Preliminary Solutions to Problems and Question
Chapter 2

Note: Printing errors and corrections are indicated in dark red. Currently none reported.

2.1 **Symmetric dielectric slab waveguide** Consider two rays such as 1 and 2′ interfering at point P in Figure 2.4 Both are moving with the same incidence angle but have different κ_m wavectors just before point P. In addition, there is a phase difference between the two due to the different paths taken to reach point P. We can represent the two waves as $E_1(y,z,t) = E_o\cos(\omega t - \kappa_m y - \beta_m z + \delta)$ and $E_2(y,z,t) = E_o\cos(\omega t + \kappa_m y - \beta_m z)$ where the $\kappa_m y$ terms have opposite signs indicating that the waves are traveling in opposite directions. δ has been used to indicate that the waves have a phase difference and travel different optical paths to reach point P. We also know that $\kappa_m = k_1\cos\theta_m$ and $\beta_m = k_1\sin\theta_m$, and obviously have the waveguide condition already incorporated into them through θ_m. Show that the superposition of E_1 and E_2 at P is given by

$$E(y,z,t) = 2E_o \cos(\kappa_m y - \tfrac{1}{2}\delta)\cos(\omega t - \beta_m z + \tfrac{1}{2}\delta)$$

What do the two cosine terms represent?

The planar waveguide is symmetric, which means that the intensity, E^2, must be either maximum (even m) or minimum (odd m) at the center of the guide. Choose suitable δ values and plot the relative magnitude of the electric field across the guide for $m = 0, 1$ and 2 for the following symmetric dielectric planar guide : $n_1 = 1.4550$, $n_2 = 1.4400$, $a = 10$ μm, $\lambda = 1.5$ μm (free space), the first three modes have $\theta_1 = 88.84°$, $\theta_2 = 87.67°$ $\theta_3 = 86.51°$. Scale the field values so that the maximum field is unity for $m = 0$ at the center of the guide. (Note: Alternatively, you can choose δ so that intensity (E^2) is the same at the boundaries at $y = a$ and $y = -a$; it would give the same distribution.)

Solution

$$E(y) = E_o \cos(\omega t - \kappa_m z - \beta_m z + \delta) + E_o \cos(\omega t + \kappa_m z - \beta_m z)$$

Use the appropriate trigonometric identity (see Appendix D) for $\cos A + \cos B$ to express it as a product of cosines $2\cos[(A+B)/2]\cos[(A-B)/2]$,

$$E(y,z,t) = 2E_o \cos(\kappa_m y - \tfrac{1}{2}\delta)\cos(\omega t - \beta_m z + \tfrac{1}{2}\delta)$$

The first cosine term represents the field distribution along y and the second term is the propagation of the field long the waveguide in the z-direction. Thus, the amplitude is

$$\text{Amplitude} = 2E_o \cos(\kappa_m y - \tfrac{1}{2}\delta)$$

The intensity is maximum or minimum at the center. We can choose $\delta = 0$ ($m = 0$), $\delta = \pi$ ($m = 1$), $\delta = 2\pi$ ($m = 2$), which would result in maximum or minimum intensity at the center. (In fact, $\delta = m\pi$). The field distributions are shown in Figure 2Q1-1.
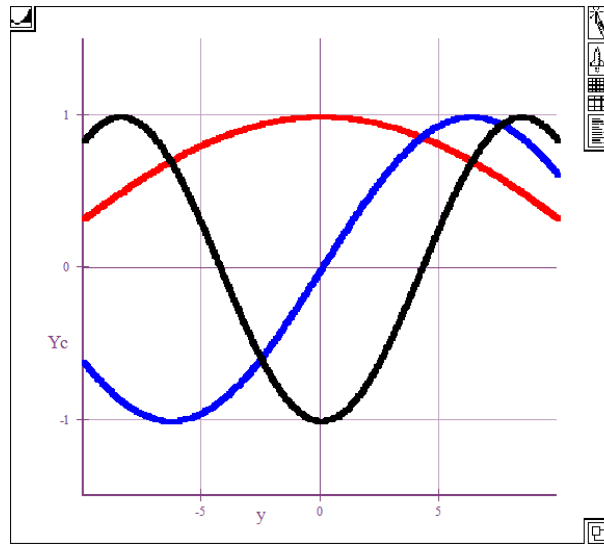
Figure 2Q1-1 Amplitude of the electric field across the planar dielectric waveguide. Red, $m = 0$; blue, $m = 1$;
black, $m = 2$.

