2.1 $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Although it is not stated, the flow is assumed to be steady so that $\frac{\partial \rho}{\partial t} = 0$.

$$\nabla \cdot (\rho \vec{V})$$

$$\left[\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}\right] \cdot \left[\rho\left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z\right)\right]$$

$$= \hat{e}_r \cdot \hat{e}_r \frac{\partial(\rho v_r)}{\partial r} + \hat{e}_r \cdot \rho v_r \frac{\partial \hat{e}_r}{\partial r}$$

$$+ \hat{e}_r \cdot \hat{e}_\theta \frac{\partial(\rho v_\theta)}{\partial r} + \hat{e}_r \cdot \rho v_\theta \frac{\partial \hat{e}_\theta}{\partial r}$$

$$+ \hat{e}_r \cdot \hat{e}_z \frac{\partial(\rho v_z)}{\partial r} + \hat{e}_r \cdot \rho v_z \frac{\partial \hat{e}_z}{\partial r}$$

$$+ \frac{\hat{e}_\theta}{r} \cdot \hat{e}_r \frac{\partial(\rho v_r)}{\partial \theta} + \frac{\hat{e}_\theta}{r} \cdot \rho v_r \frac{\partial \hat{e}_r}{\partial \theta}$$

$$+ \frac{\hat{e}_\theta}{r} \cdot \hat{e}_\theta \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\hat{e}_\theta}{r} \cdot \rho v_\theta \frac{\partial \hat{e}_\theta}{\partial \theta}$$

$$+ \frac{\hat{e}_\theta}{r} \cdot \hat{e}_z \frac{\partial(\rho v_z)}{\partial \theta} + \frac{\hat{e}_\theta}{r} \cdot \rho v_z \frac{\partial \hat{e}_z}{\partial \theta}$$

$$+ \hat{e}_z \cdot \hat{e}_r \frac{\partial(\rho v_r)}{\partial z} + \hat{e}_z \cdot \rho v_r \frac{\partial \hat{e}_r}{\partial z}$$

$$+ \hat{e}_z \cdot \hat{e}_\theta \frac{\partial(\rho v_\theta)}{\partial z} + \hat{e}_z \cdot \rho v_\theta \frac{\partial \hat{e}_\theta}{\partial z}$$

$$+ \hat{e}_z \cdot \hat{e}_z \frac{\partial(\rho v_z)}{\partial z} + \hat{e}_z \cdot \rho v_z \frac{\partial \hat{e}_z}{\partial z}$$

To evaluate the terms in the left-hand column, 2.1 contd. we note that:

$$\hat{e}_r \cdot \hat{e}_r = \hat{e}_\theta \cdot \hat{e}_\theta = \hat{e}_z \cdot \hat{e}_z = 1 \quad \text{and}$$
$$\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_\theta \cdot \hat{e}_z = \hat{e}_z \cdot \hat{e}_r = 0$$

To evaluate the terms in the right-hand column, we note that a vector can change in magnitude and/or in direction. Obviously, a unit vector cannot change in magnitude (or length). From vector calculus, the derivatives of the unit vectors in cylindrical coordinates are:

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \;;\; \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$

All other derivatives are zero. Thus, the equation becomes:

$$\nabla \cdot (\rho \vec{V}) = 0$$
$$= \frac{\partial(\rho v_r)}{\partial r} + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}$$
$$= \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} \quad QED$$

2.2 (a) We have a radial flow in which ρ = constant

Therefore, we can use the result from problem 2.1

$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \stackrel{?}{=} 0$$

Since $\vec{V} = \frac{K}{2\pi r}\hat{e}_r$, we see that $v_r = \frac{K}{2\pi r}$; $v_\theta = 0$; and $v_z = 0$. Substituting these components into the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{K}{2\pi r}\right) = 0! \quad \text{Continuity is satisfied.}$$

2.2 Contd.) (b) Let us use the continuity equation for a three-dimensional flow, i.e., equation (2.1):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \stackrel{?}{=} 0$$

For constant density flow, this equation becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Thus,

$$\frac{\partial}{\partial x}\left\{-\frac{2xyz}{(x^2+y^2)^2}U_\infty L\right\} + \frac{\partial}{\partial y}\left\{\frac{(x^2-y^2)z}{(x^2+y^2)^2}U_\infty L\right\} + \frac{\partial}{\partial z}\left\{\frac{y}{x^2+y^2}U_\infty L\right\} = 0$$

Since U_∞ and L are constants and since they appear in every term, they can be divided out leaving:

$$-\frac{2yz}{(x^2+y^2)^2} - \frac{2xyz(-2)(2x)}{(x^2+y^2)^3} - \frac{2yz}{(x^2+y^2)^2} + \frac{(x^2-y^2)z(-2)(2y)}{(x^2+y^2)^3}$$

$$= -\frac{4yz}{(x^2+y^2)^2} - \frac{-8x^2yz + 4x^2yz - 4y^3z}{(x^2+y^2)^3}$$

$$= \frac{-4x^2yz - 4y^3z + 8x^2yz - 4x^2yz + 4y^3z}{(x^2+y^2)^3} = 0$$

Therefore, the continuity equation is satisfied.

2.3)

Given: Two of three velocity components for an incompressible flow:

$$u = x^2 + 2xz \qquad v = y^2 + 2yz$$

The velocity components must satisfy the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

2.3) contd.

For incompressible flow this becomes:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Find the derivatives of the given velocity components:

$$\frac{\partial u}{\partial x}=2(x+z) \qquad \frac{\partial v}{\partial y}=2(y+z)$$

Therefore:

$$\frac{\partial w}{\partial z}=-2(x+y+2z)$$

Integrating yields:

$$w=-2z(x+y+z)=f(x,y,t)$$

Where $f(x,y,t)$ is an arbitrary function (x,y,t). Since the first two velocity components are not a function of time, it may be possible to assume the flow is steady and drop the time function from the arbitrary constant.

2.4)

Given: Velocity components for a 2D incompressible flow:

$$u=-\frac{Ky}{(x^2+y^2)} \qquad v=+\frac{Kx}{(x^2+y^2)}$$

For 2D incompressible flow the continuity equation is:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Taking the required derivatives yields:

$$\frac{\partial u}{\partial x}=\frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y}=-\frac{2xy}{(x^2+y^2)^2}$$

2.4) contd.

which shows that the flowfield satisfies continuity. Now convert to cylindrical coordinates for simplicity using:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Resulting in velocity components of:

$$u = -\frac{K\sin\theta}{r} \qquad v = +\frac{K\cos\theta}{r}$$

Now some vector information is required:

$$\vec{V} = u\hat{i} + v\hat{j} = v_r\hat{e}_r + v_\theta\hat{e}_\theta$$

$$v_r = \vec{V}\cdot\hat{e}_r = u\hat{i}\cdot\hat{e}_r + v\hat{j}\cdot\hat{e}_r \qquad \hat{i}\cdot\hat{e}_r = \cos\theta \qquad \hat{j}\cdot\hat{e}_r = \sin\theta$$

$$v_\theta = \vec{V}\cdot\hat{e}_\theta = u\hat{i}\cdot\hat{e}_\theta + v\hat{j}\cdot\hat{e}_\theta \qquad \hat{i}\cdot\hat{e}_\theta = -\sin\theta \qquad \hat{j}\cdot\hat{e}_\theta = \cos\theta$$

Resulting in:

$$v_r = u\hat{i}\cdot\hat{e}_r + v\hat{j}\cdot\hat{e}_r = -\frac{K\sin\theta\cos\theta}{r} + \frac{K\sin\theta\cos\theta}{r} = 0$$

$$v_\theta = u\hat{i}\cdot\hat{e}_\theta + v\hat{j}\cdot\hat{e}_\theta = \frac{K\sin^2\theta}{r} + \frac{K\cos^2\theta}{r} = \frac{K}{r}$$

This represents a counter-clockwise vortex flow about the origin with a velocity singularity at the origin and a circular velocity about the origin proportional to $1/r$.

2.5)

Given: Velocity components for a 2D incompressible flow:

$$u = \frac{C\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2} \qquad v = -\frac{2Cxy}{\left(x^2 + y^2\right)^2}$$

2.5) contd.

Assume 2D incompressible flow and that C is a constant. For 2D incompressible flow the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Taking the required derivatives yields:

$$\frac{\partial u}{\partial x} = C\left(y^2 - x^2\right)(-2)\left(x^2 + y^2\right)^{-3}(2x) + C(-2x)\left(x^2 + y^2\right)^{-2}$$

$$\frac{\partial v}{\partial y} = -2Cxy(-2)\left(x^2 + y^2\right)^{-3}(2y) + C(-2x)\left(x^2 + y^2\right)^{-2}$$

$$\frac{-4Cx\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^3} - \frac{2Cx}{\left(x^2 + y^2\right)^2} + \frac{8Cxy^2}{\left(x^2 + y^2\right)^3} - \frac{2Cx}{\left(x^2 + y^2\right)^2} = 0$$

after some algebra and patience!

2.6) Referring to the continuity equation for a two-dimensional, incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \frac{1}{2}\frac{a_1 y}{x^{1.5}} - \frac{3}{2}\frac{a_2 y^3}{x^{2.5}}$$

Integrating with respect to y

$$v = +\frac{1}{4}\frac{a_1 y^2}{x^{1.5}} - \frac{3}{8}\frac{a_2 y^4}{x^{2.5}} + C$$

To evaluate the constant of integration C, we note that $v = 0$ when $y = 0$. Thus, $C = 0$ and

$$v = \frac{1}{4}\frac{a_1 y^2}{x^{1.5}} - \frac{3}{8}\frac{a_2 y^4}{x^{2.5}}$$

2.7) The integral form of the continuity equation [equation (2.5)] for steady, one-dimensional flow in a streamtube yields:

$$-\iint \rho_1 V_1 \, dA_1 + \iint \rho_2 V_2 \, dA_2 = 0$$

Since the flow properties (e.g., ρ and V) are uniform across the area (the one-dimension for which the flow properties vary is the streamwise coordinate:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho V A = \text{constant}$$

Differentiating:

$$(d\rho) VA + \rho (dV) A + \rho V (dA) = 0$$

Dividing by ρVA, we obtain:

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

If the flow is incompressible, $d\rho = 0$, and

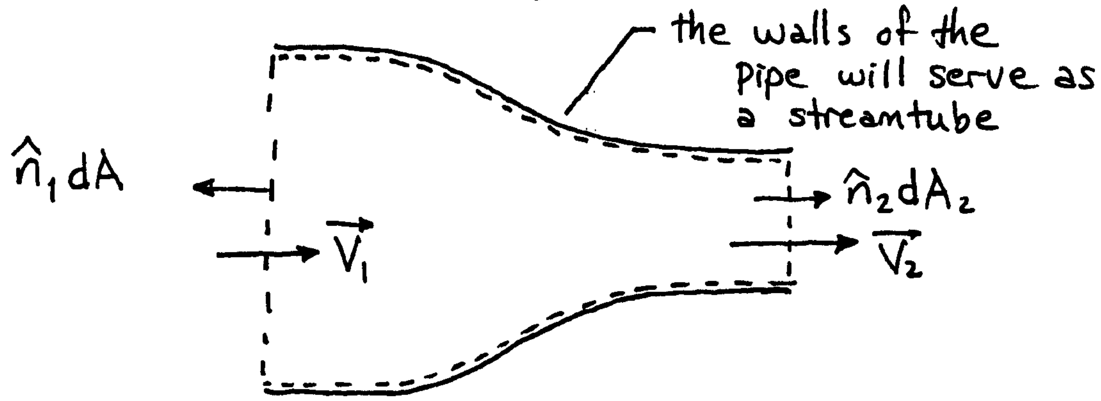
$$\frac{dV}{V} = -\frac{dA}{A}$$

Thus, if the cross-sectional area between streamlines (or of a streamtube, or of the walls of a wind tunnel) decreases, the flow accelerates. When the cross-sectional-area between streamlines increases, the velocity of the fluid particles decreases. These relations between dV and dA are not true if the flow is supersonic as will be discussed in Chapter 8.

