1.102 My Uncle Ben was born in Pogrebishte, a village near Kiev, and he claimed that his birthday was February 29, 1900. I told him that this could not be, for 1900 was not a leap year. Why was I wrong?
Solution. Even though 1900 was not a leap year in America, it was a leap year in Russia, which did not adopt the Gregorian calendar until after the Russian Revolution.

Exercises for Chapter 2

2.1 True or false with reasons.

(i) If $S \subseteq T$ and $T \subseteq X$, then $S \subseteq X$.
Solution. True.

(ii) Any two functions $f: X \to Y$ and $g: Y \to Z$ have a composite $f \circ g: X \to Z$.
Solution. False.

(iii) Any two functions $f: X \to Y$ and $g: Y \to Z$ have a composite $g \circ f: X \to Z$.
Solution. True.

(iv) For every set X, we have $X \times \varnothing = \varnothing$.
Solution. True.

(v) If $f: X \to Y$ and $j: \operatorname{im} f \to Y$ is the inclusion, then there is a surjection $g: X \to \operatorname{im} f$ with $f = j \circ g$.
Solution. True.

(vi) If $f: X \to Y$ is a function for which there is a function $g: Y \to X$ with $f \circ g = 1_Y$, then f is a bijection.
Solution. False.

(vii) The formula $f(\frac{a}{b}) = (a+b)(a-b)$ is a well-defined function $\mathbb{Q} \to \mathbb{Z}$.
Solution. False.

(viii) If $f: \mathbb{N} \to \mathbb{N}$ is given by $f(n) = n + 1$ and $g: \mathbb{N} \to \mathbb{N}$ is given by $g(n) = n^2$, then the composite $g \circ f$ is $n \mapsto n^2(n+1)$.
Solution. False.

(ix) Complex conjugation $z = a + ib \mapsto \overline{z} = a - ib$ is a bijection $\mathbb{C} \to \mathbb{C}$.
Solution. True.

2.2 If A and B are subsets of a set X, prove that $A - B = A \cap B'$, where $B' = X - B$ is the complement of B.
Solution. This is one of the beginning set theory exercises that is so easy it is difficult; the difficulty is that the whole proof turns on the meaning of the words "and" and "not." For example, let us prove that $A - B \subseteq A \cap B'$. If $x \in A - B$, then $x \in A$ and $x \notin B$; hence, $x \in A$ and $x \in B'$, and so $x \in A \cap B'$. The proof is completed by proving the reverse inclusion.

2.3 Let A and B be subsets of a set X. Prove the ***de Morgan laws***

$$(A \cup B)' = A' \cap B' \quad \text{and} \quad (A \cap B)' = A' \cup B',$$

where $A' = X - A$ denotes the complement of A.
Solution. Absent.

2.4 If A and B are subsets of a set X, define their ***symmetric difference*** (see Figure 2.5) by

$$A + B = (A - B) \cup (B - A).$$

(i) Prove that $A + B = (A \cup B) - (A \cap B)$.
Solution. Absent.

(ii) Prove that $A + A = \varnothing$.
Solution. Absent.

(iii) Prove that $A + \varnothing = A$.
Solution. Absent.

(iv) Prove that $A + (B + C) = (A + B) + C$ (see Figure 2.6).
Solution. Show that each of $A + (B + C)$ and $(A + B) + C$ is described by Figure 2.6.

(v) Prove that $A \cap (B + C) = (A \cap B) + (A \cap C)$.
Solution. Absent.

