Instructor's Solution Manual

To Accompany

Mathematics
of
Interest Rates and Finance

First Edition

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Preface

The preface of the text gives an overview of the text and its adaptability for use in college classes on any of three levels. Some would argue that our target audience is too large and that the advanced level students would find the basic level material too easy. Our response would be to have you consider an analogy. The most successful and enduring college athletic programs have been under coaches who emphasized the fundamentals of the game and drilled those fundamentals into the players before ever introducing a complex set of plays. If your advanced students spend time laying down a good foundation of concepts in interest and finance, they will find greater success and gain a better grasp of the more theoretical ideas. They will also more readily transition into other more complex forms of notation without difficulty.

Basic Level

Target Audience

This level is directed at the student who is not looking toward graduate school but who may be majoring or minoring in accounting, business, finance, or management. This level is also adequate for others majoring or minoring in related fields that require insight into business or money matters such as insurance sales, real estate, banking, interior design, financial planning, and brokerage or for those who want superior insight into their personal finances.

Course Description

The basic student needs to understand those applications inherent in all mathematics of interest processes as well as the elementary math underlying those techniques. Hence it is assumed that the basic student meets a two-year high school algebra prerequisite and understands simple algebraic solutions, exponents and logs, and the concept of a root, and has the ability to comprehend geometric series. A financial calculator similar to the TI BA II Plus or the Sharp EL 733 is absolutely necessary for this student. Students who try to get by with a scientific calculator become frustrated by Chapter 4.

Content

Cover Chapters 1 through 8, but omit sections 1.10, 2.6, 3.3, 3.8, 3.9, and 5.7. This level should concentrate on both the Concept and Calculation sections of the exercises.

Intermediate Level

Target Audience

The intermediate level student may be looking toward graduate school or doing technical undergraduate work in a field such as accounting, finance, operation research or actuarial science. Some may require certification in a technical field similar to the CPA or CFP.

Course Description

For the intermediate student the emphasis on "just solve the exercise" is extended to the understanding of the mathematics underlying those processes. Hence it is assumed that the intermediate student can handle arithmetic and geometric series, Calculus concepts of differentiation and integration, summation, and limits. These expectations would require that they meet a prerequisite of Elementary Stats and Calculus I. A programmable financial calculator similar to the TI 83 Plus or the TI 89 is absolutely necessary for this student. The solver routines will be used as a standard solving technique for the intermediate student.

Content

Cover the basic material in Chapters 1 and 2 briefly, all of Chapters 3 through 8, and the first four sections of Chapter 9. Emphasize the theory sections 1.10, 2.6, 3.3, 3.8, 3.9, 5.7, 9.1 – 9.4. Cover a representative number of the Concept and Calculation exercises and do all of the Theory and Extension exercises.

Advanced Level

Target Audience

This level is for the student looking toward graduate school in technical areas that require a strong mathematical background with an understanding of theory of interest and related financial applications. These fields might include operations research or insurance development and actuarial science; many of these students plan to take the SOA/CAS Course 2 Exam and beyond.

Course Description

The advanced student must be able to design financial processes that extend the usual processes. Hence, they ultimately must handle situations that include the concepts of continuous payments, block payments, and stochastic interest, so those new or off-the-wall situations can be met with a reasoned attack. Therefore, the approximation techniques inherent in Taylor's expansion with remainder and the stochastic processes of a mathematical statistics course are assumed for the advanced student as well as the prerequisites of Calculus I and II (through Taylor's expansion), and Linear Algebra. A programmable financial calculator similar the TI 89 or constant access to programs similar to Maple or Mathematica is absolutely necessary for this student. The solver routines on the TI 89 or Maple or Mathematica will be used as a standard solving technique for the advanced student

Content

Cover the same material as the intermediate level with only minimal time on the basic material of Chapters 1 and 2, but cover all of Chapter 9. This level should do a limited number of the Concept and Calculation exercises, all of the Theory and Extension exercises, and all of the exercises in Chapter 9.

The Society of Actuaries has graciously given permission for us to reprint some of their test questions from the May and November Course 2 exams, Copyrighted 2001 by the Society of Actuaries. These questions can be identified by the ◆ mark. We have made a few variations from the actual exam questions.

We have made every effort to insure the accuracy of this manual, but a number of the sample test questions are newly written and there will undoubtedly be some mistakes that have inadvertently slipped in. We would appreciate these errors being called to our attention.

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Serial Table—The Number of Each Day of the Year

Days	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept	Oct.	Nov.	Dec.	Days
1	1	32	60	91	121	152	182	213	244	274	305	335	**1**
2	2	33	61	92	122	153	183	214	245	275	306	336	**2**
3	3	34	62	93	123	154	184	215	246	276	307	337	**3**
4	4	35	63	94	124	155	185	216	247	277	308	338	**4**
5	5	36	64	95	125	156	186	217	248	278	309	339	**5**
6	6	37	65	96	126	157	187	218	249	279	310	340	**6**
7	7	38	66	97	127	158	188	219	250	280	311	341	**7**
8	8	39	67	98	128	159	189	220	251	281	312	342	**8**
9	9	40	68	99	129	160	190	221	252	282	313	343	**9**
10	10	41	69	100	130	161	191	222	253	283	314	344	**10**
11	11	42	70	101	131	162	192	223	254	284	315	345	**11**
12	12	43	71	102	132	163	193	224	255	285	316	346	**12**
13	13	44	72	103	133	164	194	225	256	286	317	347	**13**
14	14	45	73	104	134	165	195	226	257	287	318	348	**14**
15	15	46	74	105	135	166	196	227	258	288	319	349	**15**
16	16	47	75	106	136	167	197	228	259	289	320	350	**16**
17	17	48	76	107	137	168	198	229	260	290	321	351	**17**
18	18	49	77	108	138	169	199	230	261	291	322	352	**18**
19	19	50	78	109	139	170	200	231	262	292	323	353	**19**
20	20	51	79	110	140	171	201	232	263	293	324	354	**20**
21	21	52	80	111	141	172	202	233	264	294	325	355	**21**
22	22	53	81	112	142	173	203	234	265	295	326	356	**22**
23	23	54	82	113	143	174	204	235	266	296	327	357	**23**
24	24	55	83	114	144	175	205	236	267	297	328	358	**24**
25	25	56	84	115	145	176	206	237	268	298	329	359	**25**
26	26	57	85	116	146	177	207	238	269	299	330	360	**26**
27	27	58	86	117	147	178	208	239	270	300	331	361	**27**
28	28	59	87	118	148	179	209	240	271	301	332	362	**28**
29	29	*	88	119	149	180	210	241	272	302	333	363	**29**
30	30		89	120	150	181	211	242	273	303	334	364	**30**
31	31		90		151		212	243		304		365	**31**

*For leap years the number of any day after February 28 is 1 greater than the value given in the Serial Table.

Chapter Notes and Sample Questions

Overview

Basic level students often come to this material from a less than successful experience with high school math story problems. When they look at the text and see that nearly every exercise set consists of story problems, they relive old fears. You must set those fears to rest with a positive approach and assurances that they will be acquiring the necessary tools to become successful problem-solvers. The mathematics employed here is generally straightforward, but the development of analysis, reasoning, and organization skills may require a great deal of time and effort. Students need only learn a few basic principles to be applied in a variety of contexts. The most important concept (for all levels of students) is the time value of money due to the accrual of interest. You should strongly emphasize that they know what a formula does to certain monies and where the resulting values are located on the time line. This will take consistent effort from the teacher so the students do not short-cut the drawing and labeling of time-line diagrams. The time line and its various components will help them extract the data from the exercise and will force them to make decisions about the location of the parameters and even about the choice of a time-value formula or formulas. The development of the various time-value formulas in class will greatly benefit the students' understanding of what effect each formula has on a piece of money. The reasonableness of the formula for the future value at simple interest helps students see beyond the clutter of symbols grouped together. $S = P(1+it)$ says that a bundle of money P is multiplied by a factor $1+it$ that is greater than 1. The result S has to be a larger value than P, and that outcome is what the time value of money says happens as we move money to the future.