2.8 Using the integral form of the continuity equation for steady flow, we can use equation (2.5)

$$\frac{\partial}{\partial t}\iiint_{Vol} \rho \, d(vol) + \oiint_{Area} \rho \vec{V} \cdot \hat{n} \, dA = 0$$

to solve this problem. Let us draw a control volume between stations 1 and 2,



The vectors representing the areas ($\hat{n} dA$) are directed outward from the control volume, as shown in the sketch. The velocities represent an assumed flow from left to right. Using the vector dot products and noting that the flow properties do not vary across the cross-section, we obtain:

$$-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0$$

where V_1 and V_2 are the magnitudes of the velocity vectors,

$\vec{V}_1$ and $\vec{V}_2$, respectively. Thus,

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \text{ (and by extension)} = \rho_3 V_3 A_3$$

(Often we see the expression for steady, one-dimensional flow in a streamtube as:

$$\rho V A = \text{constant}$$

2.8 Contd.

The duct need not be straight, providing the flow is approximately one-dimensional. Thus, the equation is often applied to flow in curved pipes and elbows.)

For this flow, water can be assumed to be of constant density. Thus,

$$\rho_1 = \rho_2 = \rho_3$$

As a result,

$$V_1 A_1 = V_2 A_2 = V_3 A_3 = 0.5 \frac{m^3}{s}$$

$$V_1\left[\frac{\pi}{4}(0.4)^2\right] = V_2\left[\frac{\pi}{4}(0.2)^2\right] = V_3\left[\frac{\pi}{4}(0.6)^2\right] = 0.5$$

Solving,

$$V_1 = 3.979 \frac{m}{s} \; ; \; V_2 = 15.915 \frac{m}{s} \; ; \; V_3 = 1.768 \frac{m}{s}$$

2.9 Following the logic of Problem 2.8,

$$\rho_s V_s A_s = \rho_1 V_1 A_1 = \rho_2 \iint u_2 \, dA_2 = 10 \frac{kg}{s}$$

Note that u_2 is left in the integral, since it is not constant over the cross section, i.e.,

$$u_2 = U_0\left[1 - \frac{r^2}{R^2}\right]$$

$$V_s\left[\frac{\pi}{4}(500\,cm)^2\right] = V_1\left[\frac{\pi}{4}(20\,cm)^2\right] = \frac{10^4 \frac{gm}{s}}{0.85 \frac{gm}{cm^3}}$$

2.9 Contd.] Thus, $V_s = 0.0599 \frac{cm}{s}$; $V_1 = 37.448 \frac{cm}{s}$

Note that the velocity at which a fluid particle at the free surface moves (V_s) is very small compared to the velocity in the drain pipe. Since the flow is axisymmetric at station 2:

$$dA = 2\pi r\,dr$$

$$\rho_2 \int_0^{R_2} U_0 \left[1 - \frac{r^2}{R_2^2}\right] 2\pi r\,dr = 10^4 \frac{gm}{s}$$

$$2\pi U_0 \rho_2 \left[\frac{R_2^2}{2} - \frac{R_2^4}{4R_2^2}\right] = 10^4 \frac{gm}{s}$$

$$U_0 = \frac{10^4 \frac{gm}{s}}{2\pi(\rho_2)\left(\frac{R_2^2}{4}\right)\frac{gm}{cm}} = 299.586 \frac{cm}{s}$$

2.10] Let us use the integral form of the continuity equation. Note that the effects of viscosity are such that there is a significant reduction of the velocity in the wake of the airfoil (at station 2). Thus, for this rectangular control volume, a significant fraction of the mass influx at station 1 does not leave the control volume through station 2. Thus, some fluid must exit through planes 3 and 4. Thus, they are obviously not streamlines.

$$\frac{\partial}{\partial t}\iiint \rho\, d(vol) + \oiint \rho \vec{V}\cdot\hat{n}\,dA = 0$$

By continuity, we know that the is a v-component of velocity in the wake of the airfoil and that $v(x,y)$ in 2. Along surface 3

2.10 Contd. $\vec{V}_3 = U_\infty \hat{i} + v_\infty(x)\hat{j}$

and along surface (4)

$$\vec{V}_4 = U_\infty \hat{i} - v_\infty(x)\hat{j}$$

Since the flow is steady, the mass fluxes per unit depth in the continuity equation can be written:

$$\underbrace{\rho\int_{-H}^{+H}\left[U_\infty\hat{i}\right]\cdot\left[-\hat{i}\,dy\right]}_{1} + \underbrace{\rho\int_{-H}^{0}\left[-\frac{U_\infty y}{H}\hat{i} - v\hat{j}\right]\cdot\left[\hat{i}\,dy\right]}_{2a}$$

$$+ \underbrace{\rho\int_{0}^{H}\left[\frac{U_\infty y}{H}\hat{i} + v\hat{j}\right]\cdot\left[\hat{i}\,dy\right]}_{2b} + \underbrace{\rho\int_{0}^{L}\left[U_\infty\hat{i} + v_\infty\hat{j}\right]\cdot\left[\hat{j}\,dy\right]}_{3}$$

$$+ \rho\int_0^L\left[U_\infty\hat{i} - v_\infty\hat{j}\right]\cdot\left[-\hat{j}\,dx\right] = 0$$

Note that the vertical component of velocity does not transport fluid across the surface at station (2) and that the horizontal component of velocity does not transport fluid across the surface at station's (3) and (4). This is because these velocity components are perpendicular to the area "vectors" at the station. Thus,

$$-\rho U_\infty 2H + \rho\frac{U_\infty}{H}\left(-\frac{y^2}{2}\right|_{-H}^{0} + \rho\frac{U_\infty}{H}\left(+\frac{y^2}{2}\right|_{0}^{H}$$

$$+ \rho\int_0^L v_\infty\,dx + \rho\int_0^L v_\infty\,dx = 0$$

The last two terms represent the total mass flow across the surfaces (3) and (4). The density is common to every term. We can divide by the density to get the volumetric flow across (3) and (4) $\left[2\int_0^L v_\infty\,dx\right] = U_\infty H$

2.11) Let us apply the integral form of the continuity equation. Note that, since surfaces ③ and ④ are streamlines, flow passes through only surfaces ① and ②.

$$\cancel{\frac{\partial}{\partial t}\iiint \rho \, d(vol)}^{\text{steady}} + \oiint \rho \vec{V}\cdot\hat{n}\, dA = 0$$

Since the flow is incompressible and steady, we can write the continuity equation as

$$\underbrace{-\rho U_\infty \int_{-H_u}^{+H_u} dy}_{①} + \underbrace{\rho U_\infty \int_{-H_D}^{0}\left(-\frac{y}{H_D}\right) dy}_{2a}$$

$$+\underbrace{\rho U_\infty \int_0^{H_D}\left(\frac{y}{H_D}\right) dy}_{2b} = 0$$

(Refer to Problem 2.10 to see how to handle the u-component of velocity at station ②.)

$$-\rho U_\infty (2H_u) - \frac{\rho U_\infty}{H_D}\left(\frac{y^2}{2}\right|_{-H_D}^{0} + \frac{\rho U_\infty}{H_D}\left(\frac{y^2}{2}\right|_{0}^{H_D} = 0$$

Rearranging and dividing through by ρU_∞ (which is a common factor to every term), we obtain:

$$H_u = \frac{1}{2} H_D$$

2.12) Let us apply the integral form of the continuity equation. Note that the effects of viscosity have caused a significant reduction of velocity in the wake of the airfoil (at station ②). As a result, there is a v-component of velocity which produces a mass flux across planes ③ and ④, because they are horizontal (perpendicular to the v-component).

2.12 Contd. The flow is steady and incompressible. As a result, the integral continuity equation becomes.

$$\oiint \vec{V} \cdot \hat{n}\, dA = 0$$

$$\underbrace{\int_{-H}^{+H} \left[U_\infty \hat{i}\right] \cdot \left[-\hat{i}\, dy\right]}_{1} + \underbrace{\int_{-H}^{+H} \left[U_\infty \left(1 - 0.5 \cos \frac{\pi y}{2H}\right)\hat{i} + v\hat{j}\right] \cdot \left[\hat{i}\, dy\right]}_{2}$$

$$+ \underbrace{\int_0^L \left[U_\infty \hat{i} + v_\infty \hat{j}\right] \cdot \left[\hat{j}\, dx\right]}_{3} + \underbrace{\int_0^L \left[U_\infty \hat{i} - v_\infty \hat{j}\right] \cdot \left[-\hat{j}\, dx\right]}_{4} = 0$$

Note that the vertical component of velocity does not transport fluid across the surface at station 2 and that the horizontal component of velocity does not transport fluid across stations 3 and 4. This is because these velocity components are perpendicular to the area "vectors". Thus,

$$-U_\infty (2H) + U_\infty \left[y - 0.5 \frac{2H}{\pi} \sin \frac{\pi y}{2H} \right]\Bigg|_{-H}^{+H}$$

$$+ \int_0^L v_\infty\, dx + \int_0^L v_\infty\, dx = 0$$

The last two terms represent the total volumetric flow across surfaces 3 and 4. Since the flow is planar symmetric at stations 1 and 2, we'll assume that the volumetric flow rate across 3 is equal to that across 4.

$$\left[2 \int_0^L v_\infty\, dx\right] = 2HU_\infty - 2HU_\infty + \frac{HU_\infty}{\pi}\left[1 - (-1)\right]$$

$$\left[2 \int_0^L v_\infty\, dx\right] = \frac{2HU_\infty}{\pi}$$

2.13] Let us apply the integral form of the continuity equation. Note that, since surfaces 3 and 4 are streamlines, fluid can cross only surfaces 1 and 2. Since the flow is steady,

$$\cancel{\frac{\partial}{\partial t}\iiint \rho\, d(\text{vol})}^{\text{steady}=0} + \oiint \rho \vec{V} \cdot \hat{n}\, dA = 0$$

Because the flow is incompressible (i.e., ρ = constant),

$$-\rho U_\infty \int_{-H_u}^{+H_u} dy + \rho U_\infty \int_{-H_D}^{+H_D} \left(1 - 0.5 \cos \frac{\pi y}{2 H_D}\right) dy = 0$$

Note that we have eliminated the v-component of velocity at station 2, since it doesn't contribute to the mass flux. See the discussion of terms 2a and 2b in Problem 2.10.