2.2   Standing waves inside the core of a symmetric slab waveguide Consider a symmetric planar
dielectric waveguide. Allowed upward and downward traveling waves inside the core of the planar
waveguide set-up a standing wave along y. The standing wave can only exist if the wave can be
replicated after it has traveled along the y-direction over one round trip. Put differently, a wave starting
at A in Figure 2.51 and traveling towards the upper face will travel along y, be reflected at B, travel
down, become reflected again at A, and then it would be traveling in the same direction as it started. At
this point, it must have an identical phase to its starting phase so that it can replicate itself and not
destroy itself. Given that the wavevector along y is κ_m, derive the waveguide condition.
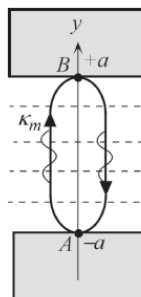

Figure 2.51 Upward and downward traveling waves along y set-up a standing wave. The condition for setting-up a
standing wave is that the wave must be identical, able to replicate itself, after one round trip along y.

Solution

From Figure 2.51 it can be seen that the optical path is

$AB + BA = 4a$

With the ray under going a phase change ϕ with each reflection the total phase change is

$\Delta\phi = 4a\kappa_m - 2\phi$

The wave will replicate itself, is the phase is same after the one round-trip, thus

$$\Delta\phi = 4a\kappa_m - 2\phi = 2\pi m$$

and since $\kappa_m = k_1 \cos\theta_m = \dfrac{2\pi n_1}{\lambda}\cos\theta_m$ we get

$$\frac{2\pi n_1(2a)}{\lambda}\cos\theta_m - \phi_m = m\pi$$

as required.

2.3   Dielectric slab waveguide

(a)   Consider the two parallel rays 1 and 2 in Figure 2.52. Show that when they meet at C at a distance y above the guide center, the phase difference is

$$\Phi_m = k_1 2(a-y)\cos\theta_m - \phi_m$$

(b)   Using the waveguide condition, show that

$$\Phi_m = \Phi_m(y) = m\pi - \frac{y}{a}(m\pi + \phi_m)$$

(c)   The two waves interfering at C can be most simply and conveniently represented as

$$E(y) = A\cos(\omega t) + A\cos[\omega t + \Phi_m(y)]$$

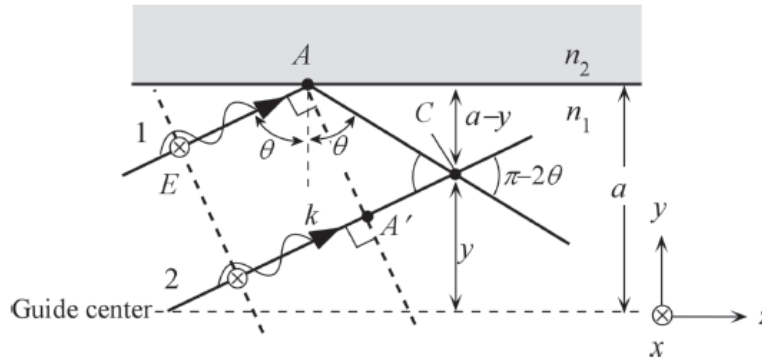Hence find the amplitude of the field variation along y, across the guide. What is your conclusion?

