Exercises for Chapter 1

Exercises for Section 1.1: Describing a Set

1.1 Only (d) and (e) are sets.

1.2 (a) $A = \{1, 2, 3\} = \{x \in S :\ x > 0\}$.

(b) $B = \{0, 1, 2, 3\} = \{x \in S :\ x \geq 0\}$.

(c) $C = \{-2, -1\} = \{x \in S :\ x < 0\}$.

(d) $D = \{x \in S : |x| \geq 2\}$.

1.3 (a) $|A| = 5$. (b) $|B| = 11$. (c) $|C| = 51$. (d) $|D| = 2$. (e) $|E| = 1$. (f) $|F| = 2$.

1.4 (a) $A = \{n \in \mathbf{Z} :\ -4 < n \leq 4\} = \{-3, -2, \ldots, 4\}$.

(b) $B = \{n \in \mathbf{Z} :\ n^2 < 5\} = \{-2, -1, 0, 1, 2\}$.

(c) $C = \{n \in \mathbf{N} :\ n^3 < 100\} = \{1, 2, 3, 4\}$.

(d) $D = \{x \in \mathbf{R} :\ x^2 - x = 0\} = \{0, 1\}$.

(e) $E = \{x \in \mathbf{R} :\ x^2 + 1 = 0\} = \{\} = \emptyset$.

1.5 (a) $A = \{-1, -2, -3, \ldots\} = \{x \in \mathbf{Z} :\ x \leq -1\}$.

(b) $B = \{-3, -2, \ldots, 3\} = \{x \in \mathbf{Z} :\ -3 \leq x \leq 3\} = \{x \in \mathbf{Z} :\ |x| \leq 3\}$.

(c) $C = \{-2, -1, 1, 2\} = \{x \in \mathbf{Z} :\ -2 \leq x \leq 2, x \neq 0\} = \{x \in \mathbf{Z} :\ 0 < |x| \leq 2\}$.

1.6 (a) $A = \{2x + 1 :\ x \in \mathbf{Z}\} = \{\cdots, -5, -3, -1, 1, 3, 5, \cdots\}$.

(b) $B = \{4n :\ n \in \mathbf{Z}\} = \{\cdots, -8, -4, 0, 4, 8, \cdots\}$.

(c) $C = \{3q + 1 :\ q \in \mathbf{Z}\} = \{\cdots, -5, -2, 1, 4, 7, \cdots\}$.

1.7 (a) $A = \{\cdots, -4, -1, 2, 5, 8, \cdots\} = \{3x + 2 :\ x \in \mathbf{Z}\}$.

(b) $B = \{\cdots, -10, -5, 0, 5, 10, \cdots\} = \{5x :\ x \in \mathbf{Z}\}$.

(c) $C = \{1, 8, 27, 64, 125, \cdots\} = \{x^3 :\ x \in \mathbf{N}\}$.

1.8 (a) $A = \{n \in \mathbf{Z} : 2 \leq |n| < 4\} = \{-3, -2, 2, 3\}$.

(b) 5/2, 7/2, 4.

(c) $C = \{x \in \mathbf{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\} = \{x \in \mathbf{R} : (x - 2)(x - \sqrt{2}) = 0\} = \{2, \sqrt{2}\}$.

(d) $D = \{x \in \mathbf{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\} = \{2\}$.

(e) $|A| = 4$, $|C| = 2$, $|D| = 1$.

1.9 $A = \{2, 3, 5, 7, 8, 10, 13\}$.

$B = \{x \in A : x = y + z,\ \text{where } y, z \in A\} = \{5, 7, 8, 10, 13\}$.

$C = \{r \in B : r + s \in B\ \text{ for some } s \in B\} = \{5, 8\}$.

Exercises for Section 1.2: Subsets

1.10 (a) $A = \{1, 2\}$, $B = \{1, 2\}$, $C = \{1, 2, 3\}$.

(b) $A = \{1\}$, $B = \{\{1\}, 2\}$. $C = \{\{\{1\}, 2\}, 1\}$.

(c) $A = \{1\}$, $B = \{\{1\}, 2\}$, $C = \{1, 2\}$.

1.11 Let $r = \min(c - a, b - c)$ and let $I = (c - r, c + r)$. Then I is centered at c and $I \subseteq (a, b)$.

1.12 $A = B = D = E = \{-1, 0, 1\}$ and $C = \{0, 1\}$.

1.13 See Figure 1.

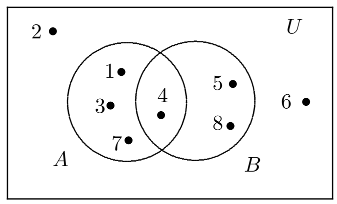

Figure 1: Answer for Exercise 1.13

1.14 (a) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$; $|\mathcal{P}(A)| = 4$.

(b) $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{1\}, \{\{a\}\}, \{\emptyset, 1\}, \{\emptyset, \{a\}\}, \{1, \{a\}\}, \{\emptyset, 1, \{a\}\}\}$; $|\mathcal{P}(A)| = 8$.

1.15 $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, A\}$.

1.16 $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$, $\mathcal{P}(\mathcal{P}(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$; $|\mathcal{P}(\mathcal{P}(\{1\}))| = 4$.

1.17 $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, A\}$; $|\mathcal{P}(A)| = 8$.

1.18 $\mathcal{P}(\{0\}) = \{\emptyset, \{0\}\}$.

$A = \{x : x = 0 \text{ or } x \in \mathcal{P}(\{0\})\} = \{0, \emptyset, \{0\}\}$.

$\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{0\}\}, \{0, \emptyset\}, \{0, \{0\}\}, \{\emptyset, \{0\}\}, A\}$.

1.19 (a) $S = \{\emptyset, \{1\}\}$.

(b) $S = \{1\}$.

(c) $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4, 5\}\}$.

(d) $S = \{1, 2, 3, 4, 5\}$.

1.20 (a) False. For example, for $A = \{1, \{1\}\}$, both $1 \in A$ and $\{1\} \in A$.

(b) Because $\mathcal{P}(B)$ is the set of all subsets of the set B and $A \subset \mathcal{P}(B)$ with $|A| = 2$, it follows that A is a proper subset of $\mathcal{P}(B)$ consisting of exactly two elements of $\mathcal{P}(B)$. Thus $\mathcal{P}(B)$ contains at least one element that is not in A. Suppose that $|B| = n$. Then $|\mathcal{P}(B)| = 2^n$. Since $2^n > 2$, it follows that $n \geq 2$ and $|\mathcal{P}(B)| = 2^n \geq 4$. Because $\mathcal{P}(B) \subset C$, it is impossible that $|C| = 4$. Suppose that $A = \{\{1\}, \{2\}\}$, $B = \{1, 2\}$ and $C = \mathcal{P}(B) \cup \{3\}$. Then $A \subset \mathcal{P}(B) \subset C$, where $|A| = 2$ and $|C| = 5$.

(c) No. For $A = \emptyset$ and $B = \{1\}$, $|\mathcal{P}(A)| = 1$ and $|\mathcal{P}(B)| = 2$.