We can divide through by ρU_∞ and obtain:

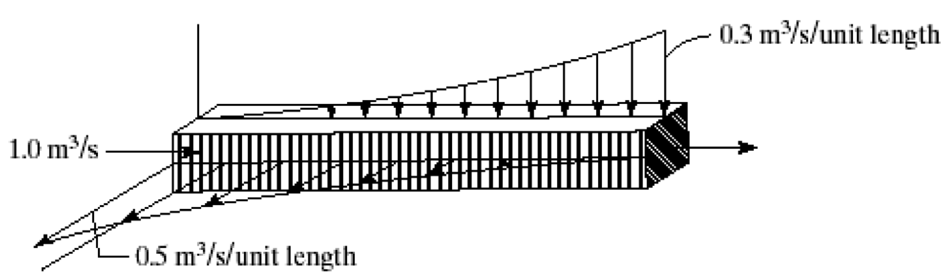
$$-2H_u + \left[y - 0.5 \frac{2 H_D}{\pi} \sin \frac{\pi y}{2 H_D} \right]_{-H_D}^{+H_D} = 0$$

$$\therefore -2H_u + \left[2 H_D - \frac{H_D}{\pi}(1+1) \right] = 0$$

$$H_u = H_D \left[1 - \frac{1}{\pi} \right] = 0.6817\, H_D$$

2.14)

Given: A rectangular duct as shown below with two porous surfaces. What is the average velocity of water leaving the duct if it is 1.0 m long and has a cross section of 0.1 m^2?



2.14) contd.

Conservation of mass requires:

$$\dot{m}_{out} = \dot{m}_{in}$$

Or, since the flow is incompressible:

$$\dot{q}_{out} = \dot{q}_{in}$$

where $\dot{q}$ is the volume flow rate. This yields:

$$\dot{q}_{in_{end}} + \dot{q}_{in_{top}} = \dot{q}_{out_{side}} + \dot{q}_{out_{end}}$$

$$1.0\frac{m^3}{s} + \int_0^1 0.3x^2 dx = \int_0^1 0.5(1-x)dx + \dot{q}_{out_{end}}$$

$$\dot{q}_{out_{end}} = 1.0\frac{m^3}{s} + 0.3\frac{x^3}{3}\bigg|_0^1 - 0.5x\big|_0^1 + 0.5\frac{x^2}{2}\bigg|_0^1$$

$$\dot{q}_{out_{end}} = 0.85\frac{m^3}{s}$$

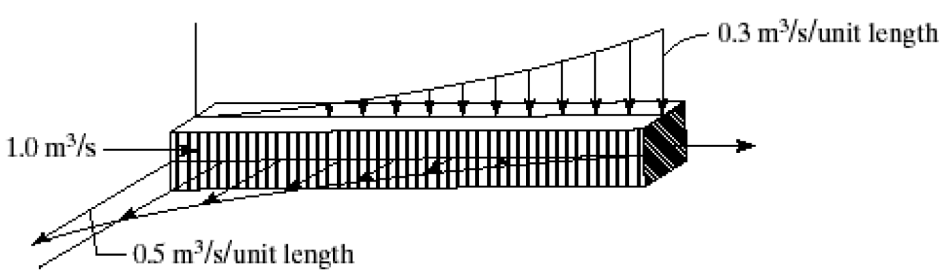
and the velocity at the outflow is:

$$V_{out_{end}} = \dot{q}_{out_{end}} / \rho A = 0.85m^3/s/\left(977.8kg/m^3 \cdot 0.1m^2\right)$$

$$V_{out_{end}} = 0.087m/s$$

2.15)

Given: The same duct as in Problem 2.14.



2.15) contd.

Using the development presented in the solution for Prob. 2.14, the volume flow rate at any station along the duct is:

$$\dot{q}_{out_{end}} = 1.0\frac{m^3}{s} + \int_0^x 0.3\xi^2 dx - \int_0^x 0.5(1-\xi)dx$$

$$\dot{q}_{out_{end}} = 1.0\frac{m^3}{s} + 0.3\frac{\xi^3}{3}\Big|_0^x - 0.5\xi\Big|_0^x + 0.5\frac{\xi^2}{2}\Big|_0^x$$

$$\dot{q}_{out_{end}} = 1.0\frac{m^3}{s} + 0.1x^3 - 0.5x + 0.25x^2$$

and the velocity at the outflow is:

$$V_x = \dot{q}_x / \rho A = \left(1.0 - 0.5x + 0.25x^2 + 0.1x^3\right) / \left(977.8 kg/m^3 \cdot 0.1m^2\right)$$

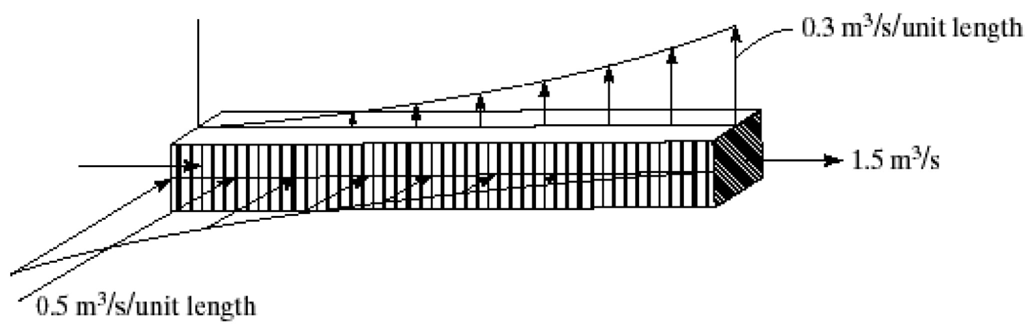
The minimum velocity is found by:

$$dV_x / dx = 3x^2 + 5x - 5 = 0$$

$$x = -\frac{5 \pm \sqrt{85}}{6} = 0.7m$$

2.16)

Given: A rectangular duct as shown below with two porous surfaces. What is the average velocity of water leaving the duct if it is 1.0 m long and has a cross section of 0.1 m^2?



2.16) contd.

Following the same procedure used in solving Prob. 2.14:

$$\dot{q}_{in_{end}} + \dot{q}_{in_{side}} = \dot{q}_{out_{top}} + \dot{q}_{out_{end}}$$

$$\dot{q}_{in_{end}} = 1.5\frac{m^3}{s} + \int_0^1 0.3x^2 dx - \int_0^1 0.5(1-x)dx$$

$$\dot{q}_{in_{end}} = 1.5\frac{m^3}{s} + 0.3\frac{x^3}{3}\Big|_0^1 - 0.5x\Big|_0^1 + 0.5\frac{x^2}{2}\Big|_0^1$$

$$\dot{q}_{out_{end}} = 1.35\frac{m^3}{s}$$

and the velocity at the outflow is:

$$V_{out_{end}} = \dot{q}_{out_{end}} / \rho A = 1.35 m^3/s / \left(977.8 kg/m^3 \cdot 0.1 m^2\right)$$

$$V_{out_{end}} = 0.014 m/s$$

2.17) $\vec{V} = -\frac{x}{2t}\hat{i}$; $\rho = \rho_0 x t$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$\vec{a} = +\frac{x}{2t^2}\hat{i} + \left[-\frac{x}{2t}\right]\left[-\frac{1}{2t}\hat{i}\right]$$

$$\vec{a} = \frac{x}{2t^2}\hat{i} + \frac{x}{4t^2}\hat{i} = \frac{3x}{4t^2}\hat{i}$$

2.18) $\vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$

$$\vec{a} = 2t\hat{i} - 10\hat{j} + \left[6 + 2xy + t^2\right]\left[2y\hat{i} - y^2\hat{j}\right]$$
$$- \left[xy^2 + 10t\right]\left[2x\hat{i} - 2xy\hat{j}\right] + 25[0]$$

when (x, y, z) is $(3, 0, 2)$ and $t = 1$

$$\vec{a} = \hat{i}\left[2 - 60\right] + \hat{j}\left[-10\right] = -58\hat{i} - 10\hat{j}$$

2.19) It will be shown that the velocity function

$$\vec{V}(r,\theta) = U_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta\,\hat{e}_r - U_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta\,\hat{e}_\theta$$

represents an inviscid, steady flow around a cylinder of radius R. To develop the expression for the acceleration:

$$\frac{d\vec{V}}{dt} = \frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

$$= v_r\frac{\partial}{\partial r}\vec{V} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta}\vec{V}$$

$$= \left[U_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta\right]\left[U_\infty\cos\theta\,\hat{e}_r\left(+\frac{2R^2}{r^3}\right) - U_\infty\sin\theta\,\hat{e}_\theta\left(-\frac{2R^2}{r^3}\right)\right]$$

$$+\left[U_\infty\left(1+\frac{R^2}{r^2}\right)\frac{\sin\theta}{r}\right]\left\{\left[U_\infty\left(1-\frac{R^2}{r^2}\right)\hat{e}_r(-\sin\theta)\right.\right.$$

$$\left.+U_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta\,\hat{e}_\theta\right] + \left[-U_\infty\left(1+\frac{R^2}{r^2}\right)\hat{e}_\theta(\cos\theta)\right.$$

$$\left.\left.+U_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta\,\hat{e}_r\right]\right\}$$

In developing this expression, we have used the fact that

$$\frac{\partial\hat{e}_r}{\partial\theta} = \hat{e}_\theta \quad \text{and} \quad \frac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{e}_r$$

Thus,

$$\frac{d\vec{V}}{dt} = \hat{e}_r\left[U_\infty^2\cos^2\theta\left(\frac{2R^2}{r^3}-\frac{2R^4}{r^5}\right) + U_\infty^2\sin^2\theta\left(\frac{1}{r}-\frac{R^4}{r^5}-\frac{1}{r}-\frac{2R^2}{r^3}-\frac{R^4}{r^5}\right)\right]$$

$$+\hat{e}_\theta\left[+U_\infty^2\sin\theta\cos\theta\left(+\frac{2R^2}{r^3}-\frac{2R^4}{r^5}-\frac{1}{r}+\frac{R^4}{r^5}+\frac{1}{r}+\frac{2R^2}{r^3}+\frac{R^4}{r^5}\right)\right]$$

2.19 Contd.

Note that when $r = R$, i.e., at points on the surface of the cylinder:

$$\frac{d\vec{V}}{dt} = \hat{e}_r \left[\frac{-4U_\infty^2 \sin^2\theta}{R} \right] + \hat{e}_\theta \left[\frac{4U_\infty^2 \sin\theta\cos\theta}{R} \right]$$

Note further that when $\theta = 0$ and when $\theta = \pi$

$$\frac{d\vec{V}}{dt} = 0$$

Thus, when $r = R$ and $\theta = 0$ and when $r = R$ and $\theta = \pi$,
$\vec{V} = 0$ (these two points are stagnation points) and
$\frac{d\vec{V}}{dt} = 0$ (the fluid particles are not accelerating at these two points). Note that $\theta = 0$ and $\theta = \pi$ represent points on the x-axis, which corresponds to the plane of symmetry for this flow.

