Instructor's Manual

Elementary Linear Algebra

A Matrix Approach

Second Edition

Spence Insel Friedberg

PEARSON
Prentice Hall

Upper Saddle River, NJ 07458

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Pearson Prentice Hall
Pearson Education, Inc.
Upper Saddle River, NJ 07458

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

ISBN 0-13-230752-9
978-0-13-230752-9

Pearson Education Ltd., *London*
Pearson Education Australia Pty. Ltd., *Sydney*
Pearson Education Singapore, Pte. Ltd.
Pearson Education North Asia Ltd., *Hong Kong*
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Preface

This Instructor's Manual contains solutions to the exercises in the second edition of *Elementary Linear Algebra: A Matrix Approach*. It is intended for the use of instructors rather than students, and so many solutions are written more succinctly than those in the Student Solutions Manual (ISBN 0-13-239734-X). In a cluster of similar exercises (such as Exercises 27–34 in Section 1.4), we usually work only one or two in detail and provide answers to the others. The Student Solutions Manual, which is available for student purchase, contains detailed solutions to selected odd-numbered exercises.

Additional materials for use with our book are available at

www.math.ilstu.edu/matrix

On this site, you will find data files for the technology exercises in our book that can be used with MATLAB or Texas Instrument calculators. There is also an appendix on mathematical proof, written by the authors, for use in a linear algebra course in which mathematical proof is an emphasis.

Other resources for an instructor are available on the publisher's website, whose address is

www.prenhall.com/spence

Planning Your Course

The chart below lists the sections of the text, categorized as essential material and supplementary material/applications. The 26 sections listed as essential material contain the material described in the Linear Algebra Curriculum Study Group's core syllabus as well as a thorough introduction to linear transformations. Some of these sections contain optional subsections (for example, Sections 3.1, 3.2, and 5.2) that can be included or excluded at the discretion of the instructor. The sections listed as supplementary material/applications may also be omitted depending on the nature and objectives of your course. In a semester course of 3 or 4 hours, there should be time to include some of the supplementary material or applications. We believe that a first course in linear algebra is strengthened significantly by the inclusion of applications and therefore recommend that, whenever possible, at least one application from each of Sections 1.5, 2.2, and 5.5 be included.

Essential Material	Supplementary Material and Applications
1.1 Matrices and Vectors	
1.2 Linear Combinations, Matrix-Vector Products, and Special Matrices	
1.3 Systems of Linear Equations	
1.4 Gaussian Elimination	
	1.5 Applications of Systems of Linear Equations
1.6 The Span of a Set of Vectors	
1.7 Linear Dependence and Linear Independence	
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	2.2 Applications of Matrix Multiplication
2.3 Invertibility and Elementary Matrices	
2.4 The Inverse of a Matrix	
	2.5 Partitioned Matrices and Block Multiplication
	2.6 The *LU* Decomposition of a Matrix
2.7 Linear Transformations and Matrices	
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3.2 Properties of Determinants	
4.1 Subspaces	
4.2 Basis and Dimension	
4.3 The Dimensions of Subspaces Associated with a Matrix	
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	4.5 Matrix Representations of Linear Operators
5.1 Eigenvalues and Eigenvectors	
5.2 The Characteristic Polynomial	
5.3 Diagonalization of Matrices	
	5.4 Diagonalization of Linear Operators
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Essential Material	**Supplementary Material and Applications**
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6.2 Orthogonal Vectors	
6.3 Orthogonal Projections	
6.4 Least-Squares Approximations and Orthogonal Projection Matrices	
6.5 Orthogonal Matrices and Operators	
6.6 Symmetric Matrices	
	6.7 Singular Value Decomposition
	6.8 Principal Component Analysis
	6.9 Rotations of $\mathcal{R}^3$ and Computer Graphics
	7.1 Vector Spaces and Their Subspaces
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	7.3 Basis and Dimension
	7.4 Matrix Representations of Linear Operators
	7.5 Inner Product Spaces
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	Appendix D MATLAB
	Appendix E The Uniqueness of the Reduced Row Echelon Form

Chapter 1

Matrices, Vectors, and Systems of Linear Equations

1.1 MATRICES AND VECTORS

1. $\begin{bmatrix} 8 & -4 & 20 \\ 12 & 16 & 4 \end{bmatrix}$ **2.** $\begin{bmatrix} -2 & 1 & -5 \\ -3 & -4 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & -4 & 24 \\ 8 & 10 & -4 \end{bmatrix}$ **4.** $\begin{bmatrix} 8 & -3 & 11 \\ 13 & 18 & 11 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 4 \\ 0 & 6 \\ -4 & 8 \end{bmatrix}$ **6.** $\begin{bmatrix} 4 & 7 \\ -1 & 10 \\ 1 & 9 \end{bmatrix}$

7. $\begin{bmatrix} 3 & -1 & 3 \\ 5 & 7 & 5 \end{bmatrix}$ **8.** $\begin{bmatrix} 4 & 7 \\ -1 & 10 \\ 1 & 9 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -1 & 7 \\ 1 & 1 & -3 \end{bmatrix}$

11. $\begin{bmatrix} -1 & -2 \\ 0 & -3 \\ 2 & -4 \end{bmatrix}$ **12.** $\begin{bmatrix} -1 & -2 \\ 0 & -3 \\ 2 & -4 \end{bmatrix}$

13. $\begin{bmatrix} -3 & 1 & -2 & -4 \\ -1 & -5 & 6 & 2 \end{bmatrix}$ **14.** $\begin{bmatrix} -12 & 0 \\ 6 & 15 \\ -3 & -9 \\ 0 & 6 \end{bmatrix}$

15. $\begin{bmatrix} -6 & 2 & -4 & -8 \\ -2 & -10 & 12 & 4 \end{bmatrix}$

16. $\begin{bmatrix} -8 & 4 & -2 & -0 \\ 0 & 10 & -6 & 4 \end{bmatrix}$ **17.** not possible

18. $\begin{bmatrix} 7 & -3 & 3 & 4 \\ 1 & 0 & -3 & -4 \end{bmatrix}$ **19.** $\begin{bmatrix} 7 & 1 \\ -3 & 0 \\ 3 & -3 \\ 4 & -4 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 1 & 4 & 12 \\ 3 & 25 & -24 & -2 \end{bmatrix}$ **21.** not possible

22. $\begin{bmatrix} 12 & 4 \\ -4 & 20 \\ 8 & -24 \\ 16 & -8 \end{bmatrix}$ **23.** $\begin{bmatrix} -7 & -1 \\ 3 & 0 \\ -3 & 3 \\ -4 & 4 \end{bmatrix}$

24. $\begin{bmatrix} -7 & -1 \\ 3 & 0 \\ -3 & 3 \\ -4 & 4 \end{bmatrix}$ **25.** -2 **26.** 0

27. $\begin{bmatrix} 3 \\ 0 \\ 2\pi \end{bmatrix}$ **28.** $\begin{bmatrix} -2 \\ 1.6 \\ 5 \end{bmatrix}$ **29.** $\begin{bmatrix} 2 \\ 2e \end{bmatrix}$

30. $\begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$ **31.** $[2 \quad -3 \quad 0.4]$ **32.** $[2e \quad 12 \quad 0]$

33. $\begin{bmatrix} 150 \\ 150\sqrt{3} \\ 10 \end{bmatrix}$ mph

34. (a) The swimmer's velocity is $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ mph.

