 + \frac{1}{4} = \frac{5}{4}$, we have $\frac{5}{4} \le a_n < \frac{7}{4}$ for all $n \ge 2$. Thus, $1 < a_n < \frac{7}{4}$ and $1 \le B \le \frac{7}{4}$.

8. Since $r^{n+1} = r \cdot r^n > r^n$ for $r > 1$, $\{a_n\}$ is strictly increasing. We show $\{a_n\}$ is not bounded above by contradiction. Thus, assume that $\{a_n\}$ is bounded above. Then, by Theorem 2.4.4, part (a), the sequence $\{a_n\}$ converges to, say, A. Taking limits of the recursion formula $a_{n+1} = ra_n$ using Remark 2.1.8, part (c), we obtain $A = rA$, which implies A must be 0. This is a contradiction, and hence, $\{a_n\}$ tends to $+\infty$.

9. (a) Since $0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{2n-1}{2n} < \frac{1}{2} \cdot 1 \cdot 1 \cdot \cdots \cdot 1 = \frac{1}{2}$ and $\{a_n\}$ is strictly decreasing, it converges to, say, A. Therefore, $0 \le A < \frac{1}{2}$.

(b) Since $0 < b_n = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n+1} < \frac{2}{3}$ and $\{b_n\}$ is strictly decreasing, it converges to, say, B. Therefore, $0 \le B < \frac{2}{3}$.

(c) Since $\{a_n\}$ and $\{b_n\}$ both converge, $\{a_n b_n\}$ also converges, by Theorem 2.2.1, part (b). Furthermore, we can write $\lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} \frac{1}{2n+1} = 0$.

(d) Since $\lim_{n \to \infty} a_n = A$, $\lim_{n \to \infty} b_n = B$ and $\lim_{n \to \infty} a_n b_n = AB$, either $A = 0$, $B = 0$, or both. But, since $a_n < b_n$ for all n, $A \le B$. Hence, A must be 0.

10. (b) $a_1 = a$ and $a_{n+1} = ra_n$, for all $n \in N$.

11. (b) By the mathematical induction we can prove that $\{a_n\}$ is increasing and bounded above by 2, see Exercise 11 from Section 1.9. Hence, $\{a_n\}$ converges to, say, A. Taking limits of the recursion formula we obtain $A = \sqrt{1 + \sqrt{A}}$. Thus, A must satisfy the equation $A^4 - 2A^2 - A + 1 = 0$.

(c) By the mathematical induction we can prove that $\{a_n\}$ is increasing and bounded above by 2. Hence,

$\{a_n\}$ converges to, say, A. Taking limits of the recursion formula we obtain $A=\sqrt{2A}$, which gives $A=0$ or $A=2$. Since the sequence begins with $\sqrt{2}>1$ and is increasing, $A=0$ is not a possibility for its limit. Hence, $\{a_n\}$ converges to 2.

(d) $\{a_n\}$ is unbounded and oscillating, and hence, divergent. Observe that taking limits of the recursion formula and obtaining $A=1$ is meaningless.

(f) $\{a_n\}$ diverges to $+\infty$.

(g) By the mathematical induction we can prove that $\{a_n\}$ is increasing and bounded above by 2. Hence, $\{a_n\}$ converges to, say, A. Taking limits of the recursion formula we obtain $A=1+\frac{1}{2}A$. Hence, $\{a_n\}$ converges to $A=2$.

(i) Since by Example 1.3.12, we have that $a_n=-1+2^n$, $\{a_n\}$ diverges to $+\infty$.

(k) Using the idea similar to the one in Example 1.3.12, we find that $a_n=\frac{1}{3^{n-1}}$. So, $\{a_n\}$ converges to 0.

(l) By the mathematical induction we can prove that $\{a_n\}$ is decreasing and bounded below by 0. Hence, $\{a_n\}$ converges to, say, A. Taking limits of the recursion formula we obtain $A=\frac{1}{3}A$, or $A=0$. In fact, $a_n=\frac{1}{3^{n-1}}$ as in parts (j) and (k).

12. By Exercises 42 and 43 of Section 1.9, $\{a_n\}$ is decreasing and bounded below by $\sqrt{A}$. Therefore, $\{a_n\}$ converges to, say, L. Taking limits of the recursion formula we obtain $L=\frac{L^2+A}{2L}$. Solving for L we obtain $L=\sqrt{A}$ or $L=-\sqrt{A}$. Since $\sqrt{A}$ is the lower bound of $\{a_n\}$, $\{a_n\}$ converges to $\sqrt{A}$.

13. By mathematical induction it can be proven that $b_n>0$ for all n. Furthermore, we have $B-(b_{n+1})^2=$

$$B-\frac{(b_n)^2\left[3B+(b_n)^2\right]^2}{\left[3(b_n)^2+B\right]^2}=\frac{\left[B-(b_n)^2\right]^3}{\left[3(b_n)^2+B\right]^2}=\left[B-(b_n)^2\right]\left[\frac{B-(b_n)^2}{3(b_n)^2+B}\right]^2, \text{ for all } n.$$

Case 1. Suppose $B>1$. Then, since $b_1=1$ we have $(b_1)^2=1$ and so $B>(b_1)^2$. Then, from the preceding formula with $n=1$, we have that $B-(b_2)^2>0$, which implies that $B>(b_2)^2$. By the mathematical induction we can prove that $B>(b_n)^2$ for all n. Therefore, $b_n<\sqrt{B}$ for all n, and thus, $\{b_n\}$ is bounded above. To show $\{b_n\}$ is strictly increasing, we multiply $(b_n)^2<B$ by 2 and add $B+(b_n)^2$ to both sides to obtain $3(b_n)^2+B<3B+(b_n)^2$. This gives $1<\frac{3B+(b_n)^2}{3(b_n)^2+B}$. Thus, $b_n<\frac{b_n\left[3B+(b_n)^2\right]}{3(b_n)^2+B}=b_{n+1}$ for all n. Therefore, $\{b_n\}$ is strictly increasing and hence, converges.