(d) Yes. There are only three distinct subsets of $\{1, 2, 3\}$ with two elements.

1.21 $B = \{1, 4, 5\}$.

Exercises for Section 1.3: Set Operations

1.22 (a) $A \cup B = \{1, 3, 5, 9, 13, 15\}$.

(b) $A \cap B = \{9\}$.

(c) $A - B = \{1, 5, 13\}$.

(d) $B - A = \{3, 15\}$.

(e) $\overline{A} = \{3, 7, 11, 15\}$.

(f) $A \cap \overline{B} = \{1, 5, 13\}$.

1.23 Let $A = \{1, 2, \ldots, 6\}$ and $B = \{4, 5, \ldots, 9\}$. Then $A - B = \{1, 2, 3\}$, $B - A = \{7, 8, 9\}$ and $A \cap B = \{4, 5, 6\}$. Thus $|A - B| = |A \cap B| = |B - A| = 3$. See Figure 2.

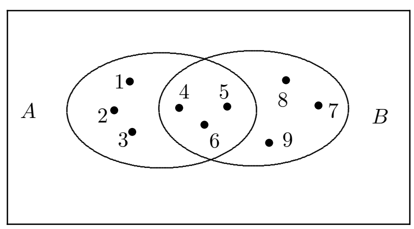

Figure 2: Answer for Exercise 1.23

1.24 Let $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 3\}$. Then $B \neq C$ but $B - A = C - A = \{3\}$.

1.25 (a) $A = \{1\}$, $B = \{\{1\}\}$, $C = \{1, 2\}$.

(b) $A = \{\{1\}, 1\}$, $B = \{1\}$, $C = \{1, 2\}$.

(c) $A = \{1\}$, $B = \{\{1\}\}$, $C = \{\{1\}, 2\}$.

1.26 (a) and (b) are the same, as are (c) and (d).

1.27 Let $U = \{1, 2, \ldots, 8\}$ be a universal set, $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Then $A - B = \{1, 2\}$, $B - A = \{5, 6\}$, $A \cap B = \{3, 4\}$ and $\overline{A \cup B} = \{7, 8\}$. See Figure 3.

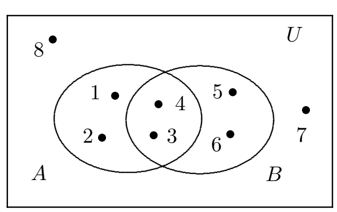

Figure 3: Answer for Exercise 1.27

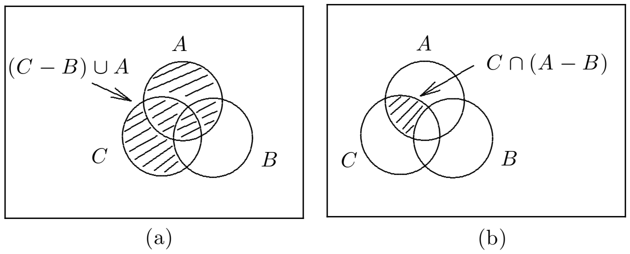

Figure 4: Answers for Exercise 1.28

1.28 See Figures 4(a) and 4(b).

1.29 (a) The sets $\emptyset$ and $\{\emptyset\}$ are elements of A.

(b) $|A| = 3$.

(c) All of $\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are subsets of A.

(d) $\emptyset \cap A = \emptyset$.

(e) $\{\emptyset\} \cap A = \{\emptyset\}$.

(f) $\{\emptyset, \{\emptyset\}\} \cap A = \{\emptyset, \{\emptyset\}\}$.

(g) $\emptyset \cup A = A$.

(h) $\{\emptyset\} \cup A = A$.

(i) $\{\emptyset, \{\emptyset\}\} \cup A = A$.

1.30 (a) $A = \{x \in \mathbf{R} : |x - 1| \leq 2\} = \{x \in \mathbf{R} : -2 \leq x - 1 \leq 2\} = \{x \in \mathbf{R} : -1 \leq x \leq 3\} = [-1, 3]$
$B = \{x \in \mathbf{R} : |x| \geq 1\} = \{x \in \mathbf{R} : x \geq 1 \text{ or } x \leq -1\} = (-\infty, -1] \cup [1, \infty)$
$C = \{x \in \mathbf{R} : |x + 2| \leq 3\} = \{x \in \mathbf{R} : -3 \leq x + 2 \leq 3\} = \{x \in \mathbf{R} : -5 \leq x \leq 1\} = [-5, 1]$

(b) $A \cup B = (-\infty, \infty) = \mathbf{R}$, $A \cap B = \{-1\} \cup [1, 3]$,
$B \cap C = [-5, -1] \cup \{1\}$, $B - C = (-\infty, -5) \cup (1, \infty)$.

1.31 $A = \{1, 2\}$, $B = \{2\}$, $C = \{1, 2, 3\}$, $D = \{2, 3\}$.

1.32 $A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$, $C = \{1, 3, 4\}$, $D = \{2, 3, 4\}$.

1.33 $A = \{1\}$, $B = \{2\}$.

1.34 $A = \{1, 2\}$, $B = \{2, 3\}$.

1.35 Let $U = \{1, 2, \ldots, 8\}$, $A = \{1, 2, 3, 5\}$, $B = \{1, 2, 4, 6\}$ and $C = \{1, 3, 4, 7\}$. See Figure 5.

Exercises for Section 1.4: Indexed Collections of Sets

1.36 $\bigcup_{\alpha \in A} S_\alpha = S_1 \cup S_3 \cup S_4 = [0, 3] \cup [2, 5] \cup [3, 6] = [0, 6]$.

$\bigcap_{\alpha \in A} S_\alpha = S_1 \cap S_3 \cap S_4 = \{3\}$.

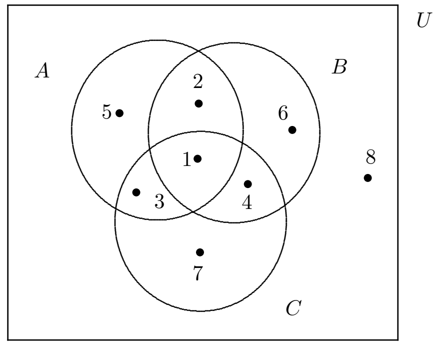

Figure 5: Answer for Exercise 1.35

1.37 $\bigcup_{X \in S} X = A \cup B \cup C = \{0, 1, 2, \ldots, 5\}$ and $\bigcap_{X \in S} X = A \cap B \cap C = \{2\}$.

1.38 (a) $\bigcup_{\alpha \in S} A_\alpha = A_1 \cup A_2 \cup A_4 = \{1\} \cup \{4\} \cup \{16\} = \{1, 4, 16\}$.
$\bigcap_{\alpha \in S} A_\alpha = A_1 \cap A_2 \cap A_4 = \emptyset$.

(b) $\bigcup_{\alpha \in S} B_\alpha = B_1 \cup B_2 \cup B_4 = [0, 2] \cup [1, 3] \cup [3, 5] = [0, 5]$.
$\bigcap_{\alpha \in S} B_\alpha = B_1 \cap B_2 \cap B_4 = \emptyset$.