2.20) From the integral form of the continuity equation:

$$uA = \text{constant} = Q$$

The cross-sectional area for a unit depth is $A = 2y(1)$
Using the boundary condition that $u = 2$ m/s and $h = 1$ m at $x = 0$. Thus, at the initial station:

$$Q\,(\text{the volumetric flow/unit depth}) = (u)(2h) = 4.0 \text{ m}^2/\text{s}$$

Thus,

$$Q/\text{depth} = (u)(2y) = u\left[2h - h\sin\left(\frac{\pi}{2}\frac{x}{L}\right)\right] = 4.0 \quad (a)$$

2.20 Contd.

Differentiating: $u\dfrac{dA}{dx} + \dfrac{du}{dx}A = 0$

or $u\dfrac{d(2y)}{dx} + \dfrac{du}{dx}(2y) = 0$

$$u\left\{-\frac{\pi}{2}\frac{h}{L}\cos\left(\frac{\pi}{2}\frac{x}{L}\right)\right\} + \frac{du}{dx}\left\{2h - h\sin\left(\frac{\pi}{2}\frac{x}{L}\right)\right\} = 0$$

$$\frac{du}{dx} = \frac{u\left\{\frac{\pi}{2}\frac{h}{L}\cos\left(\frac{\pi}{2}\frac{x}{L}\right)\right\}}{\left\{2h - h\left(\sin\left(\frac{\pi}{2}\frac{x}{L}\right)\right)\right\}} \qquad (b)$$

The acceleration $\vec{a} = \dfrac{d\vec{V}}{dt} = \dfrac{\partial\vec{V}}{\partial t} + u\dfrac{\partial\vec{V}}{\partial x} + v\dfrac{\partial\vec{V}}{\partial y} + w\dfrac{\partial\vec{V}}{\partial z}$

reduces to $\vec{a} = \dfrac{d\vec{V}}{dt} = u\dfrac{\partial u}{\partial x}\hat{i}$ for this one-dimensional, steady flow.

$$\vec{a} = u\left\{u\frac{\left[\frac{\pi}{2}\frac{h}{L}\cos\left(\frac{\pi}{2}\frac{x}{L}\right)\right]}{\left[2h - h\sin\left(\frac{\pi}{2}\frac{x}{L}\right)\right]}\right\}$$

Substituting the expression for the velocity, i.e., (a),

$$\vec{a} = \frac{16}{\left[2h - h\sin\left(\frac{\pi}{2}\frac{x}{L}\right)\right]^2}\left\{\frac{\frac{\pi}{2}\frac{h}{L}\cos\left(\frac{\pi}{2}\frac{x}{L}\right)}{\left[2h - h\sin\left(\frac{\pi}{2}\frac{x}{L}\right)\right]}\right\}\hat{i}$$

At $x=0$: $\vec{a} = \dfrac{8\pi h}{L}\dfrac{1}{8h^3}\hat{i} = \dfrac{\pi}{Lh^2}\hat{i} = \pi\,\hat{i}$

At $x = 0.5L$ $\quad \vec{a} = \dfrac{\frac{8\pi h}{L}\cos\frac{\pi}{4}}{\left[2h - h\sin\left(\frac{\pi}{4}\right)\right]^3}\hat{i} = \dfrac{8\pi h}{Lh^3}(0.3272)\hat{i}$

$$\vec{a} = 8.223\,\hat{i}\ \text{m/s}^2$$

2.21)

Given: A mass flow rate for the cabin air of:

$$\dot{m}_c = -0.040415 \frac{p_c}{\sqrt{T_c}} \left[A_{hole} \right]$$

Using the Ideal Gas Law and the definition of density:

$$p = \rho RT \qquad \rho = \frac{m}{\cancel{V}}$$

The pressure becomes:

$$p = \frac{m}{\cancel{V}} RT$$

And the mass flow rate equation can be rewritten as:

$$\dot{m}_c = -0.040415 \frac{m_c}{\cancel{V}} R_c \sqrt{T_c} \left[A_{hole} \right]$$

and:

$$\frac{\dot{m}_c}{m_c} = -\frac{0.040415}{\cancel{V}} R_c \sqrt{T_c} \left[A_{hole} \right]$$

But the mass flow rate is defined as $\dot{m} = dm/dt$ and the relationship can be integrated as:

$$\int_{m_{c_i}}^{m_{c_f}} \frac{dm}{m} = -\frac{0.040415}{\cancel{V}} R_c \sqrt{T_c} \left[A_{hole} \right] \int_0^{t_f} dt$$

where *i* represents an initial value and *f* represents a final value. Solving for the final time:

$$t_f = \frac{-\cancel{V}}{0.040415 R_c \sqrt{T_c} A_{hole}} \ln\left(\frac{m_{c_f}}{m_{c_i}} \right)$$

Since $T_c = 22^\circ C$ we see that $m_{c_f} / m_{c_i} = p_{c_f} / p_{c_i}$ and:

$$t_f = \frac{-\cancel{V}}{0.040415 R_c \sqrt{T_c} A_{hole}} \ln\left(\frac{p_{c_f}}{p_{c_i}} \right)$$

Using $\cancel{V} = 71\ m^3$ and consistent units, we get:

$$t_f = 5589 s = 1.55 hours$$

2.22)

Given: A mass flow rate for the cabin air of:

$$\dot{m}_c = -0.5318 \frac{p_c}{\sqrt{T_c}} [A_{hole}]$$

Using the Ideal Gas Law and the definition of density:

$$p = \rho R T \qquad \rho = \frac{m}{V\!\!\!/}$$

The pressure becomes:

$$p = \frac{m}{V\!\!\!/} R T$$

And the mass flow rate equation can be rewritten as:

$$\dot{m}_c = -0.5318 \frac{m_c}{V\!\!\!/} R_c \sqrt{T_c} [A_{hole}]$$

and:

$$\frac{\dot{m}_c}{m_c} = -\frac{0.5318}{V\!\!\!/} R_c \sqrt{T_c} [A_{hole}]$$

But the mass flow rate is defined as $\dot{m} = dm/dt$ and the relationship can be integrated as:

$$\int_{m_{c_i}}^{m_{c_f}} \frac{dm}{m} = -\frac{0.5318}{V\!\!\!/} R_c \sqrt{T_c} [A_{hole}] \int_0^{t_f} dt$$

where *i* represents an initial value and *f* represents a final value. Solving for the final time:

$$t_f = \frac{-V\!\!\!/}{0.5318 R_c \sqrt{T_c} A_{hole}} \ln\left(\frac{m_{c_f}}{m_{c_i}}\right)$$

Since $T_c = 22^\circ C$ we see that $m_{c_f} / m_{c_i} = p_{c_f} / p_{c_i}$ and:

$$t_f = \frac{-V\!\!\!/}{0.5318 R_c \sqrt{T_c} A_{hole}} \ln\left(\frac{p_{c_f}}{p_{c_i}}\right)$$

Using $V\!\!\!/ = 2513\ ft^3$ and consistent units, we get:

$$t_f = 5385 s = 1.50 hours$$

2.23)

Given: A mass flow rate for Oxygen of:

$$\dot{m}_{O_2} = -0.6847 \frac{p_{O_2}}{\sqrt{R_{O_2} T_{O_2}}} \left[A_{hole}\right]$$

Using the Ideal Gas Law and the definition of density:

$$p = \rho R T \qquad \rho = \frac{m}{V\!\!\!/}$$

The pressure becomes:

$$p = \frac{m}{V\!\!\!/} R T$$

And the mass flow rate equation can be rewritten as:

$$\dot{m}_{O_2} = -0.6847 \frac{m_{O_2}}{V\!\!\!/} \sqrt{R_{O_2} T_{O_2}} \left[A_{hole}\right]$$

and:

$$\frac{\dot{m}_{O_2}}{m_{O_2}} = -\frac{0.6847}{V\!\!\!/} \sqrt{R_{O_2} T_{O_2}} \left[A_{hole}\right]$$

But the mass flow rate is defined as $\dot{m} = dm / dt$ and the relationship can be integrated as:

$$\int_{m_{O_{2i}}}^{m_{O_{2f}}} \frac{dm}{m} = -\frac{0.6847}{V\!\!\!/} \sqrt{R_{O_2} T_{O_2}} \left[A_{hole}\right] \int_0^{t_f} dt$$

where *i* represents an initial value and *f* represents a final value. Solving for the final time:

$$t_f = \frac{-V\!\!\!/}{0.6847 \sqrt{R_{O_2} T_{O_2}} A_{hole}} \ln\left(\frac{m_{O_{2f}}}{m_{O_{2i}}}\right)$$

Since $T_{O_2} = 18^\circ C$ we see that $m_{O_{2f}} / m_{O_{2i}} = p_{O_{2f}} / p_{O_{2i}}$ and:

$$t_f = \frac{-V\!\!\!/}{0.6847 \sqrt{R_{O_2} T_{O_2}} A_{hole}} \ln\left(\frac{p_{O_{2f}}}{p_{O_{2i}}}\right)$$

Using $V\!\!\!/ = 1\ m^3$ and consistent units, we get:

$$t_f = 445859 s = 7431 hours = 310 days$$

For $V\!\!\!/ = 0.1\ m^3$ and consistent units, we get:

$$t_f = 31 days$$

2.24) As was done in Example 2.2, we can write that:

$$v = 0; \quad w = 0; \quad \frac{dp}{dx} = \mu \frac{d^2u}{dy^2}$$

Integrating twice: $\quad u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2 \qquad (i)$

Which is subject to the two boundary conditions:

(a) $y = 0$: $u = 0$ (the lower plate is stationary)

(b) $y = h$: $u = U_0$ (the upper plate moves with constant speed)

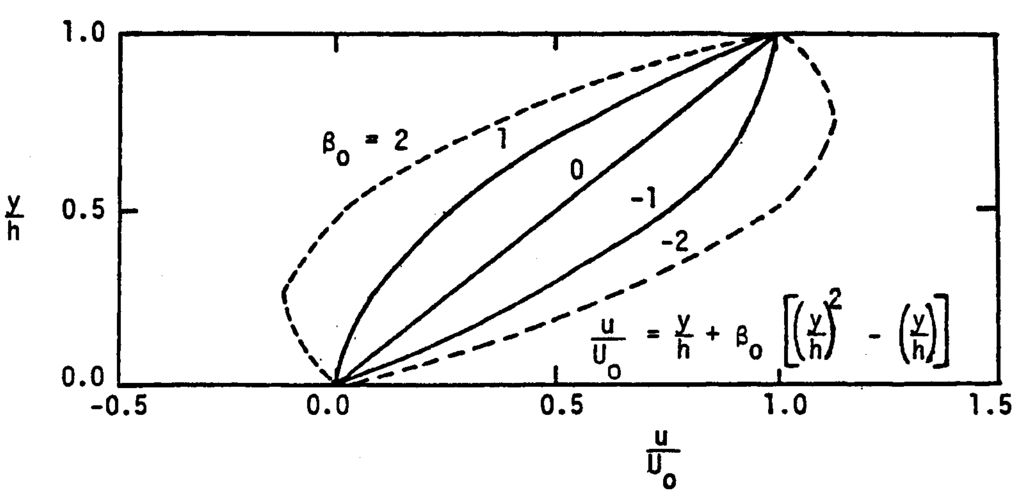
Applying these two boundary conditions;

(a) $0 = C_2$; (b) $U_0 = \frac{1}{2\mu}\frac{dp}{dx}h^2 + C_1 h$

Thus, $\quad C_1 = \frac{U_0}{h} - \frac{1}{2\mu}\frac{dp}{dx}h$

Substituting these constants into (i)

$$u = \underbrace{\frac{U_0}{h}y}_{\substack{\text{linear variation}\\ \text{due to movement}\\ \text{of the upper plate}}} + \underbrace{\frac{1}{2\mu}\frac{dp}{dx}(y^2 - yh)}_{\substack{\text{velocity variation due to the}\\ \text{existence of the pressure}\\ \text{gradient}}}$$



2.24 Contd.) The factor $\frac{h^2}{2\mu U_0}\frac{dp}{dx}$ is a constant for a given problem, which we shall call β_0. The velocity profiles (u/U_0) are presented in the sketch as a function of (y/h) for various values of β_0. Note that the profile is "fuller" when the pressure decreases in the x-direction, i.e., β_0 is negative, which is known as a favorable pressure gradient.