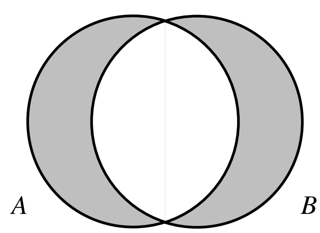

Figure 2.5 Symmetric Difference

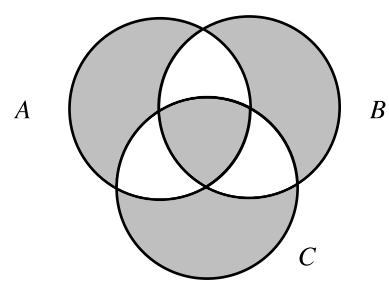

Figure 2.6 Associativity

2.5 Let A and B be sets, and let $a \in A$ and $b \in B$. Define their ***ordered pair*** as follows:

$$(a, b) = \{a, \{a, b\}\}.$$

If $a' \in A$ and $b' \in B$, prove that $(a', b') = (a, b)$ if and only if $a' = a$ and $b' = b$.

Solution. The result is obviously true if $a' = a$ and $b' = b$.

For the converse, assume that

$$\{a\{a, b\}\} = \{a'\{a', b'\}\}$$

There are two cases:

$$a = a' \text{ and } \{a, b\} = \{a', b'\};$$
$$a = \{a', b'\} \text{ and } \{a, b\} = a'.$$

If $a = a'$, we have $\{a, b\} = \{a', b'\} = \{a, b'\}$. Therefore,

$$\{a, b\} - \{a\} = \{a, b'\} - \{a\}.$$

If $a = b$, the left side is empty, hence the right side is also empty, and so $a = b'$; therefore, $b = b'$. If $a \neq b$, the the left side is $\{b\}$, and so the right side is nonempty and is equal to $\{b'\}$. Therefore, $b = b'$, as desired.

In the second case, $a = \{a', b'\} = \{\{a, b\}b'\}$. Hence,

$$a \in \{a, b\}$$

and

$$\{a, b\} \in \{\{a, b\}, b'\} = a,$$

contradicting the axiom $a \in x \in a$ being false. Therefore, this case cannot occur.

2.6 Let $\Delta = \{(x, x) : x \in \mathbb{R}\}$; thus, Δ is the line in the plane which passes through the origin and which makes an angle of 45° with the x-axis.

(i) If $P = (a, b)$ is a point in the plane with $a \neq b$, prove that Δ is the perpendicular bisector of the segment PP' having endpoints $P = (a, b)$ and $P' = (b, a)$.

Solution. The slope of Δ is 1, and the slope of PP' is $(b - a)/(a - b) = -1$. Hence, the product of the slopes is -1, and so Δ is perpendicular to the PP'. The midpoint of PP' is $M = (\frac{1}{2}(a + b), \frac{1}{2}(a + b))$, which lies on Δ, and

$$|PM| = \sqrt{[a - \tfrac{1}{2}(a + b)]^2 + [b - \tfrac{1}{2}(a + b)]^2} = |MP'|.$$

(ii) If $f: \mathbb{R} \to \mathbb{R}$ is a bijection whose graph consists of certain points (a, b) [of course, $b = f(a)$], prove that the graph of f^{-1} is

$$\{(b, a) : (a, b) \in f\}.$$

Solution. By definition, $f^{-1}(b) = a$ if and only if $b = f(a)$. Hence, the graph of f^{-1} consists of all ordered pairs

$$(b, f^{-1}(b)) = (b, a) = (f(a), a).$$

2.7 Let X and Y be sets, and let $f: X \to Y$ be a function.

(i) If S is a subset of X, prove that the restriction $f|S$ is equal to the composite $f \circ i$, where $i: S \to X$ is the inclusion map.

Solution. Both $f|S$ and $f \circ i$ have domain S and target Y. If $s \in S$, then $(f \circ i)(s) = f(s) = (f|S)(s)$. Therefore, $f|S = f \circ i$, by Proposition 2.2.

(ii) If $\operatorname{im} f = A \subseteq Y$, prove that there exists a surjection $f': X \to A$ with $f = j \circ f'$, where $j: A \to Y$ is the inclusion.

Solution. For each $x \in X$, define $f'(x) = f(x)$. Thus, f' differs from f only in its target.

