Chapter 2

Lecture Notes, Detailed Comments, and Additional Explorations

This chapter includes chapter overviews, section-by-section lecture notes, comments about the lecture notes, and additional explorations. The lecture notes include short and medium homework assignments.

The comments about the lecture notes describe typical student difficulties and what an instructor can do to help students overcome these obstacles. This chapter contains a few explorations that do not appear in the textbook. For each exploration in this manual and the textbook, there is a discussion of what to do before, during, and after it.

CHAPTER 1 OVERVIEW

In Section 1.1, students learn the meaning of *variable*, *constant*, and various types of numbers; students also graph data. In Section 1.2, students construct scatterplots of data and interpret bar graphs. In Section 1.3, students use a linear model to describe an exact linear relationship; students use such a model to make estimates and predictions and find the intercepts of the model. In Section 1.4, students perform similar tasks with a linear model that describes an approximate linear relationship.

SECTION 1.1 LECTURE NOTES

Objectives

1. Describe the meaning of *variable* and *constant*.
2. Describe the meaning of *counting numbers*, *integers*, *rational numbers*, *irrational numbers*, *real numbers*, *positive numbers*, and *negative numbers*.
3. Use a number line to describe numbers.
4. Describe a number as a decimal number.
5. Graph data.
6. Find the average (or mean) of a group of numbers.
7. Describe a concept or procedure.

Main point: Describe the meaning of *variable* and *constant*.

OBJECTIVE 1

Definition *Variable*

A **variable** is a symbol that represents a quantity that can vary.

1. Let p be the price (in dollars) to see a Modest Mouse concert. What is the meaning of $p = 60$?
2. Let t be the number of years since 2015. What is the meaning of $t = 7$? of $t = -5$?
3. Choose a symbol to represent the number of students in a class. Explain why the symbol is a variable. Give two numbers that the variable can represent and two numbers that it cannot represent.

Definition *Constant*

A **constant** is a symbol that represents a specific number (a quantity that does *not* vary).

4. A rectangle has an area of 16 square feet. Let W be the width (in feet), L be the length (in feet), and A be the area (in square feet).
   a. Sketch three possible rectangles of area 16 square feet.
   b. Which of the symbols W, L, and A are variables? Explain.
   c. Which of the symbols W, L, and A are constants? Explain.

OBJECTIVE 2

- The **counting numbers**, or **natural numbers**, are the numbers 1, 2, 3, 4, 5, ...
- The **integers** are the numbers . . . , -3, -2, -1, 0, 1, 2, 3, ...

- The **positive integers** are the numbers $1, 2, 3, \ldots$.
- The **negative integers** are the numbers $-1, -2, -3, \ldots$.
- The number 0 is neither positive nor negative.
- The **rational numbers** are the numbers that can be written in the form $\frac{n}{d}$, where n and d are integers and d is nonzero.
- The numbers represented on the number line that are *not* rational are called **irrational numbers**.
- The **real numbers** are all the numbers represented on the number line.

OBJECTIVE 3

5. Graph the integers between -4 and 2, inclusive, on one number line.

6. Graph the integers between -4 and 2 on one number line.

7. Graph all the numbers 3, -5, $\frac{7}{2}$, 0, -2.7, and -1.3 on one number line.

OBJECTIVE 4

- A rational number can be written as a decimal number that either terminates or repeats.
- An irrational number can be written as a decimal number that neither terminates nor repeats.

OBJECTIVE 5

8. The number of homes sold by a real estate agent for various months are 2, 3, 0, 7, 5. Let n be the number of homes sold by the agent in a month. Graph the agent's sales on a number line.

OBJECTIVE 6

Definition *Average, mean*

To find the **average** (or **mean**) of a group of numbers, we divide the sum of the numbers by the number of numbers in the group.

9. Use the data values in Problem 8 to find the mean sales of the agent and indicate it on the number line you sketched in Problem 8.

10. The average smartphone screen size (in inches) for the years 2007, 2008, 2009, 2010, 2011, 2012, 2013, and 2014 are 2.59, 2.67, 3.00, 3.27, 3.53, 4.03, 4.38, 4.86, respectively (Source: *PhoneArena.com*). Let A be the average smartphone screen size (in inches) in a given year.

a. Graph the data. Find the mean of the data values and indicate it on the graph.

b. Did the annual average screen size of a smartphone increase, decrease, stay approximately constant, or none of these from 2007 to 2014, inclusive? Explain.

c. Did the annual *increases* in the average size of a smartphone generally increase, decrease, stay approximately constant, or none of these from 2007 to 2014, inclusive? Explain.

When we write numbers on a number line, they should increase by a fixed amount and be equally spaced.

OBJECTIVE 2 (revisited)

- The **negative numbers** are the real numbers less than 0.
- The **positive numbers** are the real numbers greater than 0.

11. Let T be the temperature (in degrees Fahrenheit). What value of T represents the temperature 20 degrees Fahrenheit below 0? Graph the number on a number line.

OBJECTIVE 7

Discuss the "Guidelines on Writing a Good Response" on page 9.

12. Describe how to graph data values. Describe how to compute the mean of the data values. Describe the meaning of the mean.

SHORT HW 5, 13, 21, 29, 35, 37, 39, 47, 55, 61, 67, 69, 79

MEDIUM HW 5, 9, 11, 13, 21, 23, 29, 31, 35, 37, 39, 45, 47, 55, 59, 61, 67, 69, 73, 75, 79

SECTION 1.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students learn the meaning of *variable* and *constant*, which lay the foundation for many definitions to come. Students graph numbers on a number line; this skill will be needed for constructing scatterplots in Section 1.2. The section includes definitions of various types of numbers such as rational numbers; the textbook will occasionally use these terminologies. Students calculate the mean of a group of numbers; although students will rarely calculate the mean in subsequent sections, this terminology will be used throughout the textbook to describe variables for some authentic situations. Finally, students learn some guidelines on how to describe concepts or procedures; most Homework sections contain exercises that instruct students to describe such things.

OBJECTIVE 1 COMMENTS

Problem 2 in the lecture notes is good preparation for working with the explanatory variable for some time-series data. Most students think that time cannot be negative and are surprised that $t = -5$ represents the year 2010. Students will sometimes get a negative value when using a model to make an estimate of the response variable.

For Problem 3 in the lecture notes, discussing unreasonable values of the variable is a nice primer for the concept of model breakdown, which is discussed in Section 1.4.

After completing Problems 1–3 in the lecture notes, I quickly mention several more examples of variables, pointing out that variables "are all around us." I suggest that throughout the rest of the day, my students notice quantities that can be represented by variables. I tell my students that a variable is one of three main concepts in algebra (the other two being expressions and equations).

When discussing constants, I give π and a number such as 5 as examples.

Before doing Problem 4 in the lecture notes, I discuss the meaning of the area of a flat surface. Most students do not know that the area (in square inches) is the number of square inches that it takes to cover the surface. I remind my students that the area of a rectangle is equal to the rectangle's length times its width and explain this by breaking up a rectangle into unit squares.

The perimeter of a rectangle is defined in Exercises 23 and 24 of Homework 1.1. Students have difficulty with these two exercises (as well as Exercises 25–28) because they are not exactly like Problem 4 in the lecture notes.

OBJECTIVE 2 COMMENTS

When I define counting numbers and integers, I make sure that students understand the meaning of the ellipsis.

I quickly define the various types of numbers. I give especially light treatment to defining rational and irrational numbers because even though these terminologies are used in the textbook, a superficial understanding of these terminologies will suffice.

OBJECTIVE 3 COMMENTS

Most students do not understand how Problems 5 and 6 in the lecture notes differ. The word "inclusive" is used many times in the textbook. I make sure I discuss how to graph decimal numbers because this skill will come in handy throughout the course when constructing scatterplots of data (see Problem 7 in the lecture notes).

OBJECTIVE 4 COMMENTS

I skip making this distinction because it really won't come into play in the course.

OBJECTIVE 5 COMMENTS

For Problem 8 in the lecture notes, I emphasize that the variable "n" and the units of n should be included to the right of the number line.

Textbook and Workbook Exploration

The 15-minute "Reasonable values of a variable" exploration is a good primer for the concept of model breakdown, which students will study in Section 1.4.

OBJECTIVE 6 COMMENTS

I emphasize that the mean of a group of data estimates the center of the numbers graphed on a number line. This is a good primer for students who will take statistics. Means are used a number of times in this course to describe quantities, but students will rarely need to compute means (aside from in this section's Homework).

For Problem 10 in the lecture notes, I discuss the importance of using uniform scaling. This is a good primer for constructing scatterplots in Section 1.2, where some students will mistakenly use nonuniform scaling if not warned against this practice.

Workbook Exploration

In the following 20-minute exploration, groups can have a visceral sense that the mean measures the center of a group of numbers, and it can be used to compare two groups of data.

EXPLORATION *Interpreting the average of some numbers*

1. Plot the given numbers on a number line. Find the average of the given numbers. Indicate it on your number line. How does the position of the average on the number line relate to the numbers you plotted?

a. 1, 3, 5 **b.** 2, 5, 7, 10 **c.** 1, 2, 5, 9, 12, 13

2. In general, what does the average of some numbers tell you about the numbers?

3. The prices (in dollars) of hot dogs at each of the stadiums for the baseball teams in the National League Central (NLC) and the National League West (NLW) are shown in the following table.

NLC Team	Hot Dog Price (dollars)	NLW Team	Hot Dog Price (dollars)
Chicago Cubs	5.50	Arizona Diamondbacks	2.75
Cincinnati Reds	1.00	Colorado Rockies	4.75
Milwaukee Brewers	3.50	Los Angeles Dodgers	5.50
Pittsburgh Pirates	3.25	San Diego Padres	4.00
St. Louis Cardinals	4.25	San Francisco Giants	5.25

Source: *Team Marketing Report*

a. Find the average hot dog price for the NLC.

b. Find the average hot dog price for the NLW.

c. Which tends to have cheaper hot dogs, stadiums in the NLC or the NLW? Explain.

OBJECTIVE 2 (REVISITED) COMMENTS

I emphasize that 0 is neither negative nor positive.

OBJECTIVE 7 COMMENTS

If I do not have time to do Problem 12 in the lecture notes, I tell my students to read Example 10 in the textbook and the preceding "Guidelines on Writing a Good Response" at home. Students will be asked to explain concepts or procedures in most Homework sections.

SECTION 1.2 LECTURE NOTES

Objectives

1. Describe the meaning of *ordered pair*, *coordinate*, and *coordinate system*.
2. Construct scatterplots.
3. Describe the meaning of *explanatory variable* and *response variable*.
4. Read bar graphs.
5. Plot points on a coordinate system.

Main point: Construct scatterplots.

OBJECTIVE 1

1. A football team scores the following number of points (listed in chronological order) in five football games: 3, 7, 7, 14, 17. Let s be the team's score (in points) in a game. Use a number line to graph the team's scores.

There are two limitations with using one number line for Problem 1:

- The graph does not show that there were two scores of 7 points each.
- The graph does not show which was the first score, the second score, and so on.

2. Let s be the football team's score (in points) in the nth game.

   a. Discuss the meaning: $s = 14$ when $n = 4$.
   b. Use a table to describe the values of n and s.

- The *ordered pair* $(4, 14)$ means that $n = 4$ and $s = 14$.
- For $(4, 14)$, the *n-coordinate* is 4 and the *s-coordinate* is 14.
- Draw a coordinate system and indicate the axes, quadrants, and origin.

OBJECTIVE 2

3. Plot the ordered pairs described by the table you constructed in part (b) of Problem 2.

A graph of plotted ordered pairs is called a **scatterplot**.

OBJECTIVE 3

Definition *explanatory and response variables*

Assume that an authentic situation can be described by using the variables t and p, and assume that t affects (explains) p.

- We call t the **explanatory variable** (or **independent variable**).
- We call p the **response variable** (or **dependent variable**).

For each situation, identify the explanatory variable and the response variable.

4. Let n be the number of words in a document and let t be the number of minutes it takes to type the document.

5. Let L be the length (in inches) of a person's bangs at t weeks since her hair was last cut.

6. Let P be the percentage of American adults who own at least three cars and I be the annual income (in thousands of dollars) of an American adult.

For an ordered pair (a, b), we write the value of the explanatory variable in the first (left) position and the value of the response variable in the second (right) position.

7. Let n be the number of words in a document and let t be the number of minutes it takes to type the document. What does the ordered pair $(400, 8)$ represent?

8. Let L be the length (in inches) of a candle at t minutes since the candle was lit. What does the ordered pair $(30, 6)$ represent?

OBJECTIVE 2 (revisited)

Columns of Tables and Axes of Coordinate Systems

Assume that an authentic situation can be described by using two variables. Then

- For tables, the values of the explanatory variable are listed in the first column and the values of the response variable are listed in the second column. [Construct a table.]
- For coordinate systems, the values of the explanatory variable are described by the horizontal axis and the values of the response variable are described by the vertical axis. [Draw a figure.]

9. The average numbers of Internet searches per day using Google are shown in the following table for various years.

Year	Average Number of Searches Per Day (in billions)
2008	1.7
2009	2.6
2010	3.6
2011	4.7
2012	5.1
2013	5.9
2014	5.7
2015	7.8
2016	9.0

Source: *Google*

a. Let n be the average number of searches per day (in billions) at t years since 2005. Construct a scatterplot of the data.

b. Did the average number of searches per day increase, decrease, stay approximately constant, or none of these? Explain.

c. Did the annual *increase* in the average number of searches per day increase, decrease, stay approximately constant, or none of these? Explain.

10. The fatality rates from automobile crashes are shown in the following table for various speeds.

Speed Group (mph)	Speed Used to Represent Speed Group (mph)	Fatality Rate (deaths per 1000 crashes)
0–30	15	2.5
30–40	35	3.5
40–50	45	6.1
50–60	55	15.3
over 60	70	16.9

Source: *National Highway Transportation Safety Administration*

Let r be the fatality rate (deaths per 1000 crashes) for a speed of s mph. Construct a scatterplot of the data.

OBJECTIVE 4

A **bar graph** is a diagram with two axes that we can use to compare measurements of two or more items (see Fig. 2.1).

11. The annual numbers of visitors to ski resorts are illustrated in the bar graph in Fig. 2.1 for various regions in the 2015/2016 season (Source: *National Ski Areas Association*).

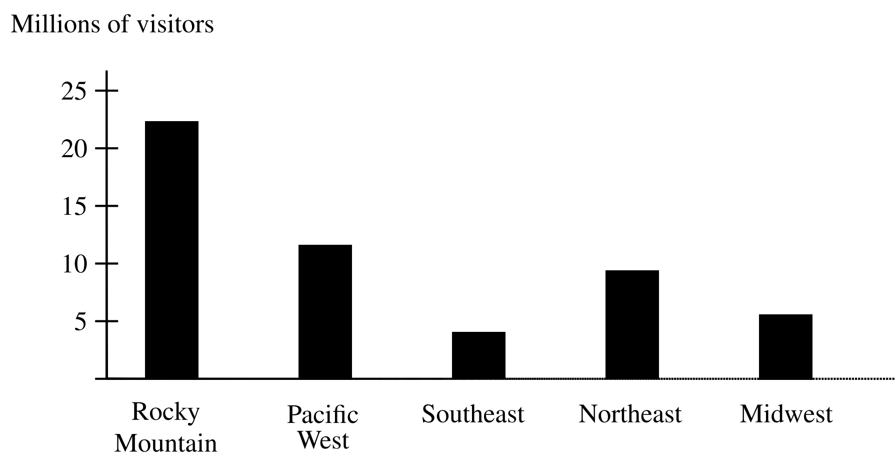


Figure 2.1: Visitors to ski resorts by region

a. For which region is the number of visitors the greatest? What is that number?

b. For which region is the number of visitors the least? What is that number?

c. Estimate the number of visitors to ski resorts in the Northeast.

OBJECTIVE 5

- When we plot points that are not being used to describe authentic situations, we call the horizontal axis the *x-axis* and the vertical axis the *y-axis*.
- In this case, x is the explanatory variable and y is the response variable.
- The ordered pair $(2, 5)$ means that $x = 2$ and $y = 5$.

12. Plot the points $(6, 2)$, $(-3, 1)$, $(4, -5)$, and $(-2, -4)$ on a coordinate system.

SHORT HW 5, 9, 17, 21, 29, 37, 39, 43, 45, 53, 61

MEDIUM HW 1, 5, 7, 9, 13, 17, 19, 21, 23, 29, 33, 35, 37, 39, 43, 45, 49, 53, 55, 61

SECTION 1.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students construct scatterplots, which is an important modeling skill of the course. Students learn the meaning of explanatory and response variables. They also interpret bar graphs, although this skill is not required for the rest of the textbook, except for the Stocks Lab in Chapter 2 (see pages 113 and 114 in the textbook).

OBJECTIVE 1 COMMENTS

Although most students have worked with coordinate systems, they have not considered the benefits of such a system (see Problem 1 and the comments that follow it).

When discussing the meaning of the ordered pair $(4, 14)$ within the context of the football team scores, I do not explain why we choose to let the first coordinate represent values of n and the second coordinate represent values of s. I first want students to have a sense of the "big picture" before discussing explanatory and response variables.

OBJECTIVE 2 COMMENTS

For Problem 3, I point out that there is a general upward trend. Sometimes I sketch a line that comes close to the points to suggest what's to come in Section 1.3; if I do this, I make sure that my students understand that the line is not part of the scatterplot.

Workbook Exploration

The following 20-minute exploration addresses an error students make more than any other error when it comes to constructing scatterplots.

Group Exploration

Constructing a scatterplot

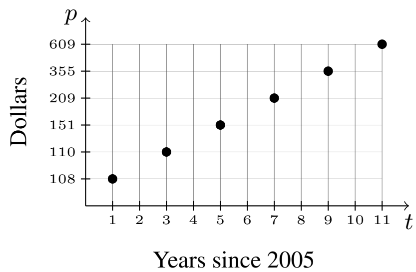
The prices of two-pack EpiPens are shown in the following table for various years.

Year	Price (dollars)
2006	108
2008	110
2010	151
2012	209
2014	355
2016	609

Source: *Elsevier*

Let p be the price (in dollars) at t years since 2005.

A student constructs the following scatterplot.



1. The student concludes that the price increased by the same amount every two years. By referring to the data table, do you agree with the student? Explain.

2. Describe all errors in the student's scatterplot. Construct a correct scatterplot.

3. Did the *increases* in the price increase, decrease, stay approximately constant, or none of these? Explain.

4. What is the main point of this exploration?

Workbook Exploration

In the following 15-minute exploration, groups can have well-needed practice with constructing a scatterplot. Many groups will likely forget to indicate the units and variables on the axes and use non-uniform scaling.

Group Exploration

Constructing and analyzing a scatterplot

Steam is a popular videogame-download store. The number of new games added to Steam are shown in the following table for various years.

Year	Number of New Games Added (thousands)
2013	0.5
2014	1.3
2015	2.7
2016	4.7
2017	7.0

Source: *SteamSpy*

1. Construct a scatterplot of the data.

2. Did the number of new games added to Steam increase, decrease, stay approximately constant, or none of these? Explain.

3. Did the *increases* in new games added to Steam increase, decrease, stay approximately constant or none of these?

4. In the first six months of 2018, 4.0 thousand new games were added to Steam. A student says that this means $2(4.0) = 8.0$ thousand new games will be added for the entire year. What would you tell the student?

OBJECTIVE 3 COMMENTS

Many students have a tough time learning this concept. However, even weaker students will eventually learn this concept because it's addressed so often in the course.

SECTION 1.3 LECTURE NOTES

Objectives

1. Describe the meaning of *linearly related*, *model*, *linear model*, *input*, and *output.*
2. Use a linear model to make estimates and predictions.
3. Use a scatterplot to help decide whether to model a situation with a linear model.
4. Find *intercepts* of a line.
5. Find intercepts of a line and a linear model.

Main point: Use a line to describe an exact linear relationship.

OBJECTIVE 1

1. A basement is flooded with water. Let w be the amount (in hundreds of cubic feet) of water that remains in the basement after t hours of pumping water out of the basement. Values of t and w are listed in the following table.

t (hours)	w (hundreds of cubic feet)
0	9
2	7
4	5
6	3
8	1

   a. Construct a scatterplot of the data.

   b. Draw the line that contains the points of the scatterplot.

Definition *Linearly related*

If two quantities of an authentic situation are described accurately by a line, then the quantities (and the variables representing those quantities) are **linearly related**.

Definition *Model*

A **model** is a mathematical description of an authentic situation. We say the description *models* the situation.

Definition *Linear model*

A **linear model** is a nonvertical line that describes the relationship between two quantities in an authentic situation.

OBJECTIVE 2

2. Use the water model to estimate the amount of water in the basement after 3 hours of pumping.

3. Use the model to estimate when there is 2 hundred cubic feet of water in the basement.

> **Definition** *Input, output*
>
> An **input** is a permitted value of the *explanatory* variable that leads to at least one **output,** which is a permitted value of the *response* variable.

For Problems 4–7, refer to the water model you sketched in part (b) of Problem 1.

4. Find an output that comes from the input $t = 5$.

5. Find an input that leads to the output $w = 8$.

6. Is $t = -2$ an input? Explain.

7. Is $w = -1$ an output? Explain.

OBJECTIVE 3

8. Consider the scatterplots of data for situations 1, 2, and 3 shown in Figs. 2.2, 2.3, and 2.4, respectively. For each situation, determine whether a linear model would describe the situation well.

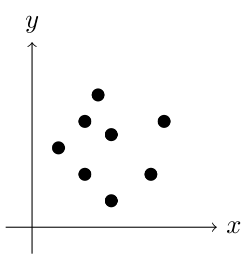


Figure 2.2: Scatterplot for situation 1

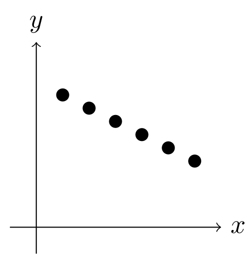


Figure 2.3: Scatterplot for situation 2

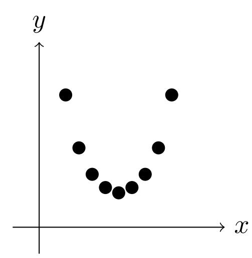


Figure 2.4: Scatterplot for situation 3

We construct a scatterplot of data to determine whether the data points lie on a line. If the points lie on a line, then we draw the line and use it to make estimates and predictions.

OBJECTIVE 4

> **Definition** *Intercepts of a line*
>
> An **intercept** of a line is any point where the line and an axis (or axes) of a coordinate system intersect. There are two types of intercepts of a line sketched on a coordinate system with an x-axis and a y-axis:
>
> - An ***x*-intercept** of a line is a point where the line and the x-axis intersect. [Sketch a figure.] The y-coordinate of an x-intercept is 0.
> - A ***y*-intercept** of a line is a point where the line and the y-axis intersect. [Sketch a figure.] The x-coordinate of a y-intercept is 0.

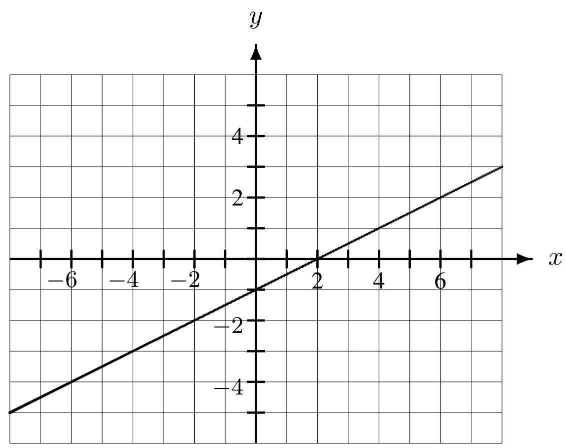
9. Refer to the following figure.

a. Find the x-intercept of the line.

b. Find the y-intercept of the line.

c. Find y when $x = -6$.

d. Find x when $y = -3$.



OBJECTIVE 5

10. Find the w-intercept of the water model. What does it mean in this situation?

11. Find the t-intercept of the water model. What does it mean in this situation?

OBJECTIVES 1–5 (revisited)

12. Let v be the value (in thousands of dollars) of a car when it is t years old. Some pairs of values of t and v are listed in the following table.

t (years)	v (thousands of dollars)
1	21
2	18
4	12
6	6
7	3

a. Construct a scatterplot of the data. Then draw a linear model.

b. Estimate the value of the car when it is 3 years old.

c. Estimate the age of the car when it is worth $9 thousand.

d. What is the v-intercept of the model? What does it mean in this situation?

e. What is the t-intercept of the model? What does it mean in this situation?

SHORT HW 1, 3, 5, 11, 13, 15, 19, 21, 27, 31, 43, 47

MEDIUM HW 1–21 odd, 25, 27, 31, 37, 39, 41, 43, 47

SECTION 1.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use a linear model to make estimates and predictions. They find intercepts of lines (nonmodels) and linear models. This section and Section 1.4 provide the "big picture" of Chapters 1–5.

OBJECTIVE 1 COMMENTS

For Problem 1, I remind my students about the concepts of explanatory and response variables and about which type of variable is represented by which axis. I also remind students to use uniform scaling and to label each axis with the appropriate variable and units.

When discussing the meaning of a model and, in particular, a linear model, I refer to the water model and give intuitive examples such as the textbook's discussion about airplane models (see the paragraph in the middle of page 27).

OBJECTIVE 2 COMMENTS

For Problems 2 and 3, I use arrows to set the stage for discussing inputs and outputs. (For an example of using arrows, see Fig. 34 on page 27 of the textbook.) I require my students to use such arrows so that I can better evaluate their work on quizzes and exams. It's difficult to award any partial credit if no arrows are included.

I explain to my students why their results of homework exercises in this section may not match exactly with the answers in the back of the text. Although I want my students to carefully construct scatterplots and linear models, I don't want them to be anxious about getting the exact results.

Problems 6 and 7 are good primers for the concept of model breakdown, which will be discussed in Section 1.4.

Textbook and Workbook Exploration

Having groups work on the 15-minute "Linear Modeling" exploration is a great way to introduce linear modeling.

OBJECTIVE 3 COMMENTS

Problem 8 helps students see that it's not a good idea to use a linear model for every situation.

OBJECTIVE 4 COMMENTS

I emphasize that students should use an ordered pair, rather than a single number, to describe an intercept. Some students have difficulties with parts (c) and (d) of Problem 9.

OBJECTIVE 5 COMMENTS

For Problems 10 and 11, some students are not sure which coordinates of the intercepts are 0. It sometimes helps to remind students that the first coordinate is a value of the explanatory variable and the second coordinate is a value of the response variable. Some students have trouble determining the meaning of the intercepts. It can help to write "t" and "w" just below the coordinates of the ordered pairs corresponding to the intercepts.

Having groups do Problem 12 serves as a great summary of the section.

SECTION 1.4 LECTURE NOTES

Objectives

1. Describe the meaning of *approximately linearly related*.
2. Use a linear model to make estimates and predictions.
3. Find errors in estimations.
4. Identify when *model breakdown* occurs.

Main point: Use a linear model to describe an approximate linear relationship.

OBJECTIVE 1

1. The revenues of IKEA are shown in the following table for various years.

Year	IKEA Revenue (billons of euros)
2009	21.8
2010	23.5
2011	25.2
2012	27.6
2013	28.5
2014	29.3
2015	32.7

Source: *IKEA*

Let r be the annual revenue (in billions of euros) of IKEA at t years since 2005.

a. Construct a scatterplot of the data.

b. Draw a line that comes close to the points in your scatterplot.

If the points in a scatterplot of data lie close to (or on) a line, we say that the relevant variables are **approximately linearly related.**

OBJECTIVE 2

2. Use the IKEA model to predict the revenue in 2020. Convert your result to U.S. dollars, assuming 1 euro will be worth 1.18 U.S. dollars.

3. Use the IKEA model to predict when the revenue will be $44 billion euros.

- We construct a scatterplot of data to determine whether the relevant variables are approximately linearly related.
- If so, we draw a line that comes close to the data points and use the line to make estimates and predictions.

OBJECTIVE 3

4. Use the IKEA model to estimate the revenue in 2015. Is your result an underestimate or an overestimate? Calculate the error in the estimate.

5. Use the IKEA model to estimate the revenue in 2014. Is your result an underestimate or an overestimate? Calculate the error in the estimate.

Underestimates and Overestimates

Suppose that an explanatory variable t and a response variable p are approximately linearly related. For a given data point $(a,\ b)$,

- If a linear model is below the data point, the model underestimates the value of p when $t = a$. [Draw a figure.]
- If a linear model is above the data point (a, b), the model overestimates the value of p when $t = a$. [Draw a figure.]

OBJECTIVE 4

6. The percentages of Apple's annual revenues that are from the iPad are shown in the following table for various years.

Year	Percent
2012	19.8
2013	18.7
2014	16.6
2015	9.9
2016	9.6
2017	7.1

Source: *Apple*

Let p be the percentage of Apple's annual revenue that is from the iPad at t years since 2010.

a. Construct a scatterplot of the data. Draw a line that comes close to the points in your scatterplot.

b. Use the model to estimate the percentage of Apple's revenue that is from the iPad in 2015. Is your result an underestimate or an overestimate?

c. Use the model to estimate when 4% of Apple's annual revenue will be from the iPad.

d. What is the p-intercept of the model? What does it mean in this situation?

e. What is the t-intercept of the model? What does it mean in this situation? [might be model breakdown]

f. Use the model to predict the percentage of Apple's revenue that will be from the iPad in 2022. [model breakdown]

Definition *Model breakdown*

When a model yields a prediction that does not make sense or an estimate that is not a good approximation, we say **model breakdown** has occurred.

SHORT HW 3, 7, 9, 11, 17, 19, 25

MEDIUM HW 1, 3, 5, 7, 9, 11, 13, 17, 19, 21, 25

SECTION 1.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students learn to use a linear model when two variables are approximately linearly related to make estimates and predictions. Students will calculate errors in estimates and determine when model breakdown has occurred. This section flows nicely from Section 1.3. Upon completion of this section, students will have the "big picture" of Chapters 1–5.

OBJECTIVE 1 COMMENTS

Before showing students the data in Problem 1, I ask them whether they think IKEA's revenue is increasing, decreasing, or neither. I play this "game" with students before revealing each data set in the course because it's a great way to engage students intellectually and sometimes generates interesting discussions.

Some students need yet more reminders about the following: explanatory and response variables, using uniform scaling, including variables and their units on the axes, and the meaning of phrases such as "at t years since 2005."

For Problem 1, I ask my students whether a line contains *all* the data points. Then I ask whether a line comes close to all the points. Then I draw various lines and discuss which ones are acceptable linear models. I emphasize that it's better for a model to come close to *all* the data points, rather than contain a *few* points and not come close to the other points. I compare a reasonable linear model that comes close to the data points, but does not contain any data points, with the highly inaccurate linear model that contains the data points $(9,\ 29.3)$ and $(10,\ 32.7)$.

Problem 1 is a good spring board for defining the terminology *approximately linearly related*.

Textbook and Workbook Exploration

If time permits, having groups of students work on the 15-minute "Approximately linearly related variables" exploration is a great way to begin this section. Without the hint to draw a *straight* line, many groups would sketch a "zigzag" curve rather than a line for Problem 2 of the exploration. Make sure to discuss this issue because the homework exercises won't include the word "straight."

OBJECTIVE 2 COMMENTS

For Problems 2 and 3, I continue the practice of using arrows to make estimates and predictions. I remind my students of the meaning of *input* and *output*. I also remind students that their answers may not match exactly with the answers in the back of the textbook; there are now three reasons: we can't draw what we intend to draw perfectly, there are many reasonable linear models when the variables are approximately linearly related, and we must estimate the values of inputs and outputs.

Workbook Exploration

The following 20-minute exploration could be assigned at the start of class. Depending on how well groups plot the data points, some groups may draw zigzag lines, which is actually a reasonable thing to do because they don't know of the advantages of using a line. Because Section 1.3 consists of a graphical approach, most groups will likely do the same for this exploration, but one or two groups may use a numerical approach. For groups that come up with simplistic approaches, challenge them to develop more sophisticated ones. For example, if a group uses the increase in Snapchat users only from 2017 to 2018 to predict the number of users in 2022, you could nudge them toward using the average of the increases from one year to the next. It may seem that a numerical approach is too far askew from the main point of this section, but this open-ended exploration prevents mimicry, supports the powerful learning process of productive struggle, and has students consider the key concept of trend. During the debrief, comparing and contrasting various groups methods can lay a great foundation for the entire chapter.

Group Exploration

Section Opener: Making a prediction

The number of daily active Snapchat users are shown in the following table for the first quarter of various years.

Predict the number of daily active Snapchat users in the first quarter of 2022.

Year	Number of Daily Active Snapchat Users (millions)
2014	46
2015	80
2016	122
2017	166
2018	191

Source: *Snap Inc.*

OBJECTIVE 3 COMMENTS

For many students, the distinction between a model and reality is foggy. For example, many students would think that the answers to Problems 4 and 5 are the actual revenues \$32.7 billion and \$29.3 billion, respectively, even if the linear model does not contain the data points $(10, 32.7)$ and $(9, 29.3)$. Making such a distinction should pay dividends both in the short and long run. For an intuitive explanation, I tell my students that a model is fiction, which we hope is very close to the truth.

After completing Problems 4 and 5, I ask my students how we could tell whether an estimate is an underestimate or an overestimate by observing whether the linear model is below or above a given data point.

I point out that for some non-time-series data, several data points might all have the same explanatory-variable coordinate; in such cases, we must specify for which data point we have calculated the error.

OBJECTIVE 4 COMMENTS

Just like when I present Problem 1, before showing students the data in Problem 6, I ask students whether they think the percentage of Apple's annual revenue that is from the iPad is increasing, decreasing, or neither. The answer to this question is not obvious, but students might be more sensitive to what's hot in tech sales than many of us. I follow up this query by asking why this information might be useful to Apple and its competitors. Such questions can attract the attention of students who may be weak in mathematics but strong in other areas, such as business.

Problem 6 serves as a nice summary of the section. Even though finding intercepts of a linear model was discussed in Section 1.3, most students will benefit from reviewing this [in parts (d) and (e)]. I remind students to include both coordinates of an intercept. Students sometimes need help determining which variable is represented by which coordinate.

Parts (e) and (f) of Problem 6 serve as a good springboard to defining model breakdown. This concept is important because most models do "break down." I tell students that when model breakdown occurs, they must say so and explain why when completing homework, quizzes, and tests. I reassure them that they do not need to be an expert in all subjects to do so. For example, I tell them whether or not they decide that model breakdown occurs in part (e) of Problem 6, I will reward them all points, provided they state the meaning of the t-intercept, point out that it's a striking result, and give a careful explanation of whether they think model breakdown has occurred.

Workbook Exploration

The following 20-minute exploration takes a closer look at the various types of errors. This activity drives home the point why modeling does not yield exact values, especially when it's done graphically by hand. If you do not plan on using many of the explorations in the textbook, it would probably be good to omit assigning this one. Many of these ideas can be touched on as you proceed through the course.