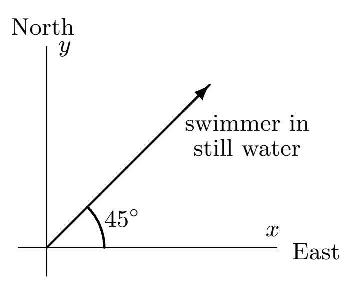

Figure for Exercise 34(a)

(b) The water's velocity is $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ mph. So the new velocity of the swimmer is $\mathbf{u} + \mathbf{v} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2}+1 \end{bmatrix}$ mph. The corresponding speed is $\sqrt{5 + 2\sqrt{2}} \approx 2.798$ mph.

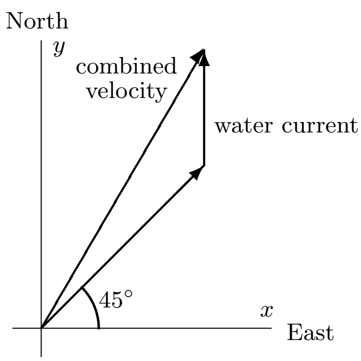

Figure for Exercise 34(b)

35. (a) $\begin{bmatrix} 150\sqrt{2} + 50 \\ 150\sqrt{2} \end{bmatrix}$ mph

(b) $50\sqrt{37 + 6\sqrt{2}} \approx 337.21$ mph

36. The three components of the vector represent, respectively, the average blood pressure, average pulse rate, and the average cholesterol reading of the 20 people.

37. True **38.** True **39.** True

40. False, a scalar multiple of the zero matrix is the zero matrix.

41. False, the transpose of an $m \times n$ matrix is an $n \times m$ matrix.

42. True

43. False, the rows of B are 1×4 vectors.

44. False, the $(3, 4)$-entry of a matrix lies in row 3 and column 4.

45. True

46. False, an $m \times n$ matrix has mn entries.

47. True **48.** True **49.** True

50. False, matrices must have the same size to be equal.

51. True **52.** True **53.** True

54. True **55.** True **56.** True

57. Suppose that A and B are $m \times n$ matrices.

(a) The jth column of $A + B$ and $\mathbf{a}_j + \mathbf{b}_j$ are $m \times 1$ vectors. The ith component of the jth column of $A + B$ is the (i, j)-entry of $A+B$, which is $a_{ij} + b_{ij}$. By definition, the ith components of $\mathbf{a}_j$ and $\mathbf{b}_j$ are a_{ij} and b_{ij}, respectively. So the ith component of $\mathbf{a}_j + \mathbf{b}_j$ is also $a_{ij} + b_{ij}$. Thus the jth columns of $A + B$ and $\mathbf{a}_j + \mathbf{b}_j$ are equal.

(b) The proof is similar to the proof of (a).

58. Since A is an $m \times n$ matrix, $0A$ is also an $m \times n$ matrix. Because the (i, j)-entry of $0A$ is $0a_{ij} = 0$, we see that $0A$ equals the $m \times n$ zero matrix.

59. Since A is an $m \times n$ matrix, $1A$ is also an $m \times n$ matrix. Because the (i, j)-entry of $1A$ is $1a_{ij} = a_{ij}$, we see that $1A$ equals A.

60. Because both A and B are $m \times n$ matrices, both $A+B$ and $B+A$ are $m \times n$ matrices. The (i, j)-entry of $A + B$ is $a_{ij} + b_{ij}$, and the (i, j)-entry of $B + A$ is $b_{ij} + a_{ij}$. Since $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ by the commutative property of addition of real numbers, the (i, j)-entries of $A+B$ and $B+A$ are equal for all i and j. Thus, since the matrices $A + B$ and $B + A$ have the same size and all pairs of corresponding entries are equal, $A+B = B + A$.

61. If O is the $m \times n$ zero matrix, then both A and $A + O$ are $m \times n$ matrices; so we need only show they have equal corresponding entries. The (i, j)-entry of $A + O$ is $a_{ij} + 0 = a_{ij}$, which is the (i, j)-entry of A.

62. The proof is similar to the proof of Exercise 61.

63. The matrices $(st)A$, tA, and $s(tA)$ are all $m \times n$ matrices; so we need only show that the corresponding entries of $(st)A$ and $s(tA)$ are equal. The (i, j)-entry of $s(tA)$ is s times the (i, j)-entry of tA, and so it equals $s(ta_{ij}) = st(a_{ij})$, which is the (i, j)-entry of $(st)A$. Therefore $(st)A = s(tA)$.

64. The matrices $(s+t)A$, sA, and tA are $m \times n$ matrices. Hence the matrices $(s+t)A$ and $sA+tA$ are $m \times n$ matrices; so we need only show they have equal corresponding entries. The (i, j)-entry of $sA + tA$ is the sum of the (i, j)-entries of sA and tA, that is, $sa_{ij} + ta_{ij}$. And the (i, j)-entry of $(s + t)A$ is $(s + t)a_{ij} = sa_{ij} + ta_{ij}$.

65. The matrices $(sA)^T$ and sA^T are $n \times m$ matrices; so we need only show they have equal corresponding entries. The (i, j)-entry of $(sA)^T$ is the (j, i)-entry of sA, which is sa_{ji}. The (i, j)-entry of sA^T is the product of s and the (i, j)-entry of A^T, which is also sa_{ji}.

66. The matrix A^T is an $n \times m$ matrix; so the matrix $(A^T)^T$ is an $m \times n$ matrix. Thus we need only show that $(A^T)^T$ and A have equal corresponding entries. The (i, j)-entry of $(A^T)^T$ is the (j,i)-entry of A^T, which in turn is the (i,j)-entry of A.

67. If $i \neq j$, then the (i,j)-entry of a square zero matrix is 0. Because such a matrix is square, it is a diagonal matrix.

68. If B is a diagonal matrix, then B is square. Hence cB is square, and the (i,j)-entry of cB is $cb_{ij} = c \cdot 0 = 0$ if $i \neq j$. Thus cB is a diagonal matrix.

69. If B is a diagonal matrix, then B is square. Since B^T is the same size as B in this case, B^T is square. If $i \neq j$, then the (i,j)-entry of B^T is $b_{ji} = 0$. So B^T is a diagonal matrix.

70. Suppose that B and C are $n \times n$ diagonal matrices. Then $B + C$ is also an $n \times n$ matrix. Moreover, if $i \neq j$, the (i,j)-entry of $B + C$ is $b_{ij} + c_{ij} = 0 + 0 = 0$. So $B + C$ is a diagonal matrix.

71. $\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$ and $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 8 \\ 6 & 8 & 4 \end{bmatrix}$

72. Let A be a symmetric matrix. Then $A = A^T$. So the (i,j)-entry of A equals the (i,j)-entry of A^T, which is the (j,i)-entry of A.

73. Let O be a square zero matrix. The (i,j)-entry of O is zero, whereas the (i,j)-entry of O^T is the (j,i)-entry of O, which is also zero. So $O = O^T$, and hence O is a symmetric matrix.

74. By Theorem 1.2(b), $(cB)^T = cB^T = cB$.

75. By Theorem 1.1(a) and Theorem 1.2(a) and (c), we have

$$(B+B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T.$$

76. By Theorem 1.2(a), $(B + C)^T = B^T + C^T = B + C$.

77. No. Consider $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 8 \\ 6 & 8 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 6 \\ 5 & 8 \end{bmatrix}$.