Our experience has shown that, for a basic level course, requiring the students to bring their calculators to class and working through examples with the teacher is invaluable. For one thing, this practice immediately indicates when certain students are misusing their calculators. This class work has a laboratory atmosphere that lets students help each other and allows the teacher to address the numerous questions that arise at a time when he or she help. It also gives a forum for students to make up questions for the class. This keeps them mentally focused on the learning at hand. Students are actually generating additional examples beyond what is in the text, that they can refer to while doing homework. The teacher can use this time to ask questions that effectively guide learning and understanding. The questioning technique must be used with caution so those who are struggling are not embarrassed.

Many students in undergraduate actuary programs hate the classic texts that are often used for theory of interest and finance. They are overwhelmed by a mass of theory that does not always seem relevant, even though they need that theory for a competent level of understanding. It is not that those students are not intelligent and competent, but they do not necessarily learn theory easily without some foundation in the basic principles. We feel that this text can give them the basic foundation quickly through the study of some practical and elementary-level material before they get to the theory. They work with actual money and interest rates, and address the issues involved with understanding the time value of money. A student who has some significant experience with the concepts at a concrete level will often enjoy the theory much more than one who has to wade into the theory and notation with no previous experience.

The mathematics of interest rates and finance provides one of the more practical and valuable courses a student may take in his or her college program. Although it generally serves the students with various business majors or an actuarial major, those with a variety of other majors can use these principles in their careers and personal finances. As a simple example, someone who brags that his mutual fund increased 100% over the last ten years would need to be reminded by a well-taught finance student that the growth rate was just a little over 7% and well below the long-term expected rate of 11%.

Appendix B (Financial Calculators and Spreadsheets) addresses only four particular tools, but we hope they are representative of the wide variety of financial tools available. The appendix help is tied to the chapters and the types of exercises within those chapters.

Chapters 1 and 2 – Simple Interest and Simple Discount

Chapters 1 and 2 are somewhat elementary but very useful foundational material. Nearly every major concept in the text is presented in embryonic form in chapters 1 and 2. Problems that are presented in these early chapters will be extended, enhanced, or modified in subsequent chapters. A deep understanding of the concepts in these chapters will reap benefits in the chapters to come. Indeed, it is the authors' opinion that the fastest path through subsequent chapters may well lie in thoroughly traveling these chapters.

Even a very brief coverage of these two chapters will greatly improve your students' understanding of the theory in Section 2.6. They introduce the time value of money and include abundant practical applications. The structure of simple interest will be used in several other chapters, so mastery of these very basic ideas enhances the students' performance in the remainder of the course. There is a tendency for students not to study due to the deception inherent in the word "simple." We like to test simple interest and discount interest together. The considerable similarities require more attention and understanding to discern the distinctives. You may even consider teaching them simultaneously to effect greater understanding of the subtleties involved. Encourage students to read the text and work through the examples so they will pick up the numerous details in finding things like the types of time and interest. The section on equations of value will be the first place where time-line diagrams play an important role in analyzing the exercises and organizing the information and structure of the questions. You may prefer a different way of laying out the information on a time line, possibly with the monies above the line and the dates below. Variety in presentation will be valuable for your students' understanding and ability to analyze.

The student must be aware of when to use each formula. The formulas for simple interest ($I = Pit$) and simple discount ($D = Sdt$) are used to calculate the ROR and the "time to" more often than they are used to calculate I or D. The formulas for present and future values, $S = P(1 + it)$, $P = S/(1 - it)$, $S = P/(1 - dt)$, and $P = S(1 - dt)$, are used to "move money" and generalize nicely to the compound interest context. Indeed, the formulas $S = P + I$ and $P = S - D$, while correct, cannot be generalized. Further, the two formulas are misleading in that it is conceptually false that $P = S - I$ and $S = P + D$. The reasons are that I cannot be calculated from S and D cannot be calculated from P.

The partial payment exercises of Section 1.9 are fairly lengthy, and you may need to scale back how many you assign. Section 1.10 on equivalent time provides the first opportunity for the teacher to introduce theory and address issues that require a little more insight. A subtle thing happens in the derivation of the average due date theorem, and it may not be apparent at first. Some of the terms of the form $P_k[1 + i(t - t_k)]$ will no doubt be negative, thus we are actually involving both simple and discount interest. This will become more apparent after Chapter 2 is completed. In spite of this theoretical license, the average due date theorem gives a final result that is quite consistent with the more cumbersome method of Example 1.10.2. The adjusted average due date method also agrees quite well with the answer in Example 1.10.1.

Emphasize that simple discount is based on the future value; otherwise students mix the formulas up with simple interest and fail to see what they have done wrong. It is key to remember that a simple discount rate stated as the same rate for simple interest always costs the borrower more in interest charges. The customer who leaves the bank with \$1000 at 6% simple discount (bank discount) for one year will pay \$63.83 in interest charges. The same loan at 6% simple interest will cost the borrower \$60. This makes Section 2.2 on comparing simple and discount interest worth the time in terms of gaining insight into the distinctives of these two methods of borrowing. Chapter 2 also provides important tools for the calculations required in the securities open market trading, where discounting is so prevalent.

Section 2.6 introduces the theory of interest for simple interest and simple discount and is very important for those using the text at the intermediate or advanced level. The ideas of accumulation function, effective rates, and equivalent rates for both simple interest and simple discount are of theoretical significance regardless of whether or not they are used in the real world of finance. Of course, simple discount is the basic tool in buying and selling securities in the secondary market. Certainly, simple interest is not used like it once was, but you can still get a simple interest loan at many banks, and add-on loans are fairly common.

Chapters 1 and 2 Sample Test Questions – Simple and Discount Interest

Multiple Choice

1. Find the exact time from November 14, 2004, to April 24, 2005.

a) 159 b) 160 c) 161 d) 162 e) none of these

2. Find the approximate time from June 24, 2005, to September 29, 2005.

a) 95 b) 96 c) 97 d) 98 e) none of these

3. How long will it take \$8000 to earn \$380 in interest at 9% per annum? Give your answer in days via the Banker's Rule.

a) 180 b) 190 c) 193 d) 682 e) none of these

4. What discount rate is equivalent to a simple interest rate of 9.5% for 180 days?

a) 9.25% b) 9.50% c) 9.97% d) 10.00% e) none of these

5. If money earns 5.5% simple interest, is it better to buy a piano for \$8200 cash or for \$8800 in 15 months?

a) \$8200 cash b) \$8800 in 15 months

6. In problem 5, what is the cash equivalent of the savings resulting from adopting the better plan?

a) \$5.00 b) \$5.36 c) \$36.25 d) \$33.92 e) none of these

7. If a builder receives a materials invoice for \$22,500 with terms 5/10, n/60, what rate of interest would he earn by paying on the 10th?

a) 2.64% b) 31.58% c) 36.00% d) 37.89% e) none of these

8. If a note is worth \$650 on 1/1/05 and \$670 on 7/1/05, what kind of interest has been calculated?

$$\text{rate} = \frac{20.00}{670(\frac{181}{360})}$$

a) simple interest rate b) discount interest rate

9. Which combination of the following is known as the Banker's Rule?

a) exact interest c) exact time e) none of these

b) ordinary interest d) approximate time

10. If a simple interest loan is paid back with several partial payments, finding the balance on the settlement date by moving all payments and debt principal to that date is called:

a) U.S. Rule
b) Merchants' Rule
c) Bankers' Rule
d) Golden Rule
e) none of these

Calculation Problems

11. Randall held a promissory note initiated on January 10, 2005, with a face value of \$15,000 and bearing interest at 9% to a maturity date of March 16, 2005. If he sells the note to the Upstate Bank, which uses a discount rate of 10.5%, how much did he get for the note on February 14, 2005?