2.8 If $f: X \to Y$ has an inverse g, show that g is a bijection.
Solution. We are told that $f \circ g = 1_Y$ and $g \circ f = 1_X$. Therefore, g has an inverse, namely, f, and so g is a bijection.

2.9 Show that if $f: X \to Y$ is a bijection, then it has exactly one inverse.
Solution. Let $g: Y \to X$ and $h: Y \to X$ both be inverses of f. Then

$$h = h1_Y = h(fg) = (hf)g = 1_X g = g.$$

2.10 Show that $f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 3x + 5$, is a bijection, and find its inverse.
Solution. The function g, defined by $g(x) = \frac{1}{3}(x - 5)$, is the inverse of f, and so f is a bijection. (Alternatively, one could prove that f is a bijection by showing directly that it is injective and surjective.)

2.11 Determine whether $f: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, given by

$$f(a/b, c/d) = (a + c)/(b + d)$$

is a function.
Solution. f is not a function: $\frac{1}{2} = \frac{2}{4}$ and $\frac{2}{6} = \frac{1}{3}$, but

$$f(\tfrac{1}{2}, \tfrac{2}{6}) = \tfrac{3}{8} \neq \tfrac{3}{7} = f(\tfrac{2}{4}, \tfrac{1}{3}).$$

2.12 Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be finite sets, where the x_i are distinct and the y_j are distinct. Show that there is a bijection $f : X \to Y$ if and only if $|X| = |Y|$; that is, $m = n$.
Solution. The hint is essentially the solution. If f is a bijection, there are m distinct elements $f(x_1), \ldots, f(x_m)$ in Y, and so $m \le n$; using the bijection f^{-1} in place of f gives the reverse inequality $n \le m$.

2.13 (***Pigeonhole Principle***)

(i) If X and Y are finite sets with the same number of elements, show that the following conditions are equivalent for a function $f : X \to Y$:

(i) f is injective;

(ii) f is bijective;

(iii) f is surjective.

Solution. Assume that X and Y have n elements. If f is injective, then there is a bijection from X to $\operatorname{im} f \subseteq Y$. Exercise 2.12 gives $|\operatorname{im} f| = n$. It follows that $\operatorname{im} f = Y$, for there can be no elements in Y outside of $\operatorname{im} f$, lest Y have more than n elements. Any bijection is surjective, and so it remains to show that if f is surjective, then it is injective. If $Y = \{y_1, \ldots, y_n\}$, then for each i, there exists $x_i \in X$ with $f(x_i) = y_i$. Were f not injective, there would be i and $x \in X$ with $x \ne x_i$ and $f(x) = f(x_i)$. This gives $n + 1$ elements in X, a contradiction.

(ii) Suppose there are 11 pigeons, each sitting in some pigeonhole. If there are only 10 pigeonholes, prove that there is a hole containing more than one pigeon.

Solution. Suppose each hole has at most one pigeon in it. If P is the set of pigeons and H is the set of holes, define $f : P \to H$ by $f : \text{pigeon} \mapsto h$, where h is the hole containing it. Since each hole contains at most one pigeon, $f(p_1) = f(p_2)$ implies $p_1 = p_2$, where p_1, p_2 are pigeons. Thus, f is an injection. By part (1), f is a bijection, giving the contradiction $11 = 10$.

2.14 Let $f : X \to Y$ and $g : Y \to Z$ be functions.

(i) If both f and g are injective, prove that $g \circ f$ is injective.

Solution. If $(g \circ f)(x) = (g \circ f)(x')$, then $g(f(x)) = g(f(x'))$. Since g is injective, $f(x) = f(x')$; since f is injective, $x = x'$. Hence, $g \circ f$ is injective.

(ii) If both f and g are surjective, prove that $g \circ f$ is surjective.