78. Let A be a diagonal matrix. If $i \neq j$, then $a_{ij} = 0$ and $a_{ji} = 0$ by definition. Also, $a_{ij} = a_{ji}$ if $i = j$. So every entry of A equals the corresponding entry of A^T. Therefore $A = A^T$.

79. The (i,i)-entries must all equal zero. By equating the (i,i)-entries of A^T and $-A$, we obtain $a_{ii} = -a_{ii}$, and so $a_{ii} = 0$.

80. Take $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. If C is any 2×2 skew-symmetric matrix, then $C^T = -C$. Therefore $c_{12} = -c_{21}$. By Exercise 79, $c_{11} = c_{22} = 0$. So

$$C = \begin{bmatrix} 0 & -c_{21} \\ c_{21} & 0 \end{bmatrix} = -c_{21} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -c_{21}B.$$

81. Let $A_1 = \frac{1}{2}(A + A^T)$ and $A_2 = \frac{1}{2}(A - A^T)$. It is easy to show that $A = A_1 + A_2$. By Exercises 75 and 74, A_1 is symmetric. Also, by Theorem 1.2(b), (a), and (c), we have

$$\begin{aligned} A_2^T &= \frac{1}{2}(A - A^T)^T = \frac{1}{2}[A^T - (A^T)^T] \\ &= \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -A_2. \end{aligned}$$

82. (a) Because the (i,i)-entry of $A+B$ is $a_{ii}+b_{ii}$, we have

$$\begin{aligned} &\text{trace}(A + B) \\ &\quad = (a_{11} + b_{11}) + \cdots + (a_{nn} + b_{nn}) \\ &\quad = (a_{11} + \cdots + a_{nn}) + (b_{11} + \cdots + b_{nn}) \\ &\quad = \text{trace}(A) + \text{trace}(B). \end{aligned}$$

(b) The proof is similar to the proof of (a).

(c) The proof is similar to the proof of (a).

83. The ith component of $a\mathbf{p}+b\mathbf{q}$ is $ap_i + bq_i$, which is nonnegative. Also, the sum of the components of $a\mathbf{p} + b\mathbf{q}$ is

$$\begin{aligned} &(ap_1 + bq_1) + \cdots + (ap_n + bq_n) \\ &\quad = a(p_1 + \cdots + p_n) + b(q_1 + \cdots + q_n) \\ &\quad = a(1) + b(1) = a + b = 1. \end{aligned}$$

84. (a) $\begin{bmatrix} 6.5 & -0.5 & -1.9 & -2.8 \\ 9.6 & -2.9 & 1.5 & -3.0 \\ 17.4 & 0.4 & -15.5 & 5.2 \\ -1.0 & -3.7 & -7.3 & 17.5 \\ 5.2 & 1.4 & 3.5 & 16.8 \end{bmatrix}$

(b) $\begin{bmatrix} -1.3 & 3.4 & -4.0 & 10.4 \\ 3.0 & 4.9 & -2.4 & 6.6 \\ -3.9 & -4.1 & 9.4 & -8.6 \\ 1.7 & -0.1 & -14.5 & -0.2 \\ -4.7 & 4.1 & -0.7 & -1.8 \end{bmatrix}$

(c) $\begin{bmatrix} 3.9 & 7.4 & 10.3 & -0.1 & 1.9 \\ 0.8 & -0.3 & -1.1 & -2.5 & 2.3 \\ -2.6 & 0.2 & -7.2 & -9.7 & 2.1 \\ 1.6 & 0.2 & 0.6 & 11.6 & 10.6 \end{bmatrix}$

1.2 LINEAR COMBINATIONS, MATRIX-VECTOR PRODUCTS, AND SPECIAL MATRICES

1. $\begin{bmatrix} 12 \\ 14 \end{bmatrix}$ **2.** $\begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix}$ **3.** $\begin{bmatrix} 9 \\ 0 \\ 10 \end{bmatrix}$ **4.** $\begin{bmatrix} 22 \\ 32 \end{bmatrix}$

5. $\begin{bmatrix} a \\ b \end{bmatrix}$ **6.** $[18]$ **7.** $\begin{bmatrix} 22 \\ 5 \end{bmatrix}$ **8.** $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

9. $\begin{bmatrix} sa \\ tb \\ uc \end{bmatrix}$ **10.** $[6]$ **11.** $\begin{bmatrix} 2 \\ -6 \\ 10 \end{bmatrix}$ **12.** $\begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$

13. $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$ **14.** $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ **15.** $\begin{bmatrix} 21 \\ 13 \end{bmatrix}$ **16.** $\begin{bmatrix} 26 \\ 9 \end{bmatrix}$

17. $\frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{e}_1$

19. $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} 3-\sqrt{3} \\ 3\sqrt{3}+1 \end{bmatrix}$

20. $\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} \sqrt{3}-2 \\ 1+2\sqrt{3} \end{bmatrix}$

21. $\frac{1}{2}\begin{bmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{bmatrix}$, $\frac{1}{2}\begin{bmatrix} \sqrt{3}-3 \\ 3\sqrt{3}+1 \end{bmatrix}$

22. $\frac{-1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\frac{-1}{\sqrt{2}}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ **23.** $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

24. $\frac{1}{2}\begin{bmatrix} 4\sqrt{3}+1 \\ \sqrt{3}-4 \end{bmatrix}$ **25.** $\frac{1}{2}\begin{bmatrix} 3-\sqrt{3} \\ 3\sqrt{3}+1 \end{bmatrix}$

26. $\frac{1}{2}\begin{bmatrix} 2-5\sqrt{3} \\ 2\sqrt{3}+5 \end{bmatrix}$ **27.** $\frac{1}{2}\begin{bmatrix} 3 \\ -3\sqrt{3} \end{bmatrix}$

28. $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ **29.** $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

30. $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ -4 \end{bmatrix}$ **31.** not possible

32. $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ **33.** not possible

34. $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

35. $\begin{bmatrix} -1 \\ 11 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

36. $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ -1 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

37. $\begin{bmatrix} 3 \\ 8 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} -2 \\ -5 \end{bmatrix}$

38. $\begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{a+2b}{3}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{a-b}{3}\right)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

39. not possible **40.** $\mathbf{u} = 4\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

41. $\mathbf{u} = 0\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$

42. $\mathbf{u} = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

43. $\mathbf{u} = (-4)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-5)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-6)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

44. $\mathbf{u} = 0\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

45. True

46. False. If the coefficients of the linear combination $3\begin{bmatrix} 2 \\ 2 \end{bmatrix} + (-6)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ were positive, the sum could not equal the zero vector.

47. True **48.** True **49.** True

50. False, the matrix-vector product of a 2×3 matrix and a 3×1 vector is a 2×1 vector.

51. False, the matrix-vector product is a linear combination of the *columns* of the matrix.

52. False, the product of a matrix and a standard vector is a column of the matrix.

53. True

54. False, the matrix-vector product of an $m \times n$ matrix and a vector in $\mathcal{R}^n$ yields a vector in $\mathcal{R}^m$.

55. False, every vector in $\mathcal{R}^2$ is a linear combination of two *nonparallel* vectors.

56. True

57. False, a standard vector is a vector with a single component equal to 1 and the others equal to 0.

58. True

59. False, consider $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

60. True

61. False, $A_\theta \mathbf{u}$ is the vector obtained by rotating $\mathbf{u}$ by a *counterclockwise* rotation of the angle θ.