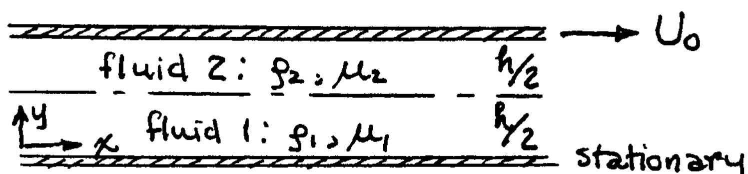
If $u = 0$, when $y = \frac{h}{2}$

$$0 = \frac{U_0}{h}\frac{h}{2} + \frac{1}{2\mu}\frac{dp}{dx}\left[\frac{h^2}{4} - \frac{h^2}{2}\right]; \quad 0 = \frac{U_0}{2} + \frac{1}{2\mu}\frac{dp}{dx}\left[-\frac{h^2}{4}\right]$$

Solving: $\frac{dp}{dx} = \frac{4\mu U_0}{h^2}$ (>0, an adverse pressure gradient)

Also, we can write: $$\frac{u}{U_0} = \frac{y}{h} + \frac{h^2}{2\mu U_0}\frac{dp}{dx}\left[\left(\frac{y}{h}\right)^2 - \left(\frac{y}{h}\right)\right]$$

2.25)



(a) τ must be constant across the fluid (including across the fluid/fluid interface)

(b) (i) $y = 0$: $u_1 = 0$ (the lower plate is stationary)

(ii) $y = h$; $u_2 = U_0$ (the upper plate moves to the right)

(iii) $y = \frac{h}{2}$: $\tau_1 = \mu_1(du_1/dy) = \mu_2(du_2/dy) = \tau_2$

(the shear is constant across the interface)

(iv) $y = \frac{h}{2}$: $u_1 = u_2$

(the velocity is continuous across the interface)

(c) For this fully-developed flow with no pressure gradient

$$\mu_1\frac{d^2u_1}{dy^2} = 0; \quad u_1 = C_1 y + C_2$$

2.25 Contd. $\mu_2 \dfrac{d^2u_2}{dy^2} = 0$; $u_2 = C_3 y + C_4$

Applying boundary condition (i): $y = 0: u_1 = 0 \Rightarrow C_2 = 0$

Applying boundary condition (ii): $y = h: u_2 = U_0$

$$U_0 = C_3 h + C_4 \quad \text{or} \quad C_4 = U_0 - C_3 h$$

Applying boundary condition (iii): $y = \dfrac{h}{2}: \tau_1 = \tau_2$

Thus, $\mu_1 C_1 = \mu_2 C_3$ or $C_3 = \dfrac{\mu_1}{\mu_2} C_1$

Applying boundary condition (iv): $y = \dfrac{h}{2}: u_1 = u_2$

$$C_1 \frac{h}{2} = C_3 \frac{h}{2} + U_0 - C_3 h$$

Rearranging: $(C_1 + C_3)\dfrac{h}{2} = U_0$

Substituting the fact that: $C_3 = \dfrac{\mu_1}{\mu_2} C_1$

$$C_1\left(1 + \frac{\mu_1}{\mu_2}\right)\frac{h}{2} = U_0$$

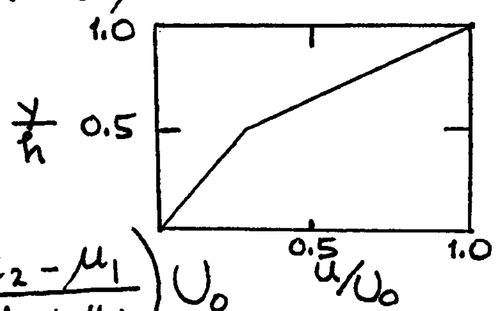
Therefore: $C_1 = \dfrac{\mu_2}{\mu_2 + \mu_1}\dfrac{2}{h}U_0$ and $C_3 = \dfrac{\mu_1}{\mu_2 + \mu_1}\dfrac{2}{h}U_0$

Then:

$$C_4 = U_0 - \frac{\mu_1}{\mu_2 + \mu_1} 2U_0 = \left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\right)U_0$$

Thus,

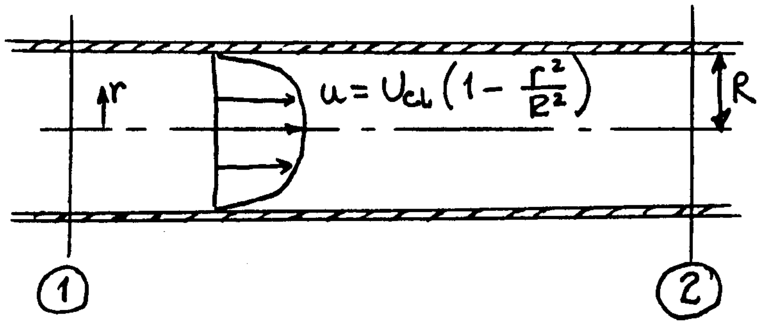
$$u_1 = \frac{\mu_2}{\mu_2 + \mu_1}\frac{2y}{h}U_0$$



$$u_2 = \frac{\mu_1}{\mu_2 + \mu_1}\frac{2y}{h}U_0 + \left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\right)U_0$$

(d) At $y = 0$: $\tau = \tau_1 = \dfrac{\mu_1 \mu_2}{\mu_1 + \mu_2}\dfrac{2U_0}{h}$

2.26)



$$\sum \vec{F} = \frac{\partial}{\partial t}\iiint \rho \vec{V}\, d(vol) + \oiint (\rho \vec{V}\cdot\hat{n}\, dA)\vec{V}$$

The first term on the right-hand side is zero for steady flow. The second term on the right-hand side is zero for fully-developed flow, since the efflux of momentum through the surface of the control volume (i.e., crossing station 2) is of equal magnitude but opposite sign to the influx of momentum through the surface of the control volume (i.e., crossing station 1). To see that this is true, let us evaluate the second term on the right-hand side:

$$\oiint (\rho \vec{V}\cdot\hat{n}\, dA)\vec{V}$$

$$= \underbrace{\int_0^R \rho \left[U_{CL}\left(1 - \frac{r^2}{R^2}\right)\right]\left[2\pi r\, dr\right]\left[U_{CL}\left(1 - \frac{r^2}{R^2}\right)\right]}_{\text{evaluated at station 2}}$$

$$- \underbrace{\int_0^R \rho \left[U_{CL}\left(1 - \frac{r^2}{R^2}\right)\right]\left[2\pi r\, dr\right]\left[U_{CL}\left(1 - \frac{r^2}{R^2}\right)\right]}_{\text{evaluated at station 1}}$$

The opposite signs result because the unit vector for area $(\hat{n})$ is directed outward for the control volume. Therefore, $\vec{V}\cdot\hat{n}\, dA > 0$ for station 2 and $\vec{V}\cdot\hat{n}\, dA < 0$ for station 1.

2.26 Contd. Since the magnitudes of the integrands are equal,

$$\oiint (\rho \vec{V} \cdot \hat{n}\, dA)\vec{V} = 0$$

Thus, $\sum F_x = p_1 A_1 - p_2 A_2 + 2\pi R(\Delta x)\tau_w = 0$

$$\tau_w = \mu \left(\frac{du}{dr}\right)_{r=R} = \mu U_{CL}\left(-\frac{2R}{R^2}\right) = -\frac{2\mu U_{CL}}{R}$$

$A_1 = A_2 = \pi R^2$. Note, that with the velocity distribution known, i.e., $u(r) = U_{CL}\left(1 - \frac{r^2}{R^2}\right)$, evaluating τ_w using the $\tau_w = \mu\left(\frac{du}{dr}\right)$ produces a negative shear force term in the momentum equation. Combining:

$$\frac{p_2 - p_1}{\Delta x} = -\frac{2}{R}\left(\frac{2\mu U_{CL}}{R}\right)$$

Thus, $\frac{dp}{dx} = -\frac{4\mu U_{CL}}{R^2}$

For U_{CL} in the direction shown, $\frac{dp}{dx} < 0$. Thus, the pressure decreases in the streamwise direction, i.e., a favorable pressure gradient exists, because of the presence of viscous forces. It is to compensate for this pressure decrease (termed a "head loss" in civil engineering terms) due to the viscous forces that pumps are needed in a pipeline. If the flow were inviscid, there would be no pressure gradient for the flow in a constant-area pipe.

The mass flow rate through the pipe is:

$$\dot{m} = \int_0^R \rho U_{CL}\left(1 - \frac{r^2}{R^2}\right) 2\pi r\, dr = \rho U_{CL} 2\pi \int_0^R \left(r - \frac{r^3}{R^2}\right) dr$$

$$\dot{m} = 2\pi \rho U_{CL}\left[\frac{R^2}{2} - \frac{R^4}{4R^2}\right] = 2\pi \rho U_{CL}\frac{R^2}{4} = \frac{\rho U_{CL} \pi R^2}{2}$$

2.26 Contd. Thus, $U_{CL} = \dfrac{2\dot{m}}{\rho \pi R^2}$

If we are to maintain the same mass flow rate (i.e., $\dot{m}_1 = \dot{m}_2$) while doubling the radius of the pipe (i.e., $R_2 = 2R_1$), then

$$\left.\frac{dp}{dx}\right|_1 = -\frac{4\mu}{R_1^2}\left(\frac{2\dot{m}_1}{\rho\pi R_1^2}\right) \quad \text{and} \quad \left.\frac{dp}{dx}\right|_2 = -\frac{4\mu}{R_2^2}\left(\frac{2\dot{m}_2}{\rho\pi R_2^2}\right)$$

Dividing one by the other and noting that $\dot{m}_1 = \dot{m}_2$:

$$\left.\frac{dp}{dx}\right|_2 = \frac{R_1^4}{R_2^4}\left.\frac{dp}{dx}\right|_1 = \frac{1}{16}\left.\frac{dp}{dx}\right|_1$$

2.27 Let us apply the integral form of the momentum equation. Since we are interested in the drag, we only need to consider the x-component of this vector equation. Refer to the solution for Problem 2.7 for the discussion of the continuity equation of this flow.

$$\Sigma F_x = \frac{\partial}{\partial t}\iiint \rho V_x \, d(vol) + \oiint (\rho \vec{V}\cdot \hat{n}\, dA) V_x$$

Since the pressure is constant over the external surface of the control volume, the only force for the left-hand side is the force of the airfoil on the fluid within the control volume, which is the negative of the drag per unit span.

$$-d = \underbrace{\rho \int_{-H}^{+H} (U_\infty \hat{i})(-\hat{i}\,dy) U_\infty}_{1}$$

$$+ \underbrace{\rho \int_{-H}^{0} \left[\left(-U_\infty \frac{y}{h}\hat{i} - v\hat{j}\right)\cdot(\hat{i}\,dy)\right] \frac{-U_\infty y}{h}}_{2}$$

$$+ \underbrace{\rho \int_{0}^{H} \left[\left(U_\infty \frac{y}{h}\hat{i} + v\hat{j}\right)\cdot(\hat{i}\,dy)\right] \frac{U_\infty y}{h}}_{2}$$

2.27 Contd.

$$+\rho\int_0^L \underbrace{\left[(U_\infty\hat{i} + v_\infty\hat{j})\cdot(\hat{j}\,dx)\right]U_\infty}_{3}$$

$$+\rho\int_0^L \underbrace{\left[(U_\infty\hat{i} - v_\infty\hat{j})\cdot(-\hat{j}\,dx)\right]U_\infty}_{4}$$

Note that because of the approximations that we have employed, the velocity at the boundaries 3 and 4 actually exceeds U_∞, while the static pressure remains unchanged. These are "second-order inconsistencies" introduced by our flow model approximations.