(c) $\bigcup_{\alpha \in S} C_\alpha = C_1 \cup C_2 \cup C_4 = (1, \infty) \cup (2, \infty) \cup (4, \infty) = (1, \infty)$.
$\bigcap_{\alpha \in S} C_\alpha = C_1 \cap C_2 \cap C_4 = (4, \infty)$.

1.39 Since $|A| = 26$ and $|A_\alpha| = 3$ for each $\alpha \in A$, we need to have at least nine sets of cardinality 3 for their union to be A; that is, in order for $\bigcup_{\alpha \in S} A_\alpha = A$, we must have $|S| \geq 9$. However, if we let $S = \{a, d, g, j, m, p, s, v, y\}$, then $\bigcup_{\alpha \in S} A_\alpha = A$. Hence the smallest cardinality of a set S with $\bigcup_{\alpha \in S} A_\alpha = A$ is 9.

1.40 (a) $\bigcup_{i=1}^{5} A_{2i} = A_2 \cup A_4 \cup A_6 \cup A_8 \cup A_{10} = \{1, 3\} \cup \{3, 5\} \cup \{5, 7\} \cup \{7, 9\} \cup \{9, 11\} = \{1, 3, 5, \ldots, 11\}$.

(b) $\bigcup_{i=1}^{5} (A_i \cap A_{i+1}) = \bigcup_{i=1}^{5} (\{i-1, i+1\} \cap \{i, i+2\}) = \bigcup_{i=1}^{5} \emptyset = \emptyset$.

(c) $\bigcup_{i=1}^{5} (A_{2i-1} \cap A_{2i+1}) = \bigcup_{i=1}^{5} (\{2i-2, 2i\} \cap \{2i, 2i+2\}) = \bigcup_{i=1}^{5} \{2i\} = \{2, 4, 6, 8, 10\}$.

1.41 (a) $\{A_n\}_{n \in \mathbf{N}}$, where $A_n = \{x \in \mathbf{R} : 0 \leq x \leq 1/n\} = [0, 1/n]$.

(b) $\{A_n\}_{n \in \mathbf{N}}$, where $A_n = \{a \in \mathbf{Z} : \ |a| \leq n\} = \{-n, -(n-1), \ldots, (n-1), n\}$.

1.42 (a) $A_n = \left[1, 2 + \frac{1}{n}\right)$, $\bigcup_{n \in \mathbf{N}} A_n = [1, 3)$ and $\bigcap_{n \in \mathbf{N}} A_n = [1, 2]$.

(b) $A_n = \left(-\frac{2n-1}{n}, 2n\right)$, $\bigcup_{n \in \mathbf{N}} A_n = (-2, \infty)$ and $\bigcap_{n \in \mathbf{N}} A_n = (-1, 2)$.

1.43 $\bigcup_{r \in \mathbf{R}^+} A_r = \bigcup_{r \in \mathbf{R}^+} (-r, r) = \mathbf{R}$;
$\bigcap_{r \in \mathbf{R}^+} A_r = \bigcap_{r \in \mathbf{R}^+} (-r, r) = \{0\}$.

1.44 For $I = \{2, 8\}$, $|\bigcup_{i \in I} A_i| = 8$. Observe that there is no set I such that $|\bigcup_{i \in I} A_i| = 10$, for in this case, we must have either two 5-element subsets of A or two 3-element subsets of A and a 4-element subset of A. In each case, not every two subsets are disjoint. Furthermore, there is no set I such that $|\bigcup_{i \in I} A_i| = 9$, for in this case, one must either have a 5-element subset of A and a 4-element subset of A (which are not disjoint) or three 3-element subsets of A. No 3-element subset of A contains 1 and only one such subset contains 2. Thus $4, 5 \in I$ but there is no third element for I.

1.45 $\bigcup_{n\in\mathbf{N}} A_n = \bigcup_{n\in\mathbf{N}}(-\frac{1}{n}, 2-\frac{1}{n}) = (-1,2)$;
$\bigcap_{n\in\mathbf{N}} A_n = \bigcap_{n\in\mathbf{N}}(-\frac{1}{n}, 2-\frac{1}{n}) = [0,1]$.

1.46 (a) $\displaystyle\bigcup_{n=1}^{\infty}\left(-\frac{1}{n},\ \frac{1}{n}\right) = (-1,1);\ \bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\ \frac{1}{n}\right) = \{0\}$

(b) $\displaystyle\bigcup_{n=1}^{\infty}\left[\frac{n-1}{n},\ \frac{n+1}{n}\right] = [0,2];\ \bigcap_{n=1}^{\infty}\left[\frac{n-1}{n},\ \frac{n+1}{n}\right] = \{1\}$

1.47 (a) $\displaystyle\bigcup_{n=1}^{\infty}\left\{\sin^2\frac{n\pi}{2} + \cos^2\frac{n\pi}{2}\right\} = \bigcap_{n=1}^{\infty}\left\{\sin^2\frac{n\pi}{2} + \cos^2\frac{n\pi}{2}\right\} = \{1\}$

(b) $\displaystyle\bigcup_{n=1}^{\infty}\left\{\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2}\right\} = \{-1,1\};\ \bigcap_{n=1}^{\infty}\left\{\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2}\right\} = \emptyset$

Exercises for Section 1.5: Partitions of Sets

1.48 (a) S_1 is a partition of A.
(b) S_2 is not a partition of A because g belongs to no element of S_2.
(c) S_3 is a partition of A.
(d) S_4 is not a partition of A because $\emptyset \in S_4$.
(e) S_5 is not a partition of A because b belongs to two elements of S_5.

1.49 (a) S_1 is not a partition of A since 4 belongs to no element of S_1.
(b) S_2 is a partition of A.
(c) S_3 is not a partition of A because 2 belongs to two elements of S_3.
(d) S_4 is not a partition of A since S_4 is not a set of subsets of A.

1.50 $S = \{\{1,2,3\},\{4,5\},\{6\}\}$; $|S| = 3$.

1.51 $A = \{1,2,3,4\}$. $S_1 = \{\{1\},\{2\},\{3,4\}\}$ and $S_2 = \{\{1,2\},\{3\},\{4\}\}$.

1.52 Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbf{N} : x > 5\}$, $A_2 = \{x \in \mathbf{N} : x < 5\}$ and $A_3 = \{5\}$.

1.53 Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbf{Q} : x > 1\}$, $A_2 = \{x \in \mathbf{Q} : x < 1\}$ and $A_3 = \{1\}$.

1.54 $A = \{1,2,3,4\}$, $S_1 = \{\{1\},\{2\},\{3,4\}\}$ and $S_2 = \{\{\{1\},\{2\}\},\{\{3,4\}\}\}$.

1.55 Let $S = \{A_1, A_2, A_3, A_4\}$, where
$A_1 = \{x \in \mathbf{Z} : x \text{ is odd and } x \text{ is positive}\}$,
$A_2 = \{x \in \mathbf{Z} : x \text{ is odd and } x \text{ is negative}\}$,
$A_3 = \{x \in \mathbf{Z} : x \text{ is even and } x \text{ is nonnegative}\}$,
$A_4 = \{x \in \mathbf{Z} : x \text{ is even and } x \text{ is negative}\}$.