Case 2. Suppose $B<1$. Following similar steps to those in case 1 we can show that $\{b_n\}$ is strictly decreasing and bounded below by $\sqrt{B}$. Hence, $\{b_n\}$ converges.

Case 3. Suppose $B=1$. Following a similar argument to the one in case 1, we can show that $b_n=1$ for all n. Therefore, $\{b_n\}$ converges.

Since in all 3 cases the sequence converges to, say, L, taking limits of the recursion formula we get that $L=0$, $L=\sqrt{B}$, or $L=-\sqrt{B}$. Due to the monotonicity and boundedness of the sequence in each case, we conclude that the limit of $\{b_n\}$ is $\sqrt{B}$.

14. Note, this is Newton's method for approximating roots of a polynomial $f(x) = x^3 - x$.

15. **(a)** Since α satisfies the equation $r^2 = 1 + r$, we can write $\alpha = \sqrt{1+\alpha} = \sqrt{1+\sqrt{1+\alpha}} = \cdots$.

(b) $\{a_n\}$ is increasing, bounded below by 1, and bounded above by 2. Thus, $\{a_n\}$ converges to, say, A. Taking limits of the recursion formula we obtain $A = \sqrt{1+A}$, which is equivalent to $A^2 - A - 1 = 0$. Thus, $A = \alpha$, since the other value of A is negative.

(c) Since β satisfies the equation $m^2 = 1 - m$, we can write $\beta = \sqrt{1-\beta} = \sqrt{1-\sqrt{1-\beta}} = \cdots$.

(d) The sequence is not monotone. It converges because it represents $\sqrt{1-\sqrt{1-\sqrt{1-\cdots}}}$ which is equal to β.

(e) This sequence has the same recursion formula as that in part (b), but the initial value b_1 is different. Thus, the sequence produced is different. In fact, it oscillates.

16. **(a)** Since $a_2 = \dfrac{a_1+b_1}{2}$ and $b_2 = \sqrt{a_1b_1}$, by Theorem 1.8.4, part (c), we have $a_2 > b_2$. Also, $a_2 = \dfrac{a_1+b_1}{2}$ implies that $2a_2 = a_1 + b_1 < a_1 + a_1 = 2a_1$, so $a_2 < a_1$. And, $b_2 = \sqrt{a_1b_1}$ implies that $(b_2)^2 = a_1b_1 > b_1b_1 = (b_1)^2$. Since $b_1 > 0$, we have $b_2 > b_1$. Therefore, $b_1 < b_2 < a_2 < a_1$. By the mathematical induction the desired inequality can be proven.

(b) Suppose the statement to be proven is $P(n)$. We will prove its validity by the mathematical induction. First observe that $P(1)$ is true because from part (a) we have $a_2 > b_2 > b_1$, which gives $0 < a_2 - b_2 = \dfrac{a_1+b_1}{2} - b_2 < \dfrac{a_1+b_1}{2} - b_1 = \dfrac{a_1-b_1}{2}$. Next, suppose $P(k)$ is true for some integer $k \in N$, that is, $0 < a_{k+1} - b_{k+1} < \dfrac{a_1-b_1}{2^k}$. We will show that $P(k+1)$ is true, that is, $0 < a_{k+2} - b_{k+2} < \dfrac{a_1-b_1}{2^{k+1}}$. To this end, we write, $0 < a_{k+2} - b_{k+2} = \dfrac{a_{k+1}+b_{k+1}}{2} - b_{k+2} < \dfrac{a_{k+1}+b_{k+1}}{2} - b_{k+1} = \dfrac{a_{k+1}-b_{k+1}}{2} < \dfrac{1}{2}\cdot\dfrac{a_1-b_1}{2^k} = \dfrac{a_1-b_1}{2^{k+1}}$. Hence, $P(n)$ is true for all n.

(c) By the sandwich theorem and part (b), we have that $\lim_{n\to\infty}(a_{n+1} - b_{n+1}) = 0$. Since each sequence converges, we obtain $0 = \lim_{n\to\infty} a_{n+1} - \lim_{n\to\infty} b_{n+1} = A - B$.

17. **(a)** By Problem 1.10.15, $0 < a < b$ implies that $a < \dfrac{2ab}{a+b} < \dfrac{a+b}{2} < b$. If $a = b_n$ and $b = a_n$, we obtain $b_n < \dfrac{2a_nb_n}{a_n+b_n} < \dfrac{a_n+b_n}{2} < a_n$ which gives $b_n < b_{n+1} < a_{n+1} < a_n$. Therefore, $\{b_n\}$ is increasing and bounded above by a_1, and thus, converges to, say, B. In addition, $\{a_n\}$ is decreasing and bounded below by b_1, and thus, converges to, say, A. Hence, taking limits of the first recursion formula we get $A = \dfrac{A+B}{2}$, which implies that $A = B$.

(b) Multiply two recursion formulas together to obtain $a_{n+1}b_{n+1} = \dfrac{a_n+b_n}{2}\cdot\dfrac{2a_nb_n}{a_n+b_n} = a_nb_n$, for all $n \ge 1$. Similarly, $a_nb_n = a_{n-1}b_{n-1}$ for all $n \ge 2$. Thus, by mathematical induction it can be proven that $a_{n+1}b_{n+1} = a_1b_1$ for all $n \ge 1$. Taking limits of both sides we get that $AB = a_1b_1$, which gives $A = \sqrt{a_1b_1}$, since $A = B$. Hence, $\lim_{n\to\infty} a_n = \sqrt{a_1b_1}$.