12. An investment requires an outlay of \$20,000 and is expected to produce returns of \$5,000 in 6 months, \$10,000 in 12 months, and \$7600 in 15 months. Find the net present value of the investment at 12% and the internal rate of return.

13. Anisha purchased a 91-day, \$500,000 Treasury Bill for a bid price of 96.752. If it is held to maturity, find the yield rate on a discount basis.

14. Suppose Anisha (problem 13) held the T-bill for 75 days and then sold it in the secondary market for \$495,750. What was her yield rate for the period the bill was held on a simple interest basis?

15. A new homeowner buys furniture costing \$11,000, making a down payment of \$4000. He agrees to pay \$2500 in 3 months and the balance in 15 months. If the lender charges 9% per annum, find the payment due in 15 months that will discharge the debt.

16. Find the average due date for the following obligations: \$550 due in 35 days, \$400 due in 48 days, \$680 due in 64 days, and \$725 due in 70 days.

17. A startup company has obligations of \$20,000 in 45 days, \$15,000 in 70 days, \$16,000 in 90 days, and \$25,000 in 120 days. When will \$75,000 satisfy these obligations if money is worth 7.5%? When will \$77,000 at 7.5% satisfy these obligations?

18. A business must spend \$50,000 initially and \$10,000 in 18 months on an investment that is expected to brings returns of \$30,000 in 9 months, \$15,000 in 21 months, and \$30,000 in 27 months. Find the net present value of this investment at 15% and the internal rate of return.

19. Find the date when \$8000 will pay off \$2000 due in 30 days, \$5000 due is 90 days, \$800 due in 120 days, and \$200 due in 180 days. Explain why the rate is irrelevant.

20. Find the date when \$8200 will pay off \$2000 due in 30 days, \$5000 due is 90 days, \$800 due in 120 days, and \$200 due in 180 days at 6%. Explain why the rate is relevant.

Theory and Extension Problems

21. Prove that the U.S. Rule and the Merchant's Rule for calculating the final payment give different results. Explain the difference.

22. Let the interest rate per payment period = i. (For example, if the yearly rate = 8% and the payments are monthly then $i = .08/12 = .0066666$.) Let there be a payment of amount R every period for n periods. Derive a nice formula for the payoff x on period $n + 1$

a. Using the Merchant's Rule,

b. Using the US Rule.

23. Program your calculator, *Excel*, *Mathematica*, or *Maple* to handle the formulas derived in the above exercise and work the following final-payment problems. Part **a** has the answers shown so you can check your programs.

	Loan	Payments	Rate	Final pmt	U.S. Rule	Merchants
a.	\$2000	3 quarterly of \$500/quarter	9%/yr	On 4th quarter	\$617.65	\$612.50
b.	\$5000	8 monthly of \$400/month	7%/yr	On 9th month		
c.	\$3000	2 yearly of \$1000/year	5%/yr	On 3rd year		

24. Use the SOLVER routine on your programmable calculator, *Excel*, or other technology to solve for the internal rate of return at simple interest for an investment of \$10,000 giving returns of \$2000 in 1 year, \$5000 in 2 and another \$8000 in 3. Show your equation and how to get the solution.

25. Use the SOLVER routine on your programmable calculator, *Excel*, or other technology to solve for the internal rate of return on a bank discount basis for an investment of \$10,000 giving returns of \$2000 in 1 year, \$5000 in 2 and another \$8000 in 3. Show your equation and how to get the solution.

26. Question 24 above is worked as a simple interest problem, and the NPV at the simple interest rate of 10% was \$2138.69.

- **a.** Find the bank discount rate equivalent to the simple interest rate of 10% for this problem.
- **b.** Tell why the conversion rate formula $d = i / (1 + it)$ cannot be used.
- **c.** Derive a conversion formula and program it to show that you get the same answer you got in part **a.**

27. Define the effective rate for the nth period and prove that the effective rate for a simple interest investment $\rightarrow 0$ as $n \rightarrow \infty$. Tell why this is intrinsically "unfair" and define a "fair" investment. Derive a recursive formula for a fair investment.

28. Suppose that $a(t) = 1 + 3t + t^2$.

- **a.** If one initially invests \$1000, tell how much is in the account after 4 years.
- **b.** Derive a formula for the effective rate i and prove that $i \rightarrow 0$ as $n \rightarrow \infty$.
- **c.** Argue that any polynomial accumulation factor is intrinsically unfair in the sense of unfair as defined in question 27.

Answers for the Sample Test Questions

1. c	**2.** a	**3.** b	**4.** e	**5.** a
6. d	**7.** d	**8.** b	**9.** b, c	**10.** b

11. MV = \$15,243.75, Proceeds = \$15,110.37 **12.** $NPV_{@12\%}$ = \$254.25, IRR = 13.5%

13. $d = 12.85\%$ **14.** Invested \$483,760, Interest \$11,990, $i = 11.9\%$

15. Merchant's Rule gives \$5062.5, U.S. Rule gives \$5076.68 **16.** $\bar{t} = 56.36$

17. $\bar{t} = 20.9$ for \$75,000 and $\bar{t} = 147.2$ for \$77,000

18. NPV@15% = \$3114.12, IRR = 20.36%

19. $\bar{t} = 80.25$ days. The rate cancels in the derivation.

20. $\bar{t} = 230.25$ days. The rate does not cancel in the derivation.

21. Use a counterexample.

Suppose we owe \$2000 and pay a partial payment of \$1000 in 1 year. Find the payoff in 2 years.

U.S. Rule: Moving \$2000 forward for 1 year = $\$2000(1+(.10)(1)) = \$2200$. Minus $\$1000 = \$1200$. Move balance forward 1 year. Payoff = $\$1200(1+(.10)(1)) = \$1320$

Merchant's: Payoff = $2000(1+(.10)(2)) - 1000(1+(.10)(1)) = \$1300$.

The Merchant's Rule uses SI for the entire term with the focal date at the payoff date. The U.S. Rule figures interest at each payment so it is essentially compound interest even though the payments may not be at periodic intervals like an annuity.

22. U.S. Rule:

1st period balance = $P(1+i) - R$

2nd period balance = $(P(1+i) - R)(1+i) - R = P(1+i)^2 - R(1+i) - R$

3rd period balance = $(P(1+i)^2 - R(1+i) - R)(1+i) - R$

$= P(1+i)^3 - R(1+i)^2 - R(1+i) - R$

*n*th period balance = $P(1+i)^n - [R(1+i)^{n-1} + R(1+i)^{n-2} + \dots + R]$

$$\text{U.S. Payoff} = P(1+i)^{n+1} - R\sum_{k=1}^{n}(1+i)^k.$$

We now use the geometric sum formula to get

$$= P(1+i)^{n+1} - \frac{R(1+i)(1-(1+i)^n)}{1-(1+i)}$$

$$\text{U.S. Payoff} = P(1+i)^{n+1} - R\frac{(1+i)^n - 1}{i}(1+i)$$

$$\text{Merchant's Rule: Payoff} = P[1+i(n+1)] - R[1+i(n)] - R[1+i(n-1)] - \cdots - R[1+i]$$

$$= P[1+i(n+1)] - R\sum_{k=1}^{n}[1-i(k)]$$

Use the arithmetic sum formula to get the Payoff $= P[1+i(n+1)] - R[2+i(n+1)]\frac{n}{2}$

23. **b.** U.S. \$1983.56, Merchant's \$1978.50 **c.** U.S. \$1320.38, Merchant's \$1300.00

24. NPV(10%) = \$2138.69, IRR = 21.54%

25. NPV(10%) = \$1400, IRR = 13.88%

26. **a.** If NPV = \$2138.69, then $d = 7.95\%$ **b.** multiple values for t

c. Let $NPV_{@i} = NPV_{@d}$ then $i = .10$ gives $d = 7.95\%$

27. Consider the following definition and proof.

The effective rate for the nth period denoted i_n is defined by

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{\text{Growth in the investment over the nth period}}{\text{Value at the start of the nth period}}.$$

For simple interest $i_n = \dfrac{P(1+in) - P(1+i(n-1))}{P(1+i(n-1))} = \dfrac{i}{1+i(n-1)} \to 0$ as $n \to \infty$.