Note also that v_∞ is some unspecified function of x. The exact functional relationship is not important. Using the result from the application of the continuity equation in Problem 2.10:

$$2\int_0^L v_\infty\,dx = U_\infty H$$

Thus,
$$-d = -\rho U_\infty^2(2H) + \rho\frac{U_\infty^2}{H^2}\left(\frac{y^3}{3}\right|_{-H}^{0}$$
$$+\rho\frac{U_\infty^2}{H^2}\left(\frac{y^3}{3}\right|_{0}^{H} + \rho U_\infty\left[2\int_0^L v_\infty\,dx\right]$$

can be written:

$$-d = -\rho U_\infty^2(2H) + \rho U_\infty^2\frac{H}{3} + \rho U_\infty^2\frac{H}{3} + \rho U_\infty^2 H$$

$$d = \frac{1}{3}\rho U_\infty^2 H$$

$$Cd = \frac{d}{\frac{1}{2}\rho U_\infty^2 c} = \frac{\frac{1}{3}\rho U_\infty^2 H}{\frac{1}{2}\rho U_\infty^2 c} = \frac{1}{60} = 0.0167$$

2.28

This is very similar to Problem 2.27, except that the side boundaries of the control volume are streamlines. Thus, instead of using the continuity equation to determine the flow through sides (3) and (4) as was done for Problem 2.27, the continuity equation must be used to determine the relation between H_U and H_D.

Again, the pressure is constant over the external surface of the control volume for this steady, incompressible flow. Thus, the only force acting on the system of the fluid particles within the control volume is the negative of the drag.

$$-d = \rho \underbrace{\int_{-H_U}^{+H_U} \left[(U_\infty \hat{i}) \cdot (-\hat{i}\, dy) \right] U_\infty}_{(1)}$$

$$+ \rho \underbrace{\int_{-H_D}^{0} \left[\left(-U_\infty \frac{y}{H_D} \hat{i} - v\hat{j}\right) \cdot (\hat{i}\, dy) \right] \left(-U_\infty \frac{y}{H_D}\right)}_{(2)}$$

$$+ \rho \underbrace{\int_{0}^{H_D} \left[\left(U_\infty \frac{y}{H_D} \hat{i} + v\hat{j}\right) \cdot (\hat{i}\, dy) \right] \left(U_\infty \frac{y}{H_D}\right)}_{(2)}$$

There is no momentum transport across boundaries (3) and (4), since they are streamlines.

$$-d = \rho U_\infty^2 \left[2H_U \right] + \rho U_\infty^2 \left[\frac{y^3}{3H_D^2} \right]_{-H_D}^{0} + \rho U_\infty^2 \left[\frac{y^3}{3H_D^2} \right]_{0}^{+H_D}$$

$$d = \rho U_\infty^2 \left[2H_U - \frac{2}{3} H_D \right]$$

2.28 Contd. We can use the integral continuity equation to determine the relation between H_U and H_D for this steady, incompressible flow;

$$+\underbrace{\int_{-H_U}^{+H_U}\left[(U_\infty\hat{i})\cdot(-\hat{i}\,dy)\right]}_{①} + \underbrace{\int_{-H_D}^{0}\left[\left(-\frac{U_\infty y}{H_D}\hat{i}-v\hat{j}\right)\cdot(\hat{i}\,dy)\right]}_{②}$$

$$+\underbrace{\int_{0}^{H_D}\left[\left(+\frac{U_\infty y}{H_D}\hat{i}+v\hat{j}\right)\cdot(\hat{i}\,dy)\right]}_{②}=0$$

Thus, $$U_\infty 2H_U - \frac{U_\infty y^2}{2H_D}\Bigg|_{-H_D}^{0} + \frac{U_\infty y^2}{2H_D}\Bigg|_{0}^{H_D} = 0$$

Therefore, $U_\infty 2H_U = U_\infty H_D$

as was shown in Problem 2.11, $H_U = \frac{1}{2}H_D$

$$d = \rho U_\infty^2\left[H_D - \frac{2}{3}H_D\right] = \frac{1}{3}\rho U_\infty^2 H_D$$

$$C_d = \frac{d}{\frac{1}{2}\rho U_\infty^2 c} = \frac{\frac{1}{3}\rho U_\infty^2\left(\frac{1}{40}c\right)}{\frac{1}{2}\rho_\infty U_\infty^2 c} = \frac{1}{60} = 0.0167$$

As one would expect, we have gotten the same result as was obtained in Problem 2.28. Therefore, the result is not dependent on the control volume.

2.29) This is the third problem in this trilogy to illustrate that the drag coefficient is not dependent on the control volume chosen in the formulation of the problem, providing the viscous boundary layer is within the bounds of the control volume.

Applying the integral momentum equation for the steady, incompressible flow with the static pressure constant over the external surface of the control volume.

$$-d = \underbrace{\rho\int_{-2H}^{+2H}\left[(U_\infty \hat{i})\cdot(-\hat{i}\,dy)\right]U_\infty}_{1}$$

$$+\underbrace{\rho\int_{-2H}^{-H}\left[(U_\infty \hat{i} - v_0\hat{j})\cdot(\hat{i}\,dy)\right]U_\infty}_{2}$$

$$+\underbrace{\rho\int_{-H}^{0}\left[\left(-\frac{U_\infty y}{H}\hat{i} - v\hat{j}\right)\cdot(\hat{i}\,dy)\right]\left(-\frac{U_\infty y}{H}\right)}_{2}$$

$$+\underbrace{\rho\int_{0}^{H}\left[\left(\frac{U_\infty y}{H}\hat{i} + v\hat{j}\right)\cdot(\hat{i}\,dy)\right]\left(\frac{U_\infty y}{H}\right)}_{2}$$

$$+\underbrace{\rho\int_{H}^{2H}\left[(U_\infty \hat{i} + v_0\hat{j})\cdot(\hat{i}\,dy)\right]U_\infty}_{2}$$

$$+\underbrace{\rho\int_{0}^{L}\left[(U_\infty \hat{i} + v_0\hat{j})\cdot(\hat{j}\,dx)\right]U_\infty}_{3}$$

$$+\underbrace{\rho\int_{0}^{L}\left[(U_\infty \hat{i} - v_0\hat{j})\cdot(-\hat{j}\,dx)\right]U_\infty}_{4}$$

Note the similarities between this expression and that of Problem 2.27. We are using v_0 in this problem, instead

2.29 Contd.

of v_∞, to represent the y-component of velocity outside of the viscous region. Note also, v_0 is some unspecified function of x. However, since we are using the integral technique, the specifics of the function will not matter. Integrating,

$$-d = -\rho U_\infty^2(4H) + \rho U_\infty^2(H) + \rho \frac{U_\infty^2}{H^2}\left(\frac{y^3}{3}\right|_{-H}^{0}$$
$$+ \rho \frac{U_\infty^2}{H^2}\left(\frac{y^3}{3}\right|_{0}^{H} + \rho U_\infty^2(H) + \rho U_\infty\left[2\int_0^L v_0\,dx\right]$$

Note that the sum of the first, second, and fifth terms on the right-hand side is $-\rho U_\infty^2(2H)$, which is the first term on the right-hand side of the corresponding equation in the solution of Problem 2.18. Thus, as we might expect, the momentum exiting the control volume between $-2H \le y \le -H$ and $H \le y \le 2H$ at station ② is exactly balanced by the influx of momentum between $-2H \le y \le -H$ and $H \le y \le 2H$ at station ①. Thus,

$$-d = -\rho U_\infty^2(2H) + \rho U_\infty^2\frac{H}{3} + \rho U_\infty^2\frac{H}{3} + \rho U_\infty\left[2\int_0^L v_0\,dx\right]$$

Using the continuity equation

$$\frac{\partial}{\partial t}\iiint \rho\, d(vol) + \oiint \rho \vec{V}\cdot\hat{n}\, dA = 0$$

The first term is zero for steady flow. The second term is:

$$-\rho U_\infty(4H) + \rho U_\infty(H) + \rho U_\infty \frac{-y^2}{2H}\Big|_{-H}^{0} + \rho U_\infty \frac{y^2}{2H}\Big|_{0}^{H}$$
$$+ \rho U_\infty(H) + 2\rho\int_0^L v_0\,dx = 0$$

2.29 Contd.

As a result:

$$2\int_0^L v_0\,dx = U_\infty H$$

As we found in Problem 2.18. Substituting this into the momentum equation,

$$C_d = \frac{d}{\frac{1}{2}\rho U_\infty^2 c} = \frac{\frac{1}{3}\rho U_\infty^2\left(\frac{1}{40}c\right)}{\frac{1}{2}\rho U_\infty^2 c} = \frac{1}{60} = 0.0167$$

Comparing the results of Problems 2.27, 2.28, and 2.29, we see that the resulting drag coefficient is the same for all three control volumes (which all enclose the viscous wake).