18. By Theorem 2.4.4, (a) $\Rightarrow$ (b). Now we prove that (b) $\Rightarrow$ (a). Let any set S of real numbers be given which is bounded above. We will show $\sup S$ exists and is finite. To do this, we form an increasing sequence $\{a_n\}$ of points in S and a decreasing sequence $\{b_n\}$ of upper bounds of S in such a way that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = A$. To obtain the desired conclusion, we will prove that $A = \sup S$.

We start with any point a_1 in S and any upper bound b_1 of S. Both exist due to our assumption and clearly, $a_1 \le b_1$. Let $c_1 = \dfrac{a_1+b_1}{2}$ be the midpoint between a_1 and b_1. We determine a_2 and b_2 as follows. If c_1 is an upper bound of S, let $a_2 = a_1$ and $b_2 = c_1$. If c_1 is not an upper bound of S, let a_2 be some point in S satisfying $a_2 \ge a_1$, and let $b_2 = b_1$. In either case, we have that $a_2 \in S$ and b_2 is an upper bound of S, $a_1 \le a_2 \le b_2 \le b_1$, and $b_2 - a_2 \le \dfrac{b_1 - a_1}{2}$. Repeat the above process using a_2 and b_2 to obtain values a_3 and b_3. Therefore, this process gives rise to two sequences $\{a_n\}$ and $\{b_n\}$, where $\{a_n\}$ is increasing and bounded above by b_1, and $\{b_n\}$ is decreasing and bounded below by a_1. In addition, by mathematical induction, we have $0 \le b_n - a_n \le \dfrac{b_1 - a_1}{2^{n-1}}$. Since the right-hand side tends to 0, by the sandwich theorem, $\lim_{n\to\infty}(b_n - a_n) = 0$, and so sequences $\{a_n\}$ and $\{b_n\}$ have the same limit, say A.

Next we show that $A = \sup S$. Let $x \in S$. Then, $b_n \ge x$ for all n because each b_n is an upper bound of S. Therefore, $A \ge x$. This proves that A is an upper bound of S. Next, if $t < A$, then $a_n > t$ for large enough n since $\lim_{n\to\infty} a_n = A$. Since $a_n \in S$, this shows that t is not an upper bound of S. Hence, A is the least upper bound of S.

19. **(a)** Use mathematical induction. Suppose $P(n)$ is the statement $n!! = 2^{\frac{n}{2}}\left(\frac{n}{2}\right)!$. Then, $P(0)$ is clearly true.

Suppose $P(k)$ is true for some even integer k, that is, $k!! = 2^{\frac{k}{2}}\left(\frac{k}{2}\right)!$. We will show that $P(k+2)$ is true, that is, $(k+2)!! = 2^{\frac{k+2}{2}}\left(\frac{k+2}{2}\right)!$. To this end, we write that $(k+2)!! = (k+2)(k!!) = (k+2)2^{\frac{k}{2}}\left(\frac{k}{2}\right)!$
$= \left(\frac{k}{2}+1\right)2\cdot 2^{\frac{k}{2}}\left(\frac{k}{2}\right)! = 2^{\frac{k+2}{2}}\left(\frac{k}{2}+1\right)! = 2^{\frac{k+2}{2}}\left(\frac{k+2}{2}\right)!$. Hence, $P(n)$ is true for all $n = 0, 2, 4, \ldots$.

(b) Use mathematical induction.

(c) Not true.

20. Not true if some values are permitted to be negative.

Section 2.5

1. If $N_\varepsilon(s_0)$ contains infinitely many points of S, it certainly contains one point different from s_0. Therefore, (b) $\Rightarrow$ (a). We prove that (a) $\Rightarrow$ (b) by contradiction. Suppose N is some neighborhood of s_0 which contains only a finite number of points of S. Let $a_1, a_2, \ldots, a_n$ be these points of $N \cap S$ which are different from s_0. Define $r = \min_{1\le k\le n}\{|s_0 - a_k|\}$. Clearly, $r > 0$. Now, $N_r(s_0)$ contains no points of S different from s_0. Hence, s_0 is not an accumulation point of S, a contradiction.

3. **(a)** If $a_n = (-1)^n \dfrac{n}{n+1}$, then accumulation points of $S = \{a_n \mid n \in N\}$ are 1 and -1.

(b) If $S = \{x \mid x \in (0,1) \cup \{2\}\}$, then 2 is not an accumulation point of S.

(c) If $S=\left\{\frac{1}{n}\middle|n\in N\right\}$, then $\sup S=1\in S$ and accumulation point $0\notin S$.

(d) If $S=\left\{\frac{(-1)^n}{n}\middle|n\in N\right\}$, then 0 is the accumulation point of S and $0\notin S$. Also, $\inf S=-1$ and $\sup S=\frac{1}{2}$ are both in S.

4. (a) ($\Rightarrow$) Suppose s_0 is an accumulation point of S. Then, every neighborhood $\left(s_0-\frac{1}{n},s_0+\frac{1}{n}\right)$ contains at least one element of S other than s_0, call it a_n. Therefore, $s_0-\frac{1}{n}<a_n<s_0+\frac{1}{n}$, which implies that $\lim_{n\to\infty}a_n=s_0$, by the sandwich theorem.

($\Leftarrow$) Suppose that $\lim_{n\to\infty}a_n=s_0$ for some sequence $\{a_n\}$ in S with $a_n\neq s_0$ for every n. Let $\varepsilon>0$ be given. Then, since $\lim_{n\to\infty}a_n=s_0$, there exists n_1 such that $|a_n-s_0|<\varepsilon$ for all $n\geq n_1$. Therefore, there exists at least one $a_m\neq s_0$ that is in this neighborhood. Hence, s_0 is an accumulation point of S.