In the above sense, a simple interest investment is intrinsically "unfair." Its per-period earnings decrease with time.

What would the effective rate have to be for the investment to be "fair"? The effective rate would have to approach a constant, j, which could well be called the steady-state return. What would the ramifications of a fair investment be?

$$i = \lim_{n\to\infty} i_n = \lim_{n\to\infty}\left(\frac{A(n)-A(n-1)}{A(n-1)}\right) = \lim_{n\to\infty}\left(\frac{A(n)}{A(n-1)}-1\right) \to 1+i = \lim_{n\to\infty}\left(\frac{A(n)}{A(n-1)}\right).$$

Hence, in the long run for an investment to be fair, $\frac{A(n)}{A(n-1)} = 1+i.$

$\to A(n) = (1+i) * A(n-1)$ with $A(0) = P$.

28. a. $A(n) = A(0)*a(n) = 1000(1 + 3t + t^2) \to A(4) = 1000(1 + 12 + 16) = \$29{,}000$

b. The effective rate for the nth period denoted i_n is defined by

$$i_n = \frac{A(n)-A(n-1)}{A(n-1)} = \frac{\text{Growth in the investment over the nth period}}{\text{Value at the start of the nth period}}.$$

For this problem $i_n = \frac{1000(1+3n+n^2)-1000(1+3(n-1)+(n-1)^2)}{1000(1+3(n-1)+(n-1)^2)}$.

$\therefore i_n = \frac{2+2n}{1+3(n-1)+(n-1)^2} \to 0$ as $n \to \infty$ using L'Hôpital's Rule.

c. $i_n = \frac{A(n)-A(n-1)}{A(n-1)} = \frac{Pa(n)-Pa(n-1)}{Pa(n-1)} = \frac{a(n)-a(n-1)}{a(n-1)}$.

If $a(n)$ is an nth degree polynomial, then $a(n) - a(n-1)$ is a polynomial of degree $< n$. Hence, by L'Hopital's Rule, $i_n \to 0$ as $n \to \infty$. Hence, all polynomial accumulation factors are "unfair."

Chapter 3 – Compound Interest

The study of compound interest is critical to understanding the rest of the material and must be covered very thoroughly. Those in a basic course should omit only Sections 3.3 and 3.9. The formula $S = P(1+i)^n$ requires logs to solve for n and fractional exponents to solve for i. Although the text gives the derivation of these concepts, in the applications we note that the financial calculators will solve for n or i without the students' having to do these steps for themselves. Consequently there are only two major formulas used in the chapter. The future value formula, $S = P(1 + i)^n$, is used to calculate S, n, or i given the other variables. The second, $P = S(1 + i)^{-n}$, is not really necessary but is used to solve for the present value P given the other variables. The negative exponent in this PV formula is a colorful way of depicting negative time.

Students in a basic course occasionally seem perplexed when asked how much interest was earned or paid. This confusion may be related to the previous chapters where formulas for the amount of interest exist, whereas no such formulas exist for compound interest, and the interest is found as the difference between the present and future values.

Although it is not necessary in a basic course to study continuous compounding, it can add much to the study of effective interest rates if the first part of Section 8 is brought into the mix while doing Section 3.4.

Section 3.3 can provide a nice analytical answer to the question of why the choice of focal dates for compound interest or compound discount equations of value does not matter. This section shows that the effective rate for compound interest or compound discount is a constant that is independent of the measurement interval. On the other hand, the effective rates for simple interest and discount interest are functions of the rate and the measurement interval. Students should have a thorough understanding of compound interest and compound discount, including the several equations that relate the two rates i and d, in order to deal with the exercises and effective rate derivations in Section 3.4. By dealing with only compound interest and not compound discount, a certain amount of theoretical elegance is lost.

Compound interest (or compound discount) should the interest of choice in that it is the first "fair" interest the student will meet. How is it fair? The basic-level elementary student should be made to feel compound interest is fair in that it is calculated on the current worth of the original investment plus its accrued interest. On the other hand, simple interest is calculated on only the worth of the original investment and simple discount is calculated on the worth of the investment at the end. To the intermediate or advanced student, this feeling of fairness is mathematically enhanced by the concept of the force of interest. The force of interest for compound interest is constant, whereas for simple interest it decreases asymptotically to zero and for simple discount it increases to infinity.

The equations of value in Section 3.7 provide an effective tool for a number of financial calculations besides net present value and internal rate of return. Requiring students to write thorough equations of value will lead them to a greater understanding of the time value of money and the theory of interest. They will need to draw quality time-line diagrams, identify a focal date, and select the right tools to move the given monies to that focal date. Students who do not understand the concepts of this section often make some real blunders when they work a problem like Example 3.7.2, where they find the amount of money distributed to several beneficiaries from an estate. One common mistake results in their finding the future value of two pieces of money the size of the given estate and using the sum of these divided by two. Carefully teaching this material, plus giving warnings of potential errors, will guide understanding and provide benefits at later points in the text.

Covering the material on continuous compounding in Section 3.8 will be essential if you wish to study annuities with continuous compounding in Section 5.7. The annual effective rate formula from a nominal rate compounded continuously makes a nice lead-in for the material in Section 3.9 dealing with the force of interest. One topic of interest that was omitted in Section 3.8 is the continuous compound discount annual effective rate formula. The direct approach would be to develop the formula using a limit like was used to find formula 3.14. We would start with the formula for the annual effective rate of compound discount 3.6, use some appropriate substitutions, and then evaluate the limit as m approaches ∞. Another approach would be to use formula (3.14) and substitute into the basic equivalence relation $d = i/(1+i)$. Since (3.14) gives $i = e^{\delta} - 1$, we get the following:

$$d = \frac{i}{1+i} = \frac{e^{\delta}-1}{1+e^{\delta}-1} = \frac{e^{\delta}-1}{e^{\delta}} = 1 - e^{-\delta}.$$

Substituting this effective rate into the compound discount formulas 3.3 and 3.4 yields the same future value and present value formulas as continuous compound interest. This should not be surprising since compound interest and compound discount are shown to have the same force of interest in Section 3.9. Solving the above formula for δ yields equation (3.15), $\delta = \ln(1+i)$.

Section 3.9 is an important section for those vitally interested in the theory of interest. By using a *rate of change* approach we have been able to make an overall comparison and contrast of the four types of interest as well as place the force of interest in a much more familiar context as the percentage rate of change. We believe that most students who have had calculus are comfortable with derivatives and will therefore have little difficulty using the rate of change approach to deal with force of interest exercises.