62. False, consider $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

63. True **64.** True

65. If $\theta = 0$, then $A_\theta = I_2$. So $A_\theta \mathbf{v} = I_2 \mathbf{v} = \mathbf{v}$ by Theorem 1.3(h).

66. We have $A_{180^\circ}\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{v} = -I_2\mathbf{v} = -\mathbf{v}$.

67. Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Then $A_\theta(A_\beta \mathbf{v})$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a\cos\beta - b\sin\beta \\ a\sin\beta + b\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos\theta\cos\beta - b\cos\theta\sin\beta \\ a\sin\theta\cos\beta - b\sin\theta\sin\beta \end{bmatrix} + \begin{bmatrix} -a\sin\theta\sin\beta - b\sin\theta\cos\beta \\ a\cos\theta\sin\beta + b\cos\theta\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos(\theta+\beta) - b\sin(\theta+\beta) \\ a\sin(\theta+\beta) + b\cos(\theta+\beta) \end{bmatrix}$$

$$= A_{\theta+\beta}\mathbf{v}.$$

68. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$. Then $A_\theta^T(A_\theta \mathbf{u})$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta - b\sin\theta\cos\theta \\ -a\sin\theta\cos\theta + b\sin^2\theta \end{bmatrix} + \begin{bmatrix} a\sin^2\theta + b\sin\theta\cos\theta \\ a\sin\theta\cos\theta + b\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} a(\sin^2\theta + \cos^2\theta) \\ b(\sin^2\theta + \cos^2\theta) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{u}.$$

Similarly, $A_\theta(A_\theta^T \mathbf{u}) = \mathbf{u}$.

69. (a) As in Example 3, the populations are given by the entries of $A\begin{bmatrix} 400 \\ 300 \end{bmatrix} = \begin{bmatrix} 349 \\ 351 \end{bmatrix}$; so there will be 349,000 people in the city and 351,000 in the suburbs.

(b) Computing $A\begin{bmatrix} 349 \\ 351 \end{bmatrix} = \begin{bmatrix} 307.180 \\ 392.820 \end{bmatrix}$, we see that there will be 307,180 people in the city and 392,820 in the suburbs.

70. $A\mathbf{u} = a\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + b\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + c\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

71. $A\mathbf{u} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$, the reflection of $\mathbf{u}$ about the y-axis

72. We have

$$A(A\mathbf{u}) = A\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{u}.$$

73. $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

74. (a) $C = A_{180^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) We have

$$A(C\mathbf{u}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}.$$

In a similar fashion, we have $C(A\mathbf{u}) = \begin{bmatrix} a \\ -b \end{bmatrix} = B\mathbf{u}$ and $B(C\mathbf{u}) = C(B\mathbf{u}) = A\mathbf{u}$.

(c) The first equation shows that reflecting about the x-axis can be accomplished by either first rotating by 180° and then reflecting about the y-axis, or first reflecting about the y-axis and then rotating by 180°.
The second equation shows that reflecting about the y-axis may be accomplished either by first rotating by 180° and then reflecting about the x-axis, or first reflecting about the x-axis and then rotating by 180°.

75. $A\mathbf{u} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, the projection of $\mathbf{u}$ on the x-axis

76. This exercise is similar to Exercise 72.

77. If $\mathbf{v} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, then $A\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = \mathbf{v}$.

78. $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

79. (a) We have

$$A(C\mathbf{u}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -a \\ 0 \end{bmatrix},$$

and

$$C(A\mathbf{u}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \end{bmatrix}.$$

(b) Rotating a vector by 180° and then projecting the result on the x-axis is equivalent to projecting a vector on the x-axis and then rotating the result by 180°.

80. The sum of the two linear combinations

$$a\mathbf{u}_1 + b\mathbf{u}_2 \quad \text{and} \quad c\mathbf{u}_1 + d\mathbf{u}_2$$

is

$$(a\mathbf{u}_1 + b\mathbf{u}_2) + (c\mathbf{u}_1 + d\mathbf{u}_2) = (a+c)\mathbf{u}_1 + (b+d)\mathbf{u}_2,$$

which is also a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

81. Write $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$ and $\mathbf{w} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2$, where a_1, a_2, b_1, and b_2 are scalars. A linear combination of $\mathbf{v}$ and $\mathbf{w}$ has the form

$$\begin{aligned} c\mathbf{v} + d\mathbf{w} &= c(a_1\mathbf{u}_1 + a_2\mathbf{u}_2) + d(b_1\mathbf{u}_1 + b_2\mathbf{u}_2) \\ &= (ca_1 + db_1)\mathbf{u}_1 + (ca_2 + db_2)\mathbf{u}_2, \end{aligned}$$

which is also a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

82. The proof is similar to that of Exercise 81.

83. We have

$$\begin{aligned} A(c\mathbf{u}) &= (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + \cdots + (cu_n)\mathbf{a}_n \\ &= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n) = c(A\mathbf{u}). \end{aligned}$$

Similarly, $(cA)\mathbf{u} = c(A\mathbf{u})$.

84. We have

$$\begin{aligned} (A+B)\mathbf{u} &= u_1(\mathbf{a}_1 + \mathbf{b}_1) + \cdots + u_n(\mathbf{a}_n + \mathbf{b}_n) \\ &= u_1\mathbf{a}_1 + u_1\mathbf{b}_1 + \cdots + u_n\mathbf{a}_n + u_n\mathbf{b}_n \\ &= (u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n) \\ &\qquad + (u_1\mathbf{b}_1 + \cdots + u_n\mathbf{b}_n) \\ &= A\mathbf{u} + B\mathbf{u}. \end{aligned}$$

85. We have $A\mathbf{e}_j =$ $0\mathbf{a}_1 + \cdots + 0\mathbf{a}_{j-1} + 1\mathbf{a}_j + 0\mathbf{a}_{j+1} + \cdots + 0\mathbf{a}_n = \mathbf{a}_j$.

86. Suppose $B\mathbf{w} = A\mathbf{w}$ for all $\mathbf{w}$. Let $\mathbf{w} = \mathbf{e}_j$. Then $B\mathbf{e}_j = A\mathbf{e}_j$. From Theorem 1.3(e), it follows that $\mathbf{b}_j = \mathbf{a}_j$ for all j. So $B = A$.

87. The vector $A\mathbf{0}$ is an $m \times 1$ vector. By definition

$$A\mathbf{0} = 0\mathbf{a}_1 + 0\mathbf{a}_2 + \cdots + 0\mathbf{a}_n = \mathbf{0}.$$

88. Every column of O is the $m \times 1$ zero vector. So

$$O\mathbf{v} = v_1\mathbf{0} + v_2\mathbf{0} + \cdots + v_n\mathbf{0} = \mathbf{0}.$$

89. The jth column of I_n is $\mathbf{e}_j$. So

$$I_n\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n = \mathbf{v}.$$

90. Using $\mathbf{p} = \begin{bmatrix} 400 \\ 300 \end{bmatrix}$, we compute $A\mathbf{p}, A(A\mathbf{p}), \ldots$ until we have ten vectors. From the final vector, we see that there will be 155,610 people living in the city and 544,389 people living in the suburbs after ten years.