1.56 Let $S = \{\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8, 9, 10\}, \{11, 12\}\}$ and $T = \{\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8, 9, 10\}\}$.

1.57 $|\mathcal{P}_1| = 2$, $|\mathcal{P}_2| = 3$, $|\mathcal{P}_3| = 5$, $|\mathcal{P}_4| = 8$, $|\mathcal{P}_5| = 13$, $|\mathcal{P}_6| = 21$.

1.58 (a) Suppose that a collection S of subsets of A satisfies Definition 1. Then every subset is nonempty. Every element of A belongs to a subset in S. If some element $a \in A$ belonged to more than one subset, then the subsets in S would not be pairwise disjoint. So the collection satisfies Definition 2.

(b) Suppose that a collection S of subsets of A satisfies Definition 2. Then every subset is nonempty and (1) in Definition 3 is satisfied. If two subsets A_1 and A_2 in S were neither equal nor disjoint, then $A_1 \neq A_2$ and there is an element $a \in A$ such that $a \in A_1 \cap A_2$, which would not satisfy Definition 2. So condition (2) in Definition 3 is satisfied. Since every element of A belongs to a (unique) subset in S, condition (3) in Definition 3 is satisfied. Thus Definition 3 itself is satisfied.

(c) Suppose that a collection S of subsets of A satisfies Definition 3. By condition (1) in Definition 3, every subset is nonempty. By condition (2), the subsets are pairwise disjoint. By condition (3), every element of A belongs to a subset in S. So Definition 1 is satisfied.

Exercises for Section 1.6: Cartesian Products of Sets

1.59 $A \times B = \{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}$.

1.60 $A \times A = \{(1,\ 1),\ (1,\ \{1\}),\ (1,\ \{\{1\}\}),\ (\{1\},\ 1),\ (\{1\},\ \{1\}),\ (\{1\},\ \{\{1\}\}),\ (\{\{1\}\},\ 1),\ (\{\{1\}\},\ \{1\}),\ (\{\{1\}\},\ \{\{1\}\})\}$.

1.61 $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, A\}$,

$A \times \mathcal{P}(A) = \{(a, \emptyset), (a, \{a\}), (a, \{b\}), (a, A), (b, \emptyset), (b, \{a\}), (b, \{b\}), (b, A)\}$.

1.62 $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, A\}$,

$A \times \mathcal{P}(A) = \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\emptyset, \{\{\emptyset\}\}), (\emptyset, A), (\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\emptyset\}, \{\{\emptyset\}\}), (\{\emptyset\}, A)\}$.

1.63 $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, A\}$, $\mathcal{P}(B) = \{\emptyset, B\}$, $A \times B = \{(1, \emptyset), (2, \emptyset)\}$,

$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset), (\emptyset, B), (\{1\}, \emptyset), (\{1\}, B), (\{2\}, \emptyset), (\{2\}, B), (A, \emptyset), (A, B)\}$.

1.64 $\{(x, y) :\ x^2 + y^2 = 4\}$, which is a circle centered at $(0,\ 0)$ with radius 2.

1.65 $S = \{(3, 0), (2, 1), (2, -1), (1, 2), (1, -2), (0, 3), (0, -3), (-3, 0), (-2, 1), (-2, -1), (-1, 2), (-1, -2)\}$. See Figure 6.

1.66 $A \times B = \{(1, 1), (2, 1)\}$,

$\mathcal{P}(A \times B) = \{\emptyset, \{(1, 1)\}, \{(2, 1)\}, A \times B\}$.

1.67 $A = \{x \in \mathbf{R} : |x - 1| \leq 2\} = \{x \in \mathbf{R} : -1 \leq x \leq 3\} = [-1, 3]$,

$B = \{y \in \mathbf{R} : |y - 4| \leq 2\} = \{y \in \mathbf{R} : 2 \leq y \leq 6\} = [2, 6]$,

$A \times B = [-1, 3] \times [2, 6]$, which is the set of all points on and within the square bounded by $x = -1$, $x = 3$, $y = 2$ and $y = 6$.

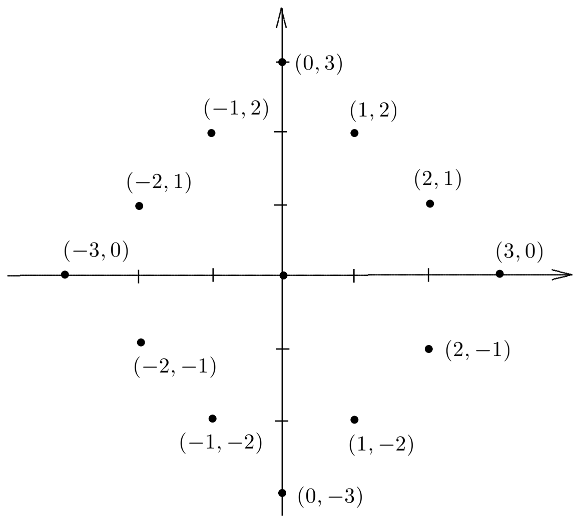

Figure 6: Answer for Exercise 1.65

1.68 $A = \{a \in \mathbf{R} : |a| \le 1\} = \{a \in \mathbf{R} : -1 \le a \le 1\} = [-1, 1]$,

$B = \{b \in \mathbf{R} : |b| = 1\} = \{-1, 1\}$,

$A \times B$ is the set of all points (x, y) on the lines $y = 1$ or $y = -1$ with $x \in [-1, 1]$, while $B \times A$ is the set of all points (x, y) on the lines $x = 1$ or $x = -1$ with $y \in [-1, 1]$. Therefore, $(A \times B) \cup (B \times A)$ is the set of all points lying on (but not within) the square bounded by $x = 1$, $x = -1$, $y = 1$ and $y = -1$.

1.69 (a)–(b) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A) = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$.

1.70 For $A = \{1, 2\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 3, 4\}$ and $D = \{2, 3\}$, it follows that

$((A \times B) \cup (C \times D)) - (D \times D) = R$.

1.71 Since $\bigcup_{i=1}^{3}(A_i \times A_i) = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$, it follows that $\left|\cup_{i=1}^{3}(A_i \times A_i)\right| = 10$.

1.72 The set $\{A \times A, A \times B, B \times A, B \times B\}$ is a partition of $S \times S$.

Chapter 1 Supplemental Exercises

1.73 (a) $A = \{4k + 3 : k \in \mathbf{Z}\} = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$.

(b) $B = \{5k - 1 : k \in \mathbf{Z}\} = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$.

1.74 (a) $A = \{x \in S : |x| \ge 1\} = \{x \in S : x \ne 0\}$.

(b) $B = \{x \in S : x \le 0\}$.

(c) $C = \{x \in S : -5 \le x \le 7\} = \{x \in S : |x - 1| \le 6\}$.

(d) $D = \{x \in S : x \ne 5\}$.