(b) The condition $a_n\neq s_0$ for every $n\in N$ can be relaxed to a condition that $a_n\neq s_0$ for some arbitrarily large values of n. We simply do not want a sequence that attains only values of s_0 after some point because in that case the set $S=\{a_n\,|n\in N\}$ would be finite. Then, the existence of such sequence would not imply that s_0 is an accumulation point of S.

5. ($\Rightarrow$) Suppose $M=\sup S$. Since for any $n\in N$, $M-\frac{1}{n}$ is not an upper bound of S, there exists $a_n\in S$ such that $M-\frac{1}{n}\leq a_n\leq M$. Hence, $\{a_n\}$ converges to M. Note that if S is finite, then $a_n\equiv M$, eventually, since $M\in S$. If S is infinite, then M may or may not belong to S.

($\Leftarrow$) Show that $M=\sup S$. We only have to show that S has no upper bound K such that $K<M$. We prove this by contradiction. Thus, suppose that K is an upper bound of S and $K<M$. Since $\lim_{n\to\infty}a_n=M$, by Theorem 2.1.12 there exists $n^*\in N$ such that $a_n>K$ for all $n\geq n^*$. Since $a_n\in S$, we conclude that K is not an upper bound of S. Contradiction. Hence, $M=\sup S$. Note that a_n need not be distinct from M. If we can find a sequence $\{a_n\}$ in S where $a_n=M$ for some n, then $M=\max S$.

6. (a) If $\sup S=\max S$, we are done. If $\sup S\neq\max S$, we need to prove that $s_0\equiv\sup S$ is an accumulation point of S. Let $\varepsilon>0$ be given. Then, $s_0-\varepsilon$ is not an upper bound of S. Therefore, there exists $x\in S$ such that $x>s_0-\varepsilon$. But, $s_0\equiv\sup S$ and $s_0\neq\max S$. Thus, $s_0\notin S$ and so $x\neq s_0$. Hence, s_0 is an accumulation point of S.

(b) Example of a set S where $\sup S=\max S$ but $\sup S$ is not an accumulation point of S is, say, $S=S_1=\{0\}$. Here, $\sup S_1=0=\max S_1$ but S_1 has no accumulation points. Example of such a set S need not be necessarily finite. Choose another set $S=S_2=\left\{\frac{1}{n}\middle|n\in N\right\}$. Then, $\sup S_2=1=\max S_2$ and 0 is an accumulation point of S_2 but, $0\neq1$.

(c) Let $S=\{1\}\cup\left\{\frac{n}{n+1}\middle|n\in N\right\}$. Then, $\sup S=\max S=1$ and 1 is one of the accumulation points of S.

7. (b) Let $\varepsilon>0$ be given. Suppose that $m>n$ and write $|a_m-a_n|=\left|\frac{m+1}{m}-\frac{n+1}{n}\right|=\frac{m-n}{mn}<\frac{m}{mn}=\frac{1}{n}$. But,

$\frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$. Thus, choose $n^* > \frac{1}{\varepsilon}$. Then, if $m > n \geq n^*$, we have $|a_m - a_n| < \varepsilon$. Therefore, $\{a_n\}$ is a Cauchy sequence by Definition 2.5.6.

(c) Let $\varepsilon > 0$ be given. Suppose that $m > n$ and write $|a_m - a_n| = \left|\frac{m}{m+1} - \frac{n}{n+1}\right| = \frac{m-n}{(m+1)(n+1)} <$ $\frac{m}{(m+1)(n+1)} < \frac{1}{n+1} < \frac{1}{n}$. But, $\frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$. Thus, choose $n^* > \frac{1}{\varepsilon}$. Then, if $m > n \geq n^*$, we have $|a_m - a_n| < \varepsilon$. Therefore, $\{a_n\}$ is a Cauchy sequence by Definition 2.5.6.

(d) Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Let $\varepsilon > 0$ be given and suppose that $m > n$. Then $a_n < a_m$, since a_m has additional positive terms. Thus, $0 < a_m - a_n = \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{m(m+1)} =$ $\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{m} - \frac{1}{m+1}\right) = \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1} < \frac{1}{n}$. Since $\frac{1}{n}$ tends to 0 as n goes to infinity, $\frac{1}{n} < \varepsilon$, eventually. Hence, the given sequence is a Cauchy sequence.

(e) Note that $\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$. Suppose $m > n > 1$, and write $|a_m - a_n| = a_m - a_n =$ $\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2} < \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) = \frac{1}{n} - \frac{1}{m} < \frac{1}{n}$. Since $\frac{1}{n}$ tends to 0 as n goes to infinity, $\frac{1}{n} < \varepsilon$, eventually, no matter what $\varepsilon > 0$ is. Hence, the given sequence is a Cauchy sequence.

(f) We will show $\{a_n\}$ is not a Cauchy sequence by finding a particular relationship between m and n for which $|a_m - a_n|$ is greater than or equal to some positive real number. To this end, if $m > n$, we can write $|a_m - a_n| = a_m - a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} > \frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m} = (m-n)\cdot\frac{1}{m} = 1 - \frac{n}{m}$. Therefore, if $m = 2n$, then $a_{2n} - a_n > \frac{1}{2}$. Hence, $\{a_n\}$ is not a Cauchy sequence.