Group Exploration

Identifying types of modeling errors

The interest rates for subsidized student loans are shown in the following table for various years.

Year	Interest Rate (percent)
2008	6.8
2009	6.0
2010	5.6
2011	4.5
2012	3.4
2013	3.4

Source: *New America Foundation*

Here you will explore possible causes of error for estimates and predictions based on a linear model for the data.

1. Let r be the interest rate (percent) for subsidized student loans at t years since 2005. Construct a scatterplot of the data.

2. Draw a line that comes close to the points in your scatterplot.

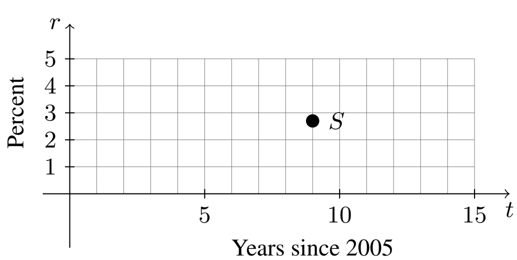
3. Use your linear model to estimate the interest rate in 2012. What is the actual interest rate? Calculate the error in your estimate. (The error is the difference between the estimated value and the actual value.)

4. Use your linear model to estimate the interest rate in 2015. The actual interest rate is 4.7%. Calculate the error in your estimate for 2015.

5. Use your linear model to predict the interest rate in 2021. Is this an accurate prediction? Explain.

6. Take another look at your sketch from Problems 1 and 2. Is the t-axis perfectly horizontal and the r-axis perfectly vertical? Are the scalings of both axes precise? Is your line straight? How might these considerations relate to the accuracy of an estimation or a prediction? Explain.

7. What is the r-coordinate of point S plotted in the following coordinate system? Do you think you have found the correct first decimal place (tenths place) for this coordinate? How about the second decimal place?



8. Problems 3–7 of this exploration suggest several possible causes of error for estimates and predictions based on a linear model. Describe the possible causes of error.

CHAPTER 2 OVERVIEW

In Section 2.1, students use expressions to describe authentic quantities and how to evaluate such expressions. In Section 2.2, students perform operations with fractions. Students add real numbers in Section 2.3 and subtract real numbers in Section 2.4. They also find the change in a quantity in Section 2.4. In Section 2.5, students work with ratios and percents. They also multiply and divide real numbers in Section 2.5. In Section 2.6, students work with exponents and use the rules for order of operations to perform computations and evaluate expressions.

SECTION 2.1 LECTURE NOTES

Objectives

1. Describe the meaning of *expression* and of *evaluate an expression*.
2. Use expressions to describe authentic quantities.
3. Evaluate expressions.
4. Translate English phrases into and from mathematical expressions.
5. Describe two roles of a variable.
6. Evaluate expressions with more than one variable.

Main point: Describe the meaning of *expression* and how to evaluate one.

OBJECTIVES 1–3

1. A person is driving 3 miles per hour over the speed limit. For each speed limit shown, find the diving speed.

a. 55 mph **b.** 70 mph **c.** s mph

We call $s + 3$ is an *expression*.

Definition *Expression*

An **expression** is a constant, a variable, or combination of constants, variables, operation symbols, and grouping symbols, such as parentheses.

Here are some examples of expressions: $x + 9$, 5, $x - 7$, π, $\dfrac{20}{x}$, xy

2. Substitute 65 for s in the expression $s + 3$ and discuss the meaning of the result (see Problem 1).

We say we have *evaluated the expression* $s + 3$ at $s = 65$.

Definition *Evaluate an expression*

We **evaluate an expression** by substituting a number for each variable in the expression and then calculating the result. If a variable appears more than once in the expression, the same number is substituted for that variable each time.

3. A certain type of pen costs \$3.

a. Complete the following table to help find an expression that describes the total cost (in dollars) of n pens. Show the arithmetic to help see the pattern.

Number of Pens	Total Cost (dollars)
5	
6	
7	
8	
n	

b. Evaluate your result for $n = 10$. What does it mean in this situation?

- We avoid using $\times$ for the multiplication operation.
- Each of the following expressions describes multiplying 3 by n: $3n$, $3 \cdot n$, $3(n)$, $(3)n$, and $(3)(n)$.

OBJECTIVE 4

Mathematics is a language.

Definition *Product, factor, and quotient*

Let a and b be numbers. Then

- The **product** of a and b is ab. We call a and b **factors** of ab.
- The **quotient** of a and b is $a \div b$, where b is not zero.

4. Use phrases such as "2 plus x" and sentences such as "Add 2 and x." to complete the second column of the following table.

Mathematical Expression	English Phrase or Sentence
$2 + x$	
$2 - x$	
$2 \cdot x$	
$2 \div x$	

Warning: Subtracting 2 from 7 is $7 - 2$, not $2 - 7$.

5. Let x be a number. Have students translate the English phrase into a mathematical expression or vice versa, as appropriate:

a. The difference of the number and 9

b. The product of 5 and the number

c. $7 + x$

d. $x \div 4$

6. Let x be a number. Translate the sentence "Subtract the number from 10." into a mathematical expression. Evaluate the expression at $x = 7$.

OBJECTIVE 5

Roles of a Variable

Here are two roles of a variable:

- A variable can represent a quantity that can vary.
- In an expression, a variable is a placeholder for a number.

OBJECTIVE 6

Recall that the area of a rectangle is equal to the length times the width of the rectangle.

7. Let W be the width (in feet) and L be the length (in feet) of a rectangle. Evaluate the expression LW for $L = 7$ and $W = 5$. What does your result mean in this situation?

SHORT HW 3, 5, 13, 17, 19, 25, 29, 33, 35, 49, 53, 57, 61, 65

MEDIUM HW 3, 5, 9, 13, 15, 17, 19, 21, 25, 27, 29, 33, 35, 39, 49, 51, 53, 55, 57, 61, 63, 65

SECTION 2.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students evaluate expressions. They use expressions to describe authentic quantities. Students translate English phrases to and from mathematical expressions. They also learn two roles of a variable. These topics will be needed throughout the course.

OBJECTIVES 1–3 COMMENTS

For part (a) of Problem 3, I emphasize that we want to show the arithmetic to help see the pattern. Insisting that students do this now will form good habits so that students will be able to find more challenging expressions of the form $mx + b$ in Section 2.6. The form of students' answers will vary for part (a), and students will be reassured to know that $3n = n(3)$. (The commutative law for multiplication will be discussed in Section 4.1.)

Textbook and Workbook Exploration

The 10-minute"Expressions" exploration is a nice way for students to discover how to use expressions to describe authentic quantities.

Textbook and Workbook Exploration

In the 10-minute "Expressions used to describe a quantity" exploration, students work backwards and find an authentic quantity that can be described by the given expression.

OBJECTIVE 4 COMMENTS

I emphasize that mathematics is a language. Most students have never thought of mathematics in this way.

Most students do not know the meanings of the words "product," "quotient," and "ratio." These words will be used extensively in the textbook.

I emphasize that "Subtracting 2 from 7" is $7 - 2$, not $2 - 7$. I point out that if you take 2 CDs *from* 7 CDs, then 5 CDs remain. This may help students see why we subtract the numbers "out of order."

OBJECTIVE 5 COMMENTS

I tell my students that sometimes a variable serves both roles. (A third role of a variable is discussed in Section 4.3:

In an equation, a variable represents any number that is a solution of the equation.)

OBJECTIVE 6 COMMENTS

Students have an easy time with this objective.

SECTION 2.2 LECTURE NOTES

Objectives

1. Describe the meaning of a fraction.
2. Explain why division by zero is undefined.
3. Describe the rules for $a \cdot 1$, $\frac{a}{1}$, and $\frac{a}{a}$.
4. Perform operations with fractions.
5. Find the prime factorization of a number.
6. Simplify fractions.

Main point: Perform operations with fractions.

OBJECTIVE 1

1. Shade a portion of a circle to illustrate the meaning of $\frac{5}{8}$ of a pizza.
2. Use two pizzas with 4 slices each to show that $\frac{8}{4} = 8 \div 4 = 2$.

The fraction $\frac{a}{b}$ means $a \div b$.

OBJECTIVE 2

Use the concept that division is repeated subtraction to show why division by 0 is undefined (see the middle of page 64 in the textbook).

Division by Zero

The fraction $\frac{a}{b}$ is undefined if $b = 0$. Division by 0 is undefined.

OBJECTIVE 3

- $a \cdot 1 = a$
- $\frac{a}{1} = a$
- $\frac{a}{a} = 1$, where a is nonzero

- When we write statements such as $a \cdot 1 = a$, we mean that if we evaluate $a \cdot 1$ and a for *any* value of a in both expressions, the results will be equal.
- We say the expressions $a \cdot 1$ and a are **equivalent expressions.**

OBJECTIVE 4

Multiplying Fractions

If b and d are nonzero, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

3. $\frac{3}{5} \cdot \frac{7}{2}$

4. $\frac{4}{9} \cdot \frac{5}{7}$

OBJECTIVE 5

Definition *Prime number*

A **prime number** is any counting number larger than 1 whose only positive factors are itself and 1.

5. Prime factor 24.

6. Prime factor 36.

OBJECTIVE 6

Use a drawing of a pizza to show that $\frac{6}{8} = \frac{3}{4}$.

Simplifying a Fraction

To simplify a fraction.

1. Find the prime factorizations of the numerator and the denominator.
2. Find an equal fraction in which the numerator and the denominator do not have common positive factors other than 1 by using the property

$$\frac{ab}{ac} = \frac{a}{a} \cdot \frac{b}{c} = 1 \cdot \frac{b}{c} = \frac{b}{c}$$

where a and c are nonzero.

7. Simplify $\frac{6}{8}$.

8. Simplify $\frac{24}{18}$.

9. Simplify $\frac{7}{49}$.

OBJECTIVE 4 (revisited)

10. $\frac{4}{25} \cdot \frac{35}{8}$

11. $\frac{14}{8} \cdot \frac{10}{21}$

Dividing Fractions

If b, c, and d are nonzero, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

12. $\frac{9}{2} \div \frac{3}{4}$

13. $\frac{45}{16} \div \frac{35}{8}$

When you use a calculator to check work with fractions, enclose each fraction in parentheses.

14. Use a drawing of a pizza to show that $\frac{3}{8} + \frac{4}{8} = \frac{7}{8}$.

Adding Fractions with the Same Denominator

If b is nonzero, then $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

15. $\frac{7}{12} + \frac{2}{12}$ **16.** $\frac{5}{6} + \frac{7}{2}$ **17.** $\frac{4}{5} + \frac{2}{3}$ **18.** $\frac{5}{6} + \frac{3}{8}$

Use a drawing of a pizza to show that $\frac{7}{9} - \frac{2}{9} = \frac{5}{9}$.

Subtracting Fractions with the Same Denominator

If b is nonzero, then $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$.

19. $\frac{5}{6} - \frac{1}{6}$ **20.** $\frac{7}{6} - \frac{2}{3}$ **21.** $\frac{5}{8} - \frac{3}{10}$

Adding (or Subtracting) Fractions with Different Denominators

To add (or subtract) two fractions with different denominators, use the fact that $\frac{a}{a} = 1$, where a is nonzero, to write an equal sum (or difference) of fractions for which each denominator is the LCD.

Throughout the course, if a result is a fraction, it must be simplified.

SHORT HW 7, 21, 25, 31, 37, 45, 49, 69, 75, 79, 81, 87, 89, 97

MEDIUM HW 1, 7, 15, 21, 25, 27, 31, 33, 37, 45, 47, 49, 51, 63, 69, 75, 79, 81, 83, 85, 87, 89, 91, 95, 97

SECTION 2.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students review the meaning of a fraction and that a division by zero is undefined. Students perform operations with fractions and simplify fractions.

Because some students have difficulty with this material, this is a good time to remind them of your availability during office hours and other forms of college support such as math center tutoring. Students' greatest difficulty is usually with adding and subtracting fractions with different denominators. So, it is wise to go quickly through the rest of the material so you have adequate time to focus on this more difficult skill.

OBJECTIVE 1 COMMENTS

Some students will need a reminder about the meaning of a fraction (see Problem 1).

OBJECTIVE 2 COMMENTS

I use repeated subtraction to show that division by zero is undefined. Most students really appreciate this explanation, because they have never been told *why* division by zero is undefined.

OBJECTIVE 3 COMMENTS

Although these rules are very basic, they are the foundation for performing operations with fractions.

OBJECTIVE 4 COMMENTS

After we have discussed how to simplify fractions (Objective 6), we will discuss how to multiply fractions which can be simplified.

OBJECTIVE 5 COMMENTS

In arithmetic, students learn a variety of ways to prime factor a number. Although I allow students to use any method, I encourage them to use the method shown in Example 2 on page 66 of the textbook, because this method clearly portrays that the factorization is equal to the original number.

OBJECTIVE 6 COMMENTS

Most students can successfully simplify a fraction, but most are not aware of the concepts involved. I make sure they understand that we use the property $\frac{a}{a} = 1$, where $a \neq 0$, to simplify a fraction. The textbook avoids using the terminology "reduce," because many professors think that this terminology suggests that reducing a fraction makes it smaller. The textbook also avoids using the technique of "canceling," because students misapply this technique in a myriad of ways.

OBJECTIVE 4 (REVISITED) COMMENTS

Once my students know how to simplify fractions, I discuss how to multiply fractions where the result needs to be simplified.

Using a drawing of a pizza to show that $\frac{3}{8} + \frac{4}{8} = \frac{7}{8}$ helps students understand why it doesn't make sense to add the denominators.

When adding fractions with different denominators, I list multiples of the denominators to find the LCD. A different (and unique) technique will be discussed in Section 10.3, but listing multiples will suffice for now. When adding fractions such as $\frac{5}{6}$ and $\frac{3}{8}$ (see Problem 18), I show at least once that we are multiplying fractions by 1 to get each denominator to be the LCD:

$$\begin{aligned}
\frac{5}{6} + \frac{3}{8} &= \frac{5}{6} \cdot 1 + \frac{3}{8} \cdot 1 \\
&= \frac{5}{6} \cdot \frac{4}{4} + \frac{3}{8} \cdot \frac{3}{3} \\
&= \frac{20}{24} + \frac{9}{24} \\
&= \frac{29}{24}
\end{aligned}$$

Textbook and Workbook Exploration

The 10-minute "Illustrations of simplifying fractions and operations with fractions" exploration is a nice way for students to learn to visualize their work with fractions.

Workbook Exploration

In the following 15-minute exploration, groups reflect on typical errors (especially Problem 4) and inefficient methods (Problem 2). As an extension (either for all groups or those that finish ahead of the rest), you could have groups draw sketches of pizza slices to illustrate why the work in Problems 1, 3, and 4 is incorrect.

Group Exploration
Performing operations with fractions

For each of Problems 1–4, a student tries to perform an operation of two fractions and then simplify the result. If the result is correct, decide whether there is a more efficient way to do the problem. If the result is incorrect, describe any errors and do the problem correctly.

1. $$\begin{aligned}\frac{3}{4}+\frac{5}{6}&=\frac{3+5}{4+6}\\&=\frac{8}{10}\\&=\frac{4}{5}\end{aligned}$$

2. $$\begin{aligned}\frac{1}{3}\cdot\frac{1}{4}&=\left(\frac{1}{3}\cdot\frac{4}{4}\right)\cdot\left(\frac{1}{4}\cdot\frac{3}{3}\right)\\&=\frac{4}{12}\cdot\frac{3}{12}\\&=\frac{12}{144}\\&=\frac{1}{12}\end{aligned}$$

3. $$\begin{aligned}4\cdot\frac{2}{3}&=\frac{4\cdot 2}{4\cdot 3}\\&=\frac{8}{12}\\&=\frac{2}{3}\end{aligned}$$

4. $$\begin{aligned}\frac{2}{3}\div\frac{4}{5}&=\frac{3}{2}\cdot\frac{4}{5}\\&=\frac{3\cdot 4}{2\cdot 5}\\&=\frac{12}{10}\\&=\frac{6}{5}\end{aligned}$$

SECTION 2.3 LECTURE NOTES

Objectives

1. Find the opposite of a number.
2. Find the opposite of the opposite of a number.
3. Find the absolute value of a number.
4. Add real numbers by thinking in terms of money, the number line, and absolute value.
5. Add real numbers pertaining to authentic situations.
6. Find an expression to model an authentic quantity.

Main point: Add real numbers.

OBJECTIVE 1

- Two number are called **opposites** of each other if they are the same distance from 0 on the number line, but are on opposite sides of 0.
- To find the opposite of a number, we reflect the number across 0 on the number line.
- The opposite of 5 is -5. The opposite of -5 is 5. [Draw a figure.]

OBJECTIVE 2

Use reflections with a number line to show that $-(-a) = a$.

1. $-(-6)$

2. $-(-(-5))$

We use parentheses to separate two opposite symbols or an operation symbol and an opposite symbol.

OBJECTIVE 3

Definition *Absolute value*

The **absolute value** of a number is the distance that the number is from 0 on the number line. [Draw a figure.]

3. $|4|$

4. $|-7|$

5. $-|8|$

6. $-|-2|$

OBJECTIVES 4 and 5

7.
- **a.** A person has a credit card balance with a 0 dollar balance. If she uses her credit card to make two purchases for \$3 and \$4, what is the new balance?
- **b.** Write a sum that is related to the computation in part (a).
- **c.** Use a number line to illustrate the sum found in part (b).

Adding Two Numbers with the Same Sign

To add two numbers with the same sign:

1. Add the absolute values of the numbers.
2. The sum of the original numbers has the same sign as the sign of the original numbers.

8. $-3+(-8)$ **9.** $-2.57+(-6.84)$ **10.** $-\frac{5}{8}+\left(-\frac{1}{8}\right)$

11. **a.** A brother owes his sister \$5. If he then pays her back \$2, how much does he still owe her?
b. Write a sum that is related to your work in part (a).
c. Use a number line to illustrate the sum you found in part (b).

Adding Two Numbers with Different Signs

To add two numbers with different signs:

1. Find the absolute values of the numbers. Then subtract the smaller absolute value from the larger absolute value.
2. The sum of the original numbers has the same sign as the original number with the larger absolute value.

12. $-9+2$ **13.** $7+(-4)$ **14.** $-\frac{7}{6}+\frac{5}{6}$

15. A person bounces several checks and is charged service fees such that the balance of the checking account is -78. The person then deposits \$250. Find the balance.

16. $-7+(-4)$ **17.** $-5.31+1.98$ **18.** $329+(-838)$ **19.** $\frac{3}{10}+\left(-\frac{5}{6}\right)$

20. Let x be a number. Translate the phrase "the number plus -5" into a mathematical expression. Then evaluate the expression for $x=-9$.

OBJECTIVE 6

21. A hardware store is offering a weekend sale of \$5 off the retail price of any of their power tools.

a. Complete the following table to help find an expression that describes the sale price (in dollars) if the retail price is r dollars. Show the arithmetic to help you see a pattern.

Retail Price (dollars)	Sale Price (dollars)
50	
75	
100	
125	
r	

b. Evaluate the expression you found in part (a) for $r = 80$. What does your result mean in this situation?

SHORT HW 1, 7, 9, 13, 21, 31, 37, 49, 53, 57, 61, 67, 75, 83

MEDIUM HW 1, 7, 9, 13, 19, 21, 27, 31, 37, 43, 47, 49, 53, 55, 57, 61, 63, 65, 67, 73, 75, 77, 83

SECTION 2.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students add real numbers and find expressions to model authentic quantities. They will use these skills throughout the course.

OBJECTIVE 1 COMMENTS

Although finding the opposite of a number is an easy topic, I discuss reflecting a number across 0 on the number line to set the stage for Objective 2.

OBJECTIVE 2 COMMENTS

This concept will help students subtract numbers in Section 2.4: $a - (-b) = a + [-(-b)] = a + b$.

OBJECTIVE 3 COMMENTS

Although absolute value is an easy topic, defining it in terms of the distance from 0 will lay a good foundation for students to solve absolute value equations and inequalities in subsequent courses.

OBJECTIVES 4 and 5 COMMENTS

Most of my students prefer to think of adding real numbers in terms of money (as opposed to the number line or absolute value). I quickly go over the rules and spend more time having my students do mixed sets of problems such as Problems 16–19.

Workbook Exploration

My students tend to be quick to pick up on how to add real numbers and why the rules make sense, so I usually skip the following 20-minute exploration. But for a weaker class, facilitating the exploration without lecturing is a great way to have students think graphically or intuitively without blindly memorizing the rules.

EXPLORATION *Adding real numbers*

1. Describe how to find the sum by thinking in terms of money. Also show how to find the sum by using a number line.

   a. $-2 + (-3)$ **b.** $-1 + (-8)$ **c.** $-3 + (-5)$

2. What is the sign of the sum of two negative numbers? Explain.

3. Describe how to find the given sum by thinking in terms of money. Also show how to find the sum by using a number line.

   a. $6 + (-2)$ **b.** $7 + (-4)$ **c.** $-3 + 5$

4. If two numbers have different signs, what is the sign of the sum of the numbers if the absolute value of the positive number is larger than the absolute value of the negative number? Explain.

5. Describe how to find the given sum by thinking in terms of money. Also show how to find the sum by using a number line.

   a. $3 + (-5)$ **b.** $1 + (-4)$ **c.** $-6 + 2$

6. If two numbers have different signs, what is the sign of the sum of the numbers if the absolute value of the positive number is smaller than the absolute value of the negative number? Explain.

Textbook and Workbook Exploration

The 10-minute "Adding a number and its opposite" exploration is a nice way for students to learn that $a + (-a) = 0$.

OBJECTIVE 6 COMMENTS

For Problem 21, I emphasize that we want to show the arithmetic. For part (a), the result can be written as $r - 5$ or $r + (-5)$. This suggests that $a - b = a + (-b)$. This could be a nice segue to Section 2.4.

SECTION 2.4 LECTURE NOTES

Objectives

1. Find the change in a quantity.
2. Subtract real numbers.
3. Find a change in elevation.
4. Determine the sign of the change for an increasing or decreasing quantity.

Main point: Find the change in a quantity and subtract real numbers.

OBJECTIVE 1

1. **a.** If the price of a basket of strawberries increases from \$2 to \$5, find the change in the price.

b. Write a difference that is related to the computation in part (a).

Change in a Quantity

The change in a quantity is the ending amount minus the beginning amount:

$$\text{Change in the quantity} = \text{Ending amount} - \text{Beginning amount}$$

2. The revenues of Facebook are shown in the following table for various years.

Year	Facebook Revenue (billions of dollars)
2011	3.71
2012	5.09
2013	7.87
2014	12.5
2015	17.9
2016	27.6
2017	40.6

Source: *Facebook*

a. For the period 2011–2017, find each of the changes in annual revenue from one year to the next.

b. From which year to the next did the annual revenue increase the most?

c. Did the changes in annual revenue from one year to the next increase, decrease, stay approximately constant, or none of these from 2011 to 2017?

OBJECTIVE 2

3. **a.** The volume of gasoline in a car's gasoline tank decreases from 6 gallons to 2 gallons. What is the change in the volume of gasoline in the tank?

b. Write a difference that is related to the computation in part (a).

c. From part (b), we found that $2 - 6 = -4$. Find $2 + (-6)$. Explain why we can conclude that $2 - 6 = 2 + (-6)$.

Subtracting a Real Number

$$a - b = a + (-b)$$

4. $3 - 7$ **5.** $-2 - 8$ **6.** $\frac{2}{9} - \frac{8}{9}$

7. **a.** The temperature increases from $-3°$F to $2°$F. Find the change in temperature.

b. Write a difference that is related to the work in part (a).

c. Find the difference obtained in part (b) by using the rule $a - b = a + (-b)$.

8. $5 - (-2)$ **9.** $-6 - (-1)$ **10.** $\frac{2}{7} - \left(-\frac{3}{7}\right)$

11. Translate the sentence "Subtract the number from 5." into a mathematical expression. Then evaluate the expression for $x = -3$.

12. $-5 + (-4)$ **13.** $4 - 11$ **14.** $-9 - 7$ **15.** $-\frac{2}{5} - \left(-\frac{7}{10}\right)$ **16.** $2.5 + (-7.1)$ **17.** $62 - (-95)$ **18.** $\frac{7}{12} - \frac{11}{15}$ **19.** $46 + (-11)$

OBJECTIVE 3

- An object that has a *positive* elevation of 500 feet is 500 feet *above* sea level.
- An object that has a *negative* elevation of -500 feet is 500 feet *below* sea level.

20. The Golden Gate Bridge has two towers that support two main cables of the bridge. The top of each tower is at an elevation of 746 feet, and the foot of each tower is at an elevation of -136 feet. Find the height of each tower.

OBJECTIVE 4

Refer to Problem 2 to motivate the following properties.

Changes of Increasing and Decreasing Quantities

- An increasing quantity has a positive change.
- A decreasing quantity has a negative change.

21. For airplanes, *available seat miles* is the number of seats available times the number of miles flown. The levels of total available seat miles for United Airlines are shown in the following table for various years.

Year	United's Total Available Seat Miles (billions of seat miles)
2011	219
2012	216
2013	213
2014	214
2015	220
2016	225
2017	235

Source: *United Airlines*

a. Find the change in United's total available seat miles from 2011 to 2012. What does it mean in this situation?

b. Find the change in United's total available seat miles from 2016 to 2017. What does it mean in this situation?

SHORT HW 1, 5, 15, 27, 31, 35, 37, 49, 53, 57, 61, 65, 73, 79, 81

MEDIUM HW 1, 5, 7, 15, 25, 27, 31, 33, 35, 37, 39, 43, 49, 53, 57, 61, 63, 65, 73, 75, 79, 81, 85, 87

SECTION 2.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students subtract real numbers, a skill that they will use throughout the course. In this section, students find the change in a quantity. This important concept is sadly overlooked in most arithmetic and algebra textbooks. This is a central concept needed to understand that in going from a point (x_1, y_1) to a point (x_2, y_2), $y_2 - y_1$ is the vertical change and $x_2 - x_1$ is the horizontal change. This gives meaning to the slope formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. The last piece of the puzzle about slope will be to discuss the meaning of ratios in Section 2.5. Understanding how to find the change in a quantity will also help students make sense of the techniques we use to subtract real numbers.

OBJECTIVE 1 COMMENTS

Problem 1 nicely motivates the change in a quantity property. Note that the results of part (a) of Problem 2 are average rates of change for various years.

Workbook Exploration

Students can complete the following 15-minute exploration with few if any hints. In particular, they are able to determine the signs of the results because they are working with authentic quantities.

Group Exploration

Section Opener: Finding the change in a quantity

1. The sales of craft soda were \$520 million in 2015 and \$541 million in 2016 (Source: *Beverage Marketing Corp*). Is the change in revenue positive or negative? Explain. Also, find the change in the revenue.

2. The enrollment at Eastern Illinois University was about 10 thousand students in 2013 and about 7 thousand students in 2016 (Source: *Eastern Illinois University*). Is the change in enrollment positive or negative? Explain. Also, find the change in enrollment.

3. The viewership of the Academy Awards was 33 million people in 2017 and 27 million people in 2018 (Source: *Nielsen*). Is the change in viewership positive or negative? Also, find the change in viewership.

4. The revenue for the top 100 North American concert tours was \$2.1 billion in 2010 and \$3.6 billion in 2017 (Source: *Pollster*). Is the change in revenue positive or negative? Explain. Also, find the change in the revenue.

5. In New York City, the average speed of taxis was 8.2 miles per hour in 2015 and 6.9 miles per hour in 2016 (Source: *Fix NYC Advisory Panel Report*). Is the change in average speed positive or negative? Explain. Also, find the change in the average speed.

6. In general, when is the change in a quantity positive? negative?

7. Describe how to compute the change in a quantity.

OBJECTIVE 2 COMMENTS

Problem 3 suggests the property $a - b = a + (-b)$. Problem 7 suggests how to subtract a negative number. Problems 12–19 serve as a summary of Sections 2.3 and 2.4.

Textbook and Workbook Exploration

The 10-minute "Subtracting numbers" exploration is a nice way for groups to discover that $a - b = a + (-b)$. Some groups will need help with Problems 4 and 5 of this exploration.

OBJECTIVE 3 COMMENTS

Although this is just a special case of computing the change in a quantity, it is nonetheless great practice for later finding the rise in going from one point to another point when calculating the slope of a line.

OBJECTIVE 4 COMMENTS

Although I address the concepts about the significance of signs of change in Objectives 1 and 2, it is nice to reinforce these important concepts at the end of this section.

SECTION 2.5 LECTURE NOTES

Objectives

1. Find the ratio of two quantities.
2. Describe the meaning of *percent*.
3. Convert percentages to and from decimal numbers.
4. Find the percentage of a quantity.
5. Multiply and divide real numbers.
6. Determine which fractions with negative signs are equal to each other.

Main point: Work with ratios, percents, and multiplying and dividing real numbers.

OBJECTIVE 1

- Suppose that there are 8 women and 4 men on a softball team. The ratio of women to men is

$$\frac{8 \text{ women}}{4 \text{ men}} = \frac{2 \text{ women}}{1 \text{ men}}$$

- This means that there are 2 women to 1 man. This ratio is called a *unit ratio*.
- A **unit ratio** is a ratio written as $\frac{a}{b}$ or as $a : b$ with $b = 1$.

1. The blink of an eye takes 400 milliseconds. One beat of a hummingbird's wings takes 20 milliseconds. Find the unit ratio of the time it takes for the blink of an eye to the time it takes for one beat of a hummingbird's wings. What does your result mean in this situation?

2. The median sales prices of existing homes and the median incomes are shown in the following table for four regions of the United States.

Region	Median Sales Price of Existing Homes (dollars)	Median Income (dollars)
Northeast	402,800	59,210
Midwest	269,700	54,267
South	257,700	49,655
West	333,900	57,688

Sources: *U.S. Census; National Association of Realtors®*

   a. Find the unit ratio of the median sales price of existing homes in the Northeast to the median income in the Northeast. What does the result mean?

   b. For each of the four regions, find the unit ratio of the median sales price of existing homes to the median income. Taking into account the median income of each region, list the regions in order of affordability of existing homes, from greatest to least.

   c. A person believes that existing homes in the South are more affordable than in the Midwest, because the median price of existing homes is lower in the South than in the Midwest. What would you tell that person?

OBJECTIVE 2

- If 57 of 100 songs are hip hop, then 57% of the songs are hip hop.
- **Percent** means "for each hundred": $a\% = \frac{a}{100}$.
- A percent is a ratio.

OBJECTIVE 3

Converting Percentages to and from Decimal Numbers

- To write a percentage as a decimal number, remove the percent symbol and divide the number by 100 (move the decimal point two places to the left).
- To write a decimal number as a percentage, multiply the number by 100 (move the decimal point two places to the right) and insert a percent symbol.

3. Write 8% as a decimal number.

4. Write 0.145 as a percent.

Warning: 8% is equal to 0.08, not 0.8.

OBJECTIVE 4

Finding the Percentage of a Quantity

To find the percentage of a quantity, multiply the decimal form of the percentage and the quantity.

5. Find 5% of 80 students.

6. Find 6.3% of \$2000.

One Hundred Percent of a Quantity

One hundred percent of a quantity is *all* of the quantity.

OBJECTIVE 5

Use repeated addition to show that $4(-3) = -12$.

Multiplying Two Numbers with Different Signs

The product of two number that have different signs is negative.

7. **a.** Find the products $3(-4) = -12$, $2(-4) = -8$, $1(-4) = -4$, $0(-4) = 0$.

b. Explain why the results of part (a) suggest that $-1(-4) = 4$, $-2(-4) = 8$, and $-3(-4) = 12$.

c. Form a theory about the sign of the product of two numbers that have the same sign.

Multiplying Two Numbers with the Same Sign

The product of two numbers that have the same sign is positive.

8. $3(-8)$ **9.** $-5(-9)$ **10.** $-0.2(-0.1)$ **11.** $-\frac{4}{9} \cdot \frac{3}{2}$

The rules for dividing numbers are similar to those for multiplying numbers, because to divide by a number, we multiply by the reciprocal of the number.

Multiplying or Dividing Real Numbers

The product or quotient of two numbers that have different signs is negative. The product or quotient of two numbers that have the same sign is positive.

12. $-12 \div (-2)$ **13.** $-35 \div 7$ **14.** $475 \div -25$ **15.** $-\frac{5}{49} \div \left(-\frac{25}{21}\right)$

16. A person has credit card balances of -3580 dollars on a Discover® account and -1590 dollars on a MasterCard® account.

a. Find the unit ratio of the Discover balance to the MasterCard balance.

b. If the person wishes to gradually pay off both accounts in the same amount of time, describe how the result in part (a) can help guide the person in making his next payment.

17. $-4-(-7)$
18. $8 \div (-2)$
19. $-15+11$
20. $(-7)(-10)$
21. $-5.8+2.9$
22. $-0.2(-0.3)$
23. $-\frac{27}{20} \div \frac{9}{8}$
24. $-\frac{7}{4} - \frac{3}{2}$

OBJECTIVE 6

Compare the results of $\frac{-6}{3}$, $\frac{6}{-3}$, and $-\frac{6}{3}$ to motivate the following property.

Equal Fractions with Negative Signs

If $b \neq 0$, then

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

25. $\frac{-32}{20}$ **26.** $\frac{7}{9} + \frac{-4}{9}$ **27.** $\frac{7}{6} - \left(\frac{5}{-4}\right)$

SHORT HW 3, 15, 17, 23, 27, 49, 51, 55, 67, 73, 91, 93, 99, 107, 115

MEDIUM HW 3, 15, 17, 19, 23, 27, 35, 41, 49, 51, 53, 55, 61, 67, 73, 83, 91, 93, 99, 105, 107, 113, 115, 121, 123, 125

SECTION 2.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students work with ratios, percents, and multiplying and dividing real numbers.

It may be tempting to skip or deemphasize ratios, but this concept, together with the concept of change, will be very helpful in introducing the slope formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ (in Section 3.3). It is also instrumental in making the connection that slope is a rate of change (in Section 3.5). Students will also work with ratios in Section 10.6, which has strong ties with Section 10.7. Most students have little experience working with ratios.

OBJECTIVE 1 COMMENTS

Most students have an easy time computing ratios but have a harder time thinking about the significance of comparing ratios such as in Problem 2, which emphasizes the important concept that comparing ratios is sometimes more meaningful than comparing quantities. This is a key concept of computing the slope of a line: it is not enough to compare the rises in gong from one point on a line to another point.

Workbook Exploration

After a quick definition of unit ratio, you could have groups work on the following 15-minute exploration, which helps groups see that sometimes comparing ratios is more meaningful than comparing quantities.

Group Exploration

Interpreting ratios

A total of 1824 adults were asked the following question: "Do you favor or oppose the death penalty for persons convicted of murder?" The following table compares the adults' responses with their ethnicities.

	African American	Caucasian	Other	Total
Favor	128	953	108	1189
Oppose	140	414	81	635
Total	268	1367	189	1824

Source: *General Social Survey*

1. Find the number of Caucasians in the survey who oppose the death penalty.
2. Find the number of African Americans in the survey who oppose the death penalty.
3. For the Caucasians in the study, find the unit ratio of the number who oppose the death penalty to the number who are in favor.
4. For the African Americans in the study, find the unit ratio of the number who oppose the death penalty to the number who are in favor.
5. A student says that Caucasians in the study are more likely to oppose the death penalty than African Americans in the study because more of the Caucasians oppose the death penalty than the African Americans. What would you tell the student?