1.75 (a) $\{0, 2, -2\}$ (b) $\{\ \}$ (c) $\{3, 4, 5\}$ (d) $\{1, 2, 3\}$

(e) $\{-2, 2\}$ (f) $\{\ \}$ (g) $\{-3, -2, -1, 1, 2, 3\}$.

1.76 (a) $|A| = 6$ (b) $|B| = 0$ (c) $|C| = 3$

(d) $|D| = 0$ (e) $|E| = 10$ (f) $|F| = 20$.

1.77 $A \times B = \{(-1, x), (-1, y), (0, x), (0, y), (1, x), (1, y)\}$.

1.78 (a) $(A \cup B) - (B \cap C) = \{1, 2, 3\} - \{3\} = \{1, 2\}$.

(b) $\overline{A} = \{3\}$.

(c) $\overline{B \cup C} = \overline{\{1, 2, 3\}} = \emptyset$.

(d) $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$.

1.79 Let $S = \{\{1\}, \{2\}, \{3, 4\}, A\}$ and let $B = \{3, 4\}$.

1.80 $\mathcal{P}(A) = \{\emptyset, \{1\}\}$, $\mathcal{P}(C) = \{\emptyset, \{1\}, \{2\}, C\}$. Let $B = \{\emptyset, \{1\}, \{2\}\}$.

1.81 Let $A = \{\emptyset\}$ and $B = \mathcal{P}(A) = \{\emptyset, \{\emptyset\}\}$.

1.82 Only $B = C = \emptyset$ and $D = E$.

1.83 $U = \{1, 2, 3, 5, 7, 8, 9\}$, $A = \{1, 2, 5, 7\}$ and $B = \{5, 7, 8\}$.

1.84 (a) A_r is the set of all points in the plane lying on the circle $x^2 + y^2 = r^2$.

$\bigcup_{r \in I} A_r = \mathbf{R} \times \mathbf{R}$ (the plane) and $\bigcap_{r \in I} A_r = \emptyset$.

(b) B_r is the set of all points lying on and inside the circle $x^2 + y^2 = r^2$.

$\bigcup_{r \in I} B_r = \mathbf{R} \times \mathbf{R}$ and $\bigcap_{r \in I} B_r = \{(0, 0)\}$.

(c) C_r is the set of all points lying outside the circle $x^2 + y^2 = r^2$.

$\bigcup_{r \in I} C_r = \mathbf{R} \times \mathbf{R} - \{(0, 0)\}$ and $\bigcap_{r \in I} C_r = \emptyset$.

1.85 Let $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{3, 5, 6\}$, $A_3 = \{1, 3\}$, $A_4 = \{1, 2, 4, 5, 6\}$. Then $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_4| = 1$, $|A_1 \cap A_3| = |A_2 \cap A_4| = 2$ and $|A_1 \cap A_4| = 3$.

1.86 (a) (i) Give an example of five sets A_i $(1 \le i \le 5)$ such that $|A_i \cap A_j| = |i - j|$ for every two integers i and j with $1 \le i < j \le 5$.

(ii) Determine the minimum positive integer k such that there exist four sets A_i $(1 \le i \le 4)$ satisfying the conditions of Exercise 1.79 and $|A_1 \cup A_2 \cup A_3 \cup A_4| = k$.

(b) (i) $A_1 = \{1, 2, 3, 4, 7, 8, 9, 10\}$
$A_2 = \{3, 5, 6, 11, 12, 13\}$
$A_3 = \{1, 3, 14, 15\}$
$A_4 = \{1, 2, 4, 5, 6, 16\}$
$A_5 = \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$.

(ii) The minimum positive integer k is 5. The example below shows that $k \leq 5$.

Let $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{1, 5\}$, $A_3 = \{1, 4\}$, $A_4 = \{1, 2, 3, 5\}$.

If $k = 4$, then since $|A_1 \cap A_4| = 3$, A_1 and A_4 have exactly three elements in common, say 1, 2, 3. So each of A_1 and A_4 is either $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$. They cannot both be $\{1, 2, 3, 4\}$. Also, they cannot both be $\{1, 2, 3\}$ because A_3 would have to contain two of 1, 2, 3 and so $|A_3 \cap A_4| \geq 2$, which is not true. So we can assume that $A_1 = \{1, 2, 3, 4\}$ and $A_4 = \{1, 2, 3\}$. However, A_2 must contain two of 1, 2, 3 and so $|A_1 \cap A_2| \geq 2$, which is impossible.

1.87 (a) $|S| = |T| = 10$.

(b) $|S| = |T| = 5$.

(c) $|S| = |T| = 6$.

1.88 Let $A = \{1, 2, 3, 4\}$, $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$, $A_3 = \{3, 4\}$. These examples show that $k \leq 4$. Since $|A_1 - A_3| = |A_3 - A_1| = 2$, it follows that A_1 contains two elements not in A_3, while A_3 contains two elements not in A_1. Thus $|A| \geq 4$ and so $k = 4$ is the smallest positive integer with this property.

1.89 (a) $S = \{(-3, 4), (0, 5), (3, 4), (4, 3)\}$.

(b) $C = \{a \in B : (a, b) \in S\} = \{3, 4\}$
$D = \{b \in A : (a, b) \in S\} = \{3, 4\}$
$C \times D = \{(3, 3), (3, 4), (4, 3), (4, 3)\}$.

1.90 $A = \{1, 2, 3\}$, $B = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $C = \{\{1\}, \{2\}, \{3\}\}$,

$D = \mathcal{P}(C) = \{\emptyset, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\{2\}, \{3\}\}, C\}$.

1.91 $S = \{x \in \mathbf{R} : x^2 + 2x - 1 = 0\} = \{-1 + \sqrt{2}, -1 - \sqrt{2}\}$.

$A_{-1+\sqrt{2}} = \{-1 + \sqrt{2}, \sqrt{2}\}$, $A_{-1-\sqrt{2}} = \{-1 - \sqrt{2} - \sqrt{2}\}$.

(a) $A_s = A_{-1-\sqrt{2}}$ and $A_t = A_{-1+\sqrt{2}}$.
$A_s \times A_t = \{(-1 - \sqrt{2}, -1 + \sqrt{2}), (-1 - \sqrt{2}, \sqrt{2}), (-\sqrt{2}, 1 + \sqrt{2}), (-\sqrt{2}, \sqrt{2})\}$.

(b) $C = \{ab : (a, b) \in B\} = \{-1, -\sqrt{2} - 2, \sqrt{2} - 2, -2\}$. The sum of the elements in C is -7.

1.92 (a) For $|A| = 2$, the largest possible value of $|A \cap \mathcal{P}(A)|$ is 2.
The set $A = \{\emptyset, \{\emptyset\}\}$ has this property.

(b) For $|A| = 3$, the largest possible value of $|A \cap \mathcal{P}(A)|$ is 3.
The set $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ has this property.

(c) For $|A| = 4$, the largest possible value of $|A \cap \mathcal{P}(A)|$ is 4.
The set $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}\}$ has this property.