91. (a) $\begin{bmatrix} 24.6 \\ 45.0 \\ 26.0 \\ -41.4 \end{bmatrix}$ (b) $\begin{bmatrix} 134.1 \\ 44.4 \\ 7.6 \\ 104.8 \end{bmatrix}$ (c) $\begin{bmatrix} 128.4 \\ 80.6 \\ 63.5 \\ 25.8 \end{bmatrix}$

(d) $\begin{bmatrix} 653.09 \\ 399.77 \\ 528.23 \\ -394.52 \end{bmatrix}$

1.3 SYSTEMS OF LINEAR EQUATIONS

1. (a) $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & 2 & 0 \\ 1 & 3 & 0 & -1 \end{bmatrix}$

2. (a) $\begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 3 & 4 \end{bmatrix}$

3. (a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \\ -3 & 4 & 1 \end{bmatrix}$

4. (a) $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 2 & -1 & 0 & 1 & 0 \end{bmatrix}$

5. (a) $\begin{bmatrix} 0 & 2 & -3 \\ -1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & -3 & 4 \\ -1 & 1 & 2 & -6 \\ 2 & 0 & 1 & 0 \end{bmatrix}$

6. (a) $\begin{bmatrix} 1 & -2 & 1 & 7 \\ 1 & -2 & 0 & 10 \\ 2 & -4 & 4 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 & 1 & 7 & 5 \\ 1 & -2 & 0 & 10 & 3 \\ 2 & -4 & 4 & 8 & 7 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 2 & -4 & 4 & 2 \\ -2 & 6 & 3 & -1 & 1 \\ 1 & -1 & 0 & 2 & -3 \end{bmatrix}$

8. $\begin{bmatrix} -3 & 3 & 0 & -6 & 9 \\ -2 & 6 & 3 & -1 & 1 \\ 0 & 2 & -4 & 4 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ 0 & 4 & 3 & 3 & -5 \\ 0 & 2 & -4 & 4 & 2 \end{bmatrix}$

10. $\begin{bmatrix} -2 & 6 & 3 & -1 & 1 \\ 1 & -1 & 0 & 2 & -3 \\ 0 & 2 & -4 & 4 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ -2 & 6 & 3 & -1 & 1 \\ 0 & 1 & -2 & 2 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ -2 & 0 & 15 & -13 & -5 \\ 0 & 2 & -4 & 4 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ -2 & 6 & 3 & -1 & 1 \\ -8 & 26 & 8 & 0 & 6 \end{bmatrix}$

14. $\begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ -2 & 6 & 3 & -1 & 1 \\ 2 & 0 & -4 & 8 & -4 \end{bmatrix}$

15. $\begin{bmatrix} -2 & 4 & 0 \\ -1 & 1 & -1 \\ 2 & -4 & 6 \\ -3 & 2 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & -2 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 2 & -4 & 6 \\ -3 & 2 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 6 \\ -3 & 2 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ 2 & -4 & 6 \\ 0 & -4 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & -2 & 0 \\ 2 & -4 & 6 \\ -1 & 1 & -1 \\ -3 & 2 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & 1 \\ 2 & -4 & 6 \\ -1 & 1 & -1 \end{bmatrix}$

21. $\begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ 2 & -4 & 6 \\ -1 & 0 & 3 \end{bmatrix}$

22. $\begin{bmatrix} -1 & 0 & -2 \\ -1 & 1 & -1 \\ 2 & -4 & 6 \\ -3 & 2 & 1 \end{bmatrix}$

23. Yes, because $1(1) - 4(-2) + 3(-1) = 6$ and $1(-5) - 2(-1) = -3$. Alternatively,

$$\begin{bmatrix} 1 & -4 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

24. No, because $1(2) - 4(0) + 3(1) = 5 \neq 6$. Alternatively, if A is the coefficient matrix, and the given vector is $\mathbf{v}$, then $A\mathbf{v} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \neq \begin{bmatrix} 6 \\ -3 \end{bmatrix}$.

25. No, because the left side of the second equation yields $1(2) - 2(1) = 0 \neq -3$. Alternatively,

$$\begin{bmatrix} 1 & -4 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 6 \\ -3 \end{bmatrix}.$$

26. Yes, the components of the vector satisfy both equations. Alternatively, if the given vector is $\mathbf{v}$, then $A\mathbf{v} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$.

27. no **28.** yes **29.** yes **30.** yes

31. yes **32.** no **33.** yes **34.** yes

35. no **36.** yes **37.** no **38.** no

39. $\begin{aligned} x_1 &= 2 + x_2 \\ x_2 &\quad \text{free} \end{aligned}$

40. $\begin{aligned} x_1 &= -4 \\ x_2 &= 5 \end{aligned}$

41. $\begin{aligned} x_1 &= 6 + 2x_2 \\ x_2 &\quad \text{free} \end{aligned}$

42. $\begin{aligned} x_1 &= 5 + 4x_2 \\ x_2 &\quad \text{free} \end{aligned}$

43. not consistent

44. $\begin{aligned} x_1 &= -6 \\ x_2 &= 3 \end{aligned}$

45. $\begin{aligned} x_1 &= 4 + 2x_2 \\ x_2 &\quad \text{free} \\ x_3 &= 3 \end{aligned}$

46. not consistent

47. $\begin{aligned} x_1 &= 3x_4 \\ x_2 &= 4x_4 \\ x_3 &= -5x_4 \\ x_4 &\quad \text{free} \end{aligned}$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$

48. $\begin{aligned} x_1 &= 9 + x_3 - 3x_4 \\ x_2 &= 8 - 2x_3 + 5x_4 \\ x_3 &\quad \text{free} \\ x_4 &\quad \text{free} \end{aligned}$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 9 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

49. $\begin{aligned} x_1 &\quad \text{free} \\ x_2 &= -3 \\ x_3 &= -4 \\ x_4 &= 5 \end{aligned}$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ -4 \\ 5 \end{bmatrix}$

50. $\begin{aligned} x_1 &= -3 + 2x_2 \\ x_2 &\quad \text{free} \\ x_3 &= -4 \\ x_4 &= 5 \end{aligned}$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ -4 \\ 5 \end{bmatrix}$$

51. $\begin{aligned} x_1 &= 6 - 3x_2 + 2x_4 \\ x_2 &\quad \text{free} \\ x_3 &= 7 - 4x_4 \\ x_4 &\quad \text{free} \end{aligned}$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

52. $\begin{aligned} x_1 &\quad \text{free} \\ x_2 &= -4 - 3x_4 \\ x_3 &= \quad 9 - 2x_4 \\ x_4 &\quad \text{free} \end{aligned}$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -3 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -4 \\ 9 \\ 0 \end{bmatrix}$$

53. not consistent

54. $\begin{aligned} x_1 &\quad \text{free} \\ x_2 &\quad \text{free} \\ x_3 &= 3x_4 - 2x_6 \\ x_4 &\quad \text{free} \\ x_5 &= \quad x_6 \\ x_6 &\quad \text{free} \end{aligned}$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

55. All variables are either free or basic, so if there are k free variables, there must be $n - k$ basic variables.

56. Because R is in reduced row echelon form, the leading entry must equal 1, and every other entry in the column must be 0. So this column equals $\mathbf{e}_4$.