OBJECTIVE 2 COMMENTS

Most students have a weak understanding of percents. Discussing percents now will lay the foundation for the interest problems and mixture problems in Section 6.5.

OBJECTIVE 3 COMMENTS

I use the definition of percent to suggest how to convert percentages to and from decimal numbers.

For Problem 3, some students think that the result is 0.8.

OBJECTIVE 4 COMMENTS

I first use the definition of percent to find the percentage of a quantity and then streamline the process.

For Problems 5 and 6, I remind students to include units in their results.

I make sure students know the meaning of 100% as this will come in handy for various curve-fitting applications later in the text.

OBJECTIVE 5 COMMENTS

Students have an easy time with these skills.

Writing $4(-3) = (-3) + (-3) + (-3) + (-3) = -12$ suggests the rule for multiplying two numbers with different signs. Problem 7 suggests the rule for multiplying two numbers with the same sign.

For Problem 10, many students think that $-0.2(-0.1) = 0.2$. For Problem 11, some students need to be reminded to simplify their result.

The textbook includes a detailed explanation about why the sign rules for division are the same as for multiplication (see page 98). Due to time constraints, I usually just say the rules for dividing numbers are similar to those for multiplying numbers, because to divide by a number, we multiply by the reciprocal of the number.

Problem 16 ties ratios in nicely with dividing two negative numbers. Many students who have credit card debt could directly benefit from learning how to do such work with credit card balances!

I budget plenty of time for students to do Problems 17–24, which serve as a good summary of Sections 2.2–2.5.

Textbook and Workbook Exploration

Groups can discover how to multiply two numbers with different signs by completing the 10 minute "Finding the product of a positive number and a negative number" exploration.

OBJECTIVE 6 COMMENTS

I emphasize the property $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$, where $b \neq 0$, because it will be useful when working with slope. It seems that no amount of emphasis is too much, because some students completely forget this property by the time we need to use it for slope (in Sections 3.3 and 3.4).

I tell my students to write their results which are negative fractions in the form $-\frac{a}{b}$ rather than $\frac{-a}{b}$ or $\frac{a}{-b}$.

SECTION 2.6 LECTURE NOTES

Objectives

1. Describe the meaning of *exponent*.
2. Use the rules for order of operations to perform computations.
3. Use the rules for order of operations to evaluate expressions.
4. Use the rules for order of operations to make predictions.

Main point: Use the rules for order of operations to perform computations.

OBJECTIVE 1

- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $4^3 = 4 \cdot 4 \cdot 4 = 64$

Definition *Exponent*

For any counting number n,

$$x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors of } x}$$

We refer to x^n as the **power**, the **nth power of x**, or **x raised to the nth power**. We call x the **base** and n the **exponent**.

1. 3^4 **2.** $(-2)^5$ **3.** $(-5)^2$ **4.** -5^2

For an expression of the of the form $-a^n$, we calculate a^n before taking the opposite.

OBJECTIVE 2

- We do operations that lie within grouping symbols before we perform other operations.
- The order of operations does matter:

$$(4+2) \cdot 5 = 6 \cdot 5 = 30$$
$$4 + (2 \cdot 5) = 4 + 10 = 14$$

- For a fraction such as $\frac{2+5}{9-4}$, the following use of parentheses is assumed:

$$\frac{2+5}{9-4} = \frac{(2+5)}{(9-4)} = \frac{7}{5}$$

5. $(4-8)(9-2)$ **6.** $\frac{2-8}{3-7}$

Order of Operations

We perform operations in the following order:

1. First, perform operations within parentheses or other grouping symbols, starting with the innermost group.
2. Then perform exponentiations.
3. Next, perform multiplications and divisions, going from left to right.
4. Last, perform additions and subtractions, going from left to right.

7. $3 - 8 \div 4$
8. $4 + (-2)^3$
9. $2(5-8) - (4-9)$
10. $3 - [6 - 2(3+4)]$
11. $3^3 - 2(3-5)^2 \div (-4)$
12. $\dfrac{9}{10} - \dfrac{3}{5} \div \dfrac{2}{7}$
13. $\dfrac{4-(-2)^3}{2+4^2}$

Operation	**Computation with 10s**
Exponentiation	$10^{10} = 10,000,000,000$
Multiplication	$10 \cdot 10 = 100$
Addition	$10 + 10 = 20$

Order of Operations and the Strengths of Operations

After we have performed operations in parentheses, the order of operations goes from the most powerful operation, exponentiation, to the next most powerful operations, multiplication and division, to the weakest operations, addition and subtraction.

OBJECTIVE 3

14. Evaluate $\dfrac{a-b}{c-d}$ for $a = -3$, $b = 5$, $c = 4$, and $d = -6$.

15. Evaluate $3x^2$ for $x = -4$.

16. Evaluate $b^2 - 4ac$ for $a = -2$, $b = -3$, and $c = 5$.

17. Let x be a number. Translate the sentence "Subtract 5 from the product of the number and -3." Then evaluate the expression for $x = -2$.

OBJECTIVE 4

18. The numbers of deaths from opioid drug poisoning in Colorado was 472 deaths in 2015 and has increased by about 32 deaths per year since then (Source: *Colorado Department of Public Health and Environment*). Complete the following table to help find an expression that stands for the number of such deaths at t years since 2015. Show the arithmetic to help see a pattern.

Years since 2015	Number of Deaths from Opioid Poisoning in Colorado
0	
1	
2	
3	
4	
t	

Evaluate your result for $t = 6$ and discuss what the result means in this situation.

19. The Academy Award viewership was 32.9 million people in 2017, and it decreased by 19.5% in 2018 (Source: *Nielsen*). What was the viewership in 2018?

SHORT HW 5, 7, 15, 21, 35, 45, 63, 67, 77, 81, 91, 95, 97, 107, 113, 117

MEDIUM HW 5, 7, 15, 21, 27, 33, 41, 45, 49, 57, 63, 67, 77, 79, 81, 85, 91, 95, 97, 99, 107, 111, 113, 117, 119

SECTION 2.6 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students work with exponents and use the rules for order of operations to perform computations and evaluate expressions. Students will use this material throughout the course.

OBJECTIVE 1 COMMENTS

It will be especially helpful for students to know the meaning of *exponent*, *base*, and *power* when using the properties of exponents in Sections 7.4–7.7. Many students think that *power* means the same thing as *exponent*. Explain that x^n is a power of x and that n is the exponent.

For Problem 4, most students are surprised that -5^2 is not equal to 25. I tell students that for an expression of the of the form $-a^n$, we calculate a^n before taking the opposite. It also helps to point out that the base of $(-5)^2$ is -5 and that the base of -5^2 is 5.

OBJECTIVE 2 COMMENTS

When discussing the order of operations, I make sure that my students understand that multiplication and division are on the same level of the hierarchy; some students think that "going from left to right" means that all multiplications should be done before all divisions, because the word "multiplications" appears to the left of the word "divisions" in the instructions for the order of operations. The same goes for addition and subtraction.

It is usually best to ignore suggestions from students to use the acronym "PEMDAS," because it lacks the detail needed for students to use it correctly. (Even with commas, the acronym "P,E,MD,AS" lacks necessary details.)

Even though order of operations is a review topic from arithmetic, some students are challenged by Problems 7–13.

OBJECTIVE 3 COMMENTS

Problem 14 is good preparation for using the slope formula in Section 3.3. Problem 16 is good preparation for using the quadratic formula in Section 9.5.

Finally, note that Exercises 81–86 of Homework 2.6 are good practice for using the slope formula. Exercise 77 is a nice primer for the quadratic formula. Exercises 87–92 are good preparation for graphing quadratic equations in two variables in Section 7.1. Exercises 117 and 118 can serve as explorations of the associative laws for multiplication and addition, respectively.

Textbook and Workbook Exploration

In the 10-minute "Order of operations" exploration, groups discover that it matters in which order we add and multiply.

OBJECTIVE 4 COMMENTS

Having students continue their work with building expressions by recognizing patterns in their arithmetic (see Problem 18) is excellent preparation for Section 3.2, where students will find and graph linear models.

Workbook Exploration

You could have students work on the following 10-exploration without first giving an example because they have done similar (simpler) work in Section 2.1. You will likely have to remind several groups to avoid performing arithmetic so they can more easily see patterns.

Group Exploration

Using an expression to model a situation

The percentage of Americans who have ever listened to a podcast was 33% in 2015 and has increased by about 3.25 percentage points per year since then (Source: *Edison Research and Triton Digital*).

1. Complete the following table to help find an expression that stands for the percentage of Americans who have ever listened to a podcast at t years since 2015. Show arithmetic to help you see a pattern.

Years since 2015	Percent
0	
1	
2	
3	
4	
t	

2. Evaluate the expression you found in Problem 1 for $t = 7$. What does your result mean in this situation?

CHAPTER 3 OVERVIEW

In Section 3.1, students graph linear equations in two variables and learn the Rule of Four for equations. In Section 3.2, students graph a linear model and perform a unit analysis of it. They also learn the Rule of Four for linear models. In Section 3.3, students calculate the slope of a line and learn the graphical significance of slope. They use slope and y-intercept to graph linear equations in two variables in Section 3.4. In Section 3.5, students find the rate of change of a quantity and use such a rate of change to help find a linear model.

SECTION 3.1 LECTURE NOTES

Objectives

1. For an equation in two variables, describe the meaning of *solution*, *satisfy*, and *solution set*.
2. Describe the meaning of the *graph* of an equation.
3. Graph equations of the form $y = mx + b$.
4. Describe the meaning of b in equations of the form $y = mx + b$.
5. Graph equations of the form $y = b$ and $x = a$.
6. Describe the Rule of Four for solutions of equations.
7. Compare finding outputs using an equation with using a graph.

Main point: Graph equations of the form $y = mx + b$.

OBJECTIVE 1

1. Show that the equation $y = x + 3$ becomes a true statement when 2 is substituted for x and 5 is substituted for y.

We say $(2, 5)$ *satisfies* $y = x + 3$ and the $(2, 5)$ is a *solution* of $y = x + 3$.

Definition *Solution, satisfy,* and *solution set* of a equation in two variables

An ordered pair (a, b) is a **solution** of an equation in terms of x and y if the equation becomes a true statement when a is substituted for x and b is substituted for y. We say (a, b) **satisfies** the equation. The **solution set** of an equation is the set of all solutions of the equation.

2. Is $(2, 7)$ a solution of $y = 4x - 1$?
3. Is $(3, 5)$ a solution of $y = 4x - 1$?
4. **a.** Find five solutions of $y = 2x - 3$ and plot them in the same coordinate system.
   b. Do the five points lie on a line? If so, sketch the line.
   c. Select another point on the line and show that the corresponding ordered pair satisfies $y = 2x - 3$.
   d. Select a point that doesn't lie on the line and show that the corresponding ordered pair does not satisfy $y = 2x - 3$.

OBJECTIVE 2

We call the line that we sketched in Problem 4 the *graph* of $y = 2x - 3$.

Definition *Graph*

The **graph** of an equation in two variables is the set of points that correspond to all solutions of the equation.

Every point on the graph of an equation represents a solution of the equation. Every point *not* on the graph represents an ordered pair that is *not* a solution.

OBJECTIVE 3

The equation $y = 2x - 3$ is of the form $y = mx + b$.

Graph of $y = mx + b$

The graph of an equation of the form $y = mx + b$, where m and b are constants, is a line.

Here are some equations whose graphs are lines:

$$y = 3x + 7 \qquad y = -6x + 2 \qquad y = 2x \qquad y = x - 4 \qquad y = 1$$

5. Graph the equation by hand. Also, find the y-intercept.

a. $y = 3x - 5$

b. $y = -2x + 4$

c. $y = -3x$

d. $y = \frac{2}{3}x + 1$

OBJECTIVE 4

Substitute 0 for x into the equation $y = mx + b$:

$$y = m(0) + b = 0 + b = b$$

The y-intercept of the graph of $y = mx + b$ is $(0, b)$.

y-Intercept of the Graph of $y = mx + b$

The graph of an equation of the form $y = mx + b$ has y-intercept $(0, b)$.

For $y = 3x - 5$, the y-intercept is $(0, -5)$ and for $y = -2x + 4$, the y-intercept is $(0, 4)$. See parts (a) and (b) of Problem 5.

OBJECTIVE 5

Find five solutions of the given equation. Then graph the equation by hand.

6. $x = 1$

7. $y = 3$

Equations for Horizontal and Vertical Lines

If a and b are constants, then

- The graph of $y = b$ is a horizontal line. [Show the graph.]
- The graph of $x = a$ is a vertical line. [Show the graph.]

Graph the equation by hand.

8. $x = 4$

9. $y = -2$

10. $x = 0$

11. $y = 0$

Equations Whose Graphs Are Lines

If an equation can be put into the form $y = mx + b$ or $x = a$, where m, a, and b are constants, then the graph of the equation is a line. We call such an equation a **linear equation in two variables**.

Here are some examples of linear equations in two variables:

$$y = -5x + 9 \qquad y = 3x - 8 \qquad y = 4 \qquad x = -2$$

OBJECTIVE 6

Rule of Four for Solutions of an Equation

We can describe some or all of the solutions of an equation in two variables with:

1. an equation,
2. a table,
3. a graph, or
4. words.

These four ways to describe solutions are known as the **Rule of Four**.

12. **a.** List some solutions of $y = 4x - 3$ by using a table.
b. Describe the solutions of $y = 4x - 3$ by using a graph.
c. Describe the solutions of $y = 4x - 3$ by using words.

OBJECTIVE 7

13. Consider the equation $y = x + 2$.

a. Substitute 3 for x and show that the input $x = 3$ leads to the output $x = 5$.
b. Graph $y = x + 2$. Then use arrows to show that the input $x = 3$ leads to the output $x = 5$.

SHORT HW 1, 5, 11, 19, 25, 27, 35, 41, 43, 49, 53, 57, 67, 69, 79

MEDIUM HW 1, 5, 11, 13, 15, 19, 25, 27, 31, 35, 41, 43, 45, 47, 49, 51, 53, 57, 61, 65, 67, 69, 71, 79

SECTION 3.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students graph equations of the form $y = mx+b$ and learn the Rule of Four for equations. Students will graph linear equations in two variables in the rest of Chapter 3 as well as in Section 5.1.

OBJECTIVE 1 COMMENTS

I emphasize these terminologies, because I use these words so much throughout the course.

Problem 4 is important, because it helps students understand the meaning of a graph. Without doing such a problem, most students will not know a graph's meaning.

OBJECTIVE 2 COMMENTS

I emphasize that every point on the graph represents a solution of the equation and that every point not on the graph represents an ordered pair that is not a solution. This important concept is used when finding equations of linear equations in Sections 5.2 and 5.3 as well as when solving systems of linear equations by graphing in Section 6.1.

OBJECTIVE 3 COMMENTS

For Problem 5, I suggest that students use three points, because the third point can serve as a check. Most students are challenged by part (d). Students will have an easier time graphing such an equation once they have learned about slope.

Workbook Exploration

In the following 15-minute exploration, groups can be introduced to graphing linear equations. Students have an easy time with this exploration.

EXPLORATION *Evaluating an expression and plotting points*

1. Evaluate the expression $2x + 1$ at each of the given values of x.

a. $x = -2$ **b.** $x = -1$ **c.** $x = 0$ **d.** $x = 1$ **e.** $x = 2$

2. Organize the results that you found in Problem 1 in the following table. The first row has been done for you.

x	$2x + 1$
-2	-3
-1	
0	
1	
2	

3. Treat the values of the expression $2x + 1$ as values of the variable y. So, from the first row of the table, we see that $y = -3$ when $x = -2$, which can be represented by the ordered pair $(-2, -3)$. List four ordered pairs that are related to the other four rows of the table.

4. Plot the ordered pair $(-2, -3)$ and the four ordered pairs you found in Problem 3 on a coordinate system. Describe any patterns in the position of the points.

5. Repeat the instructions for Problems 1–4, but use the expression $-2x + 3$.

Textbook and Workbook Exploration

Although the 15-minute "Solutions of an equation" exploration addresses the meaning of a graph (Objective 2), it is better to use this exploration once students are comfortable graphing linear equations (Objective 3). That way, groups will be able to efficiently do Problem 1 of the exploration and get to the main point of the exploration.

OBJECTIVE 4 COMMENTS

Students will need to be able to determine the y-intercept of an equation of the form $y = mx + b$ when they perform graphing in Section 3.4.

OBJECTIVE 5 COMMENTS

For Problems 6 and 7, many students are thrown for a loop, because only one variable appears in these equations. I've included two ways to address this issue below.

One way to address this issue is to begin with an application. Let n be the number of states in the United States at t years since 1960. Next, create a table of nonnegative values of t and n. Then determine that the model is $n = 50$ and graph the model. Finally, discuss for which years model breakdown occurs. (Hawaii became the 50th state on August 21, 1959.)

Another way to address the issue, when finding solutions of the equation $x = 1$ (Problem 6), is to use the following scenario:

> Suppose that a younger sibling must return home by 1 A.M. but an older sibling can return home at any time. Assuming that the younger sibling always returns home at 1 A.M., discuss possible times when the siblings will return home.

This story helps students see that the equation $x = 1$ is restricting the value of x to be 1, but the value of y can be any real number. (I'd pick an earlier curfew, but I'd rather not graph an equation such as $x = 10$ for an introductory example; plus a curfew of 10 P.M. is a setup for having to contend with the issue that 1 follows 12 with how we keep time.

For students who confuse the graphs of equations of the form $x = a$ and $y = b$, I suggest that they build a table of (at least two) solutions to tip them off whether they should graph a horizontal or vertical line.

Workbook Exploration

Not only does the following 10-minute exploration get across the idea of how to graph equations of the forms $x = a$ and $y = b$, it also encourages students to reflect on the nature of the coordinates of points on a horizontal or vertical line.

Group Exploration

Section Opener: Equations of horizontal and vertical lines

1. Sketch a horizontal line on the given coordinate system.
2. List the ordered pairs of five points that lie on the horizontal line you sketched in Problem 1.
3. What do the coordinates of the ordered pairs you listed in Problem 2 have in common?
4. Translate the response your wrote in Problem 3 into an equation by filling in the blank: $y =$ _____.
5. Graph $y = -2$.
6. Graph $x = 4$.

Workbook Exploration

Students who have seen a bit of graphing in a previous course will take for granted that graphs of equations of the form $y = mx + b$ are lines because they were told so. In the following 25-minute exploration, students see that graphs of some linear equations are lines *and* that the graphs of some nonlinear equations are not lines.

If you are worried about getting bogged down in Chapter 3, this is a good exploration to skip and just lecture about the concepts because you can communicate the concepts in much less time than it takes groups to do the exploration.

One advantage with having students do the exploration is that they get a preview of the types of curves with which they will be working in subsequent chapters. They'll also see a fuller range of what their graphing calculator can do. And they'll get hands-on experience of the concepts addressed in this exploration.

Group Exploration

Section Opener: Forms of equations whose graphs are lines

1. Use a graphing calculator to determine which of the equations that follow have graphs that are lines. (To view these graphs, use ZStandard followed by ZSquare.)

   a. $y = x$

   b. $y = x^2$ (For "x^2": Press $\boxed{\text{X,T,}\Theta,n}$ $\boxed{\wedge}$ **2** or $\boxed{\text{X,T,}\Theta,n}$ $\boxed{x^2}$.)

   c. $y = 2^x$ (For "2^x": Press **2** $\boxed{\wedge}$ $\boxed{\text{X,T,}\Theta,n}$.)

   d. $y = \dfrac{2}{x}$

   e. $y = 3x$

   f. $y = 2x + 4$

   g. $y = -2x - 8$ (For "$-2x - 8$": Press $\boxed{(-)}$ **2** $\boxed{\text{X,T,}\Theta,n}$ $\boxed{-}$ **8**.)

   h. $y = 4$

2. Make up your own example of an equation whose graph is a line. Use a graphing calculator to verify that you are right.

3. Here are some examples of equations whose graphs are lines.

$$y = 3x + 5 \qquad y = 2x - 8 \qquad y = -4x + 7 \qquad y = -7x - 6 \qquad y = 5x \qquad y = -3$$

   Form a theory about how to recognize whether an equation has a graph that is a line without graphing the equation. Test your theory. You can do this by creating equations you think have graphs that are lines and then using a graphing calculator to graph each equation to see if you are correct.

4. Is the graph of $3x^2 + y = 3x^2 + 2x + 8$ a line? Explain. Modify your theory in Problem 3, if necessary.

OBJECTIVE 6 COMMENTS

Students will be asked to demonstrate the Rule of Four in several Homework sections.

OBJECTIVE 7 COMMENTS

Problem 13 is important, because it helps students see the connection between finding inputs and outputs symbolically and graphically.

SECTION 3.2 LECTURE NOTES

Objectives

1. Graph a linear model.
2. Describe the Rule of Four for linear models.
3. Perform a unit analysis of a linear model.

Main point: Find an equation of a linear model and know the Rule of Four for linear models.

OBJECTIVE 1

- Recall that a linear model is a line that describes the relationship between two quantities in an authentic situation.
- An equation of such a line is also called a *linear model*.

1. The profit of a company is \$20 million in 2010. Each year, the profit increases by \$3 million. Let p the annual profit (in millions of dollars) at t years since 2010.
   a. Use a table to help find an equation for t and p.
   b. Substitute 5 for t into the equation. What does it mean in this situation?
   c. Graph the equation.
   d. What is the p-intercept? What does it mean in this situation?
   e. When will the annual profit be \$41 million? Explain by using arrows on your graph in part (c).

All the equations

$$p = 3t + 20 \qquad p = 20 + 3t \qquad p = t(3) + 20 \qquad p = 20 + t(3)$$

describe the same relationship between t and p and have the same graph.

OBJECTIVE 2

Recall that the Rule of Four of equations states that we can use equations, tables, graphs, and words to describe solutions of equations.

Rule of Four for Linear Models

We can describe a linear model with

1. an equation,
2. a table,
3. a graph, or
4. words.

Refer to Problem 1 to demonstrate the Rule of Four for linear models.

OBJECTIVE 3

Here is a *unit analysis* of our model $p = 3t + 20$:

$$\underbrace{p}_{\text{millions of dollars}} = \underbrace{3}_{\frac{\text{millions of dollars}}{\text{year}}} \cdot \underbrace{t}_{\text{years}} + \underbrace{20}_{\text{millions of dollars}}$$

We can use the fact that $\frac{\text{years}}{\text{year}} = 1$ to show that the units of the expressions on both sides of the equation are millions of dollars, which suggests that our equation is correct.

Definition *Unit analysis*

We perform a **unit analysis** of a model's equation by determining the units of the expressions on both sides of the equation. The units of the expressions on both sides of the equation should be the same.

We can perform a unit analysis of a model's equation to help verify the equation.

2. The balance of a person's checking account in 2010 is \$15 thousand. Each year the balance decreases by \$2 thousand. Let B be the balance (in thousands of dollars) at t years since 2010.

a. Use a table to help find an equation for t and B.

b. Perform a unit analysis of the equation you found in part (a).

c. Graph the equation.

d. When will the balance be \$3 thousand? Explain by using arrows on your graph in part (c).

SHORT HW 5, 7, 13, 17, 23, 25, 29, 31

MEDIUM HW 1, 3, 5, 7, 11, 13, 17, 21, 23, 25, 27, 29, 31

SECTION 3.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students find an equation of a linear model and make predictions. They perform a unit analysis of such a model and learn the Rule of Four for linear models. This work lays the foundation for working with rates of change in Section 3.5. Students will also graph equations of the form $x = a$ and $y = b$ in this section.

OBJECTIVE 1 COMMENTS

For part (a) of Problem 1, students' work with finding expressions in Chapter 2 should help them find the equation $p = 3t + 20$. I highlight the shift from working with expressions such as $3t + 20$ in Chapter 2 with working with equations such as $p = 3t + 20$ in Chapter 3. In the lecture notes, the profit model is written in four ways. It should be apparent to students that these four equations are equivalent from the various ways that the arithmetic used to create the table in part (a) could be arranged. (The commutative and associative laws will be discussed in Section 4.1.)

When finding the p-intercept [part (d) of Problem 1], students may have some trouble seeing that the p-intercept is $(0, 20)$ by inspecting the equation. It will help to refer to the table you found in part (a) or to the graph you found in part (c). However, it is important that students learn to find the vertical intercept from the equation alone. Remind students that the y-intercept of the graph of an equation of the form $y = mx + b$ is the point $(0, b)$ as was discussed in Section 3.1.

While doing parts (b) and (e) of Problem 1, I use the terminology of input and output.

OBJECTIVE 2 COMMENTS

I remind students about the Rule of Four for equations in two variables and use this as a segue to discuss the Rule of Four for linear models.

Workbook Exploration

The following 15-minute exploration is a nice summary of a lot of key concepts in Sections 3.1 and 3.2.

Group Exploration

Rule of four for linear models

The number of bankrupt borrowers who asked a judge to cancel their student loans was 667 borrowers in 2015 and has decreased by about 44 borrowers per year since then (Source: *Administrative Office of the U.S. Court documents*). Let n be the number of bankrupt borrowers who asked a judge to cancel their student loans at t years since 2015. Recall that we can describe a linear model using an equation, a table, a graph, or words.

1. Use a table of values of t and n to describe the situation.
2. Use an equation to describe the situation.
3. Use a graph to describe the situation.
4. Use your table, your equation, and your graph to estimate the number of borrowers who asked a judge to cancel their student loans in 2018. Compare your results.

OBJECTIVE 3 COMMENTS

I emphasize that a unit analysis of a model can be used to verify the equation of the model. This work will help students solve value problems in Section 6.5. (The number of items in a group times the value per item is equal to the total value of the group of items.) Students will also convert units of quantities in Section 10.2.

SECTION 3.3 LECTURE NOTES

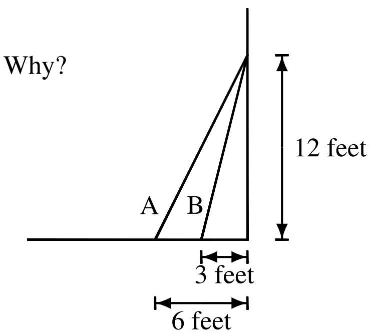
Objectives

1. Use a ratio to compare the steepness of two objects.
2. Describe the meaning of, and how to calculate, the *slope* of a nonvertical line.
3. Determine the sign of the slope of an increasing line and of a decreasing line.
4. Explain why the slope of a horizontal line is zero and that the slope of a vertical line is undefined.

Main point: Describe the meaning of, and how to calculate, the *slope* of a line.

OBJECTIVE 1

1. Ladder B is steeper than ladder A. Why?



Compare $\dfrac{\text{vertical distance}}{\text{horizontal distance}}$ for ladders A and B.

Comparing the Steepness of Two Objects

To compare the steepness of two objects, compute the unit ratio

$$\frac{\text{vertical distance}}{\text{horizontal distance}}$$

for each object. The object with the larger ratio is the steeper object.

2. While taking off, airplane A climbs steadily for 2100 feet over a horizontal distance of 7000 feet. Airplane B climbs steadily for 2850 feet over a horizontal distance of 9900 feet. Which plane is climbing at a greater incline? Explain.

OBJECTIVE 2

Definition *Slope of a nonvertical line*

$$m = \text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

3. Find the slope of the line that contains the points $(1, 5)$ and $(4, 3)$.

Calculating Slope

Let (x_1, y_1) and (x_2, y_2) be two distinct points of a nonvertical line. The slope of the line is

$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{y_2 - y_1}{x_2 - x_1}$$

When working with negative coordinates, it can help to first write

$$\frac{(\quad) - (\quad)}{(\quad) - (\quad)}$$

and then insert the coordinates into the appropriate parentheses.

Find the slope of the line that contains the given points.

4. $(1, 3)$ and $(4, 9)$

5. $(-4, 1)$ and $(2, -3)$

6. $(-5, -1)$ and $(4, 2)$

7. $(-6, -3)$ and $(-2, -5)$

8. Find the approximate slope of the line that contains the points $(-2.8, 5.9)$ and $(-1.1, -3.7)$. Round the result to the second decimal place.

OBJECTIVE 3

For Problems 4 and 6, the *increasing* lines have *positive* slopes. For Problems 5 and 7, the *decreasing* lines have *negative* slopes.

Slopes of Increasing or Decreasing Lines

- An increasing line has positive slope.
- A decreasing line has negative slope.

- Explain this property in general by discussing the signs of rises and runs for an increasing line and for a decreasing line.
- Compare the steepness of two lines with slopes 2 and 3.
- Compare the steepness of two lines with slopes -3 and 2.

Measuring the Steepness of a Line

The absolute value of the slope of a line measures the steepness of the line. The steeper the line, the larger the absolute value of its slope will be.

OBJECTIVE 4

- Draw a horizontal line and find its slope $\left(\frac{0}{\text{run}} = 0\right)$.

- Draw a vertical line and explain why its slope, $\frac{\text{rise}}{0}$, is undefined.

Slopes of Horizontal and Vertical Lines

- A horizontal line has slope equal to zero. [Draw a figure.]
- A vertical line has undefined slope. [Draw a figure.]

For Problems 9–12, find the slope of the line that contains the given points. Determine whether the line is increasing, decreasing, horizontal, or vertical.

9. $(2, -3)$ and $(2, 1)$

10. $(-3, -2)$ and $(2, 8)$

11. $(-1, -4)$ and $(3, -4)$

12. $(-5, -2)$ and $(-1, -4)$

13. Sketch an increasing line, a decreasing line, a horizontal line, and a vertical line. For each line, determine whether the slope is positive, negative, zero, or undefined (Exercise 51).

SHORT HW 1, 9, 13, 21, 27, 35, 37, 39, 49, 51, 53, 59, 61, 65, 75

MEDIUM HW 1, 5, 9, 13, 15, 21, 25, 27, 33, 35, 37, 39, 43, 49, 51, 53, 57, 59, 61, 63, 65, 73, 75

SECTION 3.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students compare the steepness of two objects and find the slope of a line. They learn about the graphical significance of slope.

OBJECTIVE 1 COMMENTS

When discussing slope, I explain why it is defined in terms of both run and rise of two points on a line and not just the rise of the two points. Students appreciate it when I discuss Problem 2 in the lecture notes. Otherwise, they would have trouble with Exercises 1–4 in Homework 3.3.

The grade of a road is a good motivator of slope (see Example 1 and the following paragraph on page 139 of the textbook.)

Workbook Exploration

The following 20-minute exploration emphasizes that when measuring the steepness of an object, it does not suffice to consider the rise of the object. Problem 2 is a challenging open-ended question. Resist the urge to help students too much. If a group's measure is not valid, point out why that's the case and challenge them to improve their measure or come up with a different one. Remember, even if a group's measure is not valid, putting thought into the concept of steepness will mentally prepare them to then learn about slope and help them appreciate how slope is defined.

Group Exploration

Section Opener: Steepness

Two ladders lean against a building. The foot of ladder A is 3 feet from the building and the top reaches a point at height 12 feet on the building. The foot of ladder B is 6 feet from the building and it reaches a point at height 18 feet on the building.

1. Draw a figure of the situation.

2. Which ladder is steeper? Explain by using calculations as well as words.
3. A student says ladder B is steeper than ladder A because ladder B reaches a point on the building higher than ladder A does. What would you tell the student?
4. For ladder A, find the unit ratio of the vertical distance to the horizontal distance. Find the unit ratio for ladder B, too. What do the unit ratios mean in this situation? Compare the two ladders' unit ratios. What does your comparison mean in this situation?
5. A student says it would be better to find the unit ratio of the horizontal distance to the vertical distance for each ladder. What would you tell the student?
6. A portion of road A climbs steadily for 105 feet over a horizontal distance of 2950 feet. A portion of road B climbs steadily for 130 feet over a horizontal distance of 4325 feet. Which road is steeper?

OBJECTIVE 2 COMMENTS

To segue from $m = \dfrac{\text{rise}}{\text{run}}$ to $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, I remind my students that the change in a quantity is the final amount minus the beginning amount (Section 2.4).

For Problem 3, I find the slope by plotting the points and using the graph to determine the rise and the run. I continue this practice for at least one of the Problems 4–7 (as well as using the slope formula).

For Problem 4, I also show my students that when we use the slope formula with two points on a line, it doesn't matter which point we choose to be (x_1, y_1) and which we choose to be (x_2, y_2).

I discuss the contents of the warning near the top of page 141 in the textbook.

Problem 8 is good preparation for finding an equation of a linear model in Section 5.3.

When some or all coordinates are negative (see Problems 5–8), a common student error is to omit at least one subtraction symbol from the slope formula or negative sign from the coordinates. The suggestion in the lecture notes to first write the parentheses addresses this type of error.

Textbook and Workbook Exploration

For the 15-minute "For a line, rise over run is constant" exploration, students discover that for a line, the ratio of the rise to the run is constant for any distinct pair of points on the line. Students are often surprised that the value of the slope is independent of which two points are chosen. A good extension of the exploration is to have groups explain why this makes sense.

OBJECTIVES 3 and 4 COMMENTS

Performing a sign analysis of an increasing line and a decreasing line will lead to the meanings of the signs of slope (see Figs. 45 and 46 on pages 141 and 142 of the textbook). Discussing the meanings of the signs of slope will help students understand the meanings of the signs of rates of change in Section 3.5.

A common error when computing the slope is for the sign to be incorrect. To check the sign, a student can quickly plot the points and determine whether the line is increasing or decreasing.

Textbook and Workbook Exploration

The 15 minute "Graphical significance of m for $y = mx$" exploration assists students in discovering the graphical significance of m for an equation of the form $y = mx$, which is a key concept of the course. After students complete the exploration, I have a short classroom discussion in which students tell me their responses for Problem 4 of the exploration. There is a lot of information in this exploration, and students have trouble seeing how it all fits together.

Logically, this exploration belongs in Section 3.4 because it's not until then that it's established that for an equation of the form $y = mx + b$, the slope of the line is m. But aside from this issue, the exploration wonderfully addresses the fundamental idea of Section 3.3.

Workbook Exploration

The following 15-minute exploration might seem redundant with other explorations that have to do with this section, but interpreting slope as a measure of steepness is a key concept of the course and one that students tend to forget without a lot of reinforcement.

EXPLORATION *Interpreting slope of a line*

1. Sketch four increasing nonparallel lines. Label your lines l_1, l_2, l_3, and l_4.
2. List your lines from the least steepest line to the steepest line.
3. Compute the slope of each of your four lines. List your lines from the line with the least slope to the one with the greatest slope.
4. Compare the lists you made in Problems 2 and 3. What does this tell you about the slope of a line?
5. Repeat Problems 1–4, but this time for four decreasing nonparallel lines.
6. Summarize what you have learned about slope in this exploration.