1.93 $\bigcup_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} S_n = \left[-\sqrt{2}, \sqrt{2}\,\right]$. First, observe that

$$\bigcap_{n=1}^{\infty} S_n = S_1 = \left\{x + y :\ x, y \in \mathbf{R}, x^2 + y^2 = 1\right\}.$$

Let $f(x) = x + y$, where $y = \pm\sqrt{1-x^2}$, say $y = \sqrt{1-x^2}$. Then $f(x) = x + \sqrt{1-x^2}$. Since $f'(x) = 1 - \frac{x}{\sqrt{1-x^2}}$, it follows that $f'(x) = 0$ when $x = \frac{1}{\sqrt{2}}$ and so the maximum value of $x+y$ is $\sqrt{2}$. Since $f(-\frac{1}{\sqrt{2}}) = 0$ and f is continuous on $\left[-\sqrt{2}, \sqrt{2}\,\right]$, it follows that f takes on all values of $\left[0, \sqrt{2}\,\right]$. If $x + y = r \in \left[0, \sqrt{2}\,\right]$, it follows that $(-x) + (-y) = -r \in \left[-\sqrt{2}, 0\,\right]$. Hence, $S_1 = \left[-\sqrt{2}, \sqrt{2}\,\right]$. If $a^2 + b^2 = r$ where $0 < r < 1$, then $a + b \in \left(-\sqrt{2}, \sqrt{2}\,\right)$ and so $\bigcup_{n=1}^{\infty} S_n = \left[-\sqrt{2}, \sqrt{2}\,\right]$ as well.

1.94 (a) No. For example, the elements 1, 2 and 5 belong to more than one subset of S.

(b) Yes. (c) Yes.

1.95 In order for $|A \times (B \cup C)| = |A \times B| + |A \times C|$, the sets B and C must be disjoint.

Exercises for Chapter 2

Exercises for Section 2.1: Statements

2.1 (a) A false statement.

(b) A true statement.

(c) Not a statement.

(d) Not a statement (an open sentence).

(e) Not a statement.

(f) Not a statement (an open sentence).

(g) Not a statement.

2.2 (a) A true statement since $A = \{3n - 2 : n \in \mathbf{N}\}$ and so $3 \cdot 9 - 2 = 25 \in A$.

(b) A false statement. Starting with the 3rd term in D, each element is the sum of the two preceding terms. Therefore, all terms following 21 exceed 33 and so $33 \notin D$.

(c) A false statement since $3 \cdot 8 - 2 = 22 \in A$.

(d) A true statement since every prime except 2 is odd.

(e) A false statement since B and D consist only of integers.

(f) A false statement since 53 is prime.

2.3 (a) False. $\emptyset$ has no elements.

(b) True.

(c) True.

(d) False. $\{\emptyset\}$ has $\emptyset$ as its only element.

(e) True.

(f) False. 1 is not a set.

2.4 (a) $x = -2$ and $x = 3$.

(b) All $x \in \mathbf{R}$ such that $x \neq -2$ and $x \neq 3$.

2.5 (a) $\{x \in \mathbf{Z} : x > 2\}$.

(b) $\{x \in \mathbf{Z} : x \leq 2\}$.

2.6 (a) A can be any of the sets $\emptyset, \{1\}, \{2\}, \{1, 2\}$, that is, A is any subset of $\{1, 2, 4\}$ that does not contain 4.

(b) A can be any of the sets $\{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{4\}$, that is, A is any subset of $\{1, 2, 4\}$ that contains 4.

(c) $A = \emptyset$ and $A = \{4\}$.

2.7 3, 5, 11, 17, 41, 59.

2.8 (a) $S_1 = \{1, 2, 5\}$ (b) $S_2 = \{0, 3, 4\}$.

2.9 $P(n)$: $\frac{n-1}{2}$ is even. $P(n)$ is true only for $n = 5$ and $n = 9$.

2.10 $P(n)$: $\frac{n}{2}$ is odd. $Q(n)$: $\frac{n^2-2n}{8}$ is even. or $Q(n)$: $n^2 + 9$ is a prime.

Exercises for Section 2.2: Negations

2.11 (a) $\sqrt{2}$ is not a rational number.

(b) 0 is a negative integer.

(c) 111 is not a prime number.

2.12 See Figure 1.

P	Q	$\sim P$	$\sim Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

Figure 1: Answer for Exercise 2.12

2.13 (a) The real number r is greater than $\sqrt{2}$.

(b) The absolute value of the real number a is at least 3.

(c) At most one angle of the triangle is 45^o.

(d) The area of the circle is less than 9π.

(e) The sides of the triangle have different lengths.

(f) The point P lies on or within the circle C.

2.14 (a) At most one of my library books is overdue.

(b) My two friends did not misplace their homework assignments.

(c) Someone expected this to happen.

(d) My instructor often teaches that course.

(e) It's not surprising that two students received the same exam score.

Exercises for Section 2.3: Disjunctions and Conjunctions

2.15 See Figure 2.

2.16 (a) True. (b) False. (c) False. (d) True. (e) True.

2.17 (a) $P \vee Q$: 15 is odd or 21 is prime. (True)

P	Q	$\sim Q$	$P \wedge (\sim Q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Figure 2: Answer for Exercise 2.15

(b) $P \wedge Q$: 15 is odd and 21 is prime. (False)

(c) $(\sim P) \vee Q$: 15 is not odd or 21 is prime. (False)

(d) $P \wedge (\sim Q)$: 15 is odd and 21 is not prime. (True)

2.18 (a) All nonempty subsets of $\{1, 3, 5\}$.

(b) All subsets of $\{1, 3, 5\}$.

(c) There are no subsets A of S for which $(\sim P(A)) \wedge (\sim Q(A))$ is true.

Exercises for Section 2.4: Implications

2.19 (a) $\sim P$: 17 is not even (or 17 is odd). (True)

(b) $P \vee Q$: 17 is even or 19 is prime. (True)

(c) $P \wedge Q$: 17 is even and 19 is prime. (False)

(d) $P \Rightarrow Q$: If 17 is even, then 19 is prime. (True)

2.20 See Figure 3.

P	Q	$\sim P$	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow (\sim P)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Figure 3: Answer for Exercise 2.20

2.21 (a) $P \Rightarrow Q$: If $\sqrt{2}$ is rational, then 22/7 is rational. (True)

(b) $Q \Rightarrow P$: If 22/7 is rational, then $\sqrt{2}$ is rational. (False)

(c) $(\sim P) \Rightarrow (\sim Q)$: If $\sqrt{2}$ is not rational, then 22/7 is not rational. (False)

(d) $(\sim Q) \Rightarrow (\sim P)$: If 22/7 is not rational, then $\sqrt{2}$ is not rational. (True)

2.22 (a) $(P \wedge Q) \Rightarrow R$: If $\sqrt{2}$ is rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational. (True)

(b) $(P \wedge Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ is rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is not rational. (True)

(c) $((\sim P) \wedge Q) \Rightarrow R$: If $\sqrt{2}$ is not rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational. (False)

(d) $(P \vee Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ is rational or $\frac{2}{3}$ is rational, then $\sqrt{3}$ is not rational. (True)

2.23 (a), (c), (d) are true.

2.24 (b), (d), (f) are true.

2.25 (a) True. (b) False. (c) True. (d) True. (e) True.