Figure 2.52 Rays 1 and 2 are initially in phase as they belong to the same wavefront. Ray 1 experiences total internal reflection at A. 1 and 2 interfere at C. There is a phase difference between the two waves.

Solution

(a)   From the geometry we have the following:

$$(a-y)/AC = \cos\theta$$

and $A'C/AC = \cos(\pi - 2\theta)$

The phase difference between the waves meeting at C is

$$\Phi = kAC - \phi - kA'C = k_1 AC - k_1 AC\cos(\pi - 2\theta) - \phi$$

$$= k_1 A C [1 - \cos(\pi - 2\theta)] - \phi = k_1 A C [1 + \cos(2\theta)] - \phi$$

$$= k_1 [(a - y)/\cos\theta][1 + 2\cos^2\theta - 1] - \phi$$

$$= k_1 [(a - y)/\cos\theta][2\cos^2\theta] - \phi$$

$$= 2k_1(a - y)\cos\theta - \phi$$

(b)   Given, $\left[\dfrac{2\pi(2a)n_1}{\lambda}\right]\cos\theta_m - \phi_m = m\pi$

$\therefore$ $\cos\theta_m = \dfrac{\lambda(m\pi + \phi_m)}{2\pi n_1(2a)} = \dfrac{m\pi + \phi_m}{k_1(2a)}$

Then, $\Phi_m = 2k_1(a - y)\cos\theta_m - \phi_m = 2k_1(a - y)\dfrac{m\pi + \phi_m}{k_1(2a)} - \phi_m$

$\therefore$ $\Phi_m = \left(1 - \dfrac{y}{a}\right)(m\pi + \phi_m) - \phi_m = m\pi - \dfrac{y}{a}(m\pi + \phi_m)$

$$\Phi_m = \Phi(y) = m\pi - \frac{y}{a}(m\pi + \phi_m) \quad \Phi_m = \Phi_m(y) = m\pi - \frac{y}{a}(m\pi + \phi_m)$$

(c)   The two waves interfering at C are out phase by Φ,

$$E(y) = A\cos(\omega t) + A\cos[\omega t + \Phi_m(y)]$$

where A is an arbitrary amplitude. Thus,

$$E = 2A\cos[\omega t + \tfrac{1}{2}\Phi_m(y)]\cos\left[\tfrac{1}{2}\Phi_m(y)\right]$$

or $E = \left\{2A\cos\left[\tfrac{1}{2}\Phi_m(y)\right]\right\}\cos(\omega t + \Phi') = E_o\cos(\omega t + \Phi')$

in which $\Phi' = \Phi_m/2$, and $\cos(\omega t + \Phi')$ is the time dependent part that represents the wave phenomenon, and the curly brackets contain the effective amplitude. Thus, the amplitude E_o is

$$E_o = 2A\cos\left[\frac{m\pi}{2} - \frac{y}{2a}(m\pi + \phi_m)\right]$$

To plot E_o as a function of y, we need to find ϕ_m for $m = 0, 1, 2\ldots$ The variation of the field is a *truncated*) cosine function with its maximum at the center of the guide. See Figure 2Q1-1.

2.4   TE field pattern in slab waveguide Consider two parallel rays 1 and 2 interfering in the guide as in Figure 2.52. Given the phase difference