SECTION 3.4 LECTURE NOTES

Objectives

1. Use the slope and the y-intercept of a line to sketch the line.
2. Describe the meaning of m for an equation of the form $y = mx + b$.
3. Graph an equation of the form $y = mx + b$ by using the line's slope and y-intercept.
4. Find an equation of a line from its graph.
5. Graph an equation of a linear model by using the model's slope and y-intercept.
6. Describe the relationship between slopes of parallel lines.
7. Describe the relationship between slopes of perpendicular lines.

Main point: Use the slope and the y-intercept to graph a linear equation.

OBJECTIVE 1

Sketch the line that has the given slope and y-intercept.

1. $m = \dfrac{2}{5}$, $(0, -3)$ **2.** $m = -\dfrac{3}{2}$, $(0, 4)$ **3.** $m = -2$, $(0, -1)$

OBJECTIVE 2

4. **a.** Use the method discussed in Section 3.1 to graph $y = 2x + 1$.

b. What is the slope of the line $y = 2x + 1$?

c. Compare the slope with the number multiplied times x in the equation $y = 2x + 1$.

Finding the Slope and y-Intercept from a Linear Equation

For a linear equation of the form $y = mx + b$,

- the slope of the line is m and
- the y-intercept of the line is $(0, b)$.

We say this equation is in **slope-intercept form**.

The graph of the equation $y = -3x + 8$ is a line with slope -3 and y-intercept $(0, 8)$.

OBJECTIVE 3

Sketch the graph of the equation by hand.

5. $y = \dfrac{3}{5}x - 1$ **6.** $y = -\dfrac{4}{3}x + 5$ **7.** $y = 3x - 4$ **8.** $y = -2x - 3$

Graphing an Equation in Slope-Intercept Form

To graph an equation of the form $y = mx + b$,

1. Plot the y-intercept $(0, b)$.
2. Use $m = \frac{\text{rise}}{\text{run}}$ to plot a second point. For example, if $m = \frac{2}{3}$, then count 3 units to the right (from the y-intercept) and 2 units up to plot another point.
3. Sketch the line that passes through the two plotted points.

OBJECTIVE 4

9. Graph $y = -\frac{2}{5}x - 1$ but do not reveal the equation to your students. Find an equation of the line.

Finding an Equation of a Line from a Graph

To find an equation of a line from a graph:

1. Determine the slope m and the y-intercept $(0, b)$ from the graph.

2. Substitute your values for m and b into the equation $y = mx + b$.

OBJECTIVE 5

10. The percentages of Americans who listened to online radio in the past week are shown in the following table for various years.

Year	Percent
2011	22
2012	29
2013	33
2014	36
2015	44
2016	50

Source: *Edison Research*

Let p be the percentage of Americans who listened to online radio in the past week at t years since 2010. A reasonable model is $p = 5.4t + 17$.

a. Graph the model by hand.

b. Use your graph to predict when all Americans will have listened to online radio in the past week. Do you have much faith in your prediction?

OBJECTIVE 6

Sketch two parallel lines and compare their slopes.

Slopes of Parallel Lines

If lines l_1 and l_2 are parallel nonvertical lines, then the slopes of the lines are equal:

$$m_1 = m_2$$

Also, if two distinct lines have equal slope, then the lines are parallel.

OBJECTIVE 7

Sketch two perpendicular lines and compare their slopes.

Slopes of Perpendicular Lines

If lines l_1 and l_2 are perpendicular nonvertical lines, then the slope of one line is the opposite of the reciprocal of the slope of the other line:

$$m_2 = -\frac{1}{m_1}$$

Also, if the slope of one line is the opposite of the reciprocal of another line's slope, then the lines are perpendicular.

Determine whether the pair of lines is parallel, perpendicular, or neither.

11. $y = -2x + 3$ and $y = 2x - 5$

12. $y = \frac{2}{7}x - 3$ and $y = -\frac{7}{2}x + 4$

SHORT HW 3, 17, 19, 29, 37, 43, 47, 51, 55, 65, 67, 71, 79, 91, 99

MEDIUM HW 1, 3, 9, 13, 15, 17, 19, 27, 29, 37, 43, 47, 49, 51, 55, 61, 65, 67, 71, 79, 81, 87, 91, 95, 99

SECTION 3.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use the slope and the y-intercept to graph equations in slope-intercept form. After students learn how to solve equations in Chapter 4, they will graph linear equations not in slope-intercept form in Section 5.1. Students tend to do better with this organization, rather than having to contend with graphing linear equations of all forms in just one section.

OBJECTIVE 1 COMMENTS

Problems 1–3 provide the "big picture" of the section.

For Problem 2, some students are unsure what to do with the negative sign of the slope. Students want to know whether they should work with the slope $\frac{-3}{2}$ or $\frac{3}{-2}$. I demonstrate that both ways give the same result; I emphasize that our main goal is to graph the line, not find two points on the line. A common error is to write $-\frac{3}{2} = \frac{-3}{-2}$.

For Problem 3, some students need a reminder that $a = \frac{a}{1}$.

OBJECTIVE 2 COMMENTS

You can address this objective by doing Problem 4 or having groups work on the following exploration.

Textbook and Workbook Exploration

The 15-minute "The meaning of m in the equation $y = mx + b$" exploration in is a great way for groups to discover that for an equation of the form $y = mx + b$, the slope of the equation is m. Many groups will need help with part (e) of Problem 4 of the exploration.

Textbook and Workbook Exploration

Students really enjoy the "Drawing lines with various slopes" exploration, which encourages students to reflect on the graphical significance of m and b of an equation of the form $y = mx + b$. Most students restrict their choices of slopes to positive integers at first and then realize that they need to use negative integers, too. Most students need hints to get them thinking about using slopes between -1 and 1.

OBJECTIVE 3 COMMENTS

This objective follows nicely from objective 1. I use a graphing calculator to verify at least one graph.

OBJECTIVE 4 COMMENTS

Before doing Problem 9, I say a line has slope $\frac{3}{7}$ and y-intercept -4 and ask my students to find the equation of the line. Then I have my students do Problem 10, which they find to be an easy task. I point out that this task is the reverse of graphing. I remind my students about the Rule of Four for equations. I say we can go from any one of these four ways to another. (Students will learn how to go from tables to equations in Section 3.5.)

Textbook and Workbook Exploration

The 10 minute "Finding an equation of a line from a graph" exploration is a fun way for students to practice finding an equation of a line from its graph.

OBJECTIVE 5 COMMENTS

For Problem 10, I tell my students that they will learn to find linear models such as $p = 5.4t + 17$ in Section 5.5. I demonstrate how to use the method in this section to sketch the model $p = 5.4t + 17$, but I also point out that it is often useful to find a point far off to the right such as $(6,\ 49.4)$ by substituting 6 for t. Finding such a third point can help students sketch an accurate line, especially when the run of their first two chosen points is small compared to the largest value of the scaling on the horizontal axis.

OBJECTIVE 6 COMMENTS

When I ask students what we can say about the slopes of two parallel lines, they are quick to say the slopes are equal.

OBJECTIVE 7 COMMENTS

When discussing the slopes of two perpendicular lines, most students will follow the verbal description that the slope of one line is the opposite reciprocal of the slope of the other line. It does not mean much to students to say the product of the slopes is -1 (as most textbooks say).

If you are tight on time, this objective can be skipped, because it will not be needed for the rest of the course.

SECTION 3.5 LECTURE NOTES

Objectives

1. Calculate the rate of change of a quantity.
2. Explain why slope is a rate of change.
3. Use rate of change to help find a linear model.
4. Describe the slope addition property.

Main point: Slope is a rate of change.

OBJECTIVE 1

1. If the temperature increased by 12°F over 3 hours, estimate how much the temperature increased per hour.
2. If temperature increased from 60°F at 7 A.M. to 68°F at 11 A.M., estimate how much the temperature increased per hour.

Formula for Rate of Change

Suppose that a quantity y changes steadily from y_1 to y_2 as a quantity x changes steadily from x_1 to x_2. Then the **rate of change** of y with respect to x is the ratio of the change in y to the change in x:

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

3. The sales of cigarettes in the United States deceased approximately linearly from 400 billion cigarettes in 2003 to 249 billion cigarettes in 2017 (Source: *Euromonitor International; Centers of Disease Control and Prevention*). Find the approximate rate of change of cigarettes sales.
4. A person of height 132.5 centimeters should use a ski pole of length 90 centimeters. A person of height 153.0 centimeters should use a ski pole of length 110 centimeters. Find the rate of change of ski pole length with respect to a person's height.

Refer to Problems 3 and 4 to motivate the following facts.

Signs of Rate of Change

Suppose that a quantity t affects (explains) a quantity p. Then

- If p increases steadily as t increases steadily, then the rate of change of p with respect to t is positive.
- If p decreases steadily as t increases steadily, then the rate of change of p with respect to t is negative.

OBJECTIVE 2

We use the formula $\dfrac{y_2 - y_1}{x_2 - x_1}$ to find rate of change and also to find slope of a line. So, slope is a rate of change.

5. Let d be the distance (in miles) that a student can drive in t hours (see the table).

t (hours)	d (miles)
0	0
1	60
2	120
3	180
4	240
5	300

a. Construct a scatterplot. Then draw a linear model.

b. Find the slope of the linear model.

c. Find the rate of change of the distance traveled in each given period. Compare each result with the slope of the linear model.

i. From $t = 3$ to $t = 4$

ii. From $t = 0$ to $t = 5$

Slope Is a Rate of Change

If there is a linear relationship between quantities t and p, and if t affects (explains) p, then the slope of the linear model is equal to the rate of change of p with respect to t.

Constant Rate of Change

Suppose that a quantity t affects (explains) a quantity p. Then

- If there is a linear relationship between t and p, then the rate of change of p with respect to t is constant.
- If the rate of change of p with respect to t is constant, then there is a linear relationship between t and p.

OBJECTIVE 3

6. A person earns a starting salary of $30 thousand at a company. Each year, she receives a $2 thousand raise. Let s be the person's salary (in thousands of dollars) after she has worked at the company for t years.

a. What is the slope of the linear model that describes this situation? What does it mean in this situation?

b. What is the s-intercept of the model? What does it mean in this situation?

c. Find an equation of the model.

d. Perform a unit analysis of the equation you found in part (c).

7. Let T be the total one-semester cost (in dollars) of tuition plus parking fee for u units (credits or hours) of classes at your college. [You could replace the parking fee with any one-time charge such as a student-services fee at your college.]

a. Find an equation of a linear model.

b. What is the total one-semester cost of tuition plus parking fee for 15 units of classes?

8. The number of multidrug-resistant tuberculosis cases in Estonia was 43 in 2015 and has decreased by about 9 cases per year (Source: *World Health Organization*).

a. Find an equation of a linear model to describe the situation. Explain what your variables represent.

b. Estimate the number of multidrug-resistant tuberculosis cases in 2018.

9. The Gallup U.S. standard of living index measures the level at which Americans are satisfied with their standard of living. The indexes are shown in the following table for various years.

Year	U.S. Standard of Living Index
2008	23
2009	26
2010	31
2011	31
2012	34
2013	38
2014	45
2015	49
2016	50
2017	54

Source: *Gallup*

Let s be the U.S. standard of living index at t years since 2000. A reasonable model is $s = 3.53t - 6.07$.

a. Use a graphing calculator to draw a scatterplot and the model in the same viewing window. Check whether the line comes close to the data points.

b. What is the slope? What does it mean in this situation?

c. Find the rates of change in the U.S. standard of living index from one year to the next. Compare the rates of change with your result in Part (b).

d. Predict the U.S. standard of living index in 2022.

OBJECTIVE 4

Use tables of solutions of $y = 2x + 3$ and $y = -3x + 20$ to motivate the slope addition property.

Slope Addition Property

For a linear equation of the form $y = mx + b$, if the value of x increases by 1, then the value of y changes by the slope m. In other words, if an input increases by 1, the output changes by the slope.

10. Four sets of points are described in the following table. For each set, decide whether there is a line that passes through every point. If so, find the slope of that line. If not, decide whether there is a line that comes close to every point.

Set 1		Set 2		Set 3		Set 4	
x	y	x	y	x	y	x	y
0	2	0	95	2	11	5	4
1	5	1	91	3	23	6	4
2	8	2	87	4	35	7	4
3	11	3	83	5	47	8	4
4	14	4	79	6	59	9	4

11. Some values of four linear equations are provided in the following table. Complete the table.

Equation 1		Equation 2		Equation 3		Equation 4	
x	y	x	y	x	y	x	y
0	3	0	79	21	25	43	83
1	8	1	72	22		44	
2		2		23		45	
3		3		24	31	46	
4		4		25		47	71

SHORT HW 3, 9, 17, 21, 23, 25, 33, 35, 39, 43, 45, 47, 51, 57

MEDIUM HW 1, 3, 9, 13, 17, 21, 23, 25, 27, 29, 33, 35, 39, 41, 43, 45, 47, 53, 57 Note: Exercises 27–30 ask students to use a graphing calculator to draw a scatterplot.

SECTION 3.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students find the rate of change of a quantity and learn that slope is a rate of change. Students then use rate of change to find a linear model. Students also learn the slope addition property. These concepts will be helpful for work in Sections 5.3, 5.4, 5.5, and 6.4.

OBJECTIVE 1 COMMENTS

If you discussed the change of a quantity in Section 2.4 and unit ratios in Section 2.5, your students should be well primed for this objective. If you didn't discuss these concepts then, spend some time discussing them now.

I tell students that 50 miles per hour is an example of rate of change. I say a rate of change is a description of how quickly a quantity changes in relation to another quantity changing. I say we often use the word "per" to describe a rate of change. I give some more examples of rates of change such as

- A student earns $8 per hour.
- The altitude of an airplane decreases by 2200 feet per minute.
- A college charges $80 per unit. (You could use the charge per unit (hours or credits) at your college.)

Then I discuss the material in the lecture notes. When doing Problems 1–4, I emphasize we should include units when computing changes in the response variable and changes in the explanatory variable. Including units throughout the calculations will help students write the correct units for their result. For Problem 4, I emphasize which change goes in the numerator and which change goes in the denominator.

Workbook Exploration

You could start class by having students work on the following 15-minute exploration. This is a great way to motivate the concept rate of change. Some groups may use a numerical approach and other groups may use a graphical approach. Some groups will likely compute changes in the response variable but not take into account changes in the explanatory variable; in fact, I selected the data set so that this approach will yield the wrong conclusion. During the debrief, comparing and contrasting groups' results should generate a productive discussion.

Group Exploration
Section Opener: Rate of change

For which type of car, domestic or imported, has fuel efficiency improved the fastest? Explain.

Year	Domestic (miles per gallon)	Year	Imported (miles per gallon)
2010	33.1	2007	32.2
2011	32.7	2009	33.8
2012	34.8	2011	33.7
2013	36.0	2013	36.6
2014	36.7	2014	36.0

Source: *U.S. Department of Transportation*

OBJECTIVE 2 COMMENTS
I point to the difference quotients used to find results for Problems 2–4 and ask students whether the difference quotients remind them of something. If no one is reminded of slope, I refer to the boxed formula

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

I then conclude that slope is a rate of change.

If you would like to do this section in one day, you will likely have to skip Problem 5, although it is a nice way to emphasize that slope is a rate of change. Due to time constraints, I usually do not discuss the boxed property with the heading "Constant Rate of Change."

Workbook Exploration
The following exploration is an excellent way for students to discover that for a linear equation in two variables, slope is a rate of change. Students will likely gain a deeper understanding of this concept by completing the exploration than by watching a lecture (which is probably true of most of the explorations, but especially this one.) This concept might be the most important one of the course for those students who will take calculus.

Students have an easy time with this exploration. To save time, I usually assign only Problems 1 and 2. Once most groups are done, I emphasize that slope is a rate of change. I remind my students of the graphical interpretations of slope we've discussed so far.

Group Exploration
Section Opener: Significance of the slope and the response variable's intercept of a model

1. A small airplane is traveling at a constant speed of 100 miles per hour. Let d be the distance (in miles) the airplane can travel in t hours.

a. Complete the following table.

Time (hours) t	Distance (miles) d
0	
1	
2	
3	
4	

b. Find an equation of a linear model.

c. Compare the slope of your model with the speed of the airplane.

d. What is the d-intercept? What does it mean in this situation?

2. In 2010, a company was worth \$10 million. Each year, its value increases by \$2 million. Let V be the company's value (in millions of dollars) at t years since 2010.

a. Complete the following table.

Year since 2010 t	Value (millions of dollars) V
0	
1	
2	
3	
4	

b. Find an equation of a linear model.

c. Compare the slope of your model with the rate at which the company's value is increasing.

d. What is the V-intercept? What does it mean in this situation?

3. A person is in a hot-air balloon at an altitude of 1600 feet. The person begins gradually letting air out of the balloon, and the balloon descends at a rate of 200 feet per minute. Let H be the balloon's altitude (in feet) after air has been released for t minutes.

a. Complete the following table.

Time (minutes) t	Altitude (feet) H
0	
1	
2	
3	
4	

b. Find an equation of a linear model.

c. Compare the slope of your model with the rate at which the balloon's altitude is changing.

d. What is the H-intercept? What does it mean in this situation?

4. In general, what is the meaning of the slope in terms of an authentic situation? What is the meaning of the response variable's intercept?

Workbook Exploration

The following 30-minute exploration is similar to the exploration with the same name in the comments about Objective 1, but this exploration provides more structure and has students consider how rate of change and slope of a linear model are connected.

EXPLORATION *Rate of change*

1. World record times for the 200-meter run are listed in the following table for various years.

Women		Men	
Year	Record Time (seconds)	Year	Record Time (seconds)
1973	22.38	1951	20.6
1974	22.21	1963	20.3
1978	22.06	1967	20.14
1984	21.71	1979	19.72
1988	21.34	1996	19.32
		2009	19.19

Source: *IAAF Statistics Handbook*

1. Which gender's record times are decreasing the most per year? Use calculations and words to explain.

2. Construct scatterplots by hand for the women's data and the men's data on the same coordinate system. Make it clear which data points are for which gender.

3. On the scatterplots you constructed in Problem 2, sketch a linear model for each gender.

4. Calculate the slope of each of your linear models. What do your results mean in this situation? Do your results support the claim you made in Problem 1? Explain.

5. A student says the men's record times are decreasing the most per year because the decrease in their record times from 1951 to 2009 is greater than the decrease in the women's record times from 1973 to 1988. What would you tell the student?

6. A student uses the 1974 and 1978 data points to find a linear model for women's record times and the 1963 and 1967 data points to find a linear model for the men's record times. She finds that both gender's record times decreased by about 0.04 second per year. The student concludes that the record times for each gender are decreasing at the same rate. What would you tell the student?

Workbook Exploration

In the following 20-minute exploration, students explore the connection between rate of change and computing the slope of a linear model where the change in the explanatory variable is different than 1. A few groups may need a reminder what "ratio" means. Once groups are over this hurdle, the rest of the exploration flows nicely.

Group Exploration

Section Opener: Slope is a ratio

Here you will work with the women's 400-meter run model $r = -0.27t + 70.45$, where r is the record time (in seconds) at t years since 1900.

1. What is the slope of the graph of $r = -0.27t + 70.45$? What does the slope mean in this situation? [**Hint:** Consider the slope addition property.]

2. According to the model, by how much should the record time change each year?

3. By how much should the record time change in two years? What is the ratio of the change in the record time to the change in calendar time?

4. By how much does the 400-meter record change in three years? What is the ratio of the change in the record time to the change in calendar time?

5. For any period, what is the ratio of the change in the record time to the corresponding change in calendar time? Explain.

OBJECTIVE 3 COMMENTS

Although students have found equations of models by recognizing patterns of arithmetic, they will initially have trouble with finding such an equation using the concepts in this section.

For Problem 6, I say that because the rate of change of salary per year is a constant, we can model the situation using a linear model. Then I write "$y = mx + b$" on the board, but point out that we must use the variables t and s. I discuss why s is the response variable and t is the explanatory variable. Then I replace y with s and x with t to get $s = mt + b$. Most students are quick to understand that m is 2, but have some trouble with seeing why b is 30. It helps to remind my students that $(0, b)$ is the y-intercept of the graph of an equation of the form $y = mx + b$. It also helps to create a table for t and s and put the number 0 beneath t and have my students tell me the value of s. But the most effective way might be to sketch a linear model on a coordinate system, which shows that the vertical intercept is 30 and, hence, the value of b is 30.

In part (a) of Problem 9, you are supposed to use a graphing calculator to draw a scatterplot. Because this section is so long, I delay discussing graphing calculator scatterplot instructions until Section 5.3, which is when students will use this skill a great deal. For part (a), I construct a scatterplot by hand, using every other point, and then graph the model.

For Exercises 27–30, I tell my students they can skip part (a), which instructs students to use a graphing calculator to draw a scatterplot. There are also two exercises in Section 4.4 and two exercises in Section 4.6 that require the use of a graphing calculator to draw scatterplots.

Textbook and Workbook Exploration

The 20 minute "Averaging rates of change" exploration has students compare the average of some rates of change with the slope of a linear model. (It is only if the data years are consecutive that we know for sure that the two quantities are equal.)

OBJECTIVE 4 COMMENTS

Students have an easy time with the slope addition property.

Workbook Exploration

The following 15-minute exploration can serve as a good exploration of the slope addition property. You could show students how to use graphing calculator tables to complete the table for Problem 1. Although students should certainly be able to complete the table without technology, a small arithmetic error might prevent a group from discovering the concept.

EXPLORATION *Slope addition property*

1. Complete the following table.

$y = 2x + 1$		$y = 3x - 5$		$y = -2x + 6$	
x	y	x	y	x	y
0		0		0	
1		1		1	
2		2		2	
3		3		3	
4		4		4	

2. In the table, the x-coordinates increase by 1 each time. For each equation, what do you notice about the y-coordinates? Compare what you notice with the coefficient of x in each equation.

3. Describe what the pattern from Problem 2 would be in general for any equation of the form $y = mx + b$.

4. Create an equation of the form $y = mx + b$, and check whether it behaves as you described in Problem 3.

5. Substitute 1 for x in the equation $y = mx + b$. Then substitute 2 for x. Then substitute 3. Explain why these results suggest that your description in Problem 3 is correct.

CHAPTER 4 OVERVIEW

In Section 4.1, students use the commutative, associate, and distributive laws to simplify linear expressions. In Section 4.2, students combine like terms to help simplify more linear expressions. They use symbolic methods, graphing, and tables to solve a linear equation in one variable in Sections 4.3 and 4.4. In Section 4.5, students compare simplifying expressions with solving equations. In Section 4.6, students solve a formula for a variable.

SECTION 4.1 LECTURE NOTES

Objectives

1. Describe the commutative, associative, and distributive laws.
2. Compare the distributive and associative laws for multiplication.
3. Describe the meaning of *equivalent expressions*.
4. *Simplify* expressions.
5. Subtract two expressions.
6. Show that a statement is false.

Main point: Simplify expressions.

OBJECTIVE 1

The equations $3+7=7+3$ and $3\cdot 7=7\cdot 3$ suggest the following laws.

Commutative Laws for Addition and Multiplication

Commutative law for addition: $a+b=b+a$
Commutative law for multiplication: $ab=ba$

Use a commutative law to write the expression in another form.

1. $5+x$

2. $x(2)$

- A **term** is a constant, variable, or a product of a constant and one or more variables raised to powers.
- Here are some terms: $6x$, 5, w, $-3pt$, $\dfrac{4y}{w}$.
- **Variable terms** are terms that contain variables.
- **Constant terms** are terms that do no contain variables.
- We usually write a sum of both variable and constant terms with the variable terms to the left of the constant terms.

3. Write $7-3x$ so that the variable term is to the left of the constant term.

The equations $(3+2)+5=3+(2+5)$ and $(3\cdot 2)\cdot 5=3\cdot(2\cdot 5)$ suggest the following laws.

Associative Laws for Addition and Multiplication

Associative law for addition: $a+(b+c)=(a+b)+c$
Associative law for multiplication: $a(bc)=(ab)c$

Use an associative law to write the expression in another form.

4. $(w+4)+6$

5. $8(4x)$

6. Rearrange the terms of $7 - 5x + 3 - 9$ so that the numbers can be added.

The equation $3(2+4) = 3 \cdot 2 + 3 \cdot 4$ suggests the following law.

Distributive Law

$$a(b+c) = ab + ac$$

Find the product.

7. $2(x+3)$

8. $3(w-5)$

9. $4(2x+8)$

10. $6(5t-3)$

11. $3(4x-7y)$

12. $-4(7-x)$

13. $(x-9)(2)$

14. $3(2p-4t+6)$

OBJECTIVE 2

Compare and contrast:

- The associative laws change the order of *operations*.
- The commutative laws change the order of *terms* or even *expressions*.

OBJECTIVE 3

15. Evaluate both of the expressions $2(x+3)$ and $2x+6$ for the values $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = 4$, and $x = 5$. (After doing a couple of evaluations by hand, use a graphing calculator table.)

Definition *Equivalent expressions*

Two or more expressions are **equivalent expressions** if, when each variable is evaluated for *any* real number (for which all the expressions are defined), the expressions all give equal results.

OBJECTIVE 4

- We **simplify** an expression by using the laws to remove any parentheses and to rearrange terms so that we can add any constant terms.
- The result of simplifying an expression is a **simplified expression**, which is equivalent to the original expression.

Simplify.

16. $2(x+3) - 8$

17. $-4(2w-5) + 3$

18. $7 - 5(x+4)$

19. $6 - 3(4x-9)$

20. Let x be a number. Translate the phrase "3, minus 2 times the difference of twice the number and 5" to an expression. Then simplify the expression.

21. Use the fact $-a = -1a$ to simplify $-(x+5)$.

OBJECTIVE 5

Use the fact $a - b = a - 1b$ to simplify the expression.

22. $5 - (2x+3)$

23. $-7 - (8t-1)$

OBJECTIVE 6
Show that the statement is false.

24. $a - b = b - a$

25. $a \div b = b \div a$

OBJECTIVE 6
Compare $a(b + c) = ab + ac$ with $a(bc) = (ab)c$.

SHORT HW 3, 9, 15, 25, 35, 49, 55, 61, 77, 79, 85, 91, 99, 103, 113

MEDIUM HW 3, 9, 15, 25, 29, 31, 35, 37, 43, 49, 55, 61, 71, 77, 79, 85, 87, 91, 95, 99, 103, 107, 109, 113

SECTION 4.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use the commutative, associative, and distributive laws to simplify expressions. These concepts will be used throughout the course. Most students will have an easy time with this section. In fact, many students will recognize this material and think, "Now *this* is algebra!"

OBJECTIVES 1 and 2 COMMENTS

When I am tight on time, I give light treatment to the commutative and associative laws and emphasize the distributive law. Either from past experiences with algebra or by way of intuition, my students seem to know how to apply the commutative and associative laws to simplify expressions. The main thing that students need to know in relation to the commutative and associative laws is that they can rearrange terms.

Textbook and Workbook Exploration

The 20-minute "Laws of operations" exploration has groups discover the three laws of operations discussed in this section. Groups have an easy time with this exploration.

OBJECTIVE 3 COMMENTS

When simplifying an expression, most students think that an equality symbol means, "here's the next step," instead of that the expressions on both sides of the equality symbol are equivalent. To spend time on Problem 15 is well worth it. In fact, this fundamental concept bears repeating multiple times throughout the course. The objective of locating errors in simplifying expressions in Section 4.2 will reinforce this concept (see pages 196 and 197 in the textbook).

Textbook and Workbook Exploration

In the 10-minute "Equivalent expressions" exploration, groups take a closer look at what it means for two expressions to be equivalent.

OBJECTIVE 4 COMMENTS

I discuss how to use the property $-a = -1a$ to simplify expressions such as $-(x+5)$: $-(x+5) = -1(x+5) = -1x + (-1)5 = -x - 5$ (see Problem 21). (The distributive law says that we can distribute a number such as -1, not an opposite symbol.) This will set the stage for Objective 4.

OBJECTIVE 5 COMMENTS

For problems such as Problems 22 and 23, most textbooks suggest to change the signs of the terms in the parentheses, but students don't seem to relate to this technique. Plus, it is not clear to students how that technique is tied to the laws of operations, whereas students can see that the technique that uses the fact $a - b = a - 1b$ is tied to the distributive law.

OBJECTIVE 6 COMMENTS

For Problem 24, showing that the statement $a - b = b - a$ is false accomplishes two things: Students are warned that there is no commutative law for subtraction, and they learn how to show that a statement is false.

Many students think that a should be distributed to both b and c in the expression $a(bc)$. In other words, students tend to confuse the distributive law and the associate law for multiplication. Addressing this issue now can help later when students simplify more complicated expressions such as $2(3\sqrt{5})$ in Section 11.2

SECTION 4.2 LECTURE NOTES

Objectives

1. Combine like terms.
2. Simplify expressions.
3. Locate errors in simplifying expressions by evaluating expressions.

Main point: Use combining like terms to simplify more expressions.

OBJECTIVE 1

- The **coefficient** of a variable term is the constant factor of the term.
- For example, the coefficient of $6x$ is 6, the coefficient of x is 1, and the coefficient of 8 is 8.
- **Like terms** are either constant terms or variable terms that contain the same variable(s) raised to exactly the same power(s).
- For example, $-8x$ and $3x$ are like terms. Also, 2 and 5 are like terms.
- If terms are not like terms, we say they are **unlike terms**.
- For example, $4x$ and $7y$ are unlike terms.
- We use the distributive law to *combine like terms*:

$$3x + 5x = (3 + 5)x = 8x$$

Combine like terms.

1. $3x + 5x$ **2.** $8k - 2k$ **3.** $4x + x$ **4.** $\frac{2}{7}x - \frac{5}{7}x$

Combining Like Terms

To combine like terms, add the coefficients of the terms and keep the same variable factors.

OBJECTIVE 2

We simplify an expression by removing parentheses and combining like terms.

Simplify.

5. $3x - 4y + 2x + 5 - 6y$

6. $2(x + 3) - 4(5x - 2)$

7. $7a - 2(3a - 5b) + 4a$

8. $2(3x - 5y) - 4(2x + 6y + 3)$

9. $3(2t - 5) - (t + 1)$

10. $5(4x - 2y) - (8x - 3y + 1)$

For Problems 11 and 12, let x be a number. Translate the English phrase into a mathematical expression. Then simplify the expression.

11. 7 times the sum of 2 and the number

12. Twice the number, minus -3 times the sum of the number and 7

For Problems 13 and 14, let x be a number. Translate the expression into an English phrase. Then simplify the expression.

13. $6(x+4)$

14. $x-4(x-2)$

OBJECTIVE 3

Consider the following incorrect work:

$$\begin{aligned} 3(x+4)-2x+5 &= 3x+4-2x+5 \\ &= 3x-2x+4+5 \\ &= x+9 \end{aligned}$$

1. Use a graphing calculator table to show that the work is incorrect.
2. Pinpoint where the error was made by evaluating the expression in each step for $x=2$.

SHORT HW 1, 7, 17, 21, 29, 37, 41, 45, 55, 63, 67, 71, 75, 87, 89

MEDIUM HW 1, 3, 7, 11, 17, 21, 23, 29, 33, 37, 41, 45, 49, 55, 59, 63, 67, 71, 73, 75, 77, 79, 85, 87, 89

SECTION 4.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students combine like terms and use the laws of operations to simplify linear expressions. These skills will be used throughout the course.

OBJECTIVE 1 COMMENTS

Students have an easy time with this skill.

Workbook Exploration

In the following 10-minute exploration, some groups may skip using the distributive law, which is the main point of the exploration.

Group Exploration

Section Opener: Combining like terms

We can write the sum $2x+5x$ as one term by using the distributive law:

$$\begin{aligned} 2x+5x &= (2+5)x && \text{Distributive law: } ac+bc=(a+b)c \\ &= 7x && \text{Add.} \end{aligned}$$

1. Simplify. Your work should include the distributive law.

   a. $3x+6x$ **b.** $7x-2x$ **c.** $4x+x$

   d. $6x+3x-5x$ **e.** $4x-2y+3-7x-y+6$ **f.** $-3.8y+9.2x+6.3-1.1x+2.5y$

2. A student tries to write $2x+3y$ as one term:

$$2x+3y=5xy$$

Is the work correct? If yes, use the distributive to explain. If no, substitute numbers for x and y on both sides of the equation to show that the result is false.

OBJECTIVE 2 COMMENTS

For Problem 6, I remind my students how to use a graphing calculator table to verify work in simplifying an expression.

Textbook and Workbook Exploration

The 5-minute "Laws of operations" exploration essentially warns groups of three types of common student errors.

OBJECTIVE 3 COMMENTS

It is unlikely that students will use this technique unless there are points attached to it on an assignment or exam. However, even if you do not wish to attach such points, a demonstration of this technique to students during lecture is a nice way to reinforce their understanding of equivalent expressions.

SECTION 4.3 LECTURE NOTES

Objectives

1. Describe a *linear equation in one variable*.
2. For an equation in one variable, describe *satisfy*, *solution*, *solution set*, and *solve*.
3. Describe three roles of variables.
4. Describe *equivalent equations*.
5. Apply the addition property of equality.
6. Apply the multiplication property of equality.
7. Solve a linear equation in one variable.
8. Solve a percentage problem.
9. Use a graph or a table to solve a linear equation in one variable.

Main point: Solve linear equations in one variable.

OBJECTIVE 1

Definition *Linear equation in one variable*

A **linear equation in one variable** is an equation that can be put into the form

$$mx + b = 0$$

where m and b are constants and $m \neq 0$.

OBJECTIVE 2

1. Show that the equation $x + 3 = 7$ becomes a true statement if 4 is substituted for x.

Definition *Solution, satisfy, solution set,* and *solve* for an equation in one variable

A number is a **solution** of an equation in one variable if the equation becomes a true statement when the number is substituted for the variable. We say the number **satisfies** the equation. The set of all solutions of the equation is called the **solution set** of the equation. We **solve** the equation by finding its solution set.

2. Is 2 a solution of the equation $2 - 4x = 3(x - 4)$?
3. Is 5 a solution of the equation $2 - 4x = 3(x - 4)$?

OBJECTIVE 3

Roles of a Variable

Here are three roles of a variable:

1. A variable represents a quantity that can vary.
2. In an expression, a variable is a placeholder for a number.
3. In an equation, a variable represents any number that is a solution of the equation.

OBJECTIVE 4

4. Show that the following equations all have the same solution set: $x = 6$, $x + 2 = 6 + 2$, and $x + 2 = 8$.

Equivalent equations are equations that have the same solution set.

OBJECTIVE 5

Addition Property of Equality

If A and B are expressions and c is a number, then the equations $A = B$ and $A + c = B + c$ are equivalent.

Solve.

5. $x - 4 = 2$
6. $x - 1.5 = 8.2$
7. $x + 3 = 7$
8. $b + 5 = 0$

9. Show that the following equations all have the same solution set: $x = 4$, $3 \cdot x = 3 \cdot 4$, and $3x = 12$.

OBJECTIVE 6

Multiplication Property of Equality

If A and B are expressions and c is a nonzero number, then the equations $A = B$ and $Ac = Bc$ are equivalent.

OBJECTIVE 7

Solve.