Solution. Let $z \in Z$. Since g is surjective, there is $y \in Y$ with $g(y) = z$; since f is surjective, there is $x \in X$ with $f(x) = y$. It follows that $(g \circ f)(x) = g(f(x)) = g(y) = z$, and so $g \circ f$ is surjective.

(iii) If both f and g are bijective, prove that $g \circ f$ is bijective.
Solution. By the first two parts, $g \circ f$ is both injective and surjective

(iv) If $g \circ f$ is a bijection, prove that f is an injection and g is a surjection.
Solution. If $h = (gf)^{-1}$, then $(hg)f = 1$ and $g(fh) = 1$. By Lemma 2.9, the first equation gives f an injection while the second equation gives g a surjection.

2.15 **(i)** If $f : (-\pi/2, \pi/2) \to \mathbb{R}$ is defined by $a \mapsto \tan a$, then f has an inverse function g; indeed, $g = \arctan$.
Solution. By calculus, $\arctan(\tan a) = a$ and $\tan(\arctan x) = x$.

(ii) Show that each of $\arcsin x$ and $\arccos x$ is an inverse function (of $\sin x$ and $\cos x$, respectively) as defined in this section.
Solution. Each of the other inverse trig functions satisfies equations analogous to $\sin(\arcsin x) = x$ and $\arcsin(\sin x) = x$, which shows that they are inverse functions as defined in this section.

2.16 **(i)** Let $f : X \to Y$ be a function, and let $\{S_i : i \in I\}$ be a family of subsets of X. Prove that

$$f\Big(\bigcup_{i \in I} S_i\Big) = \bigcup_{i \in I} f(S_i).$$

Solution. Absent.

(ii) If S_1 and S_2 are subsets of a set X, and if $f : X \to Y$ is a function, prove that $f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2)$. Give an example in which $f(S_1 \cap S_2) \neq f(S_1) \cap f(S_2)$.
Solution. Absent.

(iii) If S_1 and S_2 are subsets of a set X, and if $f : X \to Y$ is an injection, prove that $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$.
Solution. Absent.

2.17 Let $f : X \to Y$ be a function.

(i) If $B_i \subseteq Y$ is a family of subsets of Y, prove that

$$f^{-1}\Big(\bigcup_i B_i\Big) = \bigcup_i f^{-1}(B_i) \quad \text{and} \quad f^{-1}\Big(\bigcap_i B_i\Big) = \bigcap_i f^{-1}(B_i).$$

Solution. Absent.

(ii) If $B \subseteq Y$, prove that $f^{-1}(B') = f^{-1}(B)'$, where B' denotes the complement of B.

Solution. Absent.

2.18 Let $f: X \to Y$ be a function. Define a relation on X by $x \equiv x'$ if $f(x) = f(x')$. Prove that $\equiv$ is an equivalence relation. If $x \in X$ and $f(x) = y$, the equivalence class $[x]$ is denoted by $f^{-1}(y)$; it is called the ***fiber*** over y.
Solution. Absent.

2.19 Let $X = \{\text{rock, paper, scissors}\}$. Recall the game whose rules are: paper dominates rock, rock dominates scissors, and scissors dominates paper. Draw a subset of $X \times X$ showing that domination is a relation on X.
Solution. Absent.

2.20 **(i)** Find the error in the following argument which claims to prove that a symmetric and transitive relation R on a set X must be reflexive; that is, R is an equivalence relation on X. If $x \in X$ and $x \; R \; y$, then symmetry gives $y \; R \; x$ and transitivity gives $x \; R \; x$.

Solution. There may not exist $y \in X$ with $x \sim y$.

(ii) Give an example of a symmetric and transitive relation on the closed unit interval $X = [0, 1]$ which is not reflexive.

Solution. Define

$$R = \{(x, x) : 0 \le x \le \tfrac{1}{2}\}.$$

Now R is the identity on $Y = [0, \frac{1}{2}]$, so that it is symmetric and transitive. However, R does not contain the diagonal of the big square $X \times X$, and so R is not a reflexive relation on X. For example, $1 \not\sim 1$.