$$\Phi_m = \Phi_m(y) = m\pi - \frac{y}{a}(m\pi + \phi_m)$$

between the waves at C, distance y above the guide center, find the electric field pattern $E(y)$ in the guide. Recall that the field at C can be written as $E(y) = A\cos(\omega t) + A\cos[\omega t + \Phi_m(y)]$. Plot the field pattern for the first three modes taking a planar dielectric guide with a core thickness 20 μm, $n_1 = 1.455$ $n_2 = 1.440$, light wavelength of 1.3 μm.
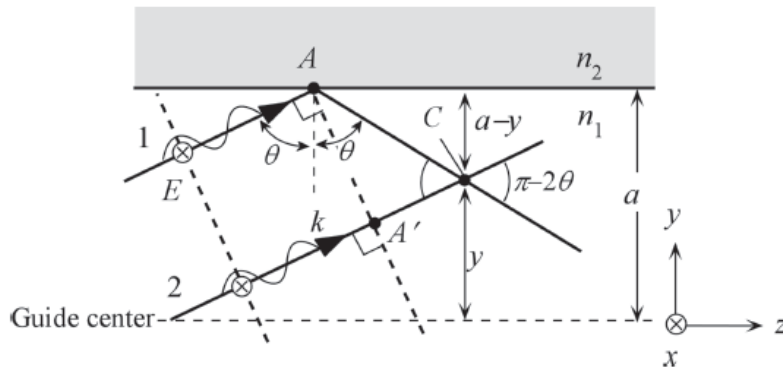
Figure 2.52 Rays 1 and 2 are initially in phase as they belong to the same wavefront. Ray 1 experiences total internal reflection at *A*. 1 and 2 interfere at *C*. There is a phase difference between the two

Solution

The two waves interfering at C are out phase by Φ,

$$E(y) = A\cos(\omega t) + A\cos[\omega t + \Phi_m(y)]$$

where *A* is an arbitrary amplitude. Thus,

$$E = 2A\cos\left[\omega t + \frac{1}{2}\Phi_m(y)\right]\cos\left[\frac{1}{2}\Phi_m(y)\right]$$

or $\qquad E = \left\{2A\cos\left[\frac{1}{2}\Phi_m(y)\right]\right\}\cos(\omega t + \Phi') = E_o\cos(\omega t + \Phi')$

in which $\Phi' = \Phi_m/2$, and $\cos(\omega t + \Phi')$ is the time dependent part that represents the wave phenomenon, and the curly brackets contain the effective amplitude. Thus, the amplitude E_o is

$$E_o = 2A\cos\left[\frac{m\pi}{2} - \frac{y}{2a}(m\pi + \phi_m)\right]$$

To plot E_o as a function of *y,* we need to find ϕ_m for $m = 0$, 1 and 2 , the first three modes. From Example 2.1.1 in the textbook, the waveguide condition is

$$(2a)k_1\cos\theta_m - m\pi = \phi_m$$

we can now substitute for ϕ_m which has different forms for TE and TM waves to find,

TE waves $\qquad \tan\left(ak_1\cos\theta_m - m\frac{\pi}{2}\right) = \dfrac{\left[\sin^2\theta_m - \left(\dfrac{n_2}{n_1}\right)^2\right]^{1/2}}{\cos\theta_m} = f_{TE}(\theta_m)$

TM waves $\quad \tan\left(ak_1\cos\theta_m - m\dfrac{\pi}{2}\right) = \dfrac{\left[\sin^2\theta_m - \left(\dfrac{n_2}{n_1}\right)^2\right]^{1/2}}{\left(\dfrac{n_2}{n_1}\right)^2\cos\theta_m} = f_{TM}(\theta_m)$

The above two equations can be solved graphically as in Example 2.1.1 to find θ_m for each choice of m. Alternatively one can use a computer program for finding the roots of a function. The above equations are functions of θ_m only for each m. Using $a = 10$ μm, $\lambda = 1.3$ μm, $n_1 = 1.455$ $n_2 = 1.440$, the results are:

TE Modes	$m = 0$	$m = 1$	$m = 2$
θ_m (degrees)	88.84	87.67	86.51
ϕ_m (degrees)	163.75	147.02	129.69
TM Modes	$m = 0$	$m = 1$	$m = 2$
θ_m (degrees)	88.84	87.67	86.51
ϕ_m (degrees)	164.08	147.66	130.60