(g) Note that $\{a_n\}$ is not monotone. Let $\varepsilon > 0$ be given and suppose that $m > n$. Then, $|a_m - a_n| =$ $\left|\frac{(-1)^{n+2}}{(n+1)!} + \frac{(-1)^{n+3}}{(n+2)!} + \cdots + \frac{(-1)^{m+1}}{m!}\right| \leq \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} =$ $\frac{1}{2^n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) < \frac{1}{2^n}\cdot 2 = \frac{1}{2^{n-1}} < \varepsilon$ if n is large. Therefore, $\{a_n\}$ is a Cauchy sequence. In fact, it converges to $1 - e^{-1}$.

(h) Let $\varepsilon > 0$ be given and suppose $m > n$. Then, $|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots +$ $|a_{n+1} - a_n| < r^{m-1} + r^{m-2} + \cdots + r^{n+1} + r^n = r^n\left(r^{m-n-1} + r^{m-n-2} + \cdots + r + 1\right) = r^n \cdot \frac{1 - r^{m-n}}{1-r} <$ $r^n \cdot \frac{1}{1-r}$, which converges to 0 since $0 \leq r < 1$, and so $\frac{r^n}{1-r} < \varepsilon$ for large enough n. Hence, $\{a_n\}$ is a Cauchy sequence.

8. (a) Choose $a_n = \sqrt{n}$. Clearly, $\{a_n\}$ diverges to $+\infty$. But, $0 \leq |a_{n+1} - a_n| = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} <$ $\frac{1}{2\sqrt{n}}$, which tends to 0 as n goes to infinity. Hence, by the sandwich theorem, $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$.

(b) Since $\{a_n\}$ converges, it is Cauchy. Therefore, for any $\varepsilon > 0$ there exists $n^* \in N$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq n^*$. Thus, in particular, pick $m = n+1$. This gives $0 \leq |a_{n+1} - a_n| < \varepsilon$ for all $n \geq n^*$. Hence, $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$.

9. Suppose $\{a_n\}$ is a Cauchy sequence. We will use a similar proof to that of Theorem 2.1.11. Since $\{a_n\}$ Cauchy implies that there exists $n^* \in N$ such that for all $m, n \geq n^*$ we have $|a_n - a_m| < \boxed{1}$. Since m and n are any values greater than or equal to n^*, choose $m = n^*$. Then we have $|a_n - a_{n^*}| < 1$ for all $n \geq n^*$. By the triangle inequality, for all $n \geq n^*$ we have $|a_n| = |a_n - a_{n^*} + a_{n^*}| \leq |a_n - a_{n^*}| + |a_{n^*}| < 1 + |a_{n^*}|$. Thus, we bounded all terms a_n starting with a_{n^*}, $n^* \in N$. Hence, $|a_n| \leq M$ for all $n \in N$ if we pick $M = \max\{1 + |a_{n^*}|, |a_1|, |a_2|, \ldots, |a_{n^*-1}|\}$.

10. Suppose $S = \{a_1, a_2, \ldots, a_k\}$ where all k elements, $k \in N$, are distinct. Let $\alpha > 0$ denote the minimum distance between any 2 elements of S. The value α exists because S is finite. Since $\{a_n\}$ is Cauchy, there exists $n_1 \in N$ such that for all $m, n \geq n_1$ we have $|a_n - a_m| < \varepsilon$ for any given $\varepsilon > 0$. In particular, if $\varepsilon = \alpha$ and $m = n_1$ we have $|a_n - a_{n_1}| < \alpha$ for all $n \geq n_1$. But, both terms a_n and a_{n_1} are in S with distance between them of α or more, unless they are equal to each other. Hence, for all $n \geq n_1$ all terms of $\{a_n\}$ must be equal. Hence, $\{a_n\}$ is constant for all $n \geq n_1$.

11. **(a)** Let $\varepsilon > 0$ be given. We need to find $n^* \in N$ such that $|(a_m + b_m) - (a_n + b_n)| < \varepsilon$ for all $m, n \geq n^*$. Since $\{a_n\}$ is Cauchy, there exists n_1 such that $|a_m - a_n| < \boxed{\frac{\varepsilon}{2}}$ for all $m, n \geq n_1$. Since $\{b_n\}$ is Cauchy, there exists n_2 such that $|b_m - b_n| < \boxed{\frac{\varepsilon}{2}}$ for all $m, n \geq n_2$. Choose $n^* = \max\{n_1, n_2\}$. Then, for all $n \geq n^*$ we have $|(a_m + b_m) - (a_n + b_n)| \leq |a_m - a_n| + |b_m - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence, $\{a_n + b_n\}$ is Cauchy.

(b) Let $\varepsilon > 0$ be given. We need to find n^* such that $|a_m b_m - a_n b_n| < \varepsilon$ for all $m, n \geq n^*$. Since $\{a_n\}$ and $\{b_n\}$ are Cauchy, by Theorem 2.5.8 they are bounded. Therefore, there exists $M > 0$ such that $|a_n| < M$ and $|b_n| < M$ for all $n \in N$. In addition, $\{a_n\}$ Cauchy, implies that there exists $n_1 \in N$ such that $|a_m - a_n| < \boxed{\frac{\varepsilon}{2M}}$ for all $m, n \geq n_1$. And $\{b_n\}$ Cauchy, implies that there exists $n_2 \in N$ such that $|b_m - b_n| < \boxed{\frac{\varepsilon}{2M}}$ for all $m, n \geq n_2$. Choose $n^* = \max\{n_1, n_2\}$. Then, for all $m, n \geq n^*$ we have
$|a_m b_m - a_n b_n| = |a_m b_m - a_n b_m + a_n b_m - a_n b_n| \leq |b_m||a_m - a_n| + |a_n||b_m - b_n| < M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon$.
Hence, $\{a_n b_n\}$ is a Cauchy sequence.