10. $\frac{5}{2}x = 3$
11. $-15 = -5x$
12. $\frac{5}{2}w = \frac{3}{4}$
13. $-t = 7$

OBJECTIVE 8

14. A person tips a waiter 15% of the bill. If the tip is \$5.10, how much is the bill?

OBJECTIVE 9

Using Graphing to Solve an Equation in One Variable

To use graphing to solve an equation $A = B$ in one variable, x, where A and B are expressions:

1. Graph the equations $y = A$ and $y = B$ on the same coordinate system. (For example, if the original equation is $5x - 9 = 3x + 7$, then we would graph the equations $y = 5x - 9$ and $y = 3x + 7$.)
2. Find all intersection points.
3. The x-coordinates of those intersection points are the solutions of the equation $A = B$.

For Problems 15 and 16, graph $y = \frac{3}{5}x + 1$ by hand. Use the graph to solve the given equation.

15. $\frac{3}{5}x + 1 = 4$

16. $\frac{3}{5}x + 1 = -2$

17. Use "intersect" to solve $\frac{2}{3}x + \frac{5}{6} = \frac{3}{2}x - \frac{5}{3}$.

18. Use a table of values of $y = 4x - 3$ to solve $4x - 3 = 5$.

Discuss solving an equation in terms of undoing operations.

SHORT HW 1, 7, 9, 19, 29, 39, 49, 57, 59, 67, 73, 79, 81, 91, 95

MEDIUM HW 1, 7, 9, 17, 19, 29, 31, 35, 39, 41, 47, 49, 57, 59, 67, 73, 75, 79, 81, 83, 91, 95, 97, 99

SECTION 4.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use symbolic methods, graphing, and tables to solve linear equations in one variable. They will use this material in many sections in the textbook.

OBJECTIVE 1 COMMENTS

I remind my students that in Chapter 3 we graphed equations in two variables. I say we will now work with linear equations in *one* variable. I provide a few examples and then define *linear equation in one variable*.

Another option is to introduce the definition after you have solved a few equations so that students know what "put into the form" means.

OBJECTIVE 2 COMMENTS

For Problem 1, I write $x + 3 = 7$ on the board and ask students what x can be. Students are quick to say x is equal to 3. I say 3 is called a *solution* and that it *satisfies* the equation. Then I define the words in the boxed statement and observe that these words mean about the same thing when applied to equations in two variables.

OBJECTIVE 3 COMMENTS

I remind students that we have discussed two roles of a variable so far and say there is a third role.

OBJECTIVE 4 COMMENTS

This objective is setting the stage for Objective 5.

OBJECTIVES 5 and 6 COMMENTS

For the addition property of equality, most students think that the entire point is that "we can add a number to both sides of an equation." Most students do not know that a solution of an equation will satisfy *all* the equations that lead to finding the solution. At various times in the course, I return to this topic to deepen students' understanding. Similar comments apply to the multiplication property of equality.

Workbook Exploration

The following 15-minute exploration could be thought of as the most important exploration in the sense that students solve so many equations throughout the course yet tend not to know that when solving an equation, all the equations are equivalent.

EXPLORATION *Addition and multiplication properties of equality*

1. The addition property of equality states that if A and B are expressions and c is a number, then the equations $A = B$ and $A + c = B + c$ are equivalent.

a. To show the addition property of equality is true when we add 5 to both sides of $x = 3$, explain why the following three equations are all equivalent.

$$x = 3$$
$$x + 5 = 3 + 5$$
$$x + 5 = 8$$

b. Here we use the addition property of equality to solve the equation $x - 4 = 6$ by adding 4 to both sides:

$$x - 4 = 6$$
$$x - 4 + 4 = 6 + 4$$
$$x + 0 = 10$$
$$x = 10$$

Show that all four equations are equivalent.

2. The multiplication property of equality states that if A and B are expressions and c is a nonzero number, then the equations $A = B$ and $Ac = Bc$ are equivalent.

a. To show the multiplication property of equality is true when we multiply both sides of $x = 2$ by 3, explain why the following three equations are all equivalent.

$$x = 2$$
$$3 \cdot x = 3 \cdot 2$$
$$3x = 6$$

b. Here we use the multiplication property of equality to solve the equation $\frac{3}{5}x = 6$ by multiplying both sides by $\frac{5}{3}$:

$$\begin{aligned} \frac{3}{5}x &= 6 \\ \frac{5}{3} \cdot \frac{3}{5}x &= \frac{5}{3} \cdot 6 \\ 1x &= 10 \\ x &= 10 \end{aligned}$$

Show that all four equations are equivalent.

OBJECTIVE 7 COMMENTS

For Problem 5, I say our objective in solving an equation is to isolate x on one side of the equation.

For Problem 7, I point out that we can still use the addition property of equality, because subtracting 3 from both sides of the equation is the same as adding -3 to both sides.

For Problem 10, I first show that the product of a fraction and its reciprocal is equal to 1. Then I use the multiplication property of equality to solve the equation $\frac{5}{2}x = 3$.

For Problem 11, I point out that we can still use the multiplication property of equality, because dividing both sides by -5 is the same as multiplying both sides by $-\frac{1}{5}$.

In solving $-t = 7$ (Problem 13), I show students how to solve the equation by dividing both sides of the equation by -1 as well as by multiplying both sides by -1. Almost all my students prefer dividing by -1 rather than multiplying by -1.

Textbook and Workbook Exploration

The 10 minute "Locating an error in solving an equation" exploration is a good reminder to students about the meaning of equivalent equations. This method of locating errors will be formally discussed in Section 4.4.

OBJECTIVE 8 COMMENTS

Perhaps the hardest part of Problem 14 is realizing that it helps to define a variable (for the amount of the bill). Such percentage problems are good preparation for harder ones in Section 4.4.

OBJECTIVE 9 COMMENTS

A discussion about inputs and outputs can be used to explain how to use graphing to solve an equation in one variable. Most students will be challenged by such an explanation. It will be easier to get across why this method works once students learn how to solve systems by using substitution in Section 6.2.

For Problem 17, I point out the advantages of using technology to solve such an equation. It is important that students learn symbolic methods of the course, but it is also important that students learn appropriate uses of technology.

SECTION 4.4 LECTURE NOTES

Objectives

1. Solve a linear equation in one variable.
2. Make estimates and predictions by solving equations.
3. Translate English sentences into and from mathematical equations.
4. Use a graph or tables to solve a linear equation in one variable.
5. Describe the meaning of *conditional equation*, *inconsistent equation*, and *identity* and how to solve these types of equations.
6. Locate errors in solving linear equations.

Main point: Solve more linear equations in one variable.

OBJECTIVE 1

Solve.

1. $7x + 2 = 23$
2. $3x - 5x + 2 = 6$
3. $2w - 9 = 6w + 23$
4. $5 - 4(2x - 3) = 3(4 - x)$
5. $2(3x + 4) - (4x - 1) = 3(x + 2)$
6. $\frac{5}{6}k - \frac{3}{2} = \frac{7}{3}$
7. $\frac{5(x-4)}{3} = -2x$
8. $\frac{4t-7}{3} = \frac{2t+1}{4}$

- It is wise to check that a result (or the results) of solving an equation does indeed satisfy the equation.
- A key step in solving an equation that contains fractions is to multiply both sides of the equation by the LCD so that there are no fractions on either side of the equation.

9. Solve $2.4(1.7t - 3.58) = 17.94$. Round the result to the second decimal place.

OBJECTIVE 2

10. The number of Bank of America branches was 4542 branches in 2016 and has decreased by about 208 branches per year (Source: *Federal Deposit Insurance Corp*). Let n be the number of Bank of America branches at t years since 2016.
    a. Find a model of the situation.
    b. Predict the number of Bank of America branches in 2022.
    c. Predict when there will be 3500 branches.

11. In 2016, the number of charges alleging sexual harassment was 6758, down 14.9% from 2010 (Source: *U.S. Equal Employment Opportunity Commission*). How many charges alleging sexual harassment were there in 2010?

OBJECTIVE 3

Here are some sentences that have the same meaning as $x = 5$:

- The number is 5.
- The number is equal to 5.
- The number is the same as 5.

For Problems 12 and 13, find the number.

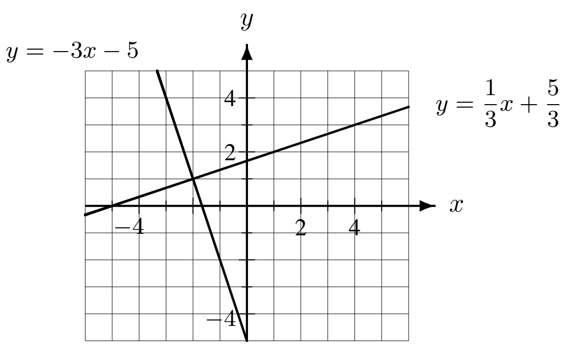
12. Seven times the difference of a number and 3 is -35.

13. Eight, minus 2 times the sum of a number and 5 is 12.

14. Let x be a number. Translate the equation $5x + 2 = 8x - 10$ into an English sentence. Then solve the equation.

OBJECTIVE 4

For Problems 15 and 16, solve the given equation by referring to the graphs of $y = \frac{1}{3}x + \frac{5}{3}$ and $y = -3x - 5$ shown below.



15. $\frac{1}{3}x + \frac{5}{3} = -3x - 5$

16. $\frac{1}{3}x + \frac{5}{3} = 2$

17. Use tables to solve the equation $3x - 2 = 6 - x$. [The solution is 2.]

OBJECTIVE 5

- A **conditional equation** is sometimes true and sometimes false, depending on which values are substituted for the variable(s).
- If an equation does not have any solutions, we call the equation **inconsistent** and say the solution set is the **empty set**.
- An equation that is true for all permissible values of the variable(s) it contains is called an **identity**.

Solve the equation. State whether the equation is a conditional equation, an inconsistent equation, or an identity.

18. $3x + 2 - 7x = 5x - 9x + 8$ **19.** $4(2x - 3) = 5 - x$ **20.** $2(x - 5) - x = x - 10$

OBJECTIVE 6

21. A student incorrectly tries to solve $2(x - 4) = 8$:

$$\begin{aligned} 2(x - 4) &= 8 \\ 2x - 4 &= 8 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

Substitute 6 for x in each of the four equations, and explain how the results help you pinpoint the error.

Locating an Error in Solving a Linear Equation

If the result of an attempt to solve a linear equation in one variable does not satisfy the equation, an error was made. To pinpoint the error, substitute the result into all the equations in the work and find which pair of consecutive equations gives a false statement followed by a true statement.

SHORT HW 13, 17, 23, 29, 39, 43, 49, 55, 65, 75, 81, 85, 93, 95, 117, 119

MEDIUM HW 1, 9, 13, 17, 23, 25, 29, 39, 43, 49, 51, 55, 65, 71, 81, 85, 93, 95, 97, 99, 109, 113, 117, 119, 121, 127

SECTION 4.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use symbolic methods, graphing, and tables to solve linear equations in one variable. Students use models to make predictions. They also learn the meaning of and how to solve conditional equations, inconsistent equations, and identities.

OBJECTIVE 1 COMMENTS

Most students are challenged by Problems 6–8. Some students try to write all the fractions with a common denominator, which is not incorrect but is inefficient, and usually such students make other errors along the way. Some students have trouble finding the LCD. Other students have trouble simplifying both sides of the equation after multiplying both sides of the equation by the LCD.

Textbook and Workbook Exploration

You could start class by having groups work on the 10-minute “Solving linear equations.”

Workbook Exploration

The following 15-minute exploration is a more demanding alternative to the exploration just described. Because all the required concepts have already been discussed, you could begin class by facilitating it.

Group Exploration

Section Opener: Solving linear equations

Solve. Check that your result satisfies the original equation.

1. $x + 3 = 11$ **2.** $2x = 8$ **3.** $2x + 3 = 11$

4. $6x - 4x + 3 = 11$ **5.** $2(x + 4) + 3x = 23$ **6.** $3(2x - 5) - (4x + 1) = 8$

OBJECTIVE 2 COMMENTS

For part (a) of Problem 10, some of my students need a reminder about how to find such a linear model (Section 3.5). For Problem 11, students tend to think that what’s required is to calculate 14.9% of the number of charges in 2016 (6758). Encourage your students to take the necessary time to understand the situation before trying to perform calculations or solve equations.

OBJECTIVE 3 COMMENTS

Most students have an easy time with this objective.

OBJECTIVE 4 COMMENTS

For Problems 15 and 16, you could verify the work using "intersect."

OBJECTIVE 5 COMMENTS

This material is not essential for the course. Although Section 4.5 will refer to identities and Chapter 6 includes inconsistent systems, students can pick up on the relevant concepts at those times.

Workbook Exploration

In the following 20-minute exploration, some groups will likely need help seeing how parts (a) and (b) of Problem 3 connect with Problems 2 and 1, respectively.

EXPLORATION *Conditional equations, inconsistent equations, and identities*

1. **a.** Can a number be 1 more than itself? How many solutions does the equation $x = x + 1$ have?

   b. If you subtract x from both sides of $x = x + 1$, is the result a true or false statement? Explain why this makes sense by referring to the results you found in part (a).

2. **a.** Can a number be equal to itself? What numbers are solutions of the equation $x = x$?

   b. If you subtract x from both sides of $x = x$, is the result a true or false statement? Explain why this makes sense by referring to the results you found in part (a).

3. Solve. Make sure that you clearly state the solutions.

   a. $2x + 6x = 8x$ **b.** $4x + 3 = 4x + 7$ **c.** $3x - 1 = 8$

   d. $3(2x - 4) + x = 7x - 10$ **e.** $8 - 2(5x - 3) = 6x + 5$ **f.** $10x - 4 - 3x = 2x - 4 + 5x$

Textbook and Workbook Exploration

The 15-minute "Any linear equation in one variable has exactly one solution" exploration not only has groups determine the number of solutions of a linear equation but also has groups work with a formula. Formulas will be discussed in Section 4.6.

OBJECTIVE 6 COMMENTS

Students will not use this strategy on their own to identify errors. If you want them to use this approach, you'll have to attach points on an exam accordingly. Even if you don't attach such points, Problem 21 does reinforce the meaning of equivalent equations.

SECTION 4.5 LECTURE NOTES

> *Objectives*
>
> **1.** Compare the meanings of *simplifying expressions* and *solving equations*.
>
> **2.** Translating English phrases and sentences into mathematical expressions and equations.

Main point: Compare the meanings of *simplifying expressions* and *solving equations*.

OBJECTIVE 1

> **Definition** *Linear expression in one variable*
>
> A **linear expression in one variable** is an expression that can be put into the form
>
> $$mx + b$$
>
> where m and b are constants and $m \neq 0$.

Recall: A *linear equation in one variable* is an equation that can be put into the form $mx + b = 0$, where m and b are constants and $m \neq 0$.

For Problems 1 and 2, is the following a linear expression or a linear equation?

1. $6(2x - 1) + 5 = 35$

2. $6(2x - 1) + 5$

3. Solve $4(x - 5) + 3 = 2x - 9$.

4. Simplify $4(x - 5) + 3$.

5. Use a graphing calculator table to check your work in Problem 3.

6. Use a graphing calculator table to check your work in Problem 4.

> **Results of Simplifying an Expression and Solving an Equation**
>
> The result of simplifying an expression is an expression. The result of solving a linear equation in one variable is a number.

> **Comparing Simplifying an Expression with Solving an Equation**
>
> - If an expression A in one variable is simplified to an expression B, then every real number (for which both expressions are defined) is a solution of the equation $A = B$. In other words, the equation $A = B$ is an identity.
> - Exactly one number is a solution of a linear equation in one variable.

7. The simplified form of $2(4x - 7) - 3x$ is $5x - 14$. What does this mean?

8. The solution of $2(4x-7)-3x=-4$ is 2. What does this mean?

Simplify the expression or solve the equation, as appropriate.

9. $2(x-5)-(6x+1)-3(2x+3)$

10. $2(x-5)-(6x+1)=3(2x+3)$

11. $\frac{3}{8}x+\frac{7}{6}-\frac{7}{4}x$

12. $\frac{3}{8}x+\frac{7}{6}=\frac{7}{4}x$

13. $\frac{5}{2}x-\frac{3}{4}=\frac{7}{8}x+\frac{1}{4}$

14. $\frac{5}{2}x-\frac{3}{4}-\frac{7}{8}x+\frac{1}{4}$

Multiplying Expressions and Both Sides of Equations by Numbers

When simplifying an expression, the only number that we can multiply the expression by or part of it by is 1. When solving an equation, we can multiply both sides of the equation by *any* number except 0.

OBJECTIVE 2

Let x be a number. Translate the following to an expression or an equation, as appropriate. Then simplify the expression or solve the equation.

15. Five, minus 3 times the sum of the number and 1

16. The sum of 4 and twice the number is 10.

17. The quotient of the number and 3 is equal to the number subtracted from 5.

18. Three, plus 5 times the difference of the number and 2

SHORT HW 5, 7, 21, 23, 29, 31, 37, 39, 41, 43, 51, 55, 59, 61, 63, 71

MEDIUM HW 5, 7, 13, 15, 21, 23, 29, 31, 37, 39, 41, 43, 51, 55, 59, 61, 63, 65, 66, 67, 71, 73

SECTION 4.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students compare the meanings of *simplifying expressions* and *solving equations*. This is excellent preparation for working with polynomial expressions and equations and working with rational expressions and equations; it is with these more complex expressions and equations that students have trouble keeping straight the meanings and methods.

Students also translate English phrases and sentences to mathematical expressions and equations in this section.

OBJECTIVE 1 COMMENTS

This objective has two purposes. Its primary function is to have students compare simplifying expressions and solving equations. A secondary function is to give weaker students another shot at mastering working with linear expressions and equations.

Most students have an easy time with Problems 1–4, 9, and 10. This means that while working with these problems, I can focus on the similarities and differences of simplifying expression and solving equations. Although I will address the same concepts when working with more challenging types of expressions and equations later in the course, it is now that the ideas can really sink in, because students are not distracted by the techniques.

For Problems 11–14, some students struggle with the techniques. Other students are ready to contemplate when they should be multiplying by 1 and when they should be multiplying by the LCD.

Textbook and Workbook Exploration

The 15-minute "Comparing expressions and equations" exploration is a nice way to lead into this section. For Problems 3 and 6 of this exploration, some groups may not be sure how to use substitutions to check their work when simplifying expressions.

Textbook and Workbook Exploration

The 10-minute "Simplifying versus solving" exploration has groups reflect on two common types of student errors.

OBJECTIVE 2 COMMENTS

By this stage of the game, most students have an easy time with Problems 15–18.

SECTION 4.6 LECTURE NOTES

Objectives

1. Determine area and perimeter formulas of a rectangle and a total-value formula.
2. Use formulas to solve various types of problems.
3. Translate an English sentence into a formula.
4. Solve a formula for a variable.

Main point: Use formulas to solve various types of problems.

OBJECTIVE 1

- Recall: A **formula** is an equation that contains two or more variables (Section 3.3).
- The **perimeter** of a polygon, which is a geometrical object such as a triangle, rectangle, or trapezoid, is the total distance around the object.
- The perimeter P of a rectangle with width W and length L is given by $P = W + L + W + L = 2L + 2W$.

Area and Perimeter of a Rectangle

For a rectangle with length L, width W, area A, and perimeter P:

- $A = LW$
- $P = 2L + 2W$

OBJECTIVE 2

1. A rectangular floor has an area of 192 square feet and a length of 16 feet. Find the width of the room.

To find a single value of a variable in a formula, we often substitute numbers for all the other variables and then solve for that one variable.

2. A photographer plans to use 76 inches of wood to make a rectangular frame for a photo. If the width of the frame is to be 14 inches, what will its length be?

For Problems 3 and 4, substitute the given values for the variables and then solve the equation for the remaining variable.

3. $A = P + Prt$; $A = 1025$, $P = 500$, $t = 15$ (simple interest)
4. $S = 2WL + 2WH + 2LH$; $S = 72$, $W = 2$, $H = 3$ (surface area of a rectangular box)

- Four dimes are worth $10 \cdot 4 = 40$ cents.
- We found the total value of the dimes by multiplying the value of one dime (10 cents) by the number of quarters (4).

Total-Value Formula

If n objects each have value v, then their total value T is given by $T = vn$.

5. Tickets for a Man Man concert cost $25 each.

- **a.** Find a formula of the total cost T (in dollars) of n of these tickets.
- **b.** Perform a unit analysis of the formula you found in part (a).
- **c.** Substitute 425 for T in the formula from part (a) and solve for n. What does your result mean in this situation?

OBJECTIVE 3

6. Let a, b, and c be the lengths (in inches) of the sides of a triangle. The average length (in inches) A of a side of a triangle is equal to the sum of the lengths of the three sides divided by 3.

- **a.** Write a formula of the average length of a side of a triangle.
- **b.** If the average length of a side of a triangle is 10 inches, and the lengths of two of the sides are 5 inches and 13 inches, find the length of the third side.

OBJECTIVE 4

For Problems 7–10, solve the equation for the specified variable.

7. $W = Fd$, for d

8. $P = 2L + 2W$, for W [Example 6]

9. $5x - 8y = 24$, for y

10. $A = P(1 + rt)$, for t [very similar to Exercise 52]

11. The number of Apple stores was 463 stores in 2015 and has increased by about 26 stores per year since then (Source: *Apple*). Let n be the number of Apple stores at t years since 2015.

- **a.** Find an equation of a linear model to describe the situation.
- **b.** Solve the equation that you found in part (a) for t.
- **c.** Estimate when there will be 600 stores.
- **d.** Use a graphing calculator table to predict in which years the numbers of stores will be 650, 700, 750, 800, and 850.

- For Problem 11, to find numbers of Apple stores, it is convenient to use $n = 26t + 463$. To find years, it is convenient to use $t = \dfrac{n - 463}{26}$.
- Solving for a variable in a formula will not change the relationship between the variables in the formula.
- To find several values of a variable in a formula, we usually solve the formula for that variable before we make any substitutions.

SHORT HW 3, 7, 13, 17, 21, 31, 35, 41, 45, 51, 63, 67, 77, 81

MEDIUM HW 3, 5, 7, 13, 15, 17, 19, 21, 25, 31, 33, 35, 37, 41, 45, 51, 57, 63, 67, 69, 75, 77, 81

SECTION 4.6 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students learn the area and perimeter formulas of a rectangle and a total-value formula. Students solve various types of problems. They also solve a formula for a variable.

OBJECTIVE 1 COMMENTS

Some students confuse area with perimeter, so I spend a little time comparing these two concepts.

If time permits, I sketch a polygon such as those in Exercises 1–6 in Homework 4.6 and have my students find a formula of the perimeter of the polygon.

OBJECTIVE 2 COMMENTS

I emphasize the arithmetic-algebra connection for the total value formula $T = vn$. The example about the four dimes in the lecture notes accomplishes this.

OBJECTIVE 3 COMMENTS

Recall that *average* was defined in Section 1.1. So, the first two sentences of Problem 6 could be replaced with "Let a, b, and c be the lengths (in inches) of the three sides of a triangle and let A be the average length (in inches) of a side."

OBJECTIVE 4 COMMENTS

Most students have a tough time with Problems 7–11. It can help to solve each formula alongside a similar equation in one variable. For example, see the solutions of Examples 5 and 6 on pages 235 and 236 of the textbook.

Problem 9 is excellent preparation for Section 5.1. I make sure that I do this problem (and assign several of Exercises 59–66) so that my students have two sections to practice this skill.

Note that part (b) of Problem 11 is essentially having students find the inverse function of the model $n = 26t + 463$. When doing part (d), I point out that it is advantageous to first solve for t before substituting the five values for n; if we substituted before solving, we'd have five equations to solve. I say that first solving for t would become a huge advantage if we wanted to make hundreds of substitutions for n.

Most textbooks seem to suggest that we always want to first solve a formula for a variable before making substitutions; this hardly makes sense if we want to find only one value of a quantity.

Workbook Exploration

The following 25-minute exploration requires a good chunk of time, but it highlights a key concept of the section and also gives groups practice with solving a formula (Problem 2 in the exploration) that is well-supported by solving equations (Problem 1 in the exploration).

Group Exploration

Determining when to substitute a value for a variable in a formula

Let F be the temperature (in Fahrenheit degrees) and C be the equivalent Celsius reading. Recall that the relationship between C and F can be described by the formula $C = \frac{5}{9}(F - 32)$.

1. Suppose the high temperature readings (in Celsius) on June 16 in five European cities are 21, 23, 24, 27, and 28. Convert the temperature readings to Fahrenheit readings by substituting each data value for C in

the formula $C = \frac{5}{9}(F - 32)$ and solving for F.

2. Solve the formula $C = \frac{5}{9}(F - 32)$ for F.

3. Perform the same conversions as you did in Problem 1, but this time use a graphing calculator table and the formula you found in Problem 2.

4. In Problem 1 you converted five Celsius readings. In Problem 2 followed by Problem 3 you converted the same five readings.

a. Which method would you use to convert 100 Celsius readings? Explain.

b. Which method would you use to convert a single Celsius reading? Explain.

CHAPTER 5 OVERVIEW

In Section 5.1, students graph linear equations in two variables. In Section 5.2, students find equations of lines. Students find equations of linear models in Section 5.3. In Section 5.4, students use linear models to make estimates and predictions about authentic situations. In Section 5.5, students solve linear inequalities in one variable and use linear inequalities to make estimates and predictions about authentic situations.

SECTION 5.1 LECTURE NOTES

Objectives

1. Graph a linear equation by solving for y.
2. Graph a linear equation by finding the intercepts of its graph.
3. Find coordinates of solutions of a linear equation.
4. Describe the difference between equations in one variable and equations in two variables.

Main point: Graph linear equations in two variables.

OBJECTIVE 1

Determine the slope and the y-intercept. Use slope and the y-intercept to graph the equation by hand.

1. $2y = 3x$ **2.** $2x + 5y = 15$ **3.** $0 = x - 3y - 6$

- Before we can use the y-intercept and the slope to graph a linear equation, we must solve for $\boldsymbol{y}$ to put it in the form $y = mx + b$.
- Warning: The slope of the line $2x + 5y = 15$ is *not* 2 (see Problem 2).

OBJECTIVE 2

Intercepts of the Graph of an Equation

For an equation containing the variables x and y,

- To find the x-coordinate of each x-intercept, substitute 0 for y and solve for x.
- To find the y-coordinate of each y-intercept, substitute 0 for x and solve for y.

For Problems 4–6, find the intercepts. Then graph the equation by hand.

4. $3x - 6y = 12$ **5.** $y = -2x + 5$ **6.** $\dfrac{x}{2} + \dfrac{y}{4} = 1$

Using Intercepts to Graph an Equation

To graph a linear equation whose graph has exactly two intercepts,

1. Find the intercepts.
2. Plot the intercepts and a third point on the line, and graph the line that contains the three points.

Find the approximate intercepts. Round coordinates to the second decimal place.

7. $3.9x - 5.8y = 23.6$ **8.** $y = 2.46x - 17.51$

OBJECTIVE 3

9. Complete the following steps to graph $3x - 5y = 20$.

a. Find y when $x = 2$.

b. Find x when $y = -2$.

c. Graph $3x - 5y = 20$ by hand.

Using Three Solutions to Graph a Linear Equation

To graph any linear equation,

1. Find three solutions of the equation.
2. Plot the three solutions and graph the line that contains them.

OBJECTIVE 4

10. Describe the solution(s) of the equation in two variables $y = 2x - 3$.

11. Describe the solution(s) of the equation in one variable $5 = 2x - 3$.

- There is a connection between the equations $y = 2x - 3$ and $5 = 2x - 3$.
- If we substitute 5 for y in the equation $y = 2x - 3$, the result is the equation $5 = 2x - 3$, whose solution is $x = 4$.
- So, $(4, 5)$ is one of an infinite number of solutions of $y = 2x - 3$. [Draw a sketch.]

Comparing Linear Equations in One and Two Variables

A linear equation in two variables has an infinite number of (ordered pair) solutions. A linear equation in one variable has exactly one (real number) solution.

SHORT HW 5, 15, 21, 25, 35, 37, 39, 43, 49, 55, 61, 71, 73, 79

MEDIUM HW 3, 5, 15, 21, 25, 29, 35, 37, 39, 43, 45, 49, 55, 61, 69, 71, 73, 79, 81, 83, 88, 89, 91

SECTION 5.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use three methods to graph linear equations in two variables. Students will use this material in Section 6.1.

OBJECTIVE 1 COMMENTS

For Problems 1–3, students combine the skill of solving a formula for a variable (Section 4.6) with the skill of graphing a linear equation in slope-intercept form (Section 3.4). The warning about 2 not being the slope of the line $2x + 5y = 15$ is worthwhile, because students have been working only with slope-intercept form until now.

Textbook and Workbook Exploration

The 15-minute “Graphing linear equations” exploration is a great introduction to Objective 1.

OBJECTIVE 2 COMMENTS

For Problems 4–8, I emphasize that an intercept is a point and has two coordinates (not one). I tell my students that after they find the intercepts of a line, that they should indicate which is the x-intercept and which is the y-intercept (assuming they are distinct).

When graphing an equation that can be put into the form $y = mx$, I warn my students that the origin is the only intercept. So, using the slope and the origin is a good way to graph the equation.

Textbook and Workbook Exploration

The 20-minute “Comparing two graphing techniques” exploration has groups compare graphing by using the slope and the y-intercept with graphing by finding the intercepts.

OBJECTIVE 3 COMMENTS

This objective has been included in the text to open students’ horizons to the concept that we can find non-intercepts of a line. I do not recommend that students use this approach to graph linear equations.

OBJECTIVE 4 COMMENTS

Because we describe the solutions of a linear equation in two variables and the solution of a linear equation in one variable in such different formats (graphs versus a statement of the form “$x = a$”), many students lose sight that in both cases, we are finding the solutions of the equations. That is, we are finding all the ordered pairs or all the numbers that satisfy the equations. The point of Problems 10 and 11 is to help students see this commonality and the connection between the two types of equations.

SECTION 5.2 LECTURE NOTES

Objectives

1. Find an equation of a line by using the slope-intercept form of a linear equation.
2. Find an equation of a line by using the *point-slope* form of a linear equation.

Main point: Find an equation of a line.

OBJECTIVE 1

Recall that an equation of a line can be put into slope-intercept form $y = mx + b$ (Section 3.4).

Use the slope-intercept form to find an equation of the line that has the given slope and contains the given point.

1. $m = 3$, $(-4, 5)$
2. $m = -2$, $(-3, -6)$
3. $m = \frac{2}{5}$; $(-3, 4)$
4. $m = -\frac{4}{3}$; $(5, -2)$
5. $m = 0$, $(-3, 2)$
6. m is undefined, $(-5, 1)$

Finding an Equation of a Line by Using the Slope, a Point, and the Slope-Intercept Form

To find an equation of a line by using the slope and a point,

1. Substitute the given value of the slope m into the equation $y = mx + b$.
2. Substitute the coordinates of the given point into the equation you found in step 1 and solve for b.
3. Substitute the value of b you found in step 2 into the equation you found in step 1.
4. Check that the graph of your equation contains the given point.

Use the slope-intercept form to find an equation of the line that contains the two given points.

7. $(2, 3)$ and $(4, 7)$ [slope is an integer]
8. $(-3, 4)$ and $(2, -6)$ [slope is an integer]
9. $(-5, -3)$ and $(2, 1)$ [slope is a noninteger]
10. $(-6, -2)$ and $(-3, -7)$ [slope is a noninteger]

Finding an Equation of a Line by Using Two Points and the Slope-Intercept Form

To find an equation of the line that passes through two given points whose x-coordinates are different,

1. Use the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to find the slope of the line containing the two points.
2. Substitute the m value you found in step 1 into the equation $y = mx + b$.
3. Substitute the coordinates of one of the given points into the equation you found in step 2 and solve for b.
4. Substitute the b value you found in step 3 into the equation you found in step 2.
5. Check that the graph of your equation contains the two given points.

11. Find an approximate equation of the line that contains the points $(-3.25, 8.11)$ and $(2.67, -1.39)$. Round the slope and the constant term to two decimal places.
12. Find an equation of the line that contains the given points.
    - **a.** $(4, -3)$ and $(4, 2)$ [slope is undefined]
    - **b.** $(-5, -2)$ and $(3, -2)$ [slope is 0]
13. Find an equation of the line that contains $(-4, 8)$ and is parallel to the line $y = -3x - 2$.
14. Find an equation of the line that contains $(5, -2)$ and is perpendicular to the line $3x + 5y = 10$.

OBJECTIVE 2 (optional)

Let (x_1, y_1) and (x, y) be two distinct points on a line with slope m. Use the true statement $\dfrac{y - y_1}{x - x_1} = m$ to derive the point-slope form.

Point-Slope Form

If a nonvertical line has slope m and contains the point (x_1, y_1), then an equation of the line is

$$y - y_1 = m(x - x_1)$$

Repeat any of Problems 1–5, 7–11, 13, and 14. Compare your results to your earlier results.

SHORT HW 1, 11, 21, 25, 29, 35, 37, 45, 55, 61, 65, 67, 71, 73, 80

MEDIUM HW 1, 7, 11, 21, 25, 29, 35, 37, 45, 49, 53, 55, 61, 65, 67, 69, 71, 73, 80, 81, 85, 87, 89

SECTION 5.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

This section discusses how to find equations of lines, both by using the slope-intercept form and the point-slope form. In subsequent sections, only the slope-intercept form will be used to find such equations. So, the point-slope form is optional. However, if some of your students will eventually take calculus, you should spend some time on the point-slope form, because this concept will likely be used in calculus.

I have chosen to emphasize finding linear equations using the slope-intercept form for several reasons:

- My students have better success with this method than with using the point-slope form, especially when the slope is a noninteger.
- Students have a better sense of why this method "works" than with using the point-slope form.
- Using two forms for linear equations in Chapters 1—6 would confuse the issue for many students.

Students will find equations of linear models in Sections 5.3 and 5.4. They will also find equations of linear models in Related Review exercises in other modeling sections.

The skills required for Problems 13 and 14 will not be needed in the rest of the text. However, the concepts about parallel lines addressed in Section 3.4 will be needed in Sections 6.1–6.3.

OBJECTIVE 1 COMMENTS

For Problem 1, when I substitute the ordered pair $(-4, 5)$ into the equation $y = 3x + b$ to find b, I remind students that an ordered pair that corresponds to a point on a graph satisfies an equation of the graph. So, making such a substitution will give a true statement, one that will allow us to find the value of b.

For Problems 3, 4, 9, 10, and 14, I make sure that students understand that they should use fractions, not decimals, when finding linear equations. Some students will have trouble working with fractions. Such work is a good primer for work with rational expressions and equations in Chapter 10. When solving an equation with fractions (to find b of $y = mx + b$), I prefer to have students multiply both sides of the equation by the LCD, because that is how students will solve rational equations in Chapter 10; also, a majority of my students prefer this approach.

For Problems 7–11, I show students how to use their graphing calculators to verify the equations.

Problem 11 is a good primer for Section 5.3, where students will find equations of linear models.