2.30 Let us apply the integral form of the momentum equation. Since we are interested in the drag acting on the airfoil, which is aligned with the x-axis, we need only consider the x-component of this vector equation.

$$\sum F_x = \frac{\partial}{\partial t}\iiint \rho V_x\,d(vol) + \oiint \rho(\vec{V}\cdot\hat{n}\,dA)V_x$$

Since atmospheric pressure acts over the entire external surface of the control volume, the only force acting on the fluid particles within the control volume (i.e., the left-hand side of this equation) is the negative of the drag. Furthermore, the flow is steady and the first term on the right-hand side is zero. The flow is incompressible (or the density is constant). Thus,

$$-d = +\rho\int_{-H}^{+H}\left[(U_\infty\hat{i})\cdot(-\hat{i}\,dy)\right]U_\infty$$

2.30 Contd.

$$+ \rho \int_{-H}^{+H} \left\{ \left[U_\infty \left(1 - 0.5\cos\frac{\pi y}{2H}\right)\hat{i} \pm v\hat{j} \right] \cdot (\hat{i}\,dy) \right\} \left[U_\infty \left(1 - 0.5\cos\frac{\pi y}{2H}\right) \right] + 2\rho \int_0^L \left[U_\infty \hat{i} + v_\infty \hat{j} \right] \cdot (\hat{j}\,dx)\, U_\infty$$

$$-d = \rho U_\infty^2 y \Big|_{-H}^{+H} + \rho U_\infty^2 \int_{-H}^{+H} \left[1 - \cos\frac{\pi y}{2H} + 0.25\cos^2\frac{\pi y}{2H} \right] dy + 2\rho U_\infty \int_0^L v_\infty\, dx$$

We can use the integral form of the continuity equation to find $\int_0^L v_\infty\, dx$

$$\rho \int_{-H}^{+H} (U_\infty \hat{i}) \cdot (-\hat{i}\,dy) + \rho \int_{-H}^{+H} \left[U_\infty \left(1 - 0.5\cos\frac{\pi y}{2H}\right)\hat{i} + v\hat{j} \right] \cdot (\hat{i}\,dy) + 2\rho \int_0^L \left[U_\infty \hat{i} + v_\infty \hat{j} \right] \cdot \hat{j}\,dx = 0$$

$$-\rho U_\infty y \Big|_{-H}^{+H} + \rho U_\infty \int_{-H}^{+H} \left(1 - 0.5\cos\frac{\pi y}{2H}\right) dy + 2\rho \int_0^L v_\infty\, dx = 0$$

$$-\rho U_\infty (2H) + \rho U_\infty (2H) - \rho U_\infty \frac{1}{2}\frac{2H}{\pi}\left[1 - (-1)\right] + 2\rho \int_0^L v_\infty\, dx = 0$$

Thus, $\quad 2\rho \int_0^L v_\infty\, dx = \rho U_\infty \frac{2H}{\pi}$

Substituting this result into the momentum equation

$$-d = -\rho U_\infty^2 (2H) + \rho U_\infty^2 \left(y\Big|_{-H}^{+H} - \frac{2H}{\pi}\sin\frac{\pi y}{2H}\Big|_{-H}^{+H} \right.$$

2.30 Contd.

$$+\frac{1}{4}\left(\frac{y}{2}+\frac{2H}{4\pi}\sin\frac{\pi y}{H}\right|_{-H}^{+H} + \rho U_\infty^2\frac{2H}{\pi}$$

$$-d = -\rho U_\infty^2(2H) + \rho U_\infty^2(2H) - \rho U_\infty^2\frac{2H}{\pi}\left[1-(-1)\right]$$

$$+\rho U_\infty^2\frac{1}{4}\left[\frac{H}{2}-\left(-\frac{H}{2}\right)\right] + \rho U_\infty^2\frac{2H}{\pi}$$

$$-d = -\rho U_\infty^2 H\left[\frac{4}{\pi}-\frac{1}{4}-\frac{2}{\pi}\right] = -\rho U_\infty^2\frac{c}{40}\left[\frac{2}{\pi}-\frac{1}{4}\right]$$

$$C_d = \frac{d}{\frac{1}{2}\rho U_\infty^2 c} = 0.01933$$

2.31 As with problem 2.30, let us apply the integral equations to solve this problem. Again, since atmospheric pressure acts over the external surface of the control volume, the only force acting on the fluid particles within the control volume, i.e., the left-hand side of the integral momentum equation, is the negative of the drag. Furthermore, the flow is steady, so that the first term on the right-hand side of the momentum equation is zero; incompressible, so that the density is constant; and surfaces 3 and 4 are streamlines. Thus, there is no flux of momentum across these streamlines.

$$-d = -\rho U_\infty^2\int_{-H_U}^{+H_U}dy + \rho\int_{-H_D}^{+H_D}\left[U_\infty\left(1-0.5\cos\frac{\pi y}{2H_D}\right)\right.$$

$$\left.\text{times } dy\right]U_\infty\left(1-0.5\cos\frac{\pi y}{2H_D}\right)$$

$$-d = -\rho U_\infty^2(2H_U) + \rho U_\infty^2\int_{-H_D}^{+H_D}\left(1-1.0\cos\frac{\pi y}{2H_D}\right.$$

$$\left.+0.25\cos^2\frac{\pi y}{2H_D}\right)dy$$

2.31 Contd.

$$-d = -\rho U_\infty^2 (2H_U) + \rho U_\infty^2 \left[y - \frac{2H_D}{\pi} \sin \frac{\pi y}{2H_D} + \frac{1}{4}\left(\frac{y}{2} + \frac{2H_D}{4\pi} \sin \frac{\pi y}{2H_D} \right)\right]\Bigg|_{-H_D}^{+H_D}$$

$$-d = -\rho U_\infty^2 (2H_U) + \rho U_\infty^2 (2H_D) - \rho U_\infty^2 \frac{2}{\pi} H_D (1+1) + \rho U_\infty^2 \frac{1}{4}\left(\frac{H_D}{2} + \frac{H_D}{2} \right)$$

Using the results from Problem 2.13 that

$$H_U = \left(1 - \frac{1}{\pi}\right) H_D$$

$$-d = -\rho U_\infty^2 H_D \left[2 - \frac{2}{\pi} - 2 + \frac{4}{\pi} - \frac{1}{4} \right]$$

$$d = \rho U_\infty^2 H_D (0.3866)$$

The drag coefficient is:

$$C_d = \frac{d}{\frac{1}{2}\rho U_\infty^2 c} = \frac{\rho U_\infty^2 (0.025c)(0.3866)}{\frac{1}{2}\rho U_\infty^2 c} = 0.01933$$

Again, comparison of the results from Problems 2.30 and 2.31 shows that the drag coefficient does not depend on the control volume chosen.

2.32 (a) Referring to Table 1.2, we find that, at sea level,

$$\rho_\infty = 0.002376 \frac{lbf \cdot s^2}{ft^4} \; ; \; \mu_\infty = 3.740 \times 10^{-7} \frac{lbf \cdot s}{ft^2}$$

Thus, $$Re_{\infty,d} = \frac{\rho_\infty U_\infty d}{\mu_\infty} = \frac{(0.002376)(200)\left(\frac{1.7}{12}\right)}{3.740 \times 10^{-7}} = 1.8 \times 10^5$$

Referring to the discussion of the drag on cylinders and

2.32 Contd. spheres in Chapter 3 (see the data presented in Fig. 3.30), we see that this Reynolds number is below the critical value. Therefore, if the golf ball were smooth, the forebody boundary layer would be laminar and the drag would be relatively high. Form drag dominates for the laminar flow over a smooth golf ball. Roughening the surface of the golf ball (such as through the use of dimples) would cause the forebody boundary layer to be turbulent, resulting in significantly delayed separation and reduced form drag.

$$M_\infty = \frac{U_\infty}{a_\infty} = \frac{200}{49.02\sqrt{519}} = 0.179$$

Problem 2.32b Solution

Given: An aircraft flying at a velocity of 1810 m/s at an altitude of 30 km with a length of 32.8 m.

The properties of air at 30 km are given in Table 1.2:

$$\rho_\infty = 0.018411 kg/m^3 \qquad \mu_\infty = 1.4753\times10^{-5} kg/s\cdot m \qquad a_\infty = 301.71 m/s$$

The Reynolds number is found from:

$$Re_{\infty,L} = \frac{\rho_\infty U_\infty L}{\mu_\infty} = \frac{(0.018411 kg/m^3)(1810 m/s)(32.8 m)}{1.4753\times10^{-5} kg/s\cdot m} = 7.405\times10^7$$

Notice that the Reynolds number for this hypersonic transport is relatively large.

The Mach number is found from:

$$M_\infty = \frac{U_\infty}{a_\infty} = \frac{1810 m/s}{301.71 m/s} \approx 6.0$$

2.33 (a) $M_\infty = 3.0$ at an altitude of 20km. Using Table 1.2

$$a_\infty = 295.069 \frac{m}{s} ; \mu_\infty = 1.4216 \times 10^{-5} \frac{kg}{s \cdot m} ; \rho_\infty = 0.0889 \frac{kg}{m^3}$$

$$Re_{\infty,L} = \frac{(0.0889)[(3.0)(295.069)](10.4)}{1.4216 \times 10^{-5}} = 5.757 \times 10^7$$

Again, the Reynolds number for a high-speed airplane is in excess of 10^7

(b) Referring to the previous problem, we found the English unit values for the density and for the viscosity at sea level. Thus,

$$Re_{\infty,L} = \frac{\left(2.376 \times 10^{-3} \frac{lbf \cdot s^2}{ft^4}\right)\left[\left(160 \frac{mi}{h}\right)\frac{5280 \frac{ft}{mi}}{3600 \frac{s}{h}}\right](4.0\,ft)}{3.740 \times 10^{-7} \frac{lbf \cdot s}{ft^2}}$$

$$Re_{\infty,L} = 5.963 \times 10^6$$

2.34 Since both cycles are reversible

$$\delta q = T ds \quad \text{and} \quad \delta w = p\, dv$$

Let us first compute the changes which occur during each portion of the cycle.

For process (i)

(a) Segment AB: $q_{AB} = \int_A^B \delta q = \int_A^B T ds$

where $ds = c_v \frac{dT}{T} + R \frac{dv}{v}$ (a)

For a perfect gas: $p = \frac{RT}{v}$

Thus, if we differentiate:

$$dp = -\frac{RT}{v^2} dv + \frac{R}{v} dT$$

2.34 Contd.

So that: $$\frac{dT}{T} = \left[dp + \frac{RT\,dv}{v^2}\right]\frac{v}{RT} = \frac{dp}{p} + \frac{dv}{v}$$

Substituting this expression into equation (a), we obtain:

$$ds = c_v \frac{dp}{p} + (c_v + R)\frac{dv}{v} = c_v \frac{dp}{p} + c_p \frac{dv}{v}$$

or $$T\,ds = c_v T \frac{dp}{p} + c_p T \frac{dv}{v}$$

Thus, $$q_{AB} = \int_A^B T\,ds = \int_A^B c_v T \frac{dp}{p} + \int_A^B c_p T \frac{dv}{v}$$

$$= 0 + \int_A^B c_p \frac{pv}{R}\frac{dv}{v} = \frac{c_p p}{R}\int_A^B dv$$

$$= \frac{c_p p_A}{R}(v_B - v_A)$$

And $$w_{AB} = \int_A^B \delta w = \int_A^B p\,dv = p_A(v_B - v_A)$$

Segment BC: $$q_{BC} = \int_B^C \delta q = \int_B^C T\,ds$$

$$q_{BC} = \int_B^C c_v T \frac{dp}{p} + \int_B^C c_p T \frac{dv}{v} = c_v \frac{v}{R}\int_B^C dp + 0$$

Thus, $$q_{BC} = \frac{c_v v_B}{R}(p_C - p_B); \quad w_{BC} = \int_B^C \delta w = \int_B^C p\,dv = 0$$

Segment CD:

$$q_{CD} = \int_C^D \delta q = \int_C^D T\,ds = \int_C^D c_v T \frac{dp}{p} + \int_C^D c_p T \frac{dv}{v}$$