12. **(a)** Proof of part (a). From the proof of Theorem 2.5.11 we have that $|a_m - a_n| \leq \frac{k^{n-1}}{1-k} \cdot |a_2 - a_1|$. Since $\lim_{m\to\infty} a_m = A$, the limits of the preceding inequality yield $|A - a_n| \leq \frac{k^{n-1}}{1-k} \cdot |a_2 - a_1|$.

Proof of part (b). It can be proven by induction that $|a_{n+p} - a_{n+p-1}| \leq k^p |a_n - a_{n-1}|$, for $n > 1$ and p nonnegative integer. Therefore, if $m > n > 1$ we have $|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_n - a_{n-1}| \leq \left(k^{m-n} + k^{m-n-1} + \cdots + k^2 + k\right)|a_n - a_{n-1}| = k \cdot \frac{1-k^{m-n}}{1-k} \cdot |a_n - a_{n-1}| <$

$\frac{k}{1-k}\cdot|a_n - a_{n-1}|$, since $0 < k < 1$. Now take limits as m tends to $+\infty$ of this inequality to obtain the desired result.

(b) In Example 2.5.13, $a_1 = 1$ makes the sequence increasing and bounded above (by 2.) If $a_1 = 2$, then $a_n \equiv 2$. In the case $a_1 = 4$, the sequence is decreasing and bounded below (by 0.) Therefore, if we used techniques from Section 2.4 to prove the convergence of $\{a_n\}$ in each of these cases, we would need to consider these cases separately. In all 3 cases, $\{a_n\}$ remains contractive and convergent. Taking limits of the recursion formula, we see that $\{a_n\}$ converges to 2 in all 3 cases.

13. Note that if $0 < a_1 \le \frac{1}{3}$ we have $a_2 = (a_1)^2 < a_1$. Also, if $a_{k+1} \le a_k$ for some $k \in N$, we have $a_{k+2} = (a_{k+1})^2 \le (a_k)^2 = a_{k+1}$. Thus, by the mathematical induction, $\{a_n\}$ is decreasing. Moreover, $a_{n+2} = (a_{n+1})^2$ and $a_{n+1} = (a_n)^2$, thus, subtracting we obtain $a_{n+2} - a_{n+1} = (a_{n+1})^2 - (a_n)^2 = (a_{n+1} - a_n)(a_{n+1} + a_n)$. Therefore, $|a_{n+2} - a_{n+1}| = |a_{n+1} - a_n||a_{n+1} + a_n| \le |a_{n+1} - a_n||a_1 + a_1| \le |a_{n+1} - a_n|\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{2}{3}|a_{n+1} - a_n|$. Hence, $\{a_n\}$ is contractive.

14. **(b)** Since $a_1 = 1$, we have $a_n > 1$ for all $n \ge 2$. In addition, $|a_{n+2} - a_{n+1}| = \left|\frac{a_{n+1} - a_n}{a_n a_{n+1}}\right| = \left|\frac{a_{n+1} - a_n}{a_n\left[1 + (a_n)^{-1}\right]}\right|$ $\le \frac{|a_{n+1} - a_n|}{|a_n + 1|} < \frac{|a_{n+1} - a_n|}{2}$, for all $n \ge 2$. Therefore, since $0 < r \equiv \frac{1}{2} < 1$, by the contraction principle, the sequence converges to, say, A.

(c) Taking limits of the recursion formula we get $A = 1 + \frac{1}{A}$, which is equivalent to $A^2 - A - 1 = 0$. Two choices for A are $\frac{1-\sqrt{5}}{2}$ or $\frac{1+\sqrt{5}}{2}$. Certainly, since the first is negative and $a_n \ge 1$ for all n, $\frac{1+\sqrt{5}}{2}$ is the correct value for the limit.

15. Since $a_{n+2} = a_n + a_{n+1}$ and $a_n > 0$ for all $n \in N$, we can divide by a_{n+1} to obtain $\frac{a_{n+2}}{a_{n+1}} = \frac{a_n}{a_{n+1}} + 1$. Thus, if $b_n = \frac{a_{n+1}}{a_n}$, we have $b_{n+1} = \frac{1}{b_n} + 1$. By Exercise 14, $\{b_n\}$ converges to $\frac{1+\sqrt{5}}{2}$.

16. **(a)** Since $a_{n+2} = \frac{a_{n+1} + a_n}{2}$, upon subtraction of a_{n+1} from both sides, we get $a_{n+2} - a_{n+1} = \frac{1}{2}(a_n - a_{n+1})$. By the contraction principle, $\{a_n\}$ converges to, say, A. We will show that $A = \frac{a_1 + 2a_2}{3}$. Taking limits of the recursion formula yields no information. We can find the explicit formula by assuming that $a_n = cr^n$, as we did in Example 1.3.12. Instead we write $a_{n+2} - a_{n+1} = \frac{1}{2}(a_n - a_{n+1})$, $a_{n+1} - a_n = \frac{1}{2}(a_{n-1} - a_n)$, ... , $a_4 - a_3 = \frac{1}{2}(a_2 - a_3)$, and $a_3 - a_2 = \frac{1}{2}(a_1 - a_2)$. Adding these together we get $a_{n+2} - a_2 = \frac{1}{2}(a_1 - a_{n+1})$. Now we take limits to obtain $A - a_2 = \frac{1}{2}(a_1 - A)$. This gives $A = \frac{1}{3}(a_1 + 2a_2)$.

(b) Let $b_n = |a_{n+1} - a_n|$. Then, since $a_{n+2} - a_{n+1} = \frac{1}{2}(a_n - a_{n+1})$, we have that $b_{n+1} = \frac{1}{2}b_n$, and therefore,

$b_{n+1} = \frac{1}{2}b_n = \frac{1}{2}\cdot\frac{1}{2}b_{n-1} = \cdots = \frac{1}{2^n}b_1$. Using the method of Example 2.5.7 we can conclude that $\{a_n\}$ is a Cauchy sequence.