There is no significant difference between the TE and TM modes (the reason is that n_1 and n_2 are very close).
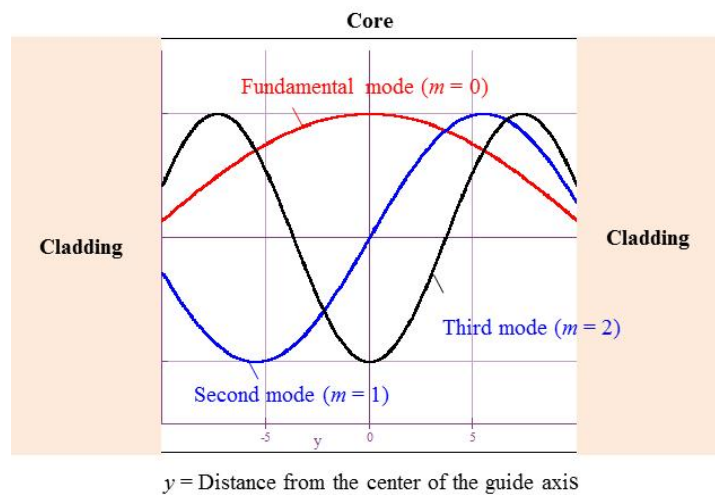

Figure 2Q4-1 Field distribution across the core of a planar dielectric waveguide

We can set $A = 1$ and plot E_o vs. y using

$$E_o = 2\cos\left[\dfrac{m\pi}{2} - \dfrac{y}{2a}(m\pi + \phi_m)\right]$$

with the ϕ_m and m values in the table above. This is shown in Figure 2Q4-1.

2.5   TE and TM Modes in dielectric slab waveguide Consider a planar dielectric guide with a core thickness 20 μm, $n_1 = 1.455$ $n_2 = 1.440$, light wavelength of 1.30 μm. Given the waveguide condition, and the expressions for phase changes ϕ and ϕ' in TIR for the TE and TM modes respectively,

$$\tan\left(\tfrac{1}{2}\phi_m\right) = \frac{\left[\sin^2\theta_m - \left(\dfrac{n_2}{n_1}\right)^2\right]^{1/2}}{\cos\theta_m} \qquad \text{and} \qquad \tan\left(\tfrac{1}{2}\phi'_m\right) = \frac{\left[\sin^2\theta_m - \left(\dfrac{n_2}{n_1}\right)^2\right]^{1/2}}{\left(\dfrac{n_2}{n_1}\right)^2 \cos\theta_m}$$

using a graphical solution find the angle θ for the fundamental TE and TM modes and compare their propagation constants along the guide.

Solution

The waveguide condition is

$$(2a)k_1 \cos\theta_m - m\pi = \phi_m$$

we can now substitute for ϕ_m which has different forms for TE and TM waves to find,

TE waves $\tan\left(ak_1\cos\theta_m - m\dfrac{\pi}{2}\right) = \dfrac{\left[\sin^2\theta_m - \left(\dfrac{n_2}{n_1}\right)^2\right]^{1/2}}{\cos\theta_m} = f_{TE}(\theta_m)$

TM waves $\tan\left(ak_1\cos\theta_m - m\dfrac{\pi}{2}\right) = \dfrac{\left[\sin^2\theta_m - \left(\dfrac{n_2}{n_1}\right)^2\right]^{1/2}}{\left(\dfrac{n_2}{n_1}\right)^2 \cos\theta_m} = f_{TM}(\theta_m)$

The above two equations can be solved graphically as in Example 2.1.1 to find θ_m for each choice of m. Alternatively one can use a computer program for finding the roots of a function. The above equations are functions of θ_m only for each m. Using $a = 10$ μm, $\lambda = 1.3$ μm, $n_1 = 1.455$ $n_2 = 1.440$, the results are:

TE Modes $m = 0$

θ_m (degrees)   88.8361

$\beta_m = k_1\sin\theta_m$ 7,030,883 m^{-1}

TM Modes $m = 0$

θ'_m (degrees)   88.8340

$\beta'_m = k_1\sin\theta_m$ 7,030,878 m^{-1}

Note that $\Delta\beta = 5.24$ m^{-1} and the β-difference is only 7.5×10^{-5} %.