57. False, the system $0x_1 + 0x_2 = 1$ has no solutions.

58. False, a system of linear equations has 0, 1, or infinitely many solutions.

59. True

60. False, the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is in row echelon form.

61. True **62.** True

63. False, the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are both row echelon forms for $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

64. True **65.** True

66. False, the system

$$\begin{aligned} 0x_1 + 0x_2 &= 1 \\ 0x_1 + 0x_2 &= 0 \end{aligned}$$

is inconsistent, but its augmented matrix is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

67. True **68.** True

69. False, the coefficient matrix of a system of m linear equations in n variables is an $m \times n$ matrix.

70. True **71.** True **72.** True

73. False, multiplying every entry of some row of a matrix by a *nonzero* scalar is an elementary row operation.

74. True

75. False, the system may be inconsistent; consider $0x_1 + 0x_2 = 1$.

76. True

77. If $[R\ \mathbf{c}]$ is in reduced row echelon form, then so is R. If we apply the same row operations to A that were applied to $[A\ \mathbf{b}]$ to produce $[R\ \mathbf{c}]$, we obtain the matrix R. So R is the reduced row echelon form of A.

78. The row operations that reduce A to R may be applied to $[A\ \mathbf{0}]$ and do not affect its last column. The resulting matrix is $[R\ \mathbf{0}]$, which is in reduced row echelon form.

79. If we let $\mathbf{0}_n$ be the $n \times 1$ zero vector, then, by Theorem 1.2(f), $A\mathbf{0}_n = \mathbf{0}$. So $\mathbf{0}_n$ is a solution of $A\mathbf{x} = \mathbf{0}$, and hence $A\mathbf{x} = \mathbf{0}$ is consistent.

80. Let R be the reduced row echelon form of A. Then by Exercise 77, $[R\ \mathbf{c}]$ is the reduced row echelon form of $[A\ \mathbf{b}]$ for some vector $\mathbf{c}$. By hypothesis, $[R\ \mathbf{c}]$ contains no row whose only nonzero entry lies in the last column. So the system $A\mathbf{x} = \mathbf{b}$ is consistent.

81. The ranks of the possible reduced row echelon forms are 0, 1, and 2. Considering each of these ranks, we see that there are 7 possible reduced row echelon forms:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

82. As in the solution to Exercise 81, there are 11 possible reduced row echelon forms:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

83. There are three cases. If the operation interchanges rows i and j of A, then interchanging rows i and j of B produces A. If the operation multiplies row i of A by the nonzero scalar c, then multiplying row i of B by $\frac{1}{c}$ produces A. Finally, if the operation adds k times row i to row j of A, then adding $-k$ times row i to row j of B produces A.

84. The system $x_1 = 1$ has only the solution 1, but the system $0x_1 = 0 \cdot 1$ has infinitely many solutions.

85. Multiplying the second equation by c produces a system whose augmented matrix is obtained from the augmented matrix of the original system by the elementary row operation of multiplying the second row by c. From the statement on page 33, the two systems are equivalent.

86. The solution is similar to that of Exercise 85.

1.4 GAUSSIAN ELIMINATION

1. $\begin{aligned} x_1 &= -2 - 3x_2 \\ x_2 &\quad \text{free} \end{aligned}$

2. $\begin{aligned} x_1 &= 3 + x_2 \\ x_2 &\quad \text{free} \end{aligned}$

3. $\begin{aligned} x_1 &= 4 \\ x_2 &= 5 \end{aligned}$

4. $\begin{aligned} x_1 &= 1 + 2x_3 \\ x_2 &= -2 - x_3 \\ x_3 &\quad \text{free} \end{aligned}$

5. not consistent

6. $\begin{aligned} x_1 &= 3 + 2x_2 + x_3 \\ x_2 &\quad \text{free} \\ x_3 &\quad \text{free} \end{aligned}$

7. $\begin{aligned} x_1 &= -1 + 2x_2 \\ x_2 &\quad \text{free} \\ x_3 &= 2 \end{aligned}$

8. $\begin{aligned} x_1 &= -1 - 4x_4 \\ x_2 &= 3x_4 \\ x_3 &= 1 - 2x_4 \\ x_4 &\quad \text{free} \end{aligned}$

9. $\begin{aligned} x_1 &= 1 + 2x_3 \\ x_2 &= -2 - x_3 \\ x_3 &\quad \text{free} \\ x_4 &= -3 \end{aligned}$

10. not consistent

11. $\begin{aligned} x_1 &= -4 - 3x_2 + x_4 \\ x_2 &\quad \text{free} \\ x_3 &= 3 - 2x_4 \\ x_4 &\quad \text{free} \end{aligned}$

12. $\begin{aligned} x_1 &= 3 + 2x_3 \\ x_2 &= -4 - 3x_3 \\ x_3 &\quad \text{free} \end{aligned}$

13. not consistent

14. not consistent

15. $\begin{aligned} x_1 &= -2 + x_5 \\ x_2 &\quad \text{free} \\ x_3 &= 3 - 3x_5 \\ x_4 &= -1 - 2x_5 \\ x_5 &\quad \text{free} \end{aligned}$

16. $\begin{aligned} x_1 &= -3 + x_2 + 2x_5 \\ x_2 &\quad \text{free} \\ x_3 &\quad \text{free} \\ x_4 &\quad -1 \qquad - 2x_5 \\ x_5 &\quad \text{free} \end{aligned}$

17. The augmented matrix can be transformed to $\begin{bmatrix} -1 & 4 & 3 \\ 0 & r+12 & 11 \end{bmatrix}$ using an elementary row operation. Therefore the system is inconsistent if $r + 12 = 0$, that is, $r = -12$.

18. The augmented matrix can be transformed to $\begin{bmatrix} -1 & 4 & 6 \\ 0 & r+12 & 16 \end{bmatrix}$ using two elementary row operations. So the system is inconsistent if $r + 12 = 0$, that is, $r = -12$.

19. The augmented matrix can be transformed to $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & r \end{bmatrix}$. So the system is inconsistent if $r \neq 0$.

20. The augmented matrix can be transformed to $\begin{bmatrix} 1 & 0 & -3 \\ 0 & r & 0 \end{bmatrix}$. So the system is inconsistent for no value of r.

21. The augmented matrix can be transformed to $\begin{bmatrix} 1 & -3 & -2 \\ 0 & r+6 & 0 \end{bmatrix}$. So the system is inconsistent for no value of r.

22. The augmented matrix is $\begin{bmatrix} -2 & 1 & 5 \\ r & 4 & 3 \end{bmatrix}$. Add $\frac{r}{2}$ times the first row to the second row to obtain $\begin{bmatrix} -2 & 1 & 5 \\ 0 & 4+\frac{r}{2} & 3+\frac{5}{2}r \end{bmatrix}$. The system is inconsistent if $4 + \frac{r}{2} = 0$ and $3 + \frac{5}{2}r \neq 0$. So $r = -8$.

23. The augmented matrix can be transformed to $\begin{bmatrix} -1 & r & 2 \\ 0 & r^2-9 & 2r+6 \end{bmatrix}$. For the system to be inconsistent, we need $r^2 - 9 = 0$ and $2r + 6 \neq 0$. So $r = \pm 3$ and $r \neq -3$. Therefore $r = 3$.

24. The argument is similar to that of Exercise 23. The system is inconsistent if $r = -4$.

25. The augmented matrix can be transformed to $\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & r+3 & -7 & -10 \end{bmatrix}$. Because this matrix does not contain a row whose only nonzero entry lies in the last column, the system is never inconsistent.

26. The augmented matrix can be transformed to $\begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 0 & r-8 & 5 \end{bmatrix}$. If $r = 8$, then this matrix contains a row whose only nonzero entry lies in the last column, and so the system is inconsistent if $r = 8$.