Students are challenged by Problems 13 and 14. One of students' difficulties is trying to track what is true for one line and what is true for the other. When doing Problem 13, it can help to use one color for all words and symbols in the problem statement and solution that describe one line and to use another color for such things that describe the other line. Similar color coding can be used for Problem 14.

Workbook Exploration

You could start class with the following 10-minute exploration, which serves as a good bridge from Section 3.5 to this section. In this exploration, students use the slope addition property to find equations. Although this skill is not necessary for this section, it does have groups reflect on how to go from tables to equations for linear functions. After completing this section, students should be able to go in all six directions between equations, graphs, and tables for linear functions.

Group Exploration

Finding an equation of a line

Some solutions of four linear equations are provided in the following table. Find an equation of each of the four lines.

Equation 1		Equation 2		Equation 3		Equation 4	
x	y	x	y	x	y	x	y
0	3	3	40	2	3	1	22
1	7	4	37	4	15	3	18
2	11	5	34	6	27	5	14
3	15	6	31	8	39	7	10

Workbook Exploration

The skill of finding an equation using two given points is introduced in the following 15-minute exploration. This exploration is more intuitive than the exploration that directly follows this one because it makes use of a graph. A few groups will have trouble with Problem 2 of the exploration. In particular, some groups will ignore Problem 2 and write $1 = m(2) + b$ in Problem 3.

Group Exploration

Section Opener: Finding linear equations

1. Find the slope of the line that contains the points $(4,\ 5)$ and $(6,\ 8)$.
2. Plot the points $(4,\ 5)$ and $(6,\ 8)$ on the same coordinate system. Then draw the line that contains the points. Finally, find the y-intercept of the line.
3. Recall that a line of the form $y = mx + b$ has slope m and y-intercept $(0,\ b)$. Use the results you found in Problems 1 and 2 to find an equation of the line that contains the points $(4,\ 5)$ and $(6,\ 8)$.
4. Substitute the coordinates of the point $(4,\ 5)$ into the equation $y = \frac{3}{2}x + b$ and solve for b. Then substitute the value for b that you found into the equation $y = \frac{3}{2}x + b$.
5. Explain why it makes sense that the equations you found in Problems 3 and 4 are the same.

Workbook Exploration

The skill of finding an equation using two given points is introduced in the following 15-minute exploration.

EXPLORATION *Finding an equation of a line*

In this exploration, you will use pencil and paper to find an equation of the line passing through the points $(2, 1)$ and $(5, 7)$.

1. Find the slope of the line containing the points $(2, 1)$ and $(5, 7)$.
2. Recall that the equation of a nonvertical line can be put into the form $y = mx + b$. Substitute your value of m from Problem 1 into $y = mx + b$. (If $m = 5$, then $y = 5x + b$, for example.)
3. To find b, recall that any point that lies on the graph of an equation must satisfy that equation. In particular, the point $(2, 1)$ should satisfy the equation from Problem 2. That is—substitute 2 for x and 1 for y, and then solve for b.
4. Write the equation of the line. (If $m = 5$ and $b = 3$, for example, then $y = 5x + 3$.)
5. Use a graphing calculator to verify that the graph of your equation passes through the points $(2, 1)$ and $(5, 7)$.

A few groups will have trouble with Problem 2 of the exploration. In particular, some groups will ignore Problem 2 and write $1 = m(2) + b$ in Problem 3.

Textbook and Workbook Exploration

The 15-minute "Finding equations of lines" exploration is a fun critical-thinking activity. Groups can complete this exploration in a timely fashion if they divide up the work. Or, this exploration is suitable to be assigned as homework.

OBJECTIVE 2 COMMENTS

This concept is optional. See my comments made earlier under "Main points."

SECTION 5.3 LECTURE NOTES

Objectives

1. Find an equation of a linear model by using data described in words.
2. Find an equation of a linear model by using data displayed in a table.

Main point: Find an equation of a linear model.

OBJECTIVE 1

Recall that in Section 5.2 we found equations of lines.

1. The number of justifiable homicides by police officers increased approximately linearly from 398 homicides in 2007 to 442 homicides in 2015 (Source: *FBI*). Let n be the number of justifiable homicides by police officers in the year that is t years since 2000. Find an equation of a linear model to describe the data. Use a graphing calculator to verify your equation.

2. The percentage of American adults who have a "great deal" or a "fair amount" of trust in the mass media has decreased approximately linearly from 43% in 2008 to 32% in 2016 (Source: *Marketingcharts*). Let p be the percentage of American adults who have a "great deal" or a "fair amount" of trust in the mass media at t years since 2000. Find an equation of a linear model to describe the data. Use a graphing calculator to verify your equation.

OBJECTIVE 2

3. The numbers of Advance Placement tests taken in statistics are shown in the following table for various years.

Year	Number of Advanced Placement Tests Taken in Statistics (thousands)
2011	143
2012	154
2013	170
2014	185
2015	196

Source: *Amstatnews*

Let n be the number (in thousands) of Advance Placement tests taken in statistics in the year that is t years since 2010. Find an equation of a linear model to describe the data. Use a graphing calculator to verify your equation.

Finding an Equation of a Linear Model

To find an equation of a linear model, given some data.

1. Construct a scatterplot of the data.
2. Determine whether there is a line that comes close to the data points. If so, choose two points (not necessarily data points) that you can use to find an equation of a linear model.
3. Find an equation of the line.
4. Use a graphing calculator to verify that the graph of your equation contains the two chosen points and comes close to all the points of the scatterplot.

4. In an attempt to raise the low graduation rates of college football and basketball players, prospective athletes must meet new grade point average (GPA) standards to play as freshmen and enrolled athletes must meet to keep playing.

SAT score	Core GPA
820	2.5
860	2.4
900	2.3
940	2.2
970	2.1
1010	2.0

Source: *NCAA*

Let G be the qualifying GPA for an SAT score of s points. Find an equation of a linear model to describe the data in the following table. Use a graphing calculator to verify your equation.

SHORT HW 1, 7, 11, 13, 15, 17, 19, 23, 33, 35

MEDIUM HW 1, 5, 7, 9, 13, 15, 17, 19, 23, 25, 27, 31, 33, 35, 37, 39

SECTION 5.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students learn to use technology to draw a scatterplot of data in this section. Then they use two "good points" to find an equation of a linear model. In Section 5.4, they will use these models to make estimates and predictions. These skills will be used throughout Chapters 5 and 6.

OBJECTIVE 1 COMMENTS

For Problems 1 and 2, I remind students once more of the roles of the explanatory variable and the response variable. It is helpful to organize the information in a table such as Table 6 on page 265 of the textbook. I write the equation $y = mx + b$ and then replace the variables x and y with the appropriate variables.

When modeling, I tell students that most of the time we will round the values of m and b to the second decimal place. Possible exceptions would be if there is a zero in the first decimal place, zeroes in the first and second decimal places, and so on.

Finally, I use a graphing calculator to verify an equation that we have found. The most efficient way to do this is to use a graphing calculator table in “Ask” mode. However, it is important that students can visualize what they have accomplished, so it’s probably best to use TRACE. If you want to avoid discussing window settings, you can use ZoomStat to draw a scatterplot of the two data points.

OBJECTIVE 2 COMMENTS

I begin solving Problem 3 by showing students how to use their graphing calculators (or other technology) to draw a scatterplot of the Advanced Placement test data. It is from this point on that graphing calculator invalid dimension errors will crop up. I describe the meaning of the message “ERR:INVALID DIM” to my students so that they know what to do when the message is displayed (see page 638 of the text). Some students may accidentally delete a column in the STAT editor; I explain how to use “SetUpEditor” to get back the column (see the first margin note on page 633 of the text).

I then number the data points (or use letters) and ask students which pairs of data points lie on a line that comes close to the data points and which pairs do not. Then I select two “good points” and derive an equation of a linear model.

When verifying a model’s equation, I explain that the line should contain the two chosen points (and come close to the other data points). I encourage students to take the time to distinguish between their chosen two points and the other data points. (By the way, most students do not realize that a point that lies close to, but not on a line, may appear to lie on the line on a graphing calculator screen.) For quizzes and exams, I have students copy their graphing calculator screen which displays how well their model fits the data; if I do not do this, some students will resist ever learning how to use a graphing calculator to draw scatterplots and to verify their models.

You can also discuss how to use two nondata points to find a linear model. I do not do this, in part, due to time constraints, but also because using data points seems to work fine in most situations. Also, by not discussing how to use nondata points, you may reduce time spent grading exams as there will be less variation in students’ derived equations for models.

I have included regression equations as answers to exercises simply because I cannot anticipate which two “good” points students will choose. So, when assigning this section’s homework, I warn students that their answers will vary from those in the textbook. I remind them that they can verify their work using a graphing calculator.

Workbook Exploration

The following 15-minute exploration about electric vehicles could be assigned at the start of class. After all, students have already learned how to find an equation of a line using two points in Section 5.2. Most, if not all, groups will forgo constructing a scatterplot and simply choose two data pairs in the table to use to find an equation. This creates the opportunity to point out the danger in doing so during the debrief. In fact, if a group uses the data pairs for 2016 and 2017, you can point to how poorly the model fits all the data points. The workbook exploration “Choosing ‘good points’ to find a model” (included later in these comments) also attends to this.

Most, if not all, groups will ignore that t is defined to be years since 2010 and will in effect, define t to be the year. During the debrief, comparing and contrasting the equations and graphs of the models for the two definitions of t can be very instructive.

Group Exploration

Section Opener: Finding an equation of a linear model

The numbers of electric vehicle models available to consumers in North America are shown in the following table for various years.

Year	Number of Electric Vehicle Models in North America
2011	10
2012	17
2013	24
2014	29
2015	39
2016	44
2017	54

Source: *Bloomberg New Energy Finance*

Let n be the number of electric vehicle models available to consumers in North America at t year since 2010. Find an equation of a linear model.

Workbook Exploration

The following 20-minute exploration about airline passengers provides more structure than the one about electric vehicles, so it's probably less valuable. But it does outline the desired approach. So, having students work on it directly after the electric-vehicle exploration can work well.

Group Exploration

Section Opener: Finding an equation of a linear model

The number of airline passengers worldwide are shown in the following table for various years.

Year	Number of Airline Passengers Worldwide (millions)
2009	2.5
2010	2.7
2011	2.9
2012	3.0
2013	3.2
2014	3.3
2015	3.6
2016	3.8

Source: *ICAO; IATA*

Let n be the number (in millions) of airline passengers worldwide in the year that is t years since 2000.

1. Use a graphing calculator to construct a scatterplot of the data. Copy the screen.
2. Imagine a line that comes close to the data points. Estimate the coordinates of two points that lie on the line. Use the coordinates to find the slope of the line.
3. Substitute the slope you found in Problem 2 for m in the equation $n = mt + b$.
4. Substitute the coordinates of one of the points you identified in Problem 2 into the equation you found in Problem 3 and solve for b.
5. Substitute the value of b you found in Problem 4 into the equation you found in Problem 3. We call such an equation a *linear model*.
6. Predict the number of airline passengers worldwide in 2018 by substituting 18 for t in your linear model.

7. International Air Transport Association predicts that there will be 4.3 million airline passengers worldwide in 2018. Is this prediction greater than, equal to, or less than the prediction you made in Problem 6?

Workbook Exploration

When finding a linear model, groups can discover the impact of their selection of a pair of points by completing the following 15-minute exploration. It is best, however, to first demonstrate how to derive an equation of a linear model before assigning this exploration, so students can focus on the main point of this exploration, which is the selection of a pair of points, not finding the equation.

You may want your students to skip Problem 7 in the exploration to save class time and/or to save time when you grade exams (as I explained before).

Group Exploration

Choosing "good points" to find a model

The revenues of IKEA are shown in the following table for various years. The table includes a first column that indicates a name for each data point. For example, point D refers to the point $(9,\ 21.8)$.

Name of Point	Years since 2000	Revenue (billions of dollars)
A	6	17.5
B	7	20.0
C	8	21.5
D	9	21.8
E	10	23.5
F	11	25.2
G	12	27.6
H	13	28.5
I	14	29.3
J	15	32.7

Source: *IKEA*

1. Let r be the annual revenue (in billions of dollars) at t years since 2000. Use a graphing calculator to draw a scatterplot of the data. Copy the screen.

2. Find an equation of the line that contains the points A and B.

3. Use a graphing calculator to verify that the graph of your equation passes through both points A and B. Copy the screen. Does the line come close to the other data points?

4. If you had used points A and H, you would have found the equation $r = 1.57t + 8.07$. Compare its graph with the graph you drew in Problem 3. Copy the screen. Explain why the graphs look so different.

5. List five pairs of points that yield equations you think would be good linear models. (You do not have to find the equations.)

6. Several pairs of points from a scatterplot yield equations that could serve as models of the data. Discuss how to choose two such data points to find an equation that comes reasonably close to all the data points.

7. It is not necessary to use data points to find an equation of a linear model. While viewing the IKEA scatterplot, use the arrow keys on a graphing calculator to identify two nondata points you feel would yield an equation of a line that is close to the data points. Find an equation of the line that contains these two points. Then use a graphing calculator to verify that the graph of your equation comes close to the data points. Copy the screen.

Textbook and Workbook Exploration

The 15-minute "Adjusting the fit of a model" exploration directs students to discover how to improve the fit of a linear model of the form $y = mx + b$ by using trial and error to adjust the values of m and b.

SECTION 5.4 LECTURE NOTES

Objectives

1. Use equations of linear models to make estimates and predictions.
2. Determine a four-step modeling process.
3. Use data described in words to make predictions.

Main point: Use linear models to make estimates and predictions about authentic situations.

OBJECTIVES 1 and 2

1. The average prices of a pint of yogurt are shown in the following table for various years.

Year	Average Price of a Pint of Yogurt (dollars)
2012	2.03
2013	2.11
2014	2.21
2015	2.23
2016	2.25
2017	2.31

Source: *USDEC, IRI*

Let p be the average price (in dollars) of a pint of yogurt at t years since 2010.

a. Use a graphing calculator to draw a scatterplot of the data.
b. Find an equation of a linear model to describe the data.
c. What is the slope? What does it mean in this situation?
d. Predict the average price in 2022.
e. Predict when the price will be $2.50.
f. What is the t-intercept? What does it mean in this situation?

2. The Department of Motor Vehicles (DMV) states that "There is no safe way to drive while under the influence [of alcohol]. Even one drink can make you an unsafe driver." The following table lists, for various weight groups, how many drinks a driver 21 years or older must drink to be likely to be driving under the influence (DUI) if it has been four hours since the first drink. (Even drivers who have had fewer drinks than those listed in the table may be DUI.)

Weight Group (pounds)	Weight Used to Represent Weight Group (pounds)	Number of Drinks Group	Number of Drinks Used to Represent Group
90–109	99.5	2	2
110–129	119.5	3	3
130–149	139.5	3	3
150–169	159.5	4	4
170–189	179.5	4–5	4.5
190–209	199.5	5	5

Source: *DMV*

Let n be the number of drinks needed for it to be likely that a driver who weighs w pounds is DUI, if it has been four hours since the person's first drink.

a. Use a graphing calculator to draw a scatterplot of the data.

b. Find an equation of a linear model to describe the data.

c. What is the slope? What does it mean in this situation?

d. Four hours after the first of 6 drinks, it is highly likely that a specific driver is DUI. According to the model, what is the greatest weight of the driver?

e. The heaviest person in the world was reported to be Jon Brower Minnoch, who weighed 1400 pounds. If Jon had drank for four hours, after how many drinks would it be highly likely that he had been under the influence?

Using an Equation of a Linear Model to Make Predictions

- To make a prediction about the response variable of a linear model, substitute a chosen value for the explanatory variable in the model's equation. Then solve for the response variable.
- To make a prediction about the explanatory variable of a linear model, substitute a chosen value for the response variable in the model's equation. Then solve for the explanatory variable.

Using an Equation of a Linear Model to Find Intercepts

If an equation of the form $p = mt + b$, where $m \neq 0$, is used to model a situation, then

- The p-intercept is $(0, b)$.
- To find the t-coordinate of the t-intercept, substitute 0 for p in the model's equation and then solve for t.

Four-Step Modeling Process

To find a linear model and then make estimates and predictions,

1. Construct a scatterplot of the data to determine whether there is a nonvertical line that comes close to the points. If so, choose two points (not necessarily data points) that you can use to find an equation of a linear model.
2. Find an equation of your model.
3. Verify your equation by checking that the graph of your model contains the two chosen points and comes close to all the data points.
4. Use the equation of your model to make estimates, make predictions, and draw conclusions.

OBJECTIVE 3

3. The total mail volume of the U.S. Postal Service has decreased from 159.8 billion units in 2012 to 149.5 billion units in 2017 (Source: *U.S. Postal Service*). Predict when the total mail volume will be 140 billion units.

4. The number of unruly airline passengers in the United States has decreased approximately linearly from 150 passengers in 2012 to 51 passengers in 2017 (Source: *Federal Aviation Administration*). Predict in which year there will be 10 unruly airline passengers in the United States.

If an exercise does not state a year from which to begin counting, you may choose any year for that purpose.

SHORT HW 1, 5, 7, 17, 21, 23, 25, 29

MEDIUM HW 1, 5, 7, 11, 13, 17, 21, 23, 25, 27, 29, 33, 35

SECTION 5.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use linear models to make estimates and predictions about authentic situations. They will use this material in Sections 5.5, 6.1, and 6.4.

OBJECTIVES 1 and 2 COMMENTS

For parts (c), (d), (e), and (f) of Problem 1, I remind my students to write complete sentences for their responses.

For part (c) of Problem 1, some students need to be reminded that slope is a rate of change. I also encourage students to write something specific such as "The average price of a pint of yogurt increases by $0.053 per year." instead of something vague such as "It increases."

For part (f) of Problem 1, I remind my students that we can find the t-intercept by substituting 0 for the other variable (p). I also remind them that an intercept has two coordinates (not one), and that we list the value of the explanatory variable as the first coordinate and list the value of the response variable as the second coordinate.

OBJECTIVE 3 COMMENTS

For Problems 3 and 4, I tell my students that it is left to them to define the variables.

SECTION 5.5 LECTURE NOTES

Objectives

1. Describe the meaning of *inequality symbols* and *inequality*.
2. Graph an inequality.
3. Apply the properties of inequalities.
4. Describe the meaning of *satisfy*, *solution*, and *solution set* for a *linear inequality in one variable*.
5. Solve a linear inequality in one variable, and graph the solution set.
6. Use linear inequalities to make predictions about authentic situations.

Main point: Solve linear inequalities in one variable.

OBJECTIVE 1

Here are the meanings of the **inequality symbols** $<$, $\leq$, $>$, and $\geq$ and some examples of *inequalities:*

Symbol	Meaning	Examples of Inequalities
$<$	less than	$3 < 8, \quad 0 < 3, \quad -5 < -1$
$\leq$	less than or equal to	$3 \leq 4, \quad 5 \leq 5, \quad -4 \leq 0$
$>$	greater than	$6 > 3, \quad -5 > -9, \quad 3 > 0$
$\geq$	greater than or equal to	$5 \geq 1, \quad 4 \geq 4, \quad -3 \geq -7$

- An **inequality** contains one of the symbols $<$, $\leq$, $>$, and $\geq$ with expressions on both sides.
- Some examples: $x < 9$, $2x - 7 \leq 5$, $3x - 5 > 4x + 1$, $9 \geq 6$

Decide whether the inequality is true or false.

1. $3 < 7$ **2.** $-6 \geq -2$ **3.** $8 > 8$ **4.** $6 \leq 6$

OBJECTIVE 2

Describe the inequality using words, a graph, and interval notation.

5. $x < 2$ **6.** $x \leq 2$ **7.** $x > 2$ **8.** $x \geq 2$

OBJECTIVE 3

Show what happens when we add a number to both sides of $5 < 7$.

Addition Property of Inequalities

If $a < b$, then $a + c < b + c$.

Similar properties hold for $\leq$, $>$, and $\geq$.

- Illustrate the addition property of inequalities by using number lines.
- Show what happens when we multiply both sides of the inequality $5 < 7$ by a number.

Multiplication Property of Inequalities

- For a *positive* number c, if $a < b$, then $ac < bc$.
- For a *negative* number c, if $a < b$, then $ac > bc$.

Similar properties hold for $\leq$, $>$, and $\geq$.

Illustrate that if $a < b$, then $-a > -b$ by using a number line.

OBJECTIVE 4

Definition *Linear inequality in one variable*

A **linear inequality in one variable** is an inequality that can be put into one of the forms

$$mx + b < 0, \qquad mx + b \leq 0, \qquad mx + b > 0, \qquad mx + b \geq 0$$

where m and b are constants and $m \neq 0$.

We say a number **satisfies** an inequality in one variable if the inequality becomes a true statement after we have substituted the number for the variable.

9. Does the number 2 satisfy the inequality $4x - 3 < 8$?

10. Does the number 5 satisfy the inequality $4x - 3 < 8$?

Definition *Solution, solution set,* and *solve* for an inequality in one variable

We say a number is a **solution** of an inequality in one variable if it satisfies the inequality. The **solution set** of an inequality is the set of all solutions of the inequality. We **solve** an inequality by finding its solution set.

OBJECTIVE 5

Solve the inequality. Describe the solution set as an inequality, in interval notation, and in a graph.

11. $-4x < 24$

12. $k + 3 \geq 5$

13. $-3x + 1 < -11$

14. $4m + 3 > 2(3m - 4)$

15. $\dfrac{2}{9} - \dfrac{5}{3}x \leq \dfrac{7}{3}$

16. $\dfrac{2p - 1}{3} - \dfrac{7p + 3}{4} \geq \dfrac{5}{6}$

OBJECTIVE 6

17. The total annual box office grosses in the United States and Canada are shown in the following table for various years.

Year	Total Box Office Grosses in the United States and Canada (billions of dollars)
1990	5.02
1995	5.27
2000	7.51
2005	8.82
2010	10.58
2015	11.12
2016	11.37

Sources: *AC Nielsen EDI, Rentrak Corporation*

Let B be the total annual box office grosses (in billions of dollars) in the United States and Canada at t years since 1990.

a. Find a linear model to describe the data.

b. In which years will the annual box office gross be more than \$13 billion?

SHORT HW 1, 13, 15, 19, 29, 45, 53, 63, 67, 73, 75, 81, 88

MEDIUM HW 1, 5, 7, 13, 15, 19, 25, 29, 39, 45, 53, 63, 67, 73, 75, 79, 81, 83, 88, 89, 91, 95, 97, 99

SECTION 5.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students solve linear inequalities in this section. Students will graph linear inequalities in two variables and solve systems of linear inequalities in Section 6.6.

OBJECTIVE 1 COMMENTS

For Problem 4, some students think that $6 \leq 6$ is a false statement, because 6 is not less than itself.

OBJECTIVE 2 COMMENTS

For Problems 5–8, I try to clear up as many questions as I can so that students can focus on the symbolic work when we reach Problems 11–16.

OBJECTIVE 3 COMMENTS

I emphasize that when we multiply or divide both sides of an inequality by a negative number, we reverse the inequality symbol. Students need several reminders about this.

Textbook and Workbook Exploration

In the 20-minute “Properties of inequalities” exploration, groups can discover properties of inequalities.

OBJECTIVE 4 COMMENTS

I observe with my students that the meanings of the words *satisfy*, *solution*, and *solution set* are similar for equations and inequalities.

OBJECTIVE 5 COMMENTS

I point out that solving a linear inequality is similar to solving a linear equation. However, I continue to heavily emphasize reversing the inequality symbol, when appropriate. When saying something like, “Here we *divide* both sides by *negative* 5," I'll raise my voice dramatically when I say "divide" and "negative." Although this is cheesy, my students get the point that they need to be vigilant about reversing the inequality symbol, when appropriate.

Many students do not understand the significance of the solution set of an inequality in one variable. For example, consider the inequality $-3x + 1 < -11$, whose solution is $x > 4$ (see Problem 13). It is instructive for students to see that numbers greater than 4 such as 4.1, 5, and 6 satisfy the inequality and numbers less than or equal to 4 such as 2, 3, and 4 do not satisfy the inequality. Exercises 83 and 84 of Homework 5.5 address this issue.

Workbook Exploration

Another way for students to learn the meaning of the solution set of an inequality is for them to complete the following 10-minute exploration.

Group Exploration

Meaning of the solution set of an inequality

We solve the inequality $-3x + 7 < 1$:

$$\begin{aligned} -3x + 7 &< 1 \\ -3x + 7 - 7 &< 1 - 7 \\ -3x &< -6 \\ \frac{-3x}{-3} &> \frac{-6}{-3} \\ x &> 2 \end{aligned}$$

1. Choose a number greater than 2. Check that your number satisfies the inequality $-3x + 7 < 1$.

2. Choose two more numbers greater than 2. Check that both of these numbers satisfy the inequality $-3x + 7 < 1$.

3. Choose three numbers that are not greater than 2. Show that each of these numbers does not satisfy the inequality $-3x + 7 < 1$

4. Explain what it means when we write $x > 2$ as the last step in solving the inequality $-3x + 7 < 1$.

OBJECTIVE 6 COMMENTS

To solve part (b) of Problem 17, I first translate the phrase "annual box office gross be more than $13 billion:"

$$B > 13$$

Then I substitute $0.27t + 4.69$ for B:

$$0.27t + 4.69 > 13$$

Many students mistakenly use equality symbols instead of inequality symbols; this is partly due to lack of reading comprehension and partly due to sloppiness in notation. Even more students have trouble interpreting the meaning of an inequality in terms of the situation. For Problem 17, students would tend to say the inequality $t > 30.78$ means that the annual box office gross will be more than $13 billion *in* 2021, rather than *after* 2021. I warn my students repeatedly about this issue. I also sketch a graph of the model and discuss the situation in detail. Unfortunately, many students still have trouble with these issues on exams.

Workbook Exploration

The following 20-minute exploration is one step closer to being authentic because groups have much more freedom than in typical problems: They can choose from three pairs of models and two types of inequalities (less than or greater than). This can make for a great debrief. If you are pressured for time, you could have groups skip Problem 1.

Group Exploration

Forming questions that can be answered by using systems and inequalities

The percentages of American adults who live in lower-income, middle-income, or upper-income households are shown in the following table for various years.

	Percent		
Year	Lower Income	Middle Income	Upper Income
1971	25	61	14
1981	26	59	15
1991	27	56	17
2001	27	54	18
2011	29	51	20
2015	29	50	21

Source: *Pew Research Center*

1. Form a question that can be solved by solving a system of linear equations. Then respond to the question by finding the system and solving it.
2. Form a question that can be solved by solving a linear inequality in one variable. Then respond to the question by forming the inequality and solving it.

CHAPTER 6 OVERVIEW

In Section 6.1, students use graphing to solve systems of linear equations, which are sometimes linear models. They do this by hand and by using technology. In Section 6.2, students solve such systems using substitution and use graphing to solve equations in one variable. Students use elimination to solve systems in Section 6.3. In Section 6.4, students use systems of two models to make estimates and predictions. In Section 6.5, they solve perimeter, value, interest, and mixture problems. In Section 6.6, students solve linear inequalities in two variables and solve systems of linear inequalities in two variables.

SECTION 6.1 LECTURE NOTES

Objectives

1. Describe a *system of linear equations in two variables*.
2. Use a graphical approach to solve systems of linear equations.
3. Use a graphing calculator to solve systems of linear equations.
4. Use graphing to make predictions about situations that can be modeled by two linear models.
5. Describe the three types of linear systems of two equations.
6. Find the solution of a system of linear equations from tables of solutions of such equations.

Main point: Solve *systems of linear equations* by graphing.

OBJECTIVE 1

A **system of linear equations in two variables** or a **linear system** for short, consists of two or more linear equations in two variables. Here is an example:

$$\begin{aligned} y &= 2x - 1 \\ y &= -x + 5 \end{aligned}$$

1. Find all ordered pairs that satisfy both of the equations in the preceding system.

Definition *Solution, solution set,* and *solve* for a system

We say an ordered pair (a, b) is a **solution** of a system of two equations in two variables if it satisfies both equations. The **solution set** of a system is the set of all solutions of the system. We **solve** a system by finding its solution set.

The solution set of a system of two linear equations can be found by locating any intersection point(s) of the graphs of the two equations.

OBJECTIVE 2

Solve the system.

2.

$$\begin{aligned} y &= 2x + 7 \\ y &= -3x - 3 \end{aligned}$$

3.

$$\begin{aligned} 2x - 5y &= 15 \\ y &= -\frac{3}{5}x + 2 \end{aligned}$$

OBJECTIVE 3

4. Use "intersect" on a graphing calculator to solve the system.

$$\begin{aligned} y &= -0.57x - 6.28 \\ y &= 1.45x + 1.31 \end{aligned}$$

OBJECTIVE 4

5. The numbers of subscribers of cable TV and satellite, Netflix, and Hulu are shown in the following table for various years.

Year	Number of Subscribers (millions)		
	Cable TV and Satellite	Netflix	Hulu
2011	101	22	1
2012	101	27	3
2013	99	33	5
2014	98	39	6
2015	97	45	10

Source: *TDG Research, scbei*

a. Let n be the number (in millions) of cable TV and satellite subscribers at t years since 2010. Find an equation of a model to describe the data.

b. Let n be the number (in millions) of Netflix subscribers at t years since 2010. Find an equation of a model to describe the data.

c. Use “intersect” on a graphing calculator to predict when the number of subscribers of cable TV and satellite will be equal to the number of subscribers of Netflix. What is that number of subscribers?

Intersection Point of the Graphs of Two Models

If the explanatory variable of two models represents time, then an intersection point of the graphs of the two models indicates when the quantities represented by the response variables were or will be equal.

OBJECTIVE 5

6. Solve the system.

$$y = \frac{1}{4}x + 2$$
$$y = \frac{1}{4}x + 3$$

A linear system whose solution set is the empty set is called an **inconsistent system**.

7. Solve the system.

$$y = 3x - 4$$
$$y = 3x - 4$$

- A linear system that has an infinite number of solutions is called a **dependent system**.
- A linear system that has one solution is called a *one-solution system*.

Types of Linear Systems

There are three types of linear systems of two equations:

1. One-solution system: The lines intersect in one point. The solution of the system is the ordered pair that corresponds to that point. [Draw a figure.]
2. Inconsistent system: The lines are parallel. The solution set of the system is the empty set. [Draw a figure.]
3. Dependent system: The lines are identical. The solution set of the system is the set of the infinite number of solutions represented by points on the same line. [Draw a figure.]

Solve the system. If the system is inconsistent or dependent, say so.

8. [dependent]

$$\begin{aligned} y &= 2x - 5 \\ 4x - 2y &= 10 \end{aligned}$$

9. [inconsistent]

$$\begin{aligned} 2x - 6y &= 12 \\ -3x + 9y &= -27 \end{aligned}$$

OBJECTIVE 6

10. Use Table 2.1 to solve the system

$$\begin{aligned} y &= -2x + 11 \\ y &= 3x - 4 \end{aligned}$$

Table 2.1: Some Solutions of $y = -2x + 11$ and $y = 3x - 4$

x	0	1	2	3	4	5
$y = -2x + 11$	11	9	7	5	3	1
$y = 3x - 4$	-4	-1	2	5	8	11

SHORT HW 3, 5, 7, 15, 19, 23, 31, 35, 37, 41, 45, 47, 53, 54

MEDIUM HW 3, 5, 7, 9, 15, 19, 23, 27, 31, 35, 37, 39, 41, 43, 45, 47, 49, 53, 54, 55, 59, 61

SECTION 6.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students use graphing and tables to solve systems of linear equations in this section. They use pencil and paper to solve some systems and technology to solve others. In particular, technology is used to solve systems of two linear models. The concepts in this section are relevant to the rest of Chapter 6.

OBJECTIVE 1 COMMENTS

When doing Problem 1, I remind students of the meaning of a graph. This will lead to finding the intersection point of the two lines. I drive this point home by selecting a point that lies on both lines, one line, or neither line and checking whether the corresponding ordered pair satisfies both equations, one equation, or neither equation, respectively.

Background Information for the Exploration That Follows

In the following 10-minute exploration, students discover why the intersection point of two nonparallel lines corresponds to the solution of the system of equations of the lines. To prepare students for this exploration, I show them how to use "intersect" on a graphing calculator. Groups' responses to Problems 2 and 3 in the exploration will reveal whether or not they remember the meaning of a graph. If they have forgotten, I emphasize this concept.

EXPLORATION *Intersection points satisfy both equations*

1. Use "intersect" on a graphing calculator to find the intersection point of the graphs of the following equations:

$$y = 3x - 7, \text{ and}$$
$$y = -2x + 8$$

(See Section A.17 for calculator instructions.)

2. Show that the ordered pair corresponding to the intersection point satisfies both equations. Explain why this makes sense.

3. Are there points other than the intersection point whose ordered pairs satisfy both equations? If so, find them. If not, explain why not.

OBJECTIVE 2 COMMENTS

When some students attempt to solve a system, they sketch the appropriate graphs but do not indicate the solution. I tell students that they must write a sentence describing the solution. Students will have an easy time with Problem 2. For Problem 3, I emphasize that it helps to first solve $2x - 5y = 15$ for y.

Workbook Exploration

In the following 10-minute exploration, students discover why the intersection point of two nonparallel lines corresponds to the solution of the system of equations of the lines. Groups' responses to Problems 2–5 in the exploration will reveal whether or not they remember the meaning of a graph. If they have forgotten, I emphasize this during the debrief.

Group Exploration

Section Opener: Using graphs to solve systems

1. Graph the equations $y = 2x - 1$ and $y = -x + 5$.

2. Find three ordered pairs that satisfy $y = 2x - 1$ but do not satisfy $y = -x + 5$.

3. Find three ordered pairs that satisfy $y = -x + 5$ but do not satisfy $y = 2x - 1$.

4. Find three ordered pairs that satisfy neither of the equations $y = 2x - 1$ and $y = -x + 5$.

5. Find all ordered pairs that satisfy both of the equations $y = 2x - 1$ and $y = -x + 5$.

OBJECTIVE 3 COMMENTS

For Problem 4, students get a kick out of learning how to use "intersect" on a graphing calculator to solve a linear system.

OBJECTIVE 4 COMMENTS

For Problem 5, most students need help with finding appropriate WINDOW settings. One way to find such settings is to use ZStandard, then use ZoomFit, and finally Zoom Out. Another way is to use ZoomStat to draw the two scatterplots of the data (in the same viewing screen), then Zoom Out twice, and finally use ZoomFit. Some students need to be reminded to find the equal number of subscribers as well as the year.

Workbook Exploration

You could start class with the following 20-minute exploration, which is an excellent way to begin this section. Groups may come up with a wide variety of approaches: numerical, graphical, and symbolic. Comparing and contrasting groups' approaches can lay a great foundation for the entire chapter.

Group Exploration

Section Opener: Estimating when quantities were equal

The total worldwide revenues from television, radio, and multimedia, and the worldwide revenues from telecommunication devices are shown in the following table for various years. Estimate when the total revenue from television, radio, and multimedia was equal to the revenue from telecommunication devices. What was that revenue?