Chapter 3 Sample Test Questions – Compound Interest

Multiple Choice

1. If a $1000 gift is invested at 8%(12) on a teen's 16th birthday, 2/14/98, how much interest will the investment have earned by graduation day 5/14/04?

   a) 640.61 b) 645.99 c) 1640.61 d) 1645.99 e) none of these

2. A promissory note with a maturity value of $6400 is sold at a bank charging 8%(4). If it is sold 24 months before the due date, what will it bring the seller?

   a) 5462.34 b) 5683.02 c) 7498.62 d) 7207.44 e) none of these

3. How long will it take an investment to increase at least 88% if it draws interest at 8%(4)? No interest is given for part of a period.

   a) 8 years b) 32 years c) $7\frac{3}{4}$ years d) 31 years e) none of these

4. Find the effective interest rate for 12%(12).

   a) 12.12% b) 11.27% c) 12.68% d) 1.0% e) none of these

5. Sales at a microchip company increased from $18.2 million in 1994 to $22.5 million in 2002. If the growth is assumed to be constant, what was the rate of increase per quarter?

   a) 0.66% b) 2.66% c) 2.69% d) 10.75% e) none of these

6. If an investment at 9%(12) is made on June 15, 1998, how many periods of interest will it accrue if it has a maturity date of January 15, 2004?

   a) 5.25 b) 60 c) 61 d) 96 e) none of these

7. Grandpa Smith invested $950 on June 1, 1940 at 3%(1) in a bank that made it through the great depression. The rate changed to 4%(2) on 6/1/1973, and to 5%(4) on 6/1/1985. To the nearest $10, how much was in his account on 6/1/1995? Assume he added no other principal.

   a) $6660 b) $4050 c) $6570 d) $4030 e) none of these

8. If a $5,000 investment accrues interest at 6%(4) for 6 years, how many days of simple interest must it earn to have at least $7225? Use Banker's Rule for simple interest.

   a) 18 days b) 66 days c) 260 days d) 72 days e) none of these

9. What is the force of interest for a $25,000 investment at 8%($\infty$)?

   a) 8.33% b) 7.00% c) 8.00% d) 7.70% e) none of these

10. How much should a father put in an investment for his newborn infant so that in 18 years college education costs of $60,000 will be available? Assume the investment pays an average of 10%(12).

    a) $6994.14 b) $9992.18 c) 10,791.53 d) 17,081.66 e) none of these

Calculation Problems

11. On September 30, 2003, a man borrowed $8800 at 6%(2). How much will he have to repay on March 30, 2004?

12. On October 10, 2003, a company invested $18,500. If the investment is expected to bring returns of $10,000 on October 10, 2004 and $12,000 on April 10, 2005, what is the NPV at 16%(2) and what is the IRR?

13. A man leaves an estate of $75,000 to his two children with the provision that they are to receive equal amounts when they reach age 21. The money is invested at 10%(4). If the children are ages 12 and 16 at the time of the endowment, how much does each receive at age 21?

14. A note was issued on January 1, 2002, for \$9500 due in 4 years at 7%(12). What was the selling price of this note on July 17, 2003, if the buyer discounted it at 8%(4) and used simple interest and Banker's Rule for part of a period?

15. A graduate student borrows \$20,000 on January 1 at 8%(12). A payment of \$5000 is made on March 1 and another for \$10,000 is made on June 1. What payment will pay off the debt on October 1?

16. If current assets of \$48,000 on February 14, 2003 need to be worth \$100,000 on May 14, 2013, what is the minimum nominal rate compounded quarterly at which they should be invested?

17. What is the value of a \$25,000 investment at 6.5%(∞) after 38 weeks?

Theory and Extension Problems

18. Prove that the effective rate for a compound interest problem is a constant i for all n and hence compound interest investments are, in this sense, fair. Derive a recursive formula for compound interest investments and solve the recurrence to gain the compound interest formula.

19. Prove that if the force of interest is a constant δ, then the investment is a compound interest investment with $i = e^{\delta} - 1$.

20. If the force of interest = $1 + 3t$ on an initial investment of \$1000, find the amount one has after 5 years.

21. If the force of interest = $1/(2t + 1)$ on an initial investment of \$1000, find the amount one has after 5 years.

22. Summarize unfair, fair, and overly generous investments using force of interest and the above exercises.

23. If a compound interest problem has a rate of 8%(4), and \$1000 is invested for 3 years and 2 months, then find the future value

    a. Using compound interest 8%(4) for whole + fractional period.

    b. Using the effective monthly rate.

    c. Using the force of interest.

    d. Prove that your answers for **a,** **b** and **c.** will always be the same for every problem.

Answers for the Sample Test Questions

1. b	**2.** a	**3.** a	**4.** c	**5.** a
6. e	**7.** a	**8.** b	**9.** a	**10.** b
11. \$13,710.11		**12.** $NPV_{16\%(2)} = -400.62$, IRR = 14.14%(2)		
13. \$73,431.17	**14.** MV = \$12,559.51, P_1 = \$10,509.24, P_2 = \$10,334.70			
15. \$5725.11	**16.** i = 1.806/q, i(4) = 7.225%(4)			**17.** \$26,216.15

18. a. The effective rate for the nth period denoted i_n is defined by

$$i_n = \frac{A(n)-A(n-1)}{A(n-1)} = \frac{\text{Growth in the investment over the nth period}}{\text{Worth at the start of the nth period}}.$$

For compound interest $i_n = \frac{P(1+i)^n - P(1+i)^{n-1}}{P(1+i)^{n-1}} = i.$

b. $i_n = \frac{A(n)-A(n-1)}{A(n-1)} = i \rightarrow \frac{A(n)}{A(n-1)} - 1 = i$

$\therefore A(n) = (1+i)A(n-1)$ with $A(0) = P$

c. $\therefore A(1) = (1+i)A(0) = (1+i)P$

$\therefore A(2) = (1+i)A(1) = (1+i)^2 P$

etc., yielding $A(n) = (1+i)^n P.$

Rename $A(n)$ as $S = P(1+i)^n$

19. If the force of interest is a constant, then $\delta_t = \frac{A'(t)}{A(t)} = \delta.$

$\therefore A'(t) = \delta A(t)$ is our differential equation $\rightarrow A(t) = Ke^{\delta t}$.

But $A(0) = P = K \rightarrow A(t) = Pe^{\delta t} = P(e^{\delta})^t = P(1+i)^t$ where $i = e^{\delta} - 1$.

It is important that we note from this and the previous exercise that for compound interest the gain is proportional to the current amount. Here that proportionality is demonstrated by $A'(t) = \delta A(t)$. In the previous exercise the proportionality was demonstrated by the difference or recursion formula $A(n) = (1+i)A(n-1)$.

20. $\delta_t = \frac{A'(t)}{A(t)} = 1 + 3t.$

$\therefore A'(t) = (1+3t)A(t)$ is our differential equation $\rightarrow A(t) = Ke^{t+3t^2/2}$.

But $A(0) = 1000 \rightarrow K = 1000 \rightarrow A(t) = 1000e^{t+3t^2/2}$.

Hence $A(5) = 1000e^{5+3*5^2/2} = \$2.867E21$. Now that is real growth of money because $\delta_t \rightarrow \infty$ as $t \rightarrow \infty$.

21. $\delta_t = \frac{A'(t)}{A(t)} = 1/(2t+1)$ is our differential equation $\rightarrow A(t) = K\sqrt{2t+1}$.

But $A(0) = 1000 \rightarrow K = 1000 \rightarrow A(t) = 1000\sqrt{2t+1}$.

Hence $A(5) = 1000\sqrt{2*5+1} = \$3316.62$. Now that's slow growth of money because $\delta_t \rightarrow 0$ as $t \rightarrow \infty$.

22. **Fair**: when $\delta_t \to$ a positive δ as $t \to \infty$ which, in the long run (steady state), says that our return at time t is proportional to the amount we have at time t.