17. **(a)** Since $a_{n+2} = \frac{1}{3}a_n + \frac{2}{3}a_{n+1}$, upon subtraction of a_{n+1} from both sides we get $a_{n+2} - a_{n+1} = \frac{1}{3}a_n - \frac{1}{3}a_{n+1} = \frac{1}{3}(a_n - a_{n+1})$. Therefore, by the contraction principle, $\{a_n\}$ converges to, say, A. Limit of the recursion formula gives no information. We find a_n in a similar way we did in Exercise 16(a). Since, $a_{n+2} - a_{n+1} = \frac{1}{3}(a_n - a_{n+1})$, $a_{n+1} - a_n = \frac{1}{3}(a_{n-1} - a_n)$, ... , $a_4 - a_3 = \frac{1}{3}(a_2 - a_3)$, and $a_3 - a_2 = \frac{1}{3}(a_1 - a_2)$. Adding these together we get $a_{n+2} - a_2 = \frac{1}{3}(a_1 - a_{n+1})$. Now we take limits to obtain $A - a_2 = \frac{1}{3}(a_1 - A)$. This gives $A = \frac{1}{4}(a_1 + 3a_2)$.

18. **(a)** $\{a_n\}$ is not monotone.

(b) Observe that $a_n \geq 1$ for all n and that $a_{n+1} = 1 + \frac{1}{1+a_n}$ and $a_{n+2} = 1 + \frac{1}{1+a_{n+1}}$ for all n. Subtracting we obtain $a_{n+2} - a_{n+1} = \frac{a_n - a_{n+1}}{(1+a_{n+1})(1+a_n)} \leq \frac{a_n - a_{n+1}}{(1+1)(1+1)}$. Therefore, $|a_{n+2} - a_{n+1}| \leq \frac{1}{4}|a_{n+1} - a_n|$. Thus, since $0 \leq k = \frac{1}{4} < 1$, by the contraction principle, the sequence $\{a_n\}$ converges to, say, A.

(c) We take limits of the recursion formula to get $A = 1 + \frac{1}{1+A}$. This gives $A = \sqrt{2}, -\sqrt{2}$. Clearly, since $a_n \geq 1$ for all n, the correct value for the limit is $\sqrt{2}$.

Section 2.6

1. **(a)** Yes, because $b_n = a_{2n-1}$.

(b) No, because $\frac{1}{\sqrt{3}}$ is a term of $\{b_n\}$ but not of $\{a_n\}$. Note that we cannot write $b_n = a_{\sqrt{n}}$ because $f(n) = \sqrt{n}$ is not a function whose range is a subset of N.

(c) No, because $\frac{1}{3}$ is a term of $\{b_n\}$ but not of $\{a_n\}$.

2. **(a)** The sequence $\{a_n\}$ is 1, 0, 1, 0, ... Subsequence $\{a_{2n}\}$ converges to 0, and $\{a_{2n-1}\}$ converges to 1. Therefore, by Theorem 2.6.5, since $0 \neq 1$, $\{a_n\}$ diverges. Subsequential limit points α are 0 and 1. Also, $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = 0$.

(b) The sequence $\{a_n\}$ is 1, 0, -1, 0, ... The subsequence $\{a_{2n-1}\}$ diverges because $a_{2n-1} = (-1)^{n+1}$. Therefore, $\{a_n\}$ contains a diverging subsequence and thus, must diverge. Subsequential limit points α are 0, 1, and -1. Also, $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$.

(c) The sequence $\{a_n\}$ converges or diverges depending on the choice of r. Subsequential limit points are $\alpha = 1$ if $r = 1$, $\alpha = 0$ if $|r| < 1$, $\alpha = 1$ and -1 if $r = -1$, and there are none if $|r| > 1$. $\limsup_{n\to\infty} a_n$ is equal to 1 if $r = 1$, 0 if $|r| < 1$, 1 if $r = -1$, and there is none if $|r| > 1$. $\liminf_{n\to\infty} a_n$ is equal to 1 if $r = 1$, 0 if $|r| < 1$, -1 if $r = -1$, and there is none if $|r| > 1$.

(d) $\{a_n\}$ diverges because subsequence $\{a_{2n}\}$ converges to 1 and subsequence $\{a_{2n-1}\}$ converges to -1, which is not equal to 1. Subsequential limit points are 1 and -1. In addition, $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$.

3. Consider the subsequence $\{a_{2^n}\}$. Since $a_{2^n} = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^{n-1}+1}+\cdots+\frac{1}{2^n}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^n}+\cdots+\frac{1}{2^n}\right) > 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}$ $= 1+n\left(\frac{1}{2}\right)$. Since $1+n\left(\frac{1}{2}\right)$ tends to $+\infty$ as n goes to infinity, by the comparison theorem, $\{a_{2^n}\}$ diverges to $+\infty$. Therefore, $\{a_n\}$ contains a subsequence which is unbounded, and so $\{a_n\}$ is not bounded and, hence, diverges.

4. We assume (a) is true and prove that if $\{a_n\}$ is any monotone (we only consider increasing) and bounded sequence, then it converges. Since (a) is true and $\{a_n\}$ is bounded, there exists a subsequence $\{a_{n_k}\}$ that converges to, say, A. This means that, if an arbitrary $\varepsilon > 0$ is given, there exists $m \in N$ such that $A-\varepsilon < a_{n_k} < A+\varepsilon$ for all $k \geq m$. But, $\{a_n\}$ is increasing. So if we choose $n^* \geq n_m$, then for all $n \geq n^*$ we have $A-\varepsilon < a_n \leq A$. Hence, $\{a_n\}$ converges to A.