Year	Revenue (billions of dollars) Television, Radio, and Multimedia	Telecommunication Devices
2011	56.3	65.5
2012	52.0	69.8
2013	47.5	73.6
2014	45.5	77.2
2015	44.0	79.9

Source: *Statista*

Textbook and Workbook Exploration

The 15-minute "Using a system to model a situation" exploration is an excellent way to address Objective 3. For professors who prefer an application-driven approach, this exploration can be used to introduce this section. Groups are already accustomed to working with linear models so they should easily progress through most of this exploration. Before groups start working on this exploration, I show them how to use "intersect" on a graphing calculator. In Problem 4, most students will need help to find appropriate WINDOW settings.

OBJECTIVE 5 COMMENTS

When solving Problems 6 and 7, I remind students of the meaning of a graph and the meaning of a solution of a system.

To ensure that my students become proficient at solving inconsistent and dependent systems in this section and Sections 6.2 and 6.3, I carefully and repeatedly discuss the relevant concepts. Students have trouble keeping straight the solutions and the names of these two types of systems.

Textbook and Workbook Exploration

In the 20-minute "Three types of systems" exploration, groups can discover the three types of systems. I encourage them to try to do Problems 5 and 6 of the exploration without graphing the equations.

OBJECTIVE 6 COMMENTS

Students will pick up on this concept quickly. Doing Problem 10 will set the stage for students to solve the more challenging Exercises 45 and 46 of Homework 6.2 and Exercises 55 and 56 of Homework 6.3. If you are tight on time, you could skip this method, because it is much more effective to use graphing, substitution, or elimination to solve linear systems.

SECTION 6.2 LECTURE NOTES

Objectives

1. Use substitution to solve a system of two linear equations.
2. Isolate a variable in an equation to help solve a system by substitution.
3. Solve inconsistent and dependent systems by substitution.

Main point: Use substitution to solve a system of two linear equations.

OBJECTIVE 1

Use substitution to solve the system.

1.
$$\begin{aligned} -3x + 2y &= 9 \\ y &= 4x + 7 \end{aligned}$$

2.
$$\begin{aligned} x &= 4y + 1 \\ -5x - 2y &= 17 \end{aligned}$$

3.
$$\begin{aligned} -4x + 3y &= -6 \\ x &= 5y - 7 \end{aligned}$$

4.
$$\begin{aligned} y &= 1 - 3x \\ 4x + 3y + 2 &= 0 \end{aligned}$$

Using Substitution to Solve a Linear System

To use **substitution** to solve a system of two linear equations,

1. Isolate a variable to one side of either equation.
2. Substitute the expression for the variable found in step 1 into the other equation.
3. Solve the equation in one variable found in step 2.
4. Substitute the solution found in step 3 into one of the original equations, and solve for the other variable.

A system of equations can be solved by substitution as well as by graphing. The methods give the same result. Use substitution to solve the system, with coordinates of solutions rounded to the second decimal place.

5.
$$\begin{aligned} y &= 1.25x - 7.49 \\ y &= -3.18x + 8.66 \end{aligned}$$

6.
$$\begin{aligned} y &= -1.42x - 6.13 \\ y &= 2.68x - 2.75 \end{aligned}$$

OBJECTIVE 2

Use substitution to solve the system.

7.
$$\begin{aligned} x - 3y &= 7 \\ -4x + 5y &= -14 \end{aligned}$$

8.
$$\begin{aligned} 4x - 3y &= 3 \\ -2x + y &= -5 \end{aligned}$$

9.
$$\begin{aligned} 2x - y &= 1 \\ 5x - 3y &= 5 \end{aligned}$$

OBJECTIVE 3

10. What happens when you solve the following inconsistent system by substitution?

$$y = 2x + 3$$
$$y = 2x + 4$$

Solving Inconsistent Systems by Substitution

If the result of applying substitution to a system of equations is a false statement, then the system is inconsistent; that is, the solution set is the empty set.

11. What happens when we solve the following dependent system by substitution? [We solved this system by graphing when we did Problem 7 in the lecture notes of Section 6.1.]

$$y = 2x - 5$$
$$4x - 2y = 10$$

Solving Dependent Systems by Substitution

If the result of applying substitution to a linear system of two equations is a true statement that can be put into the form $a = a$, then the system is dependent; that is, the solution set is the set of ordered pairs represented by every point on the (same) line.

Use substitution to solve the system.

12. [inconsistent]

$$y = 4 - 3x$$
$$15x + 5y = 17$$

13. [dependent]

$$-5x + 20y = 25$$
$$x = 4y - 5$$

14. [dependent]

$$x = 2y - 1$$
$$4x - 8y = -4$$

15. [inconsistent]

$$y = 2x - 5$$
$$-4x + 2y = -6$$

SHORT HW 1, 7, 13, 19, 25, 31, 35, 39, 45, 47, 49, 53, 55

MEDIUM HW 1, 7, 11, 13, 15, 17, 19, 25, 31, 35, 39, 45, 47, 49, 53, 55, 57, 59

SECTION 6.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use substitution to solve systems of linear equations. They will use these skills in Sections 6.3, 6.4, and 6.5.

OBJECTIVE 1 COMMENTS

For Problem 1, I say that for any solution of the system, the value of y is equal to the value of $4x + 7$. So, we substitute $4x + 7$ for y in the equation $-3x + 2y = 9$.

It is a common student error to find only one coordinate of a solution of a system. I remind my students that we found both coordinates of solutions of systems in Section 6.1.

Problems 4 and 6 are excellent preparation for Section 6.4, where students will solve systems of linear models.

Workbook Exploration

The following 15-minute exploration is an effective way to introduce substitution because students take to this method quickly.

Group Exploration

Section Opener: Using substitution to solve systems

1. Use graphing to solve the system

$$\begin{aligned} y &= 3x - 5 \\ 2x + y &= 5 \end{aligned}$$

2. Substitute $3x - 5$ for y in the equation $2x + y = 5$ and then solve for x. How is your result similar to the result you found in Problem 1?

3. Substitute the result you found for x in Problem 2 into the equation $y = 3x - 5$ and solve for y. Also substitute it for x in the equation $2x + y = 5$ and solve for y. Explain why it makes sense that your two results are equal. How do your equal results relate to the system in Problem 1? Explain why this makes sense.

Textbook and Workbook Exploration

The 25 minute "Comparing techniques for solving systems" exploration has groups compare using substitution with using graphing to solve systems. If you are tight on time, you could skip this exploration to save time for assigning the exploration in Section 6.3, which has groups compare three techniques (elimination, substitution, and graphing) to solve systems.

OBJECTIVE 2 COMMENTS

Students have an easy time with Problems 7–9. Exercises 33 and 34 of Homework 6.2 are challenging, because using substitution will require working with fractions. However, these exercises have not been included in the recommended homework assignments, because it's probably better to use elimination than substitution for these types of systems.

OBJECTIVE 3 COMMENTS

Students have a tough time with this objective. I spend a good amount of time discussing why we arrive at false statements when we use substitution to solve inconsistent systems and arrive at true statements of the form $a = a$ when we use substitution to solve dependent systems. I point out that if a student forgets what it means to get either of these types of statements, the student can sort out what is going on by solving both of the original equations for y.

I carefully distinguish between the types of systems (e.g. inconsistent) and the solution sets of systems (e.g. empty set).

Background Information for the Exploration That Follows

The following 15-minute exploration has students discover what happens when we use substitution to solve an inconsistent system or a dependent system.

EXPLORATION *Solving dependent, inconsistent, and one-point solution systems*

1. **a.** Is the following system a dependent system, an inconsistent system, or a one-solution system? Explain.

$$\begin{aligned} y &= 3x - 4 \\ y &= 3x - 5 \end{aligned}$$

b. Use substitution to solve the system. What happens?

c. Will this always happen with inconsistent systems? Test your theory.

d. What happens when you use "intersect" on a graphing calculator to try to solve the system?

2. **a.** Is the following system a dependent system, an inconsistent system, or a one-solution system? Explain.

$$y = 2x + 1$$
$$y = 2x + 1$$

b. Use substitution to solve the system. What happens?

c. Will this always happen with dependent systems? Test your theory.

d. What happens when you use "intersect" on a graphing calculator to try to solve the system?

3. Summarize your findings. What will happen when you use substitution to solve an inconsistent system? a dependent system? a one-solution system?

Although groups may give correct responses to the question "What happens?" in parts 1(b) and 2(b), they may still not get the entire point of this exploration. For example, if a group's result is $0 = 0$ for part 2(b), they may not leap to the generalization that dependent systems will result in true statements of the form $a = a$. Parts 1(c) and 2(c) are meant to have students reflect on their responses to parts 1(b) and 2(b) with a wider scope, but groups may not think of these types of generalizations, especially in part 2(c).

SECTION 6.3 LECTURE NOTES

Objectives

1. Use elimination to solve a system of two linear equations.
2. Solve inconsistent systems and dependent systems by elimination.
3. Compare three ways to solve a system.

Main point: Use elimination to solve systems of two linear equations.

OBJECTIVE 1

If we add the left sides and add the right sides of the equations $2 = 2$ and $4 = 4$, we obtain the true statement $2 + 4 = 2 + 4$.

Adding Left Sides and Right Sides of Two Equations

If $a = b$ and $c = d$, then

$$a + c = b + d$$

Solve the system by elimination.

1.

$$\begin{aligned} 5x - 2y &= -16 \\ -5x + 4y &= 22 \end{aligned}$$

2.

$$\begin{aligned} -5x + 4y &= 7 \\ 7x - 4y &= -5 \end{aligned}$$

3.

$$\begin{aligned} 2x + 5y &= 2 \\ x - 4y &= -12 \end{aligned}$$

4.

$$\begin{aligned} 4x - 3y &= 5 \\ 3x + y &= 7 \end{aligned}$$

5.

$$\begin{aligned} 4x - 3y &= 13 \\ 7x + 5y &= -8 \end{aligned}$$

6.

$$\begin{aligned} 2x - 3y &= 7 \\ 5x - 7y &= 17 \end{aligned}$$

Using Elimination to Solve a Linear System

To use **elimination** to solve a system of two linear equations,

1. Use the multiplication property of equality (Section 4.3) to get the coefficients of one variable to be equal in absolute value and opposite in sign.
2. Add the left sides and add the right sides of the equations to eliminate one of the variables.
3. Solve the equation in one variable found in step 2.
4. Substitute the solution found in step 3 into one of the original equations, and solve for the other variable.

Use elimination to solve the system, with coordinates of solutions rounded to the second decimal place.

7.

$$\begin{aligned} y &= 2.18x - 9.66 \\ y &= -4.57x + 8.05 \end{aligned}$$

8.

$$\begin{aligned} y &= -1.72x + 5.34 \\ y &= -2.39x + 8.76 \end{aligned}$$

Solve the system by elimination.

9.

$$\begin{aligned} \frac{1}{5}x - \frac{1}{2}y &= \frac{2}{5} \\ 4x - 3y &= -6 \end{aligned}$$

10.

$$\begin{aligned} \frac{1}{5}x + \frac{3}{2}y &= 7 \\ \frac{2}{5}x - \frac{9}{2}y &= -16 \end{aligned}$$

OBJECTIVE 2

Solving Inconsistent Systems and Dependent Systems by Elimination

If the result of applying elimination to a linear system of two equations is

- a false statement, the system is inconsistent; that is, the solution set is the empty set.
- a true statement that can be put into the form $a = a$, then the system is dependent; that is, the solution set is the set of ordered pairs represented by every point on the (same) line.

Solve the system by elimination. If the system is inconsistent or dependent, say so.

11. [dependent]

$$\begin{aligned} -4x + 6y &= -10 \\ 6x - 9y &= 15 \end{aligned}$$

12. [inconsistent]

$$\begin{aligned} 6x + 2y &= 6 \\ -15x - 5y &= -10 \end{aligned}$$

13. [inconsistent]

$$\begin{aligned} 4x - 2y &= 8 \\ 10x - 5y &= 17 \end{aligned}$$

14. [dependent]

$$\begin{aligned} -4x + 12y &= -8 \\ 5x - 15y &= 10 \end{aligned}$$

OBJECTIVE 3

Any linear system of two equations can be solved by graphing, substitution, or elimination. All three methods give the same result.

15. Solve the system of equations three times, once by each of the three methods—graphing by hand, substitution, and elimination. Decide which method you prefer for this system. Explain.

$$\begin{aligned} 4x - y &= 8 \\ 5x - 3y &= 3 \end{aligned}$$

SHORT HW 1, 5, 13, 15, 19, 27, 29, 31, 39, 41, 49, 51, 55, 57

MEDIUM HW 1, 5, 7, 11, 13, 15, 17, 19, 27, 29, 31, 39, 41, 47, 49, 51, 53, 55, 57, 59, 63, 65, 67

SECTION 6.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use elimination to solve systems of linear equations. They will use this skill in Sections 6.4 and 6.5.

OBJECTIVE 1 COMMENTS

While solving the system in Problem 1, I emphasize that things worked out so nicely because the coefficients of y in the original system are equal in absolute value and opposite in sign.

For Problem 3, I ask my students how we can make the coefficients of x to be equal in absolute value and opposite in sign.

I compare solving the system in Problem 5 by eliminating y with finding the sum $\frac{1}{3} + \frac{1}{5}$ (both involve finding the LCM of 3 and 5).

For Problems 7 and 8, some students are surprised that they can use elimination to solve the systems. These problems can be good preparation for working with systems in Section 6.4; however, most students will prefer to use substitution in Section 6.4.

Most students have difficulties solving the systems in Problems 9 and 10.

Textbook and Workbook Exploration

The 15-minute "Comparing techniques of solving systems" exploration has students compare the techniques of solving systems. By completing this exploration, students not only reflect on which technique is best suited to solve a given system, but they also see that each technique gives the same result.

Background Information for the Exploration That Follows

In the following 20-minute exploration, students practice using elimination and reflect on issues related to using elimination and substitution.

EXPLORATION *Solving systems using elimination*

1. Consider the system

$$x - 3y = 2$$
$$2x + 5y = 15$$

- **a.** Solve this system using elimination by first getting the coefficients of x to be equal in absolute value and opposite in sign.
- **b.** Verify that your solution to the system is correct by checking that it satisfies both equations in the system.
- **c.** Explain why it is more efficient to eliminate x rather than y in solving this system.
- **d.** Discuss whether it is more convenient to use elimination or substitution to solve this system.

2. Consider the system

$$25x - 4y = 1$$
$$37x - 6y = 1$$

- **a.** Solve this system using elimination by first getting the coefficients of y to be equal in absolute value and opposite in sign.
- **b.** Verify that your solution is correct by checking that it satisfies both equations in the system.
- **c.** Discuss which variable would be easiest to eliminate, x or y.
- **d.** Discuss whether it is more efficient to use elimination or substitution to solve this system.

OBJECTIVE 2 COMMENTS

Most students continue to be greatly challenged by dependent systems and inconsistent systems such as those in Problems 11–14. Most students arrive at a true or false statement and are unsure what this means.

Workbook Exploration

The following 15-minute exploration has students discover what happens when using substitution or elimination to solve an inconsistent system or a dependent system.

EXPLORATION *Solving dependent, inconsistent, and one-point solution systems*

1. Is the following system a dependent system, an inconsistent system, or a one-solution system? Explain.

$$y = 2x + 1$$
$$y = 2x - 4$$

- **a.** Solve the system using either elimination or substitution. What happens?
- **b.** Will this always happen with inconsistent systems? Test your theory.
- **c.** What happens when you use “intersect” on a graphing calculator to try to solve the system?

2. Is the following system a dependent system, an inconsistent system, or a one-solution system? Explain.

$$y = 2x - 3$$
$$y = 2x - 3$$

- **a.** Solve the system using either elimination or substitution. What happens?
- **b.** Will this always happen with dependent systems? Test your theory.
- **c.** What happens when you use “intersect” on a graphing calculator to try to solve the system?

3. Summarize your findings. What will happen when you solve an inconsistent system? a dependent system? a one-solution system?

OBJECTIVE 3 COMMENTS

I emphasize that any linear system of two equations can be solved by graphing, substitution, or elimination, and that all three methods will give the same result.

If I do not have time to use each of the three methods to solve the system in Problem 15, I quickly discuss what’s involved with each method and then ask my students which method might be the best to use.

SECTION 6.4 LECTURE NOTES

Objectives

1. Use substitution and elimination to make predictions about situations described by a tables of data.
2. Use substitution and elimination to make predictions about situations described by rates of change.

Main point: Use substitution and elimination to make predictions about authentic situations.

OBJECTIVE 1

- In Section 6.1 we used graphing to find the intersection point of two linear models.
- We can also find an intersection point of the graphs of two linear models by substitution or elimination.
- We will solve modeling problems in this section by using substitution or elimination. Then we will use "intersect" on a graphing calculator to verify our work.

1. In Problem 5 of the lecture notes of Section 6.1, you worked with the data in the following table.

	Number of Subscribers (millions)		
Year	Cable TV and Satellite	Netflix	Hulu
2011	101	22	1
2012	101	27	3
2013	99	33	5
2014	98	39	6
2015	97	45	10

Source: *TDG Research, scbei*

Let n be the number (in millions) of subscribers at t years since 2010. Here are reasonable models for subscribers of cable TV and satellite, and subscribers of Netflix:

$$n = -1.1t + 102.5 \quad \text{Cable TV and subscribers}$$
$$n = 5.8t + 15.8 \quad \text{Netflix}$$

Use substitution or elimination to predict when the number of subscribers of cable TV and satellite and the number of subscribers of Netflix will be equal. What is that number of subscribers?

2. The percentages of U.S. marathoners who competed in the divisions Juniors (under 20 years), Open (20–39 years), and Masters (over 39 years) are shown in the following table for various years.

	Percent		
Year	Juniors	Open	Masters
1995	2	57	41
2000	2	54	44
2005	2	54	44
2010	2	52	46
2015	5	46	49

Source: *Running USA*

a. Let p be the percentage of U.S. marathoners who competed in the Open division at t years since 1990. Find a model to describe the data.

b. Let p be the percentage of U.S. marathoners who competed in the Masters division at t years since 1990. Find a model to describe the data.

c. Estimate when the percentage of U.S. marathoners who competed in the Open division was equal to the percentage of U.S. marathoners who competed in the Masters division. What is that percentage?

OBJECTIVE 2

Recall from Section 3.5 that the slope of a linear model is a rate of change.

3. Window's global market share of corporate and government PC and tablet purchases was 87% in 2013 and has decreased by about 2.2% per year. Apple's global market share of corporate and government PC and tablet purchases was 9% in 2013 and has increased by about 0.7% per year (Source: *Forrester Research*).

a. Let s be Window's global market share of corporate and government PC and tablet purchases at t years since 2013. Find an equation of a model to describe the data.

b. Let s be Apple's global market share of corporate and government PC and tablet purchases at t years since 2013. Find an equation of a model to describe the data.

c. Predict when Window's and Apple's market shares will be equal.

4. The total annual electricity generated from coal has decreased approximately linearly from 2.0 million gigawatt hours in 2007 to 1.2 million gigawatt hours in 2016. The total annual electricity generated from natural gas has increased approximately linearly from 0.9 million gigawatt hours in 2007 to 1.4 million gigawatt hours in 2016 (Source: *Energy Department*). Estimate when the total annual electricity generated from coal was equal to the total annual electricity generated from natural gas. What is that total annual amount of electricity?

SHORT HW 1, 5, 9, 11, 13, 19, 23, 25

MEDIUM HW 1–19 odds, 23, 25, 27

SECTION 6.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use substitution or elimination to solve a system of two models to make an estimate or prediction.

OBJECTIVE 1 COMMENTS

Most students prefer to use substitution to solve modeling problems in this section.

After showing my students how to do Problem 1, they have an easy time doing Problem 2. In fact, most students can probably do Problem 1 without much help, because they have learned all the relevant skills and concepts in previous parts of the text. If I have time, I remind students how to find an equation of a linear model from a table of data.

For Problem 1, I emphasize that students are to find both the year and the equal number of subscribers; without such a warning, many students forget to find the equal number of subscribers.

I remind students to use graphing calculator tables or graphs to verify their work.

Workbook Exploration

You could start class by facilitating the following 20-minute exploration because students already know how to find equations of models and how to solve linear systems using substitution or elimination. Because groups might use different definitions for the explanatory variable, select different pairs of response variables to analyze, and use different methods to solve their systems, the debrief could be an interesting and instructive discussion.

Group Exploration
Section Opener: Forming a question that can be answered by using a system

The percentages of children living with two, one, or no parents are shown in the following table for various years.

	Percentage of Children Who Live with the Following Number of Parents		
Year	Two	One	None
1960	87	9	4
1970	81	14	4
1980	77	19	4
1990	76	20	4
2000	73	22	5
2010	70	25	5
2014	69	26	5

Source: *Pew Research Center*

Form a question that can be solved by solving a system of linear equations. Then respond to the question by finding the system and solving it using substitution or elimination.

Workbook Exploration
The following 25-minute exploration is similar to the "Using a system of equations to model a situation" exploration in Section 6.1, but now students are asked to solve the new linear system by substitution or elimination.

Group Exploration
Section Opener: Using systems to model data

Per-person daily consumptions of television and the Internet are shown in the following table for various years.

	Per-Person Daily Consumption (minutes)	
Year	Television	Internet
2009	188	48
2011	182	73
2013	176	102
2015	175	128
2017	167	149

Source: *Zenith via Recode*

1. Let c be the per-person daily consumption (in minutes) of television at t years since 2000.
2. Let c be the per-person daily consumption (in minutes) of the Internet at t years since 2000.
3. Estimate the per-person daily consumptions of television and the Internet in 2010.
4. Compare the slopes of your two models. What does your comparison tell you about this situation?
5. Explain why your work in Problems 2 and 3 suggest there may be a time when the per-person daily consumptions of television and the Internet will be equal.

6. Use substitution or elimination to predict when the per-person daily consumptions of television and the Internet will be equal.

7. Predict the total per-person daily consumption of television and the Internet in 2020.

8. Find the rate of change of the total per-person daily consumption of television and the Internet.

OBJECTIVE 2 COMMENTS

For Problem 3, if you did not assign the Related Review exercises in Sections 5.3 and 5.4, some students will need a reminder that slope is rate of change and how to use this concept to find equations of the models.

For Problem 4, if you don't have time to find one or both of the models in class, here are the models:

$$E = -0.089t + 2.62 \qquad \text{Coal}$$
$$E = 0.056t + 0.51 \qquad \text{Natural gas}$$

where E is the total annual electricity (in millions of gigawatt hours) at t years since 2000.

Textbook and Workbook Exploration

In the 15 minute "Using a difference to make a prediction" exploration, groups discover how to use a difference to solve a system. Groups also learn about the significance of a positive or negative difference; this is good preparation for Section 7.3, which uses differences of models to make predictions about authentic situations.

SECTION 6.5 LECTURE NOTES

Objectives

1. Describe a five-step problem-solving method.
2. Solve perimeter, value, interest, and mixture problems.

Main point: Solve perimeter, value, interest, and mixture problems.

OBJECTIVE 1

Five-Step Problem-Solving Method

To solve some problems in which we want to find two quantities, it is useful to perform the following five steps:

- ***Step 1: Define each variable.*** For each quantity that we are trying to find, we usually define a variable to represent that unknown quantity.
- ***Step 2: Write a system of two equations.*** We find a system of two equations by using the variables from step 1. We can usually write each equation either by translating the information stated in the problem into mathematics or by making a substitution into a formula.
- ***Step 3: Solve the system.*** We solve the system of equations from step 2.
- ***Step 4: Describe each result.*** We use a complete sentence to describe the quantities we found.
- ***Step 5: Check.*** We reread the problem and check that the quantities we found agree with the given information.

OBJECTIVE 2

Recall that the formula of the perimeter P of a rectangle with length L and width W is $P = 2L+2W$ (Section 4.6).

1. For a golden rectangle, the length is equal to about 1.62 times the width. If an architect wants to design the base of a building to be a golden rectangle, what are the dimensions of the base if the perimeter is to be 800 feet?
2. A landscaper plans to dig a rectangular garden for which the length is to be 3 feet less than twice the width. If the landscaper has 66 feet of fencing to enclose the garden, what should be the dimensions of the garden?

Recall that if n objects each have value v, then their total value T is given by $T = vn$ (Section 4.6).

3. A 10,000-seat amphitheater will sell tickets at $22 and $30 for a Sarah McLachlan concert. How many tickets should be sold at each price for a sellout performance to generate a total revenue of $232,000?
4. An auditorium has 800 balcony seats and 3000 main-level seats. If tickets for balcony seats will cost $25 less than tickets for main-level seats, what should the prices be for each type of ticket so that the total revenue from a sell-out performance will be $189,000?

- Money deposited in an account is called the **principal**.
- A person invests money in hopes of later getting back the principal plus additional money called the **interest**.

- The **annual simple interest rate** is the percentage of the principal that equals the interest earned per year.

5. A person plans to invest a total of \$9000. She will invest in both an account at 3% annual interest and an account at 7% annual interest. How much should she invest in each account so that the total interest in one year will be \$378?

6. A person plans to invest twice as much money in an account at 4% annual interest as in an account at 9% annual interest. How much should he invest in each account to earn a total of \$578 in one year?

Discuss the meaning of a 20% lime-juice solution.

7. How many quarts each of a 10% acid solution and a 25% acid solution must be mixed to make 6 quarts of a 15% acid solution?

8. How many quarts each of a 14% antifreeze solution and a 26% antifreeze solution must be mixed to make 8 quarts of a 23% antifreeze solution?

9. A chemist needs 6 ounces of a 16% alcohol solution but has only a 20% alcohol solution. How many ounces each of the 20% solution and water should she mix to make the desired 6 ounces of 16% alcohol solution?

SHORT HW 1, 5, 11, 17, 23, 25, 29, 35, 37, 45, 47, 53

MEDIUM HW 1, 5, 7, 11, 13, 17, 19, 21, 23, 25, 27, 29, 33, 35, 37, 39, 45, 47, 53, 55, 57, 59, 61

SECTION 6.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use a five-step problem-solving method to solve perimeter, value, interest, and mixture problems. Although students will not solve these types of problems in subsequent sections, they will use the five-step problem-solving method to solve area problems in Section 8.6.

OBJECTIVE 1 COMMENTS

Before solving problems, I quickly go over the five-step problem-solving method so that my students see the big picture. I tell students that they should number their work with the numbers 1 through 5 accordingly. This makes it a lot easier for me to grade their work.

OBJECTIVE 2 COMMENTS

For Problem 1, I tell my students a bit about golden rectangles (see the introduction to Example 1 on page 336 of the textbook).

For Problem 3, it helps to warm up by finding the total value of, say, 3000 \$22 tickets and 7000 \$30 tickets. This work will suggest the equation

$$22x + 30y = 232,000$$

where x and y are the number of tickets that will sell for \$22 and \$30, respectively. Performing a unit analysis of the equation can increase students' understanding of the concepts and serve as a check. When finding the system

$$\begin{aligned} 22x + 30y &= 232,000 && \text{Equation (1)} \\ x + y &= 10,000 && \text{Equation (2)} \end{aligned}$$

I emphasize that equation (1) describes the value of tickets and equation (2) describes the number of tickets.

For interest problems, I explain why investors often diversify by investing both in lower-risk and higher-risk accounts whose interest rates are lower and higher, respectively.

For Problem 5, it helps to warm up by finding the total interest from investing, say, \$4000 in the 3% account and investing \$5000 in the 7% account for one year. This work will suggest the equation

$$0.03x + 0.07y = 378$$

where x and y are the principals of the accounts at 3% and 7%, respectively. When finding the system

$$\begin{aligned} 0.03x + 0.07y &= 378 && \text{Equation (3)} \\ x + y &= 9000 && \text{Equation (4)} \end{aligned}$$

I emphasize that equation (3) describes the interest amounts and equation (4) describes the principals.

For mixture problems, most students do not know what it means to have a 20% lime-juice solution. I explain that a 100-ounce 20% lime-juice solution is composed of 20 ounces of lime juice and 80 ounces of water.

I explain why equal amounts of a 40% lime-juice solution and a 60% lime-juice solution mixed together would yield a 50% lime-juice solution (and not a 100% lime-juice solution). See Exercises 49 and 50 in Homework 6.5.

Students are usually challenged by mixture problems that involve pure water, so I take my time when going over Problem 9.

Workbook Exploration

The following 15-minute exploration walks students through the five-step problem-solving method to solve a value problem.

Group Exploration

Section Opener: Solving a value problem

1. Write an expression for the total revenue from selling x tickets at \$60.
2. Write an expression for the total revenue from selling x tickets at \$60 and y tickets at \$85.
3. Write an expression for the total *number* of tickets sold from selling x tickets at \$65 and y tickets at \$85.
4. A 12,000-seat amphitheater will sell tickets at \$60 and at \$85 for a Thundercat concert. Let x and y be the number of tickets sold for \$60 and \$85, respectively. Assuming the total revenue is \$795,000 for a sellout performance, fill in the following blanks:

$$\begin{aligned} x + y &= \underline{\hspace{2cm}} \\ 60x + 85y &= \underline{\hspace{2cm}} \end{aligned}$$

5. Solve the system you found in Problem 4.
6. What does the solution you found in Problem 5 mean in this situation?
7. Perform calculations to check that your response in Problem 6 is correct.

SECTION 6.6 LECTURE NOTES

Objectives

1. Graph a linear inequality in two variables.
2. Solve a system of linear inequalities in two variables.
3. Use a system of linear inequalities in two variables to make estimates.

Main point: Graph *linear inequalities in two variables* and solve *systems of linear inequalities in two variables*.

A **linear inequality in two variables** is an inequality of the form $y < mx + b$ or $x < a$ (or with $<$ replaced with $\leq$, $>$, or $\geq$), where m, a, and b are constants.

OBJECTIVE 1

Definition *Satisfy, solution, solution set, and solve for an inequality in two variables*

If an inequality in the two variables x and y becomes a true statement when a is substituted for x and b is substituted for y, we say the ordered pair (a, b) **satisfies** the inequality and call (a, b) a **solution** of the inequality. The **solution set** of an inequality is the set of all solutions of the inequality. We **solve** the inequality by finding its solution set.

Graph the inequality by hand.

1. $y < \frac{2}{5}x - 1$
2. $y \geq -x + 2$
3. $y \leq -\frac{1}{3}x$
4. $4x - 3y > 6$
5. $3(x - 2) - 2y \geq -4$
6. $y \leq 2$
7. $x > -3$

Graph of a Linear Inequality in Two Variables

- The graph of a linear inequality of the form $y > mx + b$ is the region above the line $y = mx + b$. The graph of a linear inequality of the form $y < mx + b$ is the region below the line $y = mx + b$. For either inequality, we use a dashed line to show that $y = mx + b$ is not part of the graph.
- The graph of a linear inequality of the form $y \geq mx + b$ is the line $y = mx + b$, as well as the region above that line. The graph of a linear inequality of the form $y \leq mx + b$ is the line $y = mx + b$, as well as the region below that line.

Warning: The graph of $4x - 3y > 6$ is not the region *above* the line $4x - 3y = 6$ (see Problem 4).

OBJECTIVE 2

A **system of linear inequalities in two variables** consists of two or more linear inequalities in two variables.

Here is an example:

$$y \le \frac{2}{7}x - 5$$
$$y > 3x - 2$$

> **Definition** *Solution, solution set,* and *solve* for a system of inequalities in two variables
>
> We say an ordered pair (a, b) is a **solution** of a system of inequalities in two variables if it satisfies all the inequalities in the system. The **solution set** of a system is the set of all solutions of the system. We **solve** a system by finding its solution set.

We can find the solution set of a system of inequalities in two variables by locating the intersection of the graphs of all the inequalities.

8.

$$y \ge -\frac{1}{3}x + 2$$
$$y < \frac{5}{3}x - 1$$

9.

$$y \ge x + 1$$
$$y \ge 4x - 5$$
$$x \ge 0$$
$$y \ge 0$$

10.

$$y > -2$$
$$y < 4$$

11.

$$3x - 2y > -8$$
$$4x + 5y < 5$$

OBJECTIVE 3

12. A person's life expectancy predicts how many *remaining* years the person will live. The life expectancies of U.S. females at birth and at age 20 years are shown in the following table for various calendar years.

	Life Expectancy (years)	
Year	At Birth	At Age 20 Years
1980	77.6	59.0
1985	78.2	59.3
1990	78.8	59.8
1995	78.9	59.9
2000	79.7	60.5
2005	80.4	61.2
2010	81.1	61.7
2015	81.6	62.2

Source: *U.S. National Center for Health Statistics*

Let L be the life expectancy (in years) of U.S. females at t years since 1980.

a. Find an equation for t and L where L is the female life expectancy at birth.

b. Find an equation for t and L where L is the female life expectancy at age 20 years.

c. Find a system of inequalities that describes the life expectancies of U.S. females from 0 years through 20 years old from 1985 to 2025.

d. Graph the solution set of the system of inequalities you found in part (c).

e. Predict the life expectancies of U.S. females from 0 years through 20 years old in 2020.

SHORT HW 3, 7, 9, 21, 25, 27, 31, 35, 39, 51, 57, 59, 63, 65, 71

MEDIUM HW
6.6 3, 7, 9, 21, 23, 25, 27, 31, 35, 39, 45, 51, 55, 57, 59, 61, 63, 65, 71, 79, 81

SECTION 6.6 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students graph linear inequalities in two variables and solve systems of linear inequalities in two variables.

OBJECTIVE 1 COMMENTS

For an inequality such as $4x - 3y > 6$ (see Problem 4), the text uses the approach of first isolating y and then determining whether the points corresponding to the solution set lie above or below the line $y = \dfrac{4}{3}x - 2$. By this approach, students must think about the meaning of an inequality and the meaning of a graph.

The text does not use the approach of using a test point to determine which half plane describes a solution. This approach does little to reinforce the meaning of an inequality or of a graph. However, the text does encourage students to substitute the coordinates of a point to *verify* that the correct half plane has been shaded.

For Problem 4, some students need a reminder that when we multiply or divide both sides of an inequality by a negative number, we reverse the inequality symbol.

Textbook and Workbook Exploration

In the 20 minute "Graphing a linear inequality in two variables" exploration, groups can discover how to graph an inequality in two variables.

OBJECTIVE 2 COMMENTS

Students tend to memorize (without understanding) that they are looking for the intersection of half planes when solving a system of linear inequalities in two variables. I remind students that when they solved systems of equations in Chapter 6, they found ordered pairs that satisfied *all* of the equations in the system, which are the points that lie on *all* of the graphs of the equations. Likewise, an ordered pair is a solution of a system of inequalities if it lies on *all* of the graphs of the inequalities.

Workbook Exploration

Most students can successfully follow a procedure to solve a system of linear equalities, but they tend to lose sight of the meaning of the region they shade. The following 20-minute exploration emphasizes that the shaded region indicates the ordered pairs that satisfy all the inequalities in a system.