Unfair: when $\delta_t \to 0$ or a negative as $t \to \infty$ which, in the long run (steady state), says that our return at time t is negligable or negative compared to the amount we have at time t.

Overly generous: when $\delta_t \to \infty$ as $t \to \infty$ which, in the long run (steady state), says that our return at time t is infinite compared to the amount we have at time t.

23. **a.** $S = P(1+i)^{n \text{ in quarters}} \to S = 1000(1+8\%/4)^{12.\bar{6}q} = \$1285.10$

b. Equivalent monthly rate $= (1+8\%/4)^{4/12} - 1 = .662270956..\%/\text{month}$

$S = P(1+i)^{n \text{ in months}} \to S = 1000(1+.662270956..\%)^{38m} = \$1285.10$

c. $\delta = \ln\left[(1+8\%/4)^4\right] = 7.9210509185\%$ is the force of interest.

$S = Pe^{\delta t \text{ in years}} \to S = 1000e^{.079210509185*\left(3\frac{1}{6}y\right)} = \$1285.10$

d. Let $i = i(m)/m$ be similar to $i = 8\%(4)/4$ in part a.

Let $j = i(k)/k$ be the equivalent amount to i similar to the equivalent monthly rate in part b. Hence $j = (1+i)^{m/k} - 1$.

Let $\delta = \ln\left((1+i)^m\right)$ or, equivalently, $e^{\delta} = (1+i)^m$ define the force of interest similar to the δ in part c.

Hence I need to show that

$$S = P(1+i)^n = P(1+j)^{n*k/m} = Pe^{\delta*n/m}.$$

Plugging the equivalent expressions for j and δ yields the obvious tautology

$$S=P(1+i)^n = P\left((1+i)^{m/k}\right)^{n*k/m} = P\left((1+i)^m\right)^{n/m}.$$

Chapter 4 – Ordinary Annuities

This chapter is pivotal for the understanding of finance and interest. There is a certain beauty in being able to use a geometric series to sum n compound interest amounts at a chosen focal date with our annuity formulas. The students must understand what these highly functional formulas are actually doing. If the student is not proficient in the concepts of ordinary annuities (annuities-immediate), he or she has no chance of understanding the generalized annuities in Chapters 5 and 9.

Students should recognize the following relationships in order to handle unit payment theory questions.

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} = \sum_{k=1}^{n}(1+i)^{k-1}$$

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} = (1+i)^{-1} + (1+i)^{-2} + \cdots + (1+i)^{-n} = \sum_{k=1}^{n}(1+i)^{-k}$$

$$a_{\overline{n}|i} = \frac{1-v^n}{i} = v + v^2 + v^3 + \cdots + v^n = \sum_{k=1}^{n} v^k$$

We have chosen to give the name *fence post principle* to the concept that there are n periods of time between $n+1$ payments on the time line. This pedagogical tool has been very successful in getting students to determine the number of payments correctly. You may have a name that suits you better, but whatever you chose to call it, keeping it before your students and illustrating why it happens will make for far fewer misunderstandings about the number of payments in an annuity. As a general rule we emphasize that an exercise with the date given for the first payment and the last payment will require the use of the fence post principle after figuring the number of interest periods using date arithmetic. Again, these calculations are best grasped by making a quality time line diagram. We require that each of the student's time-line diagrams has the following three critical elements:

a. The first several payments and the final payment (or the payment symbol) with their dates.

b. The date or location and the value or symbol of the annuity's present or future value bundle.

c. Directional arrows telling which way the bundle of money or the payments are to move to establish equivalence for the equation of value.

Using financial calculators to find the answers tends to hide the real mathematics and cloud understanding, although calculator use is well worth the risk. This need for good understanding gives further reason to emphasize throughout Chapter 4 the use of time-line diagrams.

Chapter 4 Sample Test Questions – Ordinary Annuities

Multiple Choice

1. A couple would like to save for the next 20 years for a future home estimated to cost \$150,000. If they can invest money at 10%(4), how much will their payment be?

a) $\$150{,}0000 = Rs_{\overline{20}|10\%}$ c) $\$\ 150{,}000 = Rs_{\overline{80}|2.5\%}$ e) none of these

b) $\$150{,}000 = Ra_{\overline{20}|10\%}$ d) $\$\ 150{,}000 = Ra_{\overline{80}|2.5\%}$

2. After 3 years what is the value of an annuity in which the payments are \$45 every 3 months and money can be invested at 8%(4)?

a) $\text{value} = 45s_{\overline{12}|2\%}$ c) $\text{value} = 45a_{\overline{12}|2\%}$ e) none of these

b) $\text{value} = 45s_{\overline{9}|8\%}$ d) $\text{value} = 45a_{\overline{3}|8\%}$

3. A family wishes to endow a lecture series at their favorite university for the next 50 years. The cost of the lecture series will be \$50,000 per year starting in 1 year. If money can be invested at 9%(1), how much should be deposited now?

a) $\$50{,}000 = Xs_{\overline{50}|9\%}$ c) $\$50{,}000 = Xa_{\overline{50}|9\%}$ e) none of these

b) $X = \$50{,}000s_{\overline{50}|9\%}$ d) $X = \$50{,}000a_{\overline{50}|9\%}$

4. How many \$500 monthly payments can be withdrawn from an account containing \$25,000 and invested at 8%(12)? The first draw takes place in 1 month.

a) 43 c) 61 e) none of these

b) 20 d) 85

5. How much should be invested semiannually at 9%(2) for 10 years to accumulate \$100,000?

a) $\$100{,}000 = Ra_{\overline{10}|4.5\%}$ c) $\$100{,}000 = Ra_{\overline{20}|9\%}$ e) none of these

b) $\$100{,}000 = Rs_{\overline{10}|4.5\%}$ d) $\$100{,}000 = Rs_{\overline{20}|9\%}$

6. A man buys an automobile for a cash price of $14,800 and pays $2200 down. The balance is financed at 11%(12) for 2 years. What is the monthly payment?

a) $689.80
b) $587.26
c) $554.13
d) $471.76
e) none of these

7. A couple's income can qualify them for a monthly loan payment of $580 (plus escrow costs) on a 30-year loan of $83,000. They have the down payment and closing costs saved. At what rate will the mortgage loan have to be calculated in order for them to get the loan?

a) 7.49%(12)
b) 0.624%(12)
c) 7.0%(12)
d) 0.583%(12)
e) none of these

8. A saver starts an account by depositing $65 on March 31, 1992 in a bank paying 5%(12). Including the deposit on October 31, 2005, how many deposits will have been made?

a) 162
b) 163
c) 164
d) 165
e) none of these

Calculation Problems

9. A condominium can be purchased for $15,000 down and $825 a month for 20 years. If the loan is based on a rate of 11%(12), what is the cash price of the condo?

10. On July 1, 2003, a woman had $40,000 in an account paying 8%(4). She withdrew $1800 every quarter beginning on October 1, 2003. How many full-size withdrawals could she make and what will be the smaller last withdrawal at 3 months after the last $1800?

11. Every 6 months a small businessman put $2000 in an account paying 8%(2). If the first deposit were made on January 1, 1999 and the last one on July 1, 2004, how much will be in the account when the last deposit is made?

12. A retailer wants to accumulate $28,500 to renovate his building. How many regular payments of $1200 into a savings program paying 9%(12) are necessary to reach the investment goal? Give the number of payments and the size of the concluding payment if needed.

13. A family buys an automobile with a cash price of $13,800 and pays $4000 down. The balance is financed with a 9% add-on loan at a bank. If the loan is paid monthly for 36 months, what is the annual percentage rate (APR) entered on the contract with the bank?