2.21 True or false with reasons.

(i) The symmetric group on n letters is a set of n elements.

Solution. False.

(ii) If $\sigma \in S_6$, then $\sigma^n = 1$ for some $n \ge 1$.

Solution. True.

(iii) If $\alpha, \beta \in S_n$, then $\alpha\beta$ is an abbreviation for $\alpha \circ \beta$.

Solution. True.

(iv) If α, β are cycles in S_n, then $\alpha\beta = \beta\alpha$.

Solution. False.

(v) If σ, τ are r-cycles in S_n, then $\sigma\tau$ is an r-cycle.

Solution. False.

(vi) If $\sigma \in S_n$ is an r-cycle, then $\alpha\sigma\alpha^{-1}$ is an r-cycle for every $\alpha \in S_n$.

Solution. True.

(vii) Every transposition is an even permutation.

Solution. False.

(viii) If a permutation α is a product of 3 transpositions, then it cannot be a product of 4 transpositions.

Solution. True.

(ix) If a permutation α is a product of 3 transpositions, then it cannot be a product of 5 transpositions.

Solution. False.

(x) If $\sigma\alpha\sigma^{-1} = \omega\alpha\omega^{-1}$, then $\sigma = \omega$.

Solution. False.

2.22 Find $\operatorname{sgn}(\alpha)$ and α^{-1}, where

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Solution. In cycle notation, $\alpha = (1\,9)(2\,8)(3\,7)(4\,6)$. Thus, α is even, being the product of four transpositions. Moreover, being a product of disjoint transpositions, $\alpha = \alpha^{-1}$.

2.23 If $\sigma \in S_n$ fixes some j, where $1 \le j \le n$ (that is, $\sigma(j) = j$), define $\sigma' \in S_X$, where $X = \{1, \ldots, \widehat{j}, \ldots, n\}$, by $\sigma'(i) = \sigma(i)$ for all $i \ne j$. Prove that

$$\operatorname{sgn}(\sigma') = \operatorname{sgn}(\sigma).$$

Solution. One of the cycles in the complete factorization of σ is the 1-cycle (j). Hence, if there are t cycles in the complete factorization of σ, then there are $t - 1$ cycles in the complete factorization of σ'. Therefore,

$$\operatorname{sgn}(\sigma') = (-1)^{[n-1]-[t-1]} = (-1)^{n-t} = \operatorname{sgn}(\sigma).$$

2.24 **(i)** If $1 < r \le n$, prove that there are

$$\tfrac{1}{r}[n(n-1)\cdots(n-r+1)]$$

r-cycles in S_n.

Solution. In the notation $(i_1\ i_2\ \ldots\ i_r)$, there are n choices for i_1, $n-1$ choices for $i_2, \ldots, n-(r-1) = n-r+1$ choices for i_r. We conclude that there are $n(n-1)\cdots(n-r+1)$ such notations. However, r such notations describe the same cycle:

$$(i_1\ i_2\ \ldots\ i_r) = (i_2\ i_3\ \ldots\ i_1) = \cdots = (i_r\ i_1\ \ldots\ i_{r-1}).$$

Therefore, there are $\frac{1}{r}[n(n-1)\cdots(n-r+1)]$ r-cycles in S_n.

(ii) If $kr \le n$, where $1 < r \le n$, prove that the number of permutations $\alpha \in S_n$, where α is a product of k disjoint r-cycles, is

$$\frac{1}{k!}\frac{1}{r^k}[n(n-1)\cdots(n-kr+1).]$$

Solution. Absent.

2.25 **(i)** If α is an r-cycle, show that $\alpha^r = (1)$.

Solution. If $\alpha = (i_0 \ldots i_{r-1})$, then the proof of Lemma 2.25(ii) shows that $\alpha^k(i_0) = i_k$, where the subscript is read $\mod r$. Hence, $\alpha^r(i_0) = i_0$. But the same is true if we choose the notation for α having any of the other i_j as the first entry.