The following intuitive calculation shows how the small difference between the TE and TM waves can lead to dispersion that is time spread in the arrival times of the TE and TM optical signals.

Suppose that $\Delta\tau$ is the delay time between the TE and TM waves over a length L. Then,

$$\frac{\Delta\tau}{L} = \frac{1}{v_{TE}} - \frac{1}{v_{TM}} = \frac{\beta_{TE}}{\omega} - \frac{\beta_{TM}}{\omega} \approx \frac{\Delta\beta}{\omega} = \frac{(5.24\text{ m}^{-1})}{(1.45\times10^{15}\text{ rad/s})}$$

$$= 3.6\times10^{-15}\text{ s m}^{-1} = 0.0036\text{ ps m}^{-1}.$$

Over 1 km, the TE-TM wave dispersion is ~3.6 ps. One should be cautioned that we calculated dispersion using the phase velocity whereas we should have used the group velocity.

2.6   Group velocity We can calculate the group velocity of a given mode as a function of frequency ω using a convenient math software package. It is assumed that the math-software package can carry out symbolic algebra such as partial differentiation (the author used Livemath, , though others can also be used). The propagation constant of a given mode is $\beta = k_1\sin\theta$ where β and θ imply β_m and θ_m. The objective is to express β and ω in terms of θ. Since $k_1 = n_1\omega/c$, the waveguide condition is

$$\tan\left(a\frac{\beta}{\sin\theta}\cos\theta - m\frac{\pi}{2}\right) = \frac{\left[\sin^2\theta - (n_2/n_1)^2\right]^{1/2}}{\cos\theta}$$

so that

$$\beta \approx \frac{\tan\theta}{a}\left[\arctan\left(\sec\theta\sqrt{\sin^2\theta - (n_2/n_1)^2}\right) + m(\pi/2)\right] = F_m(\theta) \tag{1}$$

where $F_m(\theta)$ and a function of θ at a given m. The frequency ω is given by

$$\omega = \frac{c\beta}{n_1\sin\theta} = \frac{c}{n_1\sin\theta}F_m(\theta) \tag{2}$$

Both β and ω are now a function of θ in Eqs (1) and (2). Then the group velocity is found by differentiating Eqs (1) and (2) with respect to θ, *i.e.*

$$v_g = \frac{d\omega}{d\beta} = \left[\frac{d\omega}{d\theta}\right]\times\left[\frac{d\theta}{d\beta}\right] = \frac{c}{n_1}\left[\frac{F_m'(\theta)}{\sin\theta} - \frac{\cos\theta}{\sin^2\theta}F_m(\theta)\right]\times\left[\frac{1}{F_m'(\theta)}\right]$$

i.e.

$$v_g = \frac{c}{n_1\sin\theta}\left[1 - \cot\theta\frac{F_m(\theta)}{F_m'(\theta)}\right] \qquad\qquad \textit{Group velocity, planar waveguide} \tag{3}$$

where $F_m' = dF_m/d\theta$ is found by differentiating the second term of Eq. (1). For a given m value, Eqs (2) and (3) can be plotted parametrically, that is, for each θ value we can calculate ω and v_g and plot v_g vs. ω. Figure 2.11 shows an example for a guide with the characteristics in the figure caption. Using a convenient math-software package, or by other means, obtain the same v_g vs. ω behavior, discuss intermodal dispersion, and whether the Equation (2.2.2) is appropriate.

Solution

The results shown in Figure 2.11, and Figure 2Q6-1 were generated by the author using LiveMath based on Eqs (1) and (3). Obviously other math software packages can also be used. In the presence of say two