We assume (b) and prove that the sequence $\{a_n\}$ has a converging subsequence. This part is easy because we used (b) to prove the Bolzano–Weierstrass theorem for sets, which in turn we used to prove the Bolzano–Weierstrass theorem for sequences which is what the goal was here.

5. Suppose $\{a_n\}$ is unbounded above. Let $M > 0$ be given. Since $\{a_n\}$ is unbounded, there exist infinitely many terms of $\{a_n\}$ larger than M. In particular, there exists $n_1 \in N$ such that $a_{n_1} > 1$. Also, there exists $n_2 > n_1$ such that $a_{n_2} > \max\{2, a_{n_1}\}$. Continue this argument to obtain $n_1 < n_2 < \cdots$ such that $a_{n_{k+1}} >$ $\max\{k+1, a_{n_k}\}$. We have constructed a subsequence that is increasing and tends to $+\infty$. Proof is similar for $\{a_n\}$ that is bounded below.

6. ($\Leftarrow$) Same as in the proof of Theorem 2.5.9. ($\Rightarrow$) If $\{a_n\}$ is a Cauchy sequence, then we want to prove that $\{a_n\}$ is convergent. First observe that by Theorem 2.5.8, A is bounded. By the Bolzano–Weierstrass theorem for sequences there exists $\{a_{n_k}\}$ which converges to, say, α. We will prove $\{a_n\}$ must also converge to α. Let $\varepsilon > 0$ be given. We want to find n^* such that for all $n \geq n^*$ we have $|a_n - \alpha| < \varepsilon$. The sequence $\{a_n\}$ Cauchy, implies that there exists n_1 such that for all $m, n \geq n_1$ we have $|a_n - a_m| < \boxed{\frac{\varepsilon}{2}}$. Since $\{a_{n_k}\}$ converges to α, there exists m such that for all $n_k \geq m$ we have $|a_{n_k} - \alpha| < \boxed{\frac{\varepsilon}{2}}$. Choose $n^* =$ $\max\{n_1, m\}$ and observe that $n_k \geq k$. Therefore, if $k \geq n^*$, then $n_k \geq n^*$. Hence, for all $n \geq n^*$ we have $|a_n - \alpha| \leq |a_n - a_{n_k}| + |a_{n_k} - \alpha| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$.

7. The proof is a special case of Exercise 4 of Section 2.5 because here the existence of a sequence from within S converging to s_0 is precisely the subsequence we are looking for. Note that $\{a_n\}$ might not converge to s_0. For example, consider a_n defined by 1 for n odd and $\frac{1}{n}$ for n even. Then $\{a_n\}$ diverges but $\{a_{2n}\}$ converges to 0, which is an accumulation point of S.

8. By Exercise 7, there exists a subsequence of $\{a_n\}$ which converges to B. Since $\{a_n\}$ converges to A, $A = B$ by Theorem 2.6.5.

9. **(a)** Since $c < 1$, $\sqrt[n]{c} < 1$ for all $n \in N$. Thus, $\{a_n\}$ is bounded and $a_{n+1} - a_n = \sqrt[n+1]{c} - \sqrt[n]{c} = \sqrt[n+1]{c}\left(1 - \sqrt[n(n+1)]{c}\right) > 0$. Hence, $\{a_n\}$ is increasing by Remark 2.4.3, part (d), and thus, converging to, say, A. To find A using subsequences, first observe that by Theorem 2.2.1, part (e), we have $\left\{\sqrt{a_n}\right\}$ converging to $\sqrt{A}$. Here, $\sqrt{a_n} = \sqrt{\sqrt[n]{c}} = \sqrt[2n]{c}$. But, $\left\{\sqrt[2n]{c}\right\}$ is a subsequence of $\left\{\sqrt[n]{c}\right\}$. Thus, $\left\{\sqrt{a_n}\right\}$ is a subsequence of $\{a_n\}$ and so it must converge to A. Therefore, $\sqrt{A} = A$ giving $A = 0$ or 1. Since $c > 0$ and $\{a_n\}$ is increasing, the limit must be 1.

(b) Since $r^{n+1} = r \cdot r^n < r^n$, the sequence $\{a_n\}$ is decreasing. Since $r^n > 0$, $\{a_n\}$ is bounded below, hence convergent to, say, A. Note that $\lim\limits_{n\to\infty} r^{n+1} = \lim\limits_{n\to\infty}\left(r \cdot r^n\right) < r \lim\limits_{n\to\infty} r^n = rA$. This means that $\{a_{n+1}\}$ converges to rA. But, $\{a_{n+1}\}$ is a subsequence of $\{a_n\}$ which converges to A. Therefore, $rA = A$. Since $r \neq 1$, A must be 0.

(c) By Exercise 11(c) of Section 2.4, $\{a_n\}$ converges to, say, A. Therefore, $\{a_{n+1}\}$ converges to A and also $\left\{\sqrt{2a_n}\right\}$ converges to $\sqrt{2A}$. Since the limit is unique, we have $A = \sqrt{2A}$ and hence, $A = 2$.

Section 2.7

1. T	9. T	17. T	25. F	33. F	41. F	49. F
2. T	10. T	18. T	26. T	34. F	42. F	50. T
3. F	11. F	19. T	27. F	35. T	43. T	
4. T	12. F	20. F	28. T	36. F	44. F	
5. F	13. F	21. F	29. F	37. F	45. T	
6. T	14. T	22. F	30. F	38. F	46. F	
7. F	15. T	23. F	31. F	39. F	47. F	
8. T	16. F	24. F	32. T	40. T	48. F	