14. If the family in problem 13 paid the loan off early with 8 payments remaining and the interest is refunded on the basis of the Rule of 78's, how much refund did they receive from the bank?

Theory and Extension Problems

15. A $24,000 loan is to be repaid with monthly payments for 4 years. If the interest rate is 6%(12) and each year the payment is increased by $50, what is the size of the first year's payments?

16. ◆ Four individuals each borrow $10,000 for 5 years. Allan's loan is a simple interest loan at 5% with the principle and interest due at the end of the term. Bob's loan is a compound discount loan at 5%(4) with the interest paid up front. Carlie's loan is a compound interest loan at 5%(4), but she pays only the interest each quarter and repays the principal at the end of the term. Denise's loan is a compound interest loan with equal payments at the end of each quarter. Find the total interest paid by the four borrowers.

17. A $15,000 car loan is financed over 4 years with monthly payments based on 8%(12). The borrower missed several payments due to an accident, but with an insurance settlement the last 33 payments and the missed payments are paid off with $12,376.93 on a payment date. How many payments were missed?

18. If at a rate i the future value of n \$1 payments is \$A and the future value of $3n$ \$1 payments is \$B, give the rate i as a function of A and B.

19. We say that the present value of an annuity, the payments in an annuity, and the future value of an annuity are all equivalent amounts. Explain what the above statement means and prove that the statement is accurate mathematically.

20. Establish the future value formula for an annuity with n equal payments of value R at compound discount rate d per period.

21. Establish the present value formula for an annuity with n equal payments of value R at compound discount rate d per period.

Answers for Chapter 4 Sample Test Questions

1. c	**2.** a	**3.** d	**4.** c	**5.** e
6. b	**7.** a	**8.** c		

9. Cost = loan + down payment = \$79,927.27 + \$15,000 = \$94,927.27

10. 29 full withdrawals of \$1800 plus a smaller last withdrawal of \$1231.93

11. 12 payments of \$2000 with a future value of \$30,051,61

12. 21 full payments of \$1200 plus a smaller last payment of \$1113.25

13. Loan amount = \$12,446, requiring 36 payments of \$345.72

$9800 = 345.72a_{\overline{36}|i} \to i = 1.3536\%$, therefore the APR = 16.24%(12)

14. $\frac{rebate}{2646} = \frac{1+2+\cdots+8}{1+2+\cdots+36} = \frac{36}{666} \to$ Rebate = \$143.03

15. $24,000 = Ra_{\overline{12}|\frac{1}{2}\%}\left[1+(1.005)^{-12}+(1.005)^{-24}+(1.005)^{-36}\right]+$

$a_{\overline{12}|\frac{1}{2}\%}\left[50(1.005)^{-12}+100(1.005)^{-24}+150(1.005)^{-36}\right]$

$24,000 - 261.16a_{\overline{12}|\frac{1}{2}\%} = Ra_{\overline{12}|\frac{1}{2}\%}(3.664735927)$

$5720.90 = Ra_{\overline{12}|\frac{1}{2}\%} \to R = \$492.38$

16.

Allan: S = 10,000(1 + .05(5)) = 12,000	I = 2500.00
Bob: P = $10,000(1 - .0125)^{20}$ = 7775.75	I = 2224.25
Carlie: I = .0125(10,000)(20) = 2500	I = 2500.00 (I = 568.20(20) – 10,000)
Denise: $10,000 = a_{\overline{20}|1.25\%} \to R = 568.20$	I = 1364.00 → T = 8588.25

17. The loan payment is computed from $15,000 = Ra_{\overline{48}|\frac{2}{3}\%} \to R = \$366.20$ *rounded up*

The PV of 32 payments = $366.29a_{\overline{32}|\frac{2}{3}\%} = \$10,521.35$, leaving \$1855.58 as the future value of the missed payments and the current payment. (This delays using an annuity due until Chapter 5.) $1855.58 = 366.20s_{\overline{n}|\frac{2}{3}\%} \to n = 5$ Therefore 4 were missed.

18. $A = s_{\overline{n}|i}$ *and* $B = s_{\overline{3n}|i}$ so $iA = (1+i)^n - 1$ and $iB = (1+i)^{3n} - 1$

$iA + 1 = (1+i)^n$ and $iB + 1 = (1+i)^{3n} \to$ $iB + 1 = (iA+1)^3$

$iB + 1 = i^3A^3 + 3i^2A^2 + 3iA + 1$

$B = i^2A^3 + 3iA^2 + 3A$

This is a quadratic equation in i. $A^3i^2 + 3A^2i + 3A - B = 0$

The positive root gives $i = -\frac{3}{2A} + \frac{1}{2A^2}\sqrt{4AB - 3A^2}$, provided 4B > 3A + 3

19. Use our basic annuity and compound interest formulas to obtain the following.

$A_n = Ra_{\overline{n}|i} = R\left(\frac{1-(1+i)^{-n}}{i}\right) = \sum_{k=1}^{n} R(1+i)^{-k}$. Hence A_n is equivalent to n equal pmts each moved back to the current date.

$S_n = Rs_{\overline{n}|i} = R\left(\frac{(1+i)^n - 1}{i}\right) = \sum_{k=1}^{n} R(1+i)^{n-k}$. Hence S_n is equivalent to n equal pmts each moved forward to the future date.

Hence A_n and S_n are both equivalent to the payments and hence equivalent to each other.

Mathematically $A_n(1+i)^n$ must = S_n.

$\therefore\ R\left(\frac{1-(1+i)^{-n}}{i}\right)(1+i)^n$ must $= R\left(\frac{(1+i)^n - 1}{i}\right)$ which it does.

20. Either use the future value formula $S = P(1-d)^{-n}$ and the geometric series as were used for annuities with compound interest on page 119 or use the equivalent compound interest rate $i = d/(1-d)$ and substitute into our S_n formula for i.

$$S_n = R\left[\frac{(1-d)^{1-n} - (1-d)}{d}\right]$$

21. Either use the present value formula $P = S(1-d)^n$ and the geometric series as were used for annuities with compound interest on page 119 or use the equivalent compound interest rate $i = d/(1-d)$ and substitute into our A_n formula for i.

$$A_n = R\left[\frac{(1-d) - (1-d)^{n+1}}{d}\right]$$

Chapter 5 – Other Annuities Certain

If the students have a good understanding of ordinary annuities, the extension of their formulas to other annuities progresses quite easily. All these other annuities could be studied without ever giving them names, but the names serve a useful role in placing monies on the time line. We use the analysis of a word problem to help them make decisions about the type of annuity and the location of the "bundle." You may have your own word for the piece of money equivalent to a sequence of payments, but we like the *money* sound of "bundle." A regular series of questions for each in-class example can establish all the pertinent information: is it present or future, is it ordinary or due, what is the number of payments, interval of deferment or forbearance?

As in ordinary annuities, a good time line will enhance the student's understanding of the problem and will lead the student to the proper context and, thus, to the proper procedure to solve the exercise. Without understanding the time line, the student often loses his way, and no calculator or computer will consistently carry him to a correct solution. This chapter extends ordinary annuities into one or more of the following contexts.

a. Annuities where we are given the payments, rate, and term and must find the equivalent bundle which can be located (valued) anywhere.

b. Annuities where we are given the bundle located (valued) somewhere, plus the rate, and must find the payment, or the term and final payment.

c. Annuities for which the number of times per year the rate is compounded does not equal the number of payments per year.

d. Annuities that have an infinite term and thus are perpetuities.

e. Combinations of the above, which we call life annuities, for example, a person pays into a retirement plan to accumulate a bundle, lets the bundle sit (mature), and then withdraws money from this retirement plan as an annuity or as a perpetuity.