Thus, $$q_{CD} = c_p \frac{p}{R}\int_C^D dv = c_p \frac{p_C}{R}(v_D - v_C)$$ and

2.34 Contd.

$$w_{CD} = \int_C^D \delta w = p_c(v_D - v_c)$$

Segment DA:

$$q_{DA} = \int_D^A \delta q = \int_D^A T ds = \int_D^A c_v T \frac{dp}{p} + \int_D^A c_p T \frac{dv}{v}$$

$$q_{DA} = c_v \frac{v}{R} \int_D^A dp = c_v \frac{v_A}{R}(p_A - p_D)$$

and $w_{DA} = \int_D^A \delta w = \int_D^A p\,dv = 0$

Let us now add up the values for each of the segments:

$$\oint_{ABCDA} \delta q = q_{AB} + q_{BC} + q_{CD} + q_{DA}$$

$$= \frac{c_p p_A}{R}(v_B - v_A) + \frac{c_v v_B}{R}(p_c - p_B)$$

$$+ \frac{c_p p_c}{R}(v_D - v_c) + \frac{c_v v_A}{R}(p_A - p_D)$$

Noting that $v_A = v_D$ and $v_B = v_c$; that $p_A = p_B$ and $p_c = p_D$

$$\oint_{ABCDA} \delta q = c_p \frac{p_A}{R}(v_B - v_A) + c_v \frac{v_B}{R}(p_c - p_A)$$

$$+ c_p \frac{p_c}{R}(v_A - v_B) + c_v \frac{v_A}{R}(p_A - p_c)$$

$$= -\frac{c_p}{R}(p_c - p_A)(v_B - v_A) + \frac{c_v}{R}(p_c - p_A)(v_B - v_A)$$

$$= (p_c - p_A)(v_B - v_A)\frac{(c_v - c_p)}{R} = (p_A - p_c)(v_B - v_A)$$

since $c_p - c_v = R$. Note also that, since $(v_B - v_A) > 0$ and since $(p_A - p_c) < 0$, $\oint_{ABCD} \delta q < 0$. Thus, heat is

2.34 Contd.

transferred to the surroundings from the air in the system.

$$\oint_{ABCDA} \delta w = W_{AB} + W_{BC} + W_{CD} + W_{DA}$$

$$= p_A(v_B - v_A) + 0 + p_C(v_D - v_C) + 0$$

$$= p_A(v_B - v_A) - p_C(v_B - v_A) = (p_A - p_C)(v_B - v_A)$$

Note that $\delta w < 0$ also. Thus, work is done by the surroundings on the air in this system.

Finally, note that

$$\oint_{ABCDA} \delta q - \oint_{ABCDA} \delta w = 0$$

as should be the case for a closed cycle.

(b) The process represented by AB is a constant pressure process in which heat is added to the system. The process represented by BC is a constant volume process in which heat is added to the system. The process represented by CD is a constant-pressure, cooling process; while DA represents a constant-volume, cooling process.

(c) and (d)

$$\oint_{ABCDA} \delta q - \oint_{ABCDA} \delta w = (p_A - p_C)(v_B - v_A) - (p_A - p_C)(v_B - v_A)$$

$= 0$. As would be expected, the first law of thermodynamics is satisfied for this process.

Process (ii)

(a) For process (ii):

$$p = C_1(v - v_A) + p_A; \text{ and } v = C_2(p - p_A) + v_A$$

2.34 Contd.

where $C_1 = \dfrac{p_c - p_A}{v_c - v_A}$ and $C_2 = \dfrac{v_c - v_A}{p_c - p_A}$

Evaluating the expression for the heat flux for segment AC:

$$q_{AC} = \int_A^C \delta q = \int_A^C T\,ds = \int_A^C c_v T \frac{dp}{p} + \int_A^C c_p T \frac{dv}{v}$$

$$= \frac{c_v}{R}\int_A^C v\,dp + \frac{c_p}{R}\int_A^C p\,dv$$

$$= \frac{c_v}{R}\int_A^C \left[C_2(p-p_A) + v_A\right]dp + \frac{c_p}{R}\int_A^C\left[C_1(v-v_A)+p_A\right]dv$$

Examining this expression, it is clear that:

$$q_{CA} = \int_C^A \delta q = -\int_A^C \delta q$$

Thus, $q_{ACA} = \int_A^C \delta q + \int_C^A \delta q = 0$

Similarly,

$$w_{AC} = \int_A^C \delta w = \int_A^C p\,dv = \int_A^C \left[C_1(v-v_A) + p_A\right] dv$$

Examining this expression, it is dear that:

$$w_{CA} = \int_C^A \delta w = -\int_A^C \delta w;\ \text{so that}$$

$$w_{ACA} = \int_A^C \delta w + \int_C^A \delta w = 0$$

(b) For the process designated AC, heat is added to the system; while the system is cooled for the segment designated CA.

2.34 Contd.

(c) and (d)

$$\oint_{ACA} \delta q - \oint_{ACA} \delta w = 0 - 0 = 0$$

consistent with the first law of thermodynamics for this process.

Note that the net heat transferred from the system to the surroundings during process (i), which is $(p_A - p_C)(v_B - v_A)$, is not equal to the net heat transferred from the system to the surroundings during process (ii), which is zero. Since the first law of thermodynamics must be satisfied (and has been shown to be satisfied), the same is true for the work done during the two processes. Both the heat transfer and the work done are path dependent phenomena.

2.35 (a) Yes, entropy is a property and is, therefore, independent of the path for the process.

(b) Even if the processes were irreversible, $s_C - s_A$ would be the same as determined in Problem (2.25). Entropy is a property and its change, therefore, depends only on the gas properties at the end points of the process. Recall that, once any two properties of a gas (which is in equilibrium) are known, the remaining properties of the gas can be determined. Thus, (p_C and v_C) and (p_A and v_A) are the same whether the process is reversible or irreversible, $s_C - s_A$ does not depend on the path of the process.

2.36) When deriving equation (2.32), we used the definitions for τ_{ij} which were given on pages 37 and 38. Thus, the fluid must satisfy the criteria given on page 37 that the stress components are a linear function of the components of the rate of strain, that the relations between the stress components and the rate-of-strain components are invariant to coordinate transformations, and that the stress components reduce to the hydrostatic pressure when all velocity gradients are zero.

In addition, we ignored effects associated with very high gas temperatures, which result in dissociation, ionization, and chemical reactions. E.g., the nitrogen molecules of air begin to dissociate at approximately 4000K. Thus, when the temperature of the gas is extremely high, one must consider additional energy transfer mechanisms, such as radiative heat transfer

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2.37) For the adiabatic, inviscid flow, the terms of the right-hand side of equation (2.32a) are zero. Thus, equation (2.32a) becomes:

$$\rho \frac{dh}{dt} - \frac{dp}{dt} = 0$$

But $T ds = du_e + p dv$

which can be rewritten using the definition for the enthalpy ($u_e = h - pv$) and the definition for the specific volume ($v = \frac{1}{\rho}$). Thus,

$$\rho T \frac{ds}{dt} = \rho \left(\frac{dh}{dt} - v \frac{dp}{dt} \right) = \rho \frac{dh}{dt} - \frac{dp}{dt}$$

which is equal to zero for this flow. Hence, $\frac{ds}{dt} = 0$.

Note that, if this flow is initially isentropic and if

2.37 Contd. the fluid along each streamline undergoes adiabatic, reversible changes, the flow is everywhere isentropic. The requirement of reversible flow implies that the flow is inviscid. Hence, the results obtained using the thermodynamic relations in the problem are consistent with those obtained using Kelvin's Theorem.

In the boundary layer near the surface, the effects of viscosity and of heat transfer produce variations in the entropy and in the stagnation enthalpy between neighboring streamlines.

2.38 At 10,000 feet, the free-stream static temperature is 483.03 °R. $U_1 = 130$ mi/h $= 190.67$ ft/s. Thus, we can use these to calculate the total enthalpy:

$$H_t = h_1 + \frac{1}{2}U_1^2$$

Assuming that the flow is a perfect gas: $H_t = c_p T_t$ and $h_1 = c_p T_1$, we can write:

$$T_t = T_1 + \frac{U_1^2}{2c_p}$$

So that:

$$T_t = 483.03\,°R + \frac{\left(190.67\,\frac{ft}{s}\right)^2}{2\left(0.2404\,\frac{Btu}{lbm\,°R}\right)\left(778.2\,\frac{ft\,lbf}{Btu}\right)\left(32.174\,\frac{ft\,lbm}{lbf\,s^2}\right)}$$

$$T_t = 483.03\,°R + 3.02\,°R = 486.05\,°R$$

The kinetic energy term is relatively small. As a result, the total (or stagnation) temperature is not much greater than the static temperature. Therefore, convective heating would not be a problem for aircraft flying at this speed.

2.39) For an airplane flying at 80,000 feet, $a_1 = 977.62 \frac{ft}{s}$ and $T_1 = 397.69\,°R$. Thus, $U_1 = M_1 a_1 = 2932.86 \frac{ft}{s}$.

Following the relations developed for the last problem:

$$T_t = T_1 + \frac{U_1^2}{2c_p} = 397.69 + \frac{(2932.86)^2}{2(0.2404)(778.2)(32.174)}$$

Thus, $T_t = 397.69 + 714.53 = 1112.22\,°R$

Convective heating could be a significant problem for aircraft flying at these speeds. The total temperature is in excess of 650 °F, which could affect the strength of many materials subjected to this environment.

The total temperature could have been calculated using the relation:

$$T_t = T_\infty\left(1 + \frac{\gamma-1}{2} M_\infty^2\right) \text{ or } T_t = T_1\left(1 + \frac{\gamma-1}{2} M_1^2\right)$$

which will be developed in the next problem.

2.40) We start with the integral form of the energy equation for a one-dimensional, steady, adiabatic flow:

$$H_t = h + \frac{U^2}{2}$$

For a perfect gas: $H_t = c_p T_t$ and $h = c_p T$. Thus, we can rewrite this equation as:

$$c_p T_t = c_p T + \frac{1}{2} U^2$$

or

$$\frac{T_t}{T} = 1 + \frac{1}{2} \frac{U^2}{c_p T}$$

But we also know that: $c_p = \frac{\gamma R}{\gamma - 1}$ for a perfect gas, 2.40 Contd. Thus, $T_t = \left(1 + \frac{\gamma-1}{2} M^2\right) T$

which defines the relation between the total temperature (T_t), the static temperature (T), and Mach number (M) for the adiabatic flow of a perfect gas.

The entropy change equation is:

$$s - s_r = c_p \ln \frac{T}{T_r} - R \ln \frac{p}{p_r}$$

For the isentropic flow of a perfect gas:

$$\frac{\gamma R}{\gamma - 1} \ln \frac{T}{T_r} - R \ln \frac{p}{p_r} = 0$$

Thus,

$$\ln \left(\frac{T}{T_r}\right)^{\gamma/(\gamma-1)} = \ln \left(\frac{p}{p_r}\right)$$

Designating the stagnation condition of a gas which is brought to rest through an isentropic process (i.e., one which is both adiabatic and reversible) represented by the symbol "t" as the reference condition "r".

$$\frac{p}{p_{t1}} = \left(\frac{T}{T_t}\right)^{\frac{\gamma}{\gamma-1}} = \left[\frac{1}{1 + \frac{\gamma-1}{2} M^2}\right]^{\frac{\gamma}{\gamma-1}}$$

This equation can be used to calculate the static pressure of a perfect gas which has undergone an isentropic expansion from a stagnant gas (whose pressure is p_{t1} and whose temperature is T_t) to a Mach number M_1. Conditions are those for the isentropic flow of a perfect gas.