(ii) If α is an r-cycle, show that r is the least positive integer k such that $\alpha^k = (1)$.

Solution. Use Proposition 2.24. If $k < r$, then $\alpha^k(i_0) = i_k \ne i_0$, so that $\alpha^k \ne 1$.

2.26 Show that an r-cycle is an even permutation if and only if r is odd.

Solution. In the proof of Proposition 2.35, we showed that any r-cycle α is a product of $r-1$ transpositions. The result now follows from Proposition 2.39, for $\operatorname{sgn}(\alpha) = (-1)^{r-1} = -1$.

2.27 Given $X = \{1, 2, \ldots, n\}$, let us call a permutation τ of X an ***adjacency*** if it is a transposition of the form $(i\ i+1)$ for $i < n$. If $i < j$, prove that $(i\ j)$ is a product of an odd number of adjacencies.

Solution. We prove the result by induction on $j - i \ge 1$. The base step is clear, for then τ is already an adjacency, and so it is a product of 1 adjacency. For the inductive step, we have

$$\tau = (i\ j) = (i\ i+1)(i+1\ j)(i\ i+1),$$

by Proposition 2.32, for $j - i \ge 2$ implies $j \ne i+1$. By induction, $(i+1\ j)$ is a product of an odd number of adjacencies, and so τ is also such a product.

2.28 Define $f\colon \{0, 1, 2, \ldots, 10\} \to \{0, 1, 2, \ldots, 10\}$ by

$$f(n) = \text{the remainder after dividing } 4n^2 - 3n^7 \text{ by } 11.$$

(i) Show that f is a permutation.

Solution. Here is the two-rowed notation for f:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 0 & 1 & 6 & 9 & 5 & 3 & 10 & 2 & 8 & 4 & 7 \end{pmatrix}.$$

It follows that f is a permutation. (The reader is expected to use his knowledge of congruences to facilitate the calculations.)

(ii) Compute the parity of f.
Solution. $f = (2\ 6\ 10\ 7)(3\ 9\ 4\ 5)$. Since 4-cycles are odd, f is even.

(iii) Compute the inverse of f.
Solution. $f^{-1} = (7\ 10\ 6\ 2)(5\ 4\ 9\ 3)$.

2.29 **(i)** Prove that α is regular if and only if α is a power of an n-cycle.
Solution. If $\alpha = (a_1\ a_2 \cdots a_k)(b_1\ b_2 \cdots b_k) \cdots (c_1\ c_2 \cdots c_k)$ is a product of disjoint k-cycles involving all the numbers between 1 and n, show that $\alpha = \beta^k$, where

$$\beta = (a_1\ b_1 \cdots z_1\ a_2\ b_2 \cdots z_2\ \ldots a_k\ b_k \cdots z_k).$$

(ii) Prove that if α is an r-cycle, then α^k is a product of (r, k) disjoint cycles, each of length $r/(r, k)$.
Solution. Absent.

(iii) If p is a prime, prove that every power of a p-cycle is either a p-cycle or (1).
Solution. Absent.

(iv) How many regular permutations are there in S_5? How many regular permutations are there in S_8?
Solution. Absent.

2.30 **(i)** Prove that if α and β are (not necessarily disjoint) permutations that commute, then $(\alpha\beta)^k = \alpha^k\beta^k$ for all $k \geq 1$.
Solution. We prove first, by induction on $k \geq 1$, that $\beta\alpha^k = \alpha^k\beta$. The base step is true because α and β commute. For the inductive step,

$$\begin{aligned} \beta\alpha^{k+1} &= \beta\alpha^k\alpha \\ &= \alpha^k\beta\alpha \qquad \text{(inductive hypothesis)} \\ &= \alpha^k\alpha\beta \\ &= \alpha^{k+1}\beta. \end{aligned}$$