2.26 (a) False. (b) True. (c) True. (d) False.

2.27 Cindy and Don attended the talk.

2.28 (b), (d), (f), (g) are true.

2.29 Only (c) implies that $P \vee Q$ is false.

Exercises for Section 2.5: More on Implications

2.30 (a) $P(n) \Rightarrow Q(n)$: If $5n + 3$ is prime, then $7n + 1$ is prime.

(b) $P(2) \Rightarrow Q(2)$: If 13 is prime, then 15 is prime. (False)

(c) $P(6) \Rightarrow Q(6)$: If 33 is prime, then 43 is prime. (True)

2.31 (a) $P(x) \Rightarrow Q(x)$: If $|x| = 4$, then $x = 4$.

$P(-4) \Rightarrow Q(-4)$ is false.

$P(-3) \Rightarrow Q(-3)$ is true.

$P(1) \Rightarrow Q(1)$ is true.

$P(4) \Rightarrow Q(4)$ is true.

$P(5) \Rightarrow Q(5)$ is true.

(b) $P(x) \Rightarrow Q(x)$: If $x^2 = 16$, then $|x| = 4$. True for all $x \in S$.

(c) $P(x) \Rightarrow Q(x)$: If $x > 3$, then $4x - 1 > 12$. True for all $x \in S$.

2.32 (a) All $x \in S$ for which $x \neq 7$.

(b) All $x \in S$ for which $x > -1$.

(c) All $x \in S$.

(d) All $x \in S$.

2.33 (a) True for $(x, y) = (3, 4)$ and $(x, y) = (5, 5)$ and false for $(x, y) = (1, -1)$.

(b) True for $(x, y) = (1, 2)$ and $(x, y) = (6, 6)$ and false for $(x, y) = (2, -2)$.

(c) True for $(x, y) \in \{(1, -1), (-3, 4), (1, 0)\}$ and false for $(x, y) = (0, -1)$.

2.34 (a) If the x-coordinate of a point on the straight line with equation $2y + x - 3 = 0$ is an integer, then its y-coordinate is also an integer. Or: If $-2n + 3 \in \mathbf{Z}$, then $n \in \mathbf{Z}$.

(b) If n is an odd integer, then n^2 is an odd integer.

(c) Let $n \in \mathbf{Z}$. If $3n + 7$ is even, then n is odd.

(d) If $f(x) = \cos x$, then $f'(x) = -\sin x$.

(e) If a circle has circumference 4π, then its area is also 4π.

(f) Let $n \in \mathbf{Z}$. If n^3 is even, then n is even.

Exercises for Section 2.6: Biconditionals

2.35 $P \Leftrightarrow Q$: 18 is odd if and only if 25 is even. (True)

2.36 The integer x is odd if and only if x^2 is odd.

That the integer x is odd is a necessary and sufficient condition for x^2 to be odd.

2.37 Let $x \in \mathbf{R}$. Then $|x - 3| < 1$ if and only if $x \in (2, 4)$.

For $x \in \mathbf{R}$, $|x - 3| < 1$ is a necessary and sufficient condition for $x \in (2, 4)$.

2.38 (a) $\sim P(x)$: $x \neq -2$. True if $x = 0, 2$.

(b) $P(x) \vee Q(x)$: $x = -2$ or $x^2 = 4$. True if $x = -2, 2$.

(c) $P(x) \wedge Q(x)$: $x = -2$ and $x^2 = 4$. True if $x = -2$.

(d) $P(x) \Rightarrow Q(x)$: If $x = -2$, then $x^2 = 4$. True for all x.

(e) $Q(x) \Rightarrow P(x)$: If $x^2 = 4$, then $x = -2$. True if $x = 0, -2$.

(f) $P(x) \Leftrightarrow Q(x)$: $x = -2$ if and only if $x^2 = 4$. True if $x = 0, -2$.

2.39 (a) True for all $x \in S - \{-4\}$.

(b) True for $x \in S - \{3\}$.

(c) True for $x \in S - \{-4, 0\}$.

2.40 (a) True for $(x, y) \in \{(3, 4), (5, 5)\}$.

(b) True for $(x, y) \in \{(1, 2), (6, 6)\}$.

(c) True for $(x, y) \in \{(1, -1), (1, 0)\}$.

2.41 True if $n = 3$.

2.42 True if $n = 3$.

2.43 $P(1) \Rightarrow Q(1)$ is false (since $P(1)$ is true and $Q(1)$ is false).

$Q(3) \Rightarrow P(3)$ is false (since $Q(3)$ is true and $P(3)$ is false).

$P(2) \Leftrightarrow Q(2)$ is true (since $P(2)$ and $Q(2)$ are both true).

2.44 (i) $P(1) \Rightarrow Q(1)$ is false;

(ii) $Q(4) \Rightarrow P(4)$ is true;

(iii) $P(2) \Leftrightarrow R(2)$ is true;

(iv) $Q(3) \Leftrightarrow R(3)$ is false.

2.45 True for all $n \in S$.

Exercises for Section 2.7: Tautologies and Contradictions

2.46 The compound statement $P \Rightarrow (P \vee Q)$ is a tautology since it is true for all combinations of truth values for the component statements P and Q. See the truth table below.

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
T	T	T	**T**
T	F	T	**T**
F	T	T	**T**
F	F	F	**T**

2.47 The compound statements $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ and $(P \Rightarrow \sim Q) \wedge (P \wedge Q)$ are contradictions. See the truth table below.

P	Q	$\sim Q$	$P \wedge Q$	$P \wedge (\sim Q)$	$(P \wedge (\sim Q)) \wedge (P \wedge Q)$	$P \Rightarrow \sim Q$	$(P \Rightarrow \sim Q) \wedge (P \wedge Q)$
T	T	F	T	F	**F**	F	**F**
T	F	T	F	T	**F**	T	**F**
F	T	F	F	F	**F**	T	**F**
F	F	T	F	F	**F**	T	**F**

2.48 The compound statement $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology since it is true for all combinations of truth values for the component statements P and Q. See the truth table below.

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$: If P and P implies Q, then Q.

2.49 The compound statement $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology since it is true for all combinations of truth values for the component statements P, Q and R. See the truth table below.

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	**T**
T	F	T	F	T	F	T	**T**
F	T	T	T	T	T	T	**T**
F	F	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	F	F	T	F	F	**T**
F	T	F	T	F	F	T	**T**
F	F	F	T	T	T	T	**T**

$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$: If P implies Q and Q implies R, then P implies R.

2.50 (a) $R \vee S$ is a tautology. (b) $R \wedge S$ is a contradiction.

(c) $R \Rightarrow S$ is a contradiction. (d) $S \Rightarrow R$ is a tautology.

2.51 The compound statement $(P \vee Q) \vee (Q \Rightarrow P)$ is a tautology.

P	Q	$P \vee Q$	$Q \Rightarrow P$	$(P \vee Q) \vee (Q \Rightarrow P)$
T	T	T	T	**T**
T	F	T	T	**T**
F	T	T	F	**T**
F	F	F	T	**T**

2.52 The compound statement $R = ((P \Rightarrow Q) \Rightarrow P) \Rightarrow (P \Rightarrow (Q \Rightarrow P))$ is a tautology.