27. The augmented matrix can be transformed to $\begin{bmatrix} 1 & r & 5 \\ 0 & 6-3r & s-15 \end{bmatrix}$.

(a) We need $6 - 3r = 0$ and $s - 15 \neq 0$. So $r = 2$ and $s \neq 15$.

(b) We need $6 - 3r \neq 0$, that is, $r \neq 2$.

(c) We need $6 - 3r = 0$ and $s - 15 = 0$. So $r = 2$ and $s = 15$.

28. The augmented matrix can be transformed to $\begin{bmatrix} -1 & 4 & s \\ 0 & r+8 & 6+2s \end{bmatrix}$.

(a) We need $r + 8 = 0$ and $6 + 2s \neq 0$. So $r = -8$ and $s \neq -3$.

(b) We need $r + 8 \neq 0$, that is, $r \neq -8$.

(c) We need $r + 8 = 0$ and $6 + 2s = 0$. So $r = -8$ and $s = -3$.

29. (a) $r = -8$, $s \neq -2$ (b) $r \neq -8$
(c) $r = -8$, $s = -2$

30. (a) $r = -12$, $s \neq 2$ (b) $r \neq -12$
(c) $r = -12$, $s = 2$

31. (a) $r = \frac{5}{2}$, $s \neq -6$ (b) $r \neq \frac{5}{2}$
(c) $r = \frac{5}{2}$, $s = -6$

32. (a) $r = -2$, $s \neq -15$ (b) $r \neq -2$
(c) $r = -2$, $s = -15$

33. (a) $r = 3$, $s \neq \frac{2}{3}$ (b) $r \neq 3$
(c) $r = 3$, $s = \frac{2}{3}$

34. (a) $r = -2$, $s \neq 6$ (b) $r \neq -2$
(c) $r = -2$, $s = 6$

35. The reduced row echelon form of the matrix is

$$R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the given matrix equals the number of nonzero rows in R, which is 3. The nullity of the given matrix equals its number of columns minus its rank, which is $4 - 3 = 1$.

36. The rank is 2, and the nullity is 2.

37. The rank is 2, and the nullity is 3.

38. The rank is 4, and the nullity is 2.

39. The rank is 3, and the nullity is 1.

40. The rank is 3, and the nullity is 2.

41. The rank is 2, and the nullity is 3.

42. The rank is 3, and the nullity is 3.

43. Let x_1, x_2, and x_3 be the number of days that mines 1, 2, and 3, respectively, must operate to supply the desired amounts.

(a) The requirements may be written as the matrix equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 80 \\ 100 \\ 40 \end{bmatrix}.$$

The reduced row echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 25 \end{bmatrix}.$$

So $x_1 = 10$, $x_2 = 20$, $x_3 = 25$.

(b) A system of equations similar to that in (a) yields the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 1 & -15 \end{bmatrix}.$$

Because $x_3 = -15$ is impossible for this problem, these amounts cannot be supplied.

44. Let x_1, x_2, and x_3 denote the number of pounds of the three types of fertilizer, respectively, needed to satisfy the requirements.

(a) The given requirements yield the system

$$\begin{aligned} x_1 + \quad x_2 + \quad x_3 &= 600 \\ .10x_1 + .08x_2 + .06x_3 &= .075(600) \\ .03x_1 + .06x_2 + .01x_3 &= .05(600). \end{aligned}$$

This system has the solution $x_1 = -18.75$, $x_2 = 487.5$, and $x_3 = 131.25$. So this mixture is impossible.

(b) A similar approach yields the solution $x_1 = 375$, $x_2 = 150$, and $x_3 = 75$.

45. Let x_1, x_2, and x_3 be the amounts of the three supplements, respectively, that must be used.

(a) The given requirements yield the system

$$\begin{aligned} 10x_1 + 15x_2 + 36x_3 &= 660 \\ 10x_1 + 20x_2 + 44x_3 &= 820 \\ 15x_1 + 15x_2 + 42x_3 &= 750, \end{aligned}$$

which has the solution

$$\begin{aligned} x_1 &= \ 18 - 1.2x_3 \\ x_2 &= \ 32 - 1.6x_3 \\ x_3 & \quad \text{free.} \end{aligned}$$

Because the solution must be nonnegative, we need $x_3 \leq 15$ and $x_3 \leq 20$. This yields a maximum value of $x_3 = 15$.

(b) No. A similar approach yields an inconsistent system.

46. Let x_1, x_2, and x_3 be the amounts of A, B, and C, respectively, that must be blended.

(a) The given requirements yield the system

$$\begin{aligned} x_1 + \quad x_2 + \quad x_3 &= \quad 100 \\ 40x_1 + 32x_2 + 24x_3 &= 35(100) \\ 30x_1 + 62x_2 + 94x_3 &= 50(100), \end{aligned}$$

which has the solution

$$\begin{aligned} x_1 &= 37.5 + \ x_3 \\ x_2 &= 62.5 - 2x_3 \\ x_3 & \quad \text{free.} \end{aligned}$$

Letting $x_3 = 0$, we obtain $x_1 = 37.5$ and $x_2 = 62.5$.

(b) In order that x_1 and x_2 be nonnegative, we need $x_3 \geq 0$ and $2x_3 \leq 62.5$. So we take $x_3 = 31.25$ for a maximum value of x_3.

47. We need $f(-1) = 14$, $f(1) = 4$, and $f(3) = 10$. These conditions produce the system

$$\begin{aligned} a - \ \ b + c &= 14 \\ a + \ \ b + c &= \ \ 4 \\ 9a + 3b + c &= 10. \end{aligned}$$

This system has the solution $a = 2$, $b = -5$, $c = 7$. So $f(x) = 2x^2 - 5x + 7$.

48. $f(x) = -3x^2 + 8x - 5$

49. $f(x) = 4x^2 - 7x + 2$

50. $f(x) = -x^3 + 6x^2 + 4x - 12$.

51. Column j is $\mathbf{e}_3$. Each pivot column has exactly one nonzero entry, which is 1, and hence it is a standard vector. Also because of the definition of the reduced row echelon form, the sequence of pivot columns must be $\mathbf{e}_1, \mathbf{e}_2, \ldots$. Hence the third pivot column must be $\mathbf{e}_3$.

52. As noted in the solution to Exercise 51, column j equals $\mathbf{e}_4$, and because $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are among the previous columns, it follows that $j \geq 4$. Because the fourth component of column j is 1, the only nonzero entry, it follows that $i = 4$.

53. True

54. False. For example, the matrix $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ can be reduced to I_2 by interchanging its rows and then multiplying the first row by $\frac{1}{2}$, or by multiplying the second row by $\frac{1}{2}$ and then interchanging rows.

55. True **56.** True **57.** True **58.** True

59. False. By definition, rank A + nullity A equals the number of columns of A. So, for a 5×8 matrix, we have $3 + 2 \neq 8$.

60. False, we need only repeat one equation to produce an equivalent system with a different number of equations.

61. True **62.** True **63.** True

64. False, the augmented matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ contains a zero row, but the corresponding system has the unique solution $x_1 = 2$, $x_3 = 3$.

65. False, the augmented matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ contains a zero row, but the system is inconsistent.

66. True **67.** True

68. False, the sum of the rank and nullity of a matrix equals the number of *columns* in the matrix.

69. True **70.** True

71. False, the third pivot position in a matrix may be in any column to the right of column 2.

72. True

73. If the rank of a matrix is 0, then its reduced row echelon form has only zero rows, which means that the original matrix must have only zero rows, and hence must be the zero matrix.