Savings accounts, like Exercise 21 on page 154, pose an interesting context for interpreting the meaning of making payments at the beginning of each interest period. Even though we typically associate payments at the beginning of the interest periods with annuities due, all savings accounts are started by making a payment followed by an interest period and another payment, and so on. Whether a savings account is treated as the future value of an annuity due depends on where we choose to locate the accumulated value of the fund. The answer key reflects the annuity due context for Exercise 15 so that the saver makes 36 deposits of $500 and one deposit of $459.61, then lets the fund accrue interest for one month to have exactly $20,000. An ordinary annuity interpretation, where the future value is at the last payment, requires 36 deposits of $500 and a final deposit of $542.60. We could also make 37 deposits of $500 and let the fund grow for one month to give a future value of $20,040.56. This then is also the future value of an annuity due.

We have chosen to use compound interest for part of a period for general case annuities. Using simple interest for part of a period requires the method of equivalent payments instead of equivalent interest rates. The equivalent payment method is time-consuming, cumbersome, and more difficult to teach. Experience has shown that the equivalent payment method coupled with simple interest for part of a period produces answers that are nearly the same as using equivalent rates. The method of equivalent rates has proved to be valuable for student's understanding effective rate calculations as well as doing the general case exercises.

Chapter 5 Sample Test Questions – Other Annuities Certain

Multiple Choice

Starting July 1, 1970 a man paid $75 per month into a retirement fund which paid 5%(4). On the date his last payment was made, December 1, 1999, the fund was invested at 9%(4). How much can he expect as annual income beginning on June 1, 2010 for the following 30 years? How much can he expect as annual income if he never uses any of the principal? Assume compound interest for part of a period.

1. How many payments did the man make into the fund?

a) 118 c) 353 e) none of these
b) 119 d) 354

2. To find the value of the fund when the rate changes we would use what formula?

a) $S_n = Rs_{\overline{n}|i}$ c) $S_n = Rs_{\overline{n}|i}(1+i)$ e) none of these
b) $A_n = Ra_{\overline{n}|i}$ d) $A_n = Ra_{\overline{n}|i}(1+i)$

3. How many periods does the fund sit before income starts?

a) 38 c) 42 e) none of these
b) 39 d) 43

4. What is the rate per payment period during the man's savings years?

a) 1.25% c) 5% e) none of these
b) .4149% d) 3.797%

5. Knowing the value of the fund on 6/1/2010, which formula would be used to find the annual income that starts on that date and lasts for 30 years?

a) $A_n = Ra_{\overline{n}|i}$
c) $A_n = Ra_{\overline{n}|i}(1+i)$
e) none of these
b) $S_n = Rs_{\overline{n}|i}$
d) $S_n = Rs_{\overline{n}|i}(1+i)$

6. What is the rate per payment period during the years of income?

a) 2.25%
c) 0.558%
e) none of these
b) 9%
d) 9.31%

7. Knowing the value of the fund on 6/1/2010, which formula would be used to find the annual income that starts on that date and goes on in perpetuity?

a) $R = .0225A_\infty$
c) $R = .022A_\infty$
e) none of these
b) $R = [(1.0225)^4 - 1]A_\infty$
d) $R = [1 - (.977995)^4]A_\infty$

Calculation Problems

8. On June 1, 1995, Jerry wants to set aside enough to provide $900 a month from June 1, 2000 to May 1, 2010 inclusive. If money earns 8%(12), what would this plan have cost on June 1, 1995?

9. On July 1, 2003 Rachel began making $500 deposits in a savings and loan association that pays 6%(4). She continues to make deposits once every 3 months until October 1, 2005, at which time she makes the last one. After interest is credited on December 31, 2005 how much will be in her account?

10. Find the value of a perpetuity which provides a payment of $2500 at the end of each year, if the interest rate is 7% converted: a) annually b) quarterly.

11. Provident Savings and Loan Association pays 5%(4). Interest dates are March 31, June 30, September 30, and December 31. Deposits in by the 10th of the month earn interest from the first. On July 4, 2002, Stefan opens a Freedom account with a deposit of $550. He continues to make quarterly deposits of $550 through July 4, 2004. Find the amount in his account on December 31, 2010.

12. The premium on a life insurance policy is $32 per month, payable in advance. Find the equivalent annual premium if the insurance company charges 6%(12) and the annual premium is paid at the beginning of the year.

13. A retired gentleman gives an endowment of $225,000 to provide an annual scholarship in perpetuity. If the endowment earns 9%(4) and the first scholarship is due a year after the gift, how much will the yearly scholarship be?

14. Find the value of an annuity that will provide an income of $2400 per quarter for 7 years if money can be invested at 8%(12).

15. Find the value at the last payment of a savings program in which payments of $200 per month were made for 8 years. The program pays 8%(4), and compound interest is paid on money left for part of a period.

16. Find the value of a savings program requiring $500 per quarter for 10 years if the force of interest $\delta = .055$.

17. If monthly income is withdrawn from a $350,000 trust fund invested at 6%(∞), what will the income be if taken over 25 years?

Theory and Extension

18. Show that the future value of a general case annuity with m interest periods per year, p payments per year of $1 each, and a total of n payments can be written as the following quotient. $\dfrac{s_{\overline{kn}|i}}{s_{\overline{k}|i}}$, where $k = \dfrac{m}{p}$ and $i = \dfrac{i(m)}{m}$

19. Show that the present value of a general case annuity with m interest periods per year, p payments per year of $1 each, and a total of n payments can be written as the following quotient. $\dfrac{a_{\overline{kn}|i}}{s_{\overline{k}|i}}$, where $k = \dfrac{m}{p}$ and $i = \dfrac{i(m)}{m}$

20. What annual effective rate of compound interest does a perpetuity pay if $200,000 produces $20,000 of income every 2 years?

21. If $\ddot{a}_{\overline{n}|i} = 68.112$ and $\ddot{a}_{\overline{n+1}|i} = 68.827$, find the effective rate of compound interest.

22. Show that $\dfrac{1}{d(m)} - \dfrac{1}{i(m)} = \dfrac{1}{m}$.

23. ◆ Suppose an endowment provides an ordinary perpetuity that pays $\$R$ per year at rate i per year. The father in a family receives the first n payments, the son the next n payments and the grandson all payments after that. If the father's share of the original endowment is 40%, what is the grandson's share?

24. ◆ Assume that the present value of the following three annuities are equal and find n.

a. A perpetuity-immediate paying $1 per year, at an annual effective rate of 7.25%

b. A 50-year annuity-immediate paying $1 per year at an annual effective rate of j%

c. An n-year annuity-immediate paying $1 per year at an annual effective rate of $j - 1$%

25. ◆ At an annual effective rate $i > 0\%$, the present value of a perpetuity paying $10 at the end of each 3-year period is 32. The first payment is at the end of year 6. A second perpetuity invested at the same annual effective rate i has present value X and pays $1 at the end of each 4-month period. Assume that the investment pays compound interest for part of a period and find X.

Extension of Calculator Problems

26. **a.** Program one expression that will relate all of the following symbols:

PV (or A), which is the present value;

FV (or S), which is the future value;

PMT (or R), which is the payment per period;

i, which is the rate per period as a decimal;

n, which is the number of payments;

m, which is the deferred period; and

p, which is the forborne period.

b. If $m = p = 0$,show you are solving an *ordinary annuity*.

c. If $m = -1$ and $p = 0$ and $FV = 0$, show you are solving ________________.

d. If $m = 0$ and $p = 1$ and $PV = 0$, show you are solving ________________.

e. If $m > 0$ and $p = 0$ and $FV = 0$, show you are solving ________________.

f. If $m = 0$ and $p > 0$ and $PV = 0$, show you are solving ________________.

g. If $n = 0$, you are solving ________________.