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow P$	$Q \Rightarrow P$	$P \Rightarrow (Q \Rightarrow P)$	R
T	T	T	T	T	T	**T**
T	F	F	T	T	T	**T**
F	T	T	F	F	T	**T**
F	F	T	F	T	T	**T**

Exercises for Section 2.8: Logical Equivalence

2.53 (a) See the truth table below.

P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$(\sim P) \Rightarrow (\sim Q)$
T	T	F	F	**T**	**T**
T	F	F	T	**F**	**T**
F	T	T	F	**T**	**F**
F	F	T	T	**T**	**T**

Since $P \Rightarrow Q$ and $(\sim P) \Rightarrow (\sim Q)$ do not have the same truth values for all combinations of truth values for the component statements P and Q, the compound statements $P \Rightarrow Q$ and $(\sim P) \Rightarrow (\sim Q)$ are not logically equivalent. Note that the last two columns in the truth table are not the same.

(b) The implication $Q \Rightarrow P$ is logically equivalent to $(\sim P) \Rightarrow (\sim Q)$.

2.54 (a) See the truth table below.

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim (P \vee Q)$	$(\sim P) \vee (\sim Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**T**
F	T	T	F	T	**F**	**T**
F	F	T	T	F	**T**	**T**

Since $\sim (P \vee Q)$ and $(\sim P) \vee (\sim Q)$ do not have the same truth values for all combinations of truth values for the component statements P and Q, the compound statements $\sim (P \vee Q)$ and $(\sim P) \vee (\sim Q)$ are not logically equivalent.

(b) The biconditional $\sim (P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$ is not a tautology as there are instances when this biconditional is false.

2.55 (a) The statements $P \Rightarrow Q$ and $(P \wedge Q) \Leftrightarrow P$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements P and Q. See the truth table.

P	Q	$P \Rightarrow Q$	$P \wedge Q$	$(P \wedge Q) \Leftrightarrow P$
T	T	**T**	T	**T**
T	F	**F**	F	**F**
F	T	**T**	F	**T**
F	F	**T**	F	**T**

(b) The statements $P \Rightarrow (Q \vee R)$ and $(\sim Q) \Rightarrow ((\sim P) \vee R)$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements P, Q and R. See the truth table.

P	Q	R	$\sim P$	$\sim Q$	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$(\sim P) \vee R$	$(\sim Q) \Rightarrow ((\sim P) \vee R)$
T	T	T	F	F	T	**T**	T	**T**
T	F	T	F	T	T	**T**	T	**T**
F	T	T	T	F	T	**T**	T	**T**
F	F	T	T	T	T	**T**	T	**T**
T	T	F	F	F	T	**T**	F	**T**
T	F	F	F	T	F	**F**	F	**F**
F	T	F	T	F	T	**T**	T	**T**
F	F	F	T	T	F	**T**	T	**T**

2.56 The statements Q and $(\sim Q) \Rightarrow (P \wedge (\sim P))$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements P and Q. See the truth table below.

P	Q	$\sim P$	$\sim Q$	$P \wedge (\sim P)$	$(\sim Q) \Rightarrow (P \wedge (\sim P))$
T	**T**	F	F	F	**T**
T	**F**	F	T	F	**F**
F	**T**	T	F	F	**T**
F	**F**	T	T	F	**F**

2.57 The statements $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements P, Q and R. See the truth table.

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	**T**	T	T	**T**
T	F	T	T	**T**	T	T	**T**
F	T	T	T	**T**	T	T	**T**
F	F	T	F	**T**	T	T	**T**
T	T	F	T	**F**	F	F	**F**
T	F	F	T	**F**	F	T	**F**
F	T	F	T	**F**	T	F	**F**
F	F	F	F	**T**	T	T	**T**

2.58 If S and T are not logically equivalent, there is some combination of truth values of the component statements P, Q and R for which S and T have different truth values.

2.59 Since there are only four different combinations of truth values of P and Q for the second and third rows of the statements S_1, S_2, S_3, S_4 and S_5, at least two of these must have identical truth tables and so are logically equivalent.

Exercises for Section 2.9: Some Fundamental Properties of Logical Equivalence

2.60 (a) The statement $P \vee (Q \wedge R)$ is logically equivalent to $(P \vee Q) \wedge (P \vee R)$ since the last two columns in the truth table in Figure 4 are the same.

(b) The statement $\sim (P \vee Q)$ is logically equivalent to $(\sim P) \wedge (\sim Q)$ since the last two columns in the truth table in Figure 5 are the same.

2.61 (a) Both $x \neq 0$ and $y \neq 0$.

(b) Either the integer a is odd or the integer b is odd.

P	Q	R	$P \vee Q$	$P \vee R$	$Q \wedge R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	**T**	**T**
T	F	T	T	T	F	**T**	**T**
F	T	T	T	T	T	**T**	**T**
F	F	T	F	T	F	**F**	**F**
T	T	F	T	T	F	**T**	**T**
T	F	F	T	T	F	**T**	**T**
F	T	F	T	F	F	**F**	**F**
F	F	F	F	F	F	**F**	**F**

Figure 4: Answer for Exercise 2.60(a)

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim (P \vee Q)$	$(\sim P) \wedge (\sim Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

Figure 5: Answer for Exercise 2.60(b)

2.62 (a) x and y are even only if xy is even.

(b) If xy is even, then x and y are even.

(c) Either at least one of x and y is odd or xy is even.

(d) x and y are even and xy is odd.

2.63 Either $x^2 = 2$ and $x \neq \sqrt{2}$ or $x = \sqrt{2}$ and $x^2 \neq 2$.

2.64 The statement $[(P \vee Q) \wedge \sim (P \wedge Q)]$ is logically equivalent to $\sim (P \Leftrightarrow Q)$ since the last two columns in the truth table below are the same.

P	Q	$P \vee Q$	$P \wedge Q$	$\sim (P \wedge Q)$	$P \Leftrightarrow Q$	$(P \vee Q) \wedge \sim (P \wedge Q)$	$\sim (P \Leftrightarrow Q)$
T	T	T	T	F	T	**F**	**F**
T	F	T	F	T	F	**T**	**T**
F	T	T	F	T	F	**T**	**T**
F	F	F	F	T	T	**F**	**F**

2.65 If $3n + 4$ is odd, then $5n - 6$ is odd.

2.66 n^3 is odd if and only if $7n + 2$ is even.

Exercises for Section 2.10: Quantified Statements

2.67 $\forall x \in S, P(x)$: For every odd integer x, the integer $x^2 + 1$ is even.

$\exists x \in S,\ Q(x)$: There exists an odd integer x such that x^2 is even.

2.68 Let $R(x)$: $x^2 + x + 1$ is even. and let $S = \{x \in \mathbf{Z} : x \text{ is odd}\}$.

$\forall x \in S,\ R(x)$: For every odd integer x, the integer $x^2 + x + 1$ is even.

$\exists x \in S,\ R(x)$: There exists an odd integer x such that $x^2 + x + 1$ is even.