74. The 4×7 zero matrix has rank 0, and the rank of any matrix must be nonnegative. Hence the smallest possible rank is 0.

75. The largest possible rank is 4. The reduced row echelon form is a 4×7 matrix and hence has at most 4 nonzero rows. So the rank must be less than or equal to 4. On the other hand, the 4×7 matrix whose first four columns are $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$ has rank 4.

76. The largest possible rank is 4. By the first boxed result on page 48, the rank of a matrix equals the number of its pivot columns. Clearly a 7×4 matrix can have at most 4 pivot columns.

77. The smallest possible nullity is 3. Note that if the rank of a 4×7 matrix A equals 4, then its nullity is $7 - \text{rank}\, A = 7 - 4 = 3$. On the other hand, from the solution to Exercise 75, we see that rank $A \leq 4$. So

$$\text{nullity}\, A = 7 - \text{rank}\, A \geq 7 - 4 = 3.$$

78. The smallest possible nullity is 0. The solution is similar to that of Exercise 77.

79. The largest possible rank is the minimum of m and n. If $m \le n$, the solution is similar to that of Exercise 75. If $n \le m$, the solution is similar to that of Exercise 76.

80. The smallest possible nullity is $n - m$ if $m \le n$ and 0 if $m > n$. By Exercise 79, the rank of a matrix A equals the minimum p of m and n. So $\text{nullity}\, A = n - \text{rank}\, A = n - p$. If $m \le n$, then $p = m$, so $\text{nullity}\, A = n - m$. If $n < m$, then $p = n$; so $\text{nullity}\, A = 0$.

81. No. Let R be the reduced row echelon form of A. By Exercise 79, $\text{rank}\, A \le 3$; so R has a zero row. Thus we can choose c so that $[R\ \ \mathbf{c}]$ has a row equal to $[0\ \ 0\ \ 0\ \ 1]$. By appropriate elementary row operations, we can transform $[R\ \ \mathbf{c}]$ into a matrix of the form $[A\ \ \mathbf{b}]$. So, by Theorem 1.5, the system $A\mathbf{x} = \mathbf{b}$ is not consistent.

82. For the solution to be unique, the solution must have no free variables; so $\text{nullity}\, A = 0$. Therefore $\text{rank}\, A = n - \text{nullity}\, A = n$.

83. There are either no solutions or infinitely many solutions. Let the system be $A\mathbf{x} = \mathbf{b}$, and let R be the reduced row echelon form of A. Each nonzero row of R corresponds to a basic variable. Since there are fewer equations than variables, if the system is consistent, there must be free variables. Therefore the system is either inconsistent or has infinitely many solutions.

84. (a) $\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 3 \\ x_1 + x_2 &= 4 \end{aligned}$ (b) $\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + x_2 &= 4 \\ 3x_1 + x_2 &= 5 \end{aligned}$

(c) $\begin{aligned} x_1 + \ \ x_2 &= 3 \\ 2x_1 + 2x_2 &= 6 \\ 3x_1 + 3x_2 &= 9 \end{aligned}$

85. Let $[R\ \ \mathbf{c}]$ denote the reduced row echelon form of $[A\ \ \mathbf{b}]$. Then R is the reduced row echelon form of A. If $\text{rank}\, A = m$, then R contains no nonzero rows. Hence $[R\ \ \mathbf{c}]$ contains no row in which the only nonzero entry lies in the last column. So $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ by Theorem 1.5.

86. Let $[R\ \ \mathbf{c}]$ denote the reduced row echelon form of $[A\ \ \mathbf{b}]$. Then R is the reduced row echelon form of A. If $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $[R\ \ \mathbf{c}]$ contains the row $[0\ \ 0\ \ \dots\ \ 0\ \ 1]$. The corresponding row of R is a zero row, and every other nonzero row of $[R\ \ \mathbf{c}]$ corresponds to a nonzero row of R. Thus $\text{rank}\,[A\ \ \mathbf{b}] = 1 + \text{rank}\ A$; so the ranks of $[A\ \ \mathbf{b}]$ and A are not equal.

Conversely, the reduced row echelon form of A equals the reduced row echelon form of $[A\ \ \mathbf{b}]$ with its last column removed. Thus if the ranks of these matrices are not equal, we must have $\text{rank}\,[A\ \ \mathbf{b}] = 1 + \text{rank}\ A$. This can happen only if $[R\ \ \mathbf{c}]$ contains the row $[0\ \ 0\ \ \dots\ \ 0\ \ 1]$. So the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

87. Yes, $A(c\mathbf{u}) = c(A\mathbf{u}) = c \cdot \mathbf{0} = \mathbf{0}$; so $c\mathbf{u}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

88. Yes, $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = \mathbf{0}+\mathbf{0} = \mathbf{0}$; so $\mathbf{u}+\mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

89. We have $A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$; so $\mathbf{u} - \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{0}$.

90. We have $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$; so $\mathbf{u} + \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

91. If $A\mathbf{x} = \mathbf{b}$ is consistent, then there exists a vector $\mathbf{u}$ such that $A\mathbf{u} = \mathbf{b}$. So $A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{b}$. Hence $c\mathbf{u}$ is a solution of $A\mathbf{x} = c\mathbf{b}$, and therefore $A\mathbf{x} = c\mathbf{b}$ is consistent.

92. As in Exercise 87, there exist vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that $A\mathbf{u}_1 = \mathbf{b}_1$ and $A\mathbf{u}_2 = \mathbf{b}_2$. Therefore $A(\mathbf{u}_1 + \mathbf{u}_2) = A\mathbf{u}_1 + A\mathbf{u}_2 = \mathbf{b}_1 + \mathbf{b}_2$. Hence $A\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2$ is consistent.

93. No. If $\mathbf{u} + \mathbf{v}$ were a solution of $A\mathbf{x} = \mathbf{b}$, then

$$\mathbf{b} = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{b} + \mathbf{b} = 2\mathbf{b};$$

so $\mathbf{b} = \mathbf{0}$. Therefore the result is not true if $\mathbf{b} \neq \mathbf{0}$.

94. $\begin{aligned} x_1 &= \ \ 4.9927 + 1.1805x_4 + \ \ 8.5341x_5 \\ x_2 &= \ \ 7.1567 + 3.0513x_4 + 15.3103x_5 \\ x_3 &= -2.5738 + 5.2366x_4 + 15.1360x_5 \\ x_4 &\ \ \text{free} \\ x_5 &\ \ \text{free} \end{aligned}$

95. $\begin{aligned} x_1 &= \ \ 2.32 + 0.32x_5 \\ x_2 &= -6.44 + 0.56x_5 \\ x_3 &= \ \ 0.72 - 0.28x_5 \\ x_4 &= \ \ 5.92 + 0.92x_5 \\ x_5 &\ \ \text{free} \end{aligned}$

96. The system is not consistent.

97. 3, 2 **98.** 5, 0 **99.** 4, 1

1.5 APPLICATIONS OF SYSTEMS OF LINEAR EQUATIONS

1. True **2.** True

3. False, the net production vector is $\mathbf{x} - C\mathbf{x}$. The vector $C\mathbf{x}$ is the total output of the economy that is consumed during the production process.

4. False, see Kirchoff's voltage law.

5. True **6.** True

7. \$50(.22) = \$11 million