We now prove the result by induction on $k \geq 1$. The base step is obviously true. For the inductive step,

$$\begin{aligned} (\alpha\beta)^{k+1} &= \alpha\beta(\alpha\beta)^k \\ &= \alpha\beta\alpha^k\beta^k \qquad \text{(inductive hypothesis)} \\ &= \alpha\alpha^k\beta\beta^k \qquad \text{(proof above)} \\ &= \alpha^{k+1}\beta^{k+1}. \end{aligned}$$

(ii) Give an example of two permutations α and β for which $(\alpha\beta)^2 \neq \alpha^2\beta^2$.

Solution. There are many examples. One is $\alpha = (1\ 2)$ and $\beta = (1\ 3)$. Since both α and β are transpositions, $\alpha^2 = (1) = \beta^2$, and so $\alpha^2\beta^2 = (1)$. On the other hand, $\alpha\beta = (1\ 3\ 2)$, and $(\alpha\beta)^2 = (1\ 3\ 2)^2 = (1\ 2\ 3) \neq (1)$.

2.31 **(i)** Prove, for all i, that $\alpha \in S_n$ moves i if and only if α^{-1} moves i.

Solution. Since α is surjective, there is k with $\alpha k = i$. If $k = i$, then $\alpha i = i$ and $\alpha i = j \neq i$, a contradiction; hence, $k \neq i$. But $\alpha^{-1} i = k$, and so α^{-1} moves i. The converse follows by replacing α by α^{-1}, for $(\alpha^{-1})^{-1} = \alpha$.

(ii) Prove that if $\alpha, \beta \in S_n$ are disjoint and if $\alpha\beta = (1)$, then $\alpha = (1)$ and $\beta = (1)$.

Solution. By (i), if α and β are disjoint, then α^{-1} and β are disjoint: if β moves some i, then α^{-1} must fix i. But $\alpha\beta = (1)$ implies $\alpha^{-1} = \beta$, so that there can be no i moved by β. Therefore, $\beta = (1) = \alpha$.

2.32 If $n \geq 2$, prove that the number of even permutations in S_n is $\frac{1}{2}n!$.

Solution. Let $\tau = (1\ 2)$, and define $f : A_n \to O_n$, where A_n is the set of all even permutations in S_n and O_n is the set of all odd permutations, by

$$f : \alpha \mapsto \tau\alpha.$$

If σ is even, then $\tau\sigma$ is odd, so that the target of f is, indeed, O_n. The function f is a bijection, for its inverse is $g : O_n \to A_n$, which is given by $g : \alpha \mapsto \tau\alpha$.

2.33 Give an example of $\alpha, \beta, \gamma \in S_5$, none of which is the identity (1), with $\alpha\beta = \beta\alpha$ and $\alpha\gamma = \gamma\alpha$, but with $\beta\gamma \neq \gamma\beta$.

Solution. Set $\alpha = (1\ 2)$, $\beta = (3\ 4)$, and $\gamma = (3\ 5)$. Then $\alpha\beta = \beta\alpha$, $\alpha\gamma = \gamma\alpha$, and $\beta\gamma \neq \gamma\beta$.

2.34 If $n \geq 3$, show that if $\alpha \in S_n$ commutes with every $\beta \in S_n$, then $\alpha = (1)$.

Solution. If $\alpha \neq (1)$, then it moves some i; say, $\alpha i = j \neq i$. There is β with $\beta j = j$ and $\beta i = k \neq i$. Then $\beta\alpha i = \beta j = j$, while $\alpha\beta i = \alpha k \neq j$ (for α is injective, and so $k \neq i$ implies $\alpha k \neq \alpha i = j$).

2.35 Can the following 15-puzzle be won?

4	10	9	1
8	2	15	6
12	5	11	3
7	14	13	#

Solution. No, because the associated permutation is odd.