EXPLORATION *Meaning of a solution of a system of linear inequalities*

1. Graph the inequalities $y \geq \dfrac{1}{2}x + 3$ and $y < -\dfrac{5}{2}x + 1$ on the following coordinate system. Also plot the points $(2,\ 5)$, $(4,\ 5)$, $(4,\ 1)$, $(2,\ -4)$, $(-2,\ -3)$, $(-4,\ 1)$, $(-4,\ 5)$, and $(-2,\ 6)$.

2. Which of the points that you plotted in Problem 1 satisfy $y \geq \dfrac{1}{2}x + 3$? Give two explanations: by performing calculations and by referring to the graph.

3. Which of the points that you plotted in Problem 1 satisfy $y < -\frac{5}{2}x + 1$? Give two explanations: by performing calculations and by referring to the graph.

4. Which of the points that you plotted in Problem 1 satisfy both $y \geq \frac{1}{2}x + 3$ and $y < -\frac{5}{2}x + 1$?

5. Create a system of two linear inequalities such that the ordered pair $(2,\ -2)$ is a solution of the system but the ordered pairs $(2,\ 2)$, $(-2,\ 2)$, and $(-2,\ -2)$ are not solutions of the system. Also, graph the linear inequalities.

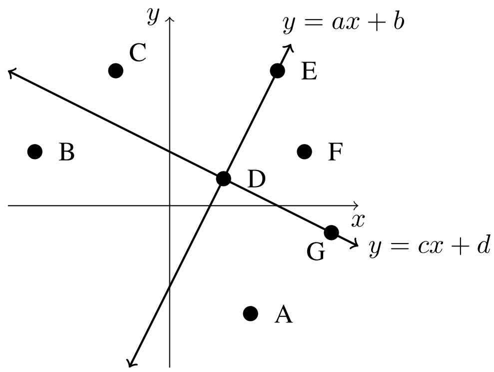
Textbook and Workbook Exploration

The following 10-minute exploration has students reflect on the connection between the meaning of a graph of an inequality, the meaning of a solution of an inequality, and the meaning of a system of inequalities.

Group Exploration

Meaning of solution of a system of linear inequalities in two variables

The graphs of $y = ax + b$ and $y = cx + d$ are sketched below.



1. For each part, decide which one or more of the points A, B, C, D, E, F, and G represent ordered pairs that

a. satisfy the inequality $y < ax + b$

b. satisfy the inequality $y \geq cx + d$

c. are solutions of the system of inequalities

$$y < ax + b$$
$$y \geq cx + d$$

2. Write a system of inequalities in terms of a, b, c, d, x, and y such that points A, D, and G are solutions and points B, C, E, and F are not solutions.

3. Write a system of inequalities in terms of a, b, c, d, x, and y such that points A and G are solutions and points B, C, D, E, and F are not solutions.

OBJECTIVE 3 COMMENTS

For parts (a) and (b) of Problem 12, here are equations of the linear models in case you are tight on time:

$L = 0.115t + 77.53$	Life expectancy at birth
$L = 0.093t + 58.82$	Life expectancy at age 20 years

CHAPTER 7 OVERVIEW

In Section 7.1, students graph quadratic equations in two variables. In Section 7.2, students use quadratic models to make estimates and predictions. They add and subtract polynomials and use a sum or difference of two polynomials to model an authentic situation in Section 7.3. In Section 7.4, students multiply polynomials and use a product of two polynomials to model an authentic situation. In Section 7.5, students simplify the square of a binomial and find the product of two binomial conjugates. They use properties of exponents to simplify power expressions in Sections 7.6 and 7.7. In Section 7.7, students learn the meaning of a negative-integer exponent and use scientific notation. They divide polynomials in Section 7.8.

SECTION 7.1 LECTURE NOTES

Objectives

1. Graph *quadratic equations in two variables*.
2. Find any intercepts of a parabola.

Main point: Graph quadratic equations in two variables.

OBJECTIVE 1

Some examples of *polynomial equations in two variables:*

$$y = 3x^2 - 10x + 2 \qquad y = x^3 - 6x^2 + 7x + 5 \qquad y = x^4 - 5x^2 + x$$

1. Use a graphing calculator to graph the above equations.

Definition *Quadratic equation in two variables*

A **quadratic equation in two variables** is an equation that can be put into the form

$$y = ax^2 + bx + c,$$

where a, b, and c are constants and $a \neq 0$. This form is called **standard form**.

The following quadratic equations are in standard form:

$$y = 3x^2 - 5x + 4 \qquad y = -2x^2 + 9 \qquad y = x^2$$

2. Graph $y = x^2$ by hand.

Draw sketches to illustrate the following:

- The graph of a quadratic equation in two variables is called a **parabola**.
- The lowest point of a parabola that *opens upward* is called the **minimum point**.
- The highest point of a parabola that *opens downward* is called the **maximum point**.
- The minimum point or the maximum point of a parabola is called the **vertex** of the parabola.
- The vertical line that passes through a parabola's vertex is called the **axis of symmetry**.
- The part of the parabola that lies to the left of the axis of symmetry is the mirror reflection of the part that lies to the right.

Graph the equation by hand. To begin, substitute the values -3, -2, -1, 0, 1, 2, and 3 for x. Make other substitutions as necessary. Give the coordinates of the vertex.

3. $y = 3x^2 - 8$
4. $y = 2x^2 - 4x - 1$
5. $y = -2x^2 - 8x - 5$

OBJECTIVE 2

- An **x-intercept of a curve** is a point where the curve and the x-axis intersect.
- A **y-intercept of a curve** is a point where the curve and the y-axis intersect.

For Problems 6–13, refer to the graph sketched in the following figure.

6. Find y when $x = 2$.

7. Find x when $y = 3$.

8. Find x when $y = 4$.

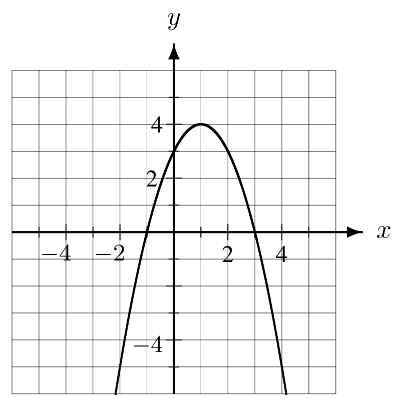
9. Find x when $y = 5$.

10. Find the y-intercept(s).

11. Find the x-intercept(s).

12. Find the maximum point.

13. Find the vertex.



By considering the shape of any parabola that opens upward or downward, we see that each (output) value of y originates from either two, one, or no (input) values of x.

SHORT HW 1, 9, 15, 17, 19, 21, 23, 25, 27, 37, 39, 41, 43, 47

MEDIUM HW 1, 7, 9, 15–31 odds, 37, 39, 41, 43, 47, 51, 53, 55, 57, 59

SECTION 7.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students graph quadratic equations in standard form. They also find the vertex and all intercepts of the graph of a quadratic equation in standard form.

OBJECTIVE 1 COMMENTS

Many students initially have trouble with Problem 4 due to the complexity of evaluating $2x^2 - 4x - 1$ for the various suggested values of x. However, with practice, most students can become quite proficient at this skill. I tell my students that we are hunting for the vertex, because once we have found the vertex, things start to fall in place. If students keep in mind the general shape of a parabola, they can identify any points that are not plotted correctly.

Textbook and Workbook Exploration

You could begin class with the 15-minute "Graphing quadratic equations in two variables" exploration. The main point is that for any unfamiliar type of equation, a good place to start is to substitute values for x and solve for y to find ordered-pair solutions.

Textbook and Workbook Exploration

Groups are capable of learning the numerous concepts of the 25-minute "Significance of a, h, and k for $y = a(x-h)^2 + k$" exploration. However, these concepts will not be further pursued in subsequent sections of the textbook.

Workbook Exploration

The following 15-minute exploration reminds students of the meaning of a graph.

EXPLORATION *Sketching graphs of quadratic equations*

Here you will explore the equation $y = x^2 + 1$.

1. Complete Table 2.2 of ordered-pair solutions of the equation $y = x^2 + 1$.

Table 2.2: Solutions of $y = x^2 + 1$

x	y
-2	5
-1	
0	
1	
2	

2. Use a graphing calculator to draw a scatterplot of the five points in the table. What do you notice about the placement of these points? Find one more solution of the equation $y = x^2 + 1$. Does the graph of this point fit the pattern as well? Does it suggest another solution?

3. Use a graphing calculator to draw the graph of $y = x^2 + 1$ and the scatterplot in the same viewing window. What do you observe?

4. Use a graphing calculator to pick a point that lies on the graph of $y = x^2 + 1$ but is different from the points plotted in the scatterplot. Verify that the ordered pair for this point satisfies the equation $y = x^2 + 1$ and is indeed a solution of the equation.

5. Use a graphing calculator to pick a point that does not lie on the graph of $y = x^2 + 1$. Does the ordered pair for this point satisfy the equation $y = x^2 + 1$?

6. What is the relationship between the equation $y = x^2 + 1$ and its graph? Verify that your theory is true.

OBJECTIVE 2 COMMENTS

I compare the results of Problems 7, 8, and 9 and why we get two, one, and no values of x, respectively. This sets the stage for discussing in Section 8.5 why quadratic equations in one variable have two, one, or no solutions.

Problems 6–13 are good preparation for Section 7.2, where students will perform similar tasks with quadratic models.

SECTION 7.2 LECTURE NOTES

Objectives

1. Use a quadratic model to make estimates and predictions.
2. Determine whether to model a situation by using a linear model, a quadratic model, or neither.
3. Use a graphing calculator to make predictions with a quadratic model.

Main point: Use a quadratic model to make estimates and predictions.

OBJECTIVE 1

Recall that a quadratic equation in two variables is an equation that can put into the form $y = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$.

Definition *Quadratic model*

A **quadratic model** is a quadratic equation in two variables that describes the relationship between two quantities for an authentic situation. We also refer to the quadratic equation's graph (a parabola) as a quadratic model.

1. The numbers of illegal immigrant apprehensions by the U.S. border patrol are shown in the following table for various years.

Year	Number of Illegal Immigrant Apprehensions by the U.S. Border Patrol (millions)
2000	1.7
2004	1.1
2008	0.7
2012	0.4
2016	0.4

Source: *Customs and border protection*

Let n be the number (in millions) of illegal immigrant apprehensions by the U.S. border patrol in the year that is t years since 2000. A quadratic model is sketched in Fig. 2.5

Figure 2.5: Illegal Immigrant Apprehension Model

a. Carefully trace the graph in Fig. 2.5 onto a piece of paper. Then on the same graph construct a scatterplot of the data. Does the model fit the data well?

b. What is the n-intercept of the model? What does it mean in this situation?

c. Estimate the number of illegal immigrant apprehensions in 2002.

d. What is the vertex of the model? What does it mean in this situation?

e. Estimate in which year there were 0.5 million illegal immigrant apprehensions.

OBJECTIVE 2

By constructing a scatterplot of some data, we can determine whether to use a linear model, a quadratic model, or neither.

2. Consider the scatterplots of data shown in Figs. 2.6, 2.7, 2.8 for situations 1, 2, and 3, respectively. For each situation, determine whether a linear model, quadratic model, or neither would describe the situation well.

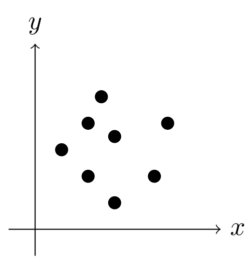


Figure 2.6: Scatterplot for situation 1

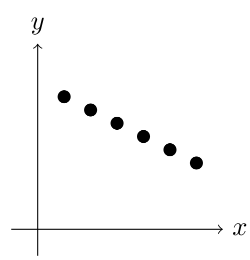


Figure 2.7: Scatterplot for situation 2

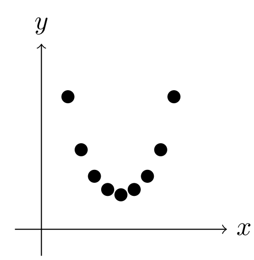


Figure 2.8: Scatterplot for situation 3

OBJECTIVE 3

3. The average number of points in a football game scored on a drive are shown in the following table for various distances from the opposing team's goal line that the drive began.

Distance From Opposing Team's Goal Line (yards)	Average Points Scored
99	0.7
90	1.0
80	1.2
70	1.6
60	1.9
50	2.0
40	2.6
30	3.3
20	3.6
10	4.3

Source: *N.F.L*

Let A be the average number of points in a football game scored on a drive that began x yards from the opposing team's goal line. A linear model of the situation is $A = -0.039x + 4.37$. A quadratic model of the situation is $A = 0.00024x^2 - 0.066x + 4.9$.

a. Which of the two models comes closer to the points in the scatterplot of the data?

b. Is the graph of the quadratic model an increasing or decreasing curve for the values of x between 0 and 100? What does this mean in this situation?

c. Substitute a value for one of the variables in the quadratic model to estimate the average score for drives that began 75 yards from the opposing team's goal line.

d. Substitute a value for one of the variables in the quadratic model to estimate the average score for drives that began 74 yards from the opposing team's goal line.

e. Punt and kick returners fight tooth and nail for every last inch. Do spots within one yard of each other have much effect on how many points the team will score? (See your results of parts (c) and (d).)

f. Use "maximum" on a graphing calculator to find the vertex of the quadratic model. What does it mean in this situation?

g. Use TRACE on a graphing calculator together with the quadratic model to estimate at what distance from the opposing team's goal line is an average of 3 points scored on a drive. Does this suggest when a team should attempt a field goal, which is worth 3 points? Explain, including any assumptions you have made.

SHORT HW 1, 3, 5, 9, 13, 15, 17, 19, 21, 23

MEDIUM HW 1–29 odds

SECTION 7.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use quadratic models to make estimates and predictions. Students will continue to work with quadratic models in Sections 8.6 and 9.6.

OBJECTIVE 1 COMMENTS

I tell my students that after working with linear models in the past six chapters, it is time to work with a new type of model called a *quadratic model*.

Most students have an easy time with Problem 1. I observe that for part (e), there are two points on the curve in which the n-coordinate is 0.5 and that's because the model is quadratic rather than linear. This observation is a good primer for work with quadratic models in Sections 8.6 and 9.6. However, I point out that part (e) requests a year in the past, so the result should be just one value (2013). Plus, on the basis of the decreasing trend of the data, we have little faith in the part of the model that lies to the right of the vertex.

OBJECTIVE 2 COMMENTS

I say that now that we are familiar with quadratic models, the purpose of constructing a scatterplot has expanded to determining whether to use a linear model, quadratic model, or neither of these types of models.

OBJECTIVE 3 COMMENTS

For part (f) of Problem 3, I show students how to use "maximum" on a graphing calculator. To find the maximum point, students will first need to Zoom Out or manually change the WINDOW settings accordingly.

For part (g), instead of using TRACE, students can use "intersect" to find the intersection points of the graph of the quadratic model and the line $A = 3$. However, at this early stage of getting acquainted with quadratic models, it may be better to have students use TRACE, which keeps students focused on the shape of a parabola.

In general, the most challenging aspect of using a graphing calculator in this section is to find useful WINDOW settings. A good starting point is to use ZoomStat to draw a scatterplot. Then, students can Zoom Out or manually adjust the WINDOW settings, if necessary.

SECTION 7.3 LECTURE NOTES

Objectives

1. Describe the meaning of *term*, *monomial*, *polynomial*, *degree*, *coefficient*, and *like terms*.
2. Combine like terms.
3. Add polynomials.
4. Subtract polynomials.
5. Use a sum or a difference of two polynomials to model an authentic situation.

Main point: Add and subtract polynomials.

OBJECTIVE 1

- A **term** is a constant, a variable, or a product of a constant and one or more variables raised to powers.
- A **monomial** is a constant, a variable, or a product of a constant and one or more variables raised to *counting number* powers.
- A **polynomial**, or **polynomial expression**, is a monomial or a sum of monomials.
- We usually write polynomials in one variable so that the exponents of the terms decrease from left to right, an arrangement called **descending order**.
- The **degree of a term** in one variable is the exponent on the variable. The **degree of a term in two or more variables** is the sum of the exponents on the variables.
- The **degree of a polynomial** is the largest degree of any nonzero term of the polynomial.
- First degree: linear, second degree: quadratic, third degree: cubic

Use words such as *linear*, *quadratic*, *cubic*, *polynomial*, *degree*, *one variable*, and *two variables* to describe the expression.

1. $5x^2 - 2x + 8$ **2.** $-7x^3 + x^2 - 9$ **3.** $2p^4q - 5p^2q^2 - 3q^3$

OBJECTIVE 2

- The **coefficient** of a term is the constant factor of the term.
- The **leading coefficient** of a polynomial is the coefficient of the term with the largest degree.
- **Like terms** are either constant terms or variable terms that contain the same variable(s) raised to exactly the same power(s).
- **Unlike terms** are terms that are not like terms.

Here we use the distributive law to combine like terms: $3x^4 + 5x^4 = (3 + 5)x^4 = 8x^4$.

Combining Like Terms

To combine like terms, add the coefficients of the terms.

Combine like terms.

4. $5x^2 + 3x^2$

5. $4x^3y^2 + 5x^2y^3$

6. $2p^3 - 7p^2 + 4p^3 + 2 + 8p^2 - p^3$

7. $3a^3b + 4a^2b^2 - 7a^2b^2 - 5a^3b$

OBJECTIVE 3

Adding Polynomials

To add polynomials, combine like terms.

Find the sum.

8. $(4x^2 - 8x^2) + (-2x^3 - 5x^2)$

9. $(2x^3 - 7x^2 - 9x + 1) + (5x^3 - 3x^2 + 6x - 4)$

10. $(5x^3 - x) + (7x^2 - 2x)$

11. $(3m^2 + 5mn - 8n^2) + (-6m^2 - 2mn - 4n^2)$

OBJECTIVE 4

We can use the fact $a - b = a - 1b$ to help us subtract two polynomials.

Subtracting Polynomials

To subtract polynomials, first distribute -1 and then combine like terms.

Find the difference.

12. $(2x^2 - 7x) - (4x^3 + 5x)$

13. $(6x^2 - 4x + 8) - (9x^2 + x - 2)$

14. $(3a^2 + 7ab - 5b^2) - (5a^2 - 2ab + 4b^2)$

15. $(2.7k^2 - 5.3k + 9.44) - (4.1k^2 + 2.5k - 7.18)$

16. Subtract $(4x^2 - 5x + 7)$ from $(3x^2 - 2x + 1)$.

17. Perform the operations: $(5x^3 - 3x) - (4x^3 - 7x^2 + x) + (2x^3 - 6x^2)$.

OBJECTIVE 5

The Meaning of the Sign of a Difference

If a difference $A - B$ is positive, then A is more than B. If a difference $A - B$ is negative, then A is less than B.

18. In Problem 5 of the lecture notes of Section 6.1, you worked with the data in the following table.

	Number of Subscribers (millions)		
Year	Cable TV and Satellite	Netflix	Hulu
2011	101	22	1
2012	101	27	3
2013	99	33	5
2014	98	39	6
2015	97	45	10

Source: *TDG Research, scbei*

Let n be the number (in millions) of subscribers at t years since 2010. Here are reasonable models for subscribers of cable TV and satellite, and subscribers of Netflix:

$$n = -1.1t + 102.5 \qquad \text{Cable TV and subscribers}$$
$$n = 5.8t + 15.8 \qquad \text{Netflix}$$

a. Find the sum of the polynomials $-1.1t + 102.5$ and $5.8t + 15.8$. What does the result represent?

b. Evaluate the result in part (a) for $t = 11$. What does your result mean in this situation?

c. Find the difference of the polynomials $-1.1t + 102.5$ and $5.8t + 15.8$. What does the result represent?

d. Evaluate the result in part (c) for $t = 11$. What does your result mean in this situation?

SHORT HW 3, 7, 13, 23, 29, 39, 43, 49, 55, 63, 67, 71, 77, 85

MEDIUM HW 1, 3, 7, 13, 15, 23, 29, 39, 43, 45, 49, 55, 57, 61, 63, 65, 67, 69, 71, 77, 83, 85, 87, 89

SECTION 7.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students add and subtract polynomials and use a sum or difference of polynomials to model an authentic situation. Students will add and subtract polynomials at many times in the rest of the course.

OBJECTIVE 1 COMMENTS

I usually quickly "define" all the polynomial terminology by way of examples. Knowing the definitions will help students communicate with you and other students, and it will help students complete Exercises 1–6 of Homework 7.3 as well as many Expressions, Equations, and Graphs exercises in Chapters 7–11.

OBJECTIVE 2 COMMENTS

Students have an easy time combining like terms.

Workbook Exploration

Although students can easily combine like terms, they forget that the distributive law is what allows them to do so. The following 10-minute exploration is a good reminder of this.

Group Exploration

Section Opener: Combining like terms

Recall that $2x + 6x$ can be written as one monomial by using the distributive law: $2x + 6x = (2 + 6)x = 8x$.

1. Can the given polynomial be written as one monomial by using the distributive law? If yes, do so. If no, explain why not. [**Hint:** When possible, make sure to show the step that uses the distributive law.]

a. $3x^4 + 5x^4$

b. $7x^3y^2 - 5x^3y^2$

c. $4x^5 + 2x^3$

d. $5x^4y^2 - 9x^2y^4$

e. $2x^3y^7 + x^3y^7 + 3x^3y^7$

2. When the sum or difference of two or more monomials can be written as one monomial, we say they are *like terms*. When are monomials like terms? When are they unlike terms?

OBJECTIVES 3 and 4 COMMENTS

Although many texts suggest that students should subtract two polynomials by changing signs, I find that my students do better by writing a difference of polynomials $P - Q$ as $P - 1Q$ and then distributing the -1; however, some students forget to distribute the -1 to all the terms in the polynomial Q.

Once we have discussed finding products in Section 7.4, some students will treat a difference as if it was a product. For example, students will treat the difference $(6x^2 - 4x + 8) - (9x^2 + x - 2)$ as if it were the product $(6x^2 - 4x + 8)(9x^2 + x - 2)$. I caution my students to take a moment to establish which operation is between two polynomials before diving in to perform the assumed operation.

For Problem 16, most students mistakenly subtract $(3x^2 - 2x + 1)$ from $(4x^2 - 5x + 7)$, rather than subtract $(4x^2 - 5x + 7)$ from $(3x^2 - 2x + 1)$.

Textbook and Workbook Exploration

Groups have an easy time with the 15 minute "The degree of a sum of polynomials" exploration. This is a good critical-thinking activity but the content is not essential to the course.

OBJECTIVE 5 COMMENTS

It may seem unnecessary to discuss the meaning of the signs of a difference $A - B$, but most students do not know such concepts, especially if the difference is negative. Working with specific values of A and B such as $A = 3$ and $B = 5$ helps get across the concepts to students.

Most students have trouble interpreting their results of parts (c) and (d) of Problem 18.

Textbook and Workbook Exploration

The 15-minute "Combining expressions that represent authentic quantities" exploration can be used to introduce Objective 4, or it can be used as a second example after going over Problem 18.

SECTION 7.4 LECTURE NOTES

Objectives

1. Describe the meaning of *binomial* and *trinomial*.
2. Apply the product property for exponents.
3. Multiply monomials.
4. Multiply a monomial and a polynomial.
5. Multiply two polynomials.
6. Use a product of two polynomials to model an authentic situation.

Main point: Multiply polynomials and use a product of two polynomials to model an authentic situation.

OBJECTIVE 1

We refer to a polynomial as a monomial, a **binomial**, or a **trinomial** depending on whether it has one, two, or three nonzero terms, respectively:

Name	**Examples**	**Meaning**
monomial	$5x^3, \quad x^2, \quad -3x^3y, \quad 79$	one nonzero term
binomial	$4x^3+6x, \quad -5x^3-7x, \quad 7y^2-2$	two nonzero terms
trinomial	$-x^3-5x+1, \quad 3x^2-2x+8$	three nonzero terms

OBJECTIVE 2

Show that $x^2 \cdot x^4 = x^{2+4}$:

$$\begin{aligned} x^2 \cdot x^4 &= (x \cdot x)(x \cdot x \cdot x \cdot x) \\ &= x \cdot x \cdot x \cdot x \cdot x \cdot x \\ &= x^6 \end{aligned}$$

Product Property for Exponents

If n and m are counting numbers, then

$$x^m x^n = x^{m+n}$$

For example, $x^3x^5 = x^{3+5} = x^8$. Also, $x^4x = x^4x^1 = x^{4+1} = x^5$.

OBJECTIVE 3

Find the product.

1. $3x(7x^4)$ **2.** $4x^5(-6x^3)$ **3.** $-5p^3q(-3p^2q^5)$

Warning: $4x^5(-6x^3)$ is a product, not a difference.

OBJECTIVE 4

Find the product.

4. $5x(4x+7)$ **5.** $3t^2(4t^3-7t)$ **6.** $-2xy^2(3x^2-5xy+y^2)$

OBJECTIVE 5

Use arrows to help show students how to use the distributive law to find the product $(a+b)(c+d)$:

$$\begin{aligned}(a+b)(c+d) &= a(c+d)+b(c+d)\\ &= ac+ad+bc+bd\end{aligned}$$

> **Multiplying Two Polynomials**
>
> To multiply two polynomials, multiply each term in the first polynomial by each term in the second polynomial. Then combine like terms, if possible.

Find the product.

7. $(x+4)(x+6)$

8. $(2x-5y)(3x+7y)$

9. $(5x-3)(5x-3)$

10. $(4x-7)(4x+7)$

11. $(4w^2+2)(3w-5)$

12. $(2x-5)(x^2-3x+4)$

13. $(2x^2-3)(4x^2+3x-5)$

14. $(3p^2-p+2)(p^2+3p-4)$

OBJECTIVE 6

15. The number n (in millions) of cows in the United States can be modeled by the equation $n = 0.033t+9.14$, where t is the number of years since 2010. The annual milk yield per cow p (in thousands of pounds per cow) can be modeled by the equation $p = 0.27t+21.1$, where t is the number of years since 2010 (see the following table).

Year	Number of Cows (millions)	Average Milk Yield Per Cow (thousands of pounds)
2010	9.12	21.1
2011	9.20	21.3
2012	9.24	21.7
2013	9.22	21.8
2014	9.26	22.3
2015	9.31	22.4
2016	9.33	22.8
2017	9.39	22.9

Source: *USDA*

a. Check that the models fit the data well.

b. Perform a unit analysis of the polynomial $(0.033t+9.14)(0.27t+21.1)$.

c. Find the product of the polynomials $0.033t+9.14$ and $0.27t+21.1$. What does the result represent?

d. Evaluate the result from part (c) for $t = 12$. What does your result mean in this situation?

SHORT HW 13, 17, 23, 29, 31, 41, 49, 61, 63, 67, 73, 75, 79, 83, 89

MEDIUM HW 1, 13, 17, 23, 29, 31, 41, 45, 49, 53, 59, 63, 67, 73, 75, 79, 81, 83, 85, 89, 91, 93, 99, 101, 103, 105

SECTION 7.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students multiply polynomials and use a product of two polynomials to model an authentic situation. Students will find products of polynomials at many times in the rest of the course.

OBJECTIVE 1 COMMENTS

I point out that we know that the prefixes mono, bi, and tri are used to mean one, two, and three, respectively, for words such as monorail, bicycle, and tricycle.

OBJECTIVE 2 COMMENTS

When discussing the product rule for exponents $x^m x^n = x^{m+n}$, I emphasize that even though we are adding the exponents m and n, we are nonetheless multiplying the powers x^m and x^n. It helps to show the multiplication operation: $x^m \cdot x^n = x^{m+n}$. Many students are unclear on which operation is involved where.

OBJECTIVE 3 COMMENTS

The warning about $4x^5(-6x^3)$ just after Problems 1–3 in the lecture notes is useful, because some students do, in fact, think that the expression is a difference.

OBJECTIVES 4 and 5 COMMENTS

This is an excellent time to remind students of the meaning of equivalent expressions. Many students treat the equality symbol as "Here's the next step." To help students understand the meaning of the equality symbol, I choose an expression such as $(x + 2)(x + 3)$ and show that this expression and the product $x^2 + 5x + 6$ both give the same result when evaluated for some real number a. Then I use a graphing calculator table to show that $P(a) = Q(a)$, where $P(x) = (x + 2)(x + 3)$ and $Q(x) = x^2 + 5x + 6$. Then I use tables to show that $P(k) = Q(k)$ for various values of k. Finally, I show that the graphs of P and Q are the same and point out that this makes sense, because the tables are the same.

I emphasize that students can verify their work by comparing graphing calculator tables or graphs. My students are often reluctant to verify their work, but it can improve their performance on exams and it is worth encouraging them to take advantage of this. To alter my students' perspective on verifying their work on exams, I often ask, "If I gave you an answer key during a test, would you refer to it?"

Workbook Exploration

Unless you are going to do an extremely large number of explorations, the following 20-minute exploration probably should be skipped because the concepts are not needed for the course.

Group Exploration

The degree of a product of polynomials

In this exploration, you will explore the degree of the product of two polynomials.

1. Find the degrees of the given polynomials. Next, multiply the polynomials. Then determine the degree of the product.

   a. $5x + 4$ and $2x - 3$

   b. $2x - 3$ and $x^2 + 2x - 1$

c. $x^2 + x + 3$ and $x^2 - 2x + 1$

2. What is the degree of the product of a polynomial of degree 1 and a polynomial of degree 1? Explain.

3. What is the degree of the product of a polynomial of degree 1 and a polynomial of degree 2? Explain.

4. What is the degree of the product of a polynomial of degree 2 and a polynomial of degree 2? Explain.

5. Without multiplying the polynomials $2x^8 - 5x^6 + 7x^3 + 8$ and $7x^5 + 2x^4 - 6x + 4$, find the degree of the product of the polynomials. Explain.

6. What is the degree of the product of a polynomial of degree m and a polynomial of degree n? Explain.

OBJECTIVE 6 COMMENTS

Part (b) of Problem 15 should help students see the meaning of the product

$$(0.033t + 9.14)(0.27t + 21.1)$$

in part (c). After doing part (c), I point out that that finding the linear models in order to find the product of the right-hand sides of the models' equations paid big dividends in finding the useful polynomial

$$0.00891t^2 + 3.1641t + 192.854$$

which would be difficult to find directly.

SECTION 7.5 LECTURE NOTES

Objectives

1. Raise a product to a power.
2. Find the power of a monomial.
3. Simplify the square of a binomial.
4. Put a quadratic equation in standard form, $y = ax^2 + bx + c$.
5. Find the product of two *binomial conjugates*.

Main point: Simplify powers of polynomials and find products of binomial conjugates.

OBJECTIVE 1

We can write $(xy)^4$ as a product of powers:

$$\begin{aligned}(xy)^4 &= (xy)(xy)(xy)(xy) && \text{Write power without exponents.}\\ &= (x \cdot x \cdot x \cdot x)(y \cdot y \cdot y \cdot y) && \text{Rearrange factors.}\\ &= x^4y^4 && \text{Simplify.}\end{aligned}$$

Raising a Product to a Power

If n is a counting number, then

$$(xy)^n = x^n y^n$$

For example, $(xy)^7 = x^7y^7$

OBJECTIVE 2

Perform the indicated operation.

1. $(xy)^5$ **2.** $(2x)^4$ **3.** $(-4p)^3$

Warning: The polynomial $(3x)^2$ is equivalent to $9x^2$, not $3x^2$.

OBJECTIVE 3

Simplify $(A + B)^2$:

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= A^2 + AB + BA + B^2\\ &= A^2 + 2AB + B^2\end{aligned}$$

Squaring a Binomial

$$\begin{aligned}(A+B)^2 &= A^2 + 2AB + B^2 && \text{Square of a sum}\\ (A-B)^2 &= A^2 - 2AB + B^2 && \text{Square of a difference}\end{aligned}$$

Simplify.

4. $(x+5)^2$

5. $(x-3)^2$

6. $(2k+5)^2$

7. $(3x-6)^2$

8. $(2t+4w)^2$

9. $(4a-5b)^2$

- A **perfect-square trinomial** is a trinomial that is equivalent to the square of a binomial (see Problems 4–9).
- Compare $(AB)^2 = A^2B^2$ and $(A+B)^2 = A^2 + 2AB + B^2$.
- Warning: When simplifying $(A+B)^2$, don't omit the middle term $2AB$ of $A^2 + 2AB + B^2$.

OBJECTIVE 4

Write the equation in standard form.

10. $y = (x-5)^2 + 3$

11. $y = -2(x+3)^2 - 4$

OBJECTIVE 5

- We say the sum of two terms and the difference of the same two terms are **binomial conjugates**.
- Simplify $(A+B)(A-B)$:

$$\begin{aligned}(A+B)(A-B) &= A^2 - AB + AB - B^2 \\ &= A^2 - B^2\end{aligned}$$

Product of Binomial Conjugates

$$(A+B)(A-B) = A^2 - B^2$$

Find the product.

12. $(x+4)(x-4)$

13. $(5x-3)(5x+3)$

14. $(8a+6b)(8a-6b)$

Perform the indicated operation.

15. $(3a^2 - 6ab + 2b^2) - (5a^2 + 3ab - 7b^2)$

16. $(2x-5)(3x^2 + 4x - 2)$

17. $(c+7)^2$

18. $(x+5)(x-9)$

19. $5ab^2(-3a^2b)$

20. $(3t-5w)^2$

21. $(4m^2 - 2m + 8) + (-7m^2 - m + 3)$

22. $-5xy(3x^2 - 7xy + 9y^2)$

23. $(4p+8q)(4p-8q)$

24. $(5x-4y)(3x-6y)$

SHORT HW 1, 9, 17, 23, 31, 39, 41, 53, 57, 75, 77, 85, 87, 93, 95

MEDIUM HW 1, 9, 17, 19, 23, 31, 39, 41, 49, 53, 57, 61, 69, 75, 77, 85, 87, 91, 93, 95, 99, 105, 107, 109

SECTION 7.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students simplify powers of polynomials and find the product of two binomial conjugates. Students will use these skills throughout most of the rest of the text.

OBJECTIVE 1 COMMENTS

Students have an easy time following the explanation of why $(xy)^4$ is equivalent to x^4y^4.

OBJECTIVE 2 COMMENTS

The warning about $(3x)^2$ is important, because many students forget to square the coefficient 3 of the base. This type of error is especially prevalent in more complicated power expressions.

OBJECTIVE 3 COMMENTS

When simplifying polynomials of the form $(A+B)^2$, most of my students omit the middle term $2AB$ of $A^2 + 2AB + B^2$. When simplifying polynomials of the form $(A-B)^2$, many students write $A^2 - B^2$ or $A^2 + B^2$. Because so many students make these types of errors, I do all the following to counteract this behavior:

- For each of the equations $(A+B)^2 = A^2 + B^2$, $(A-B)^2 = A^2 - B^2$, and $(A-B)^2 = A^2 + B^2$, I evaluate both sides of the equation to show that the statement is incorrect.
- I assign many or all of Exercises 85–92 and 95 of Homework 7.5.
- Throughout the rest of the course, when I find the square of a binomial, I underline the middle term several times and remind students to include it.

If I employ these tactics, my students rarely make these types of errors on exams.

For Problems 4–9, I continue to use arrows as shown in the solutions of Examples 2 and 3 on page 398 and of the textbook until students get the hang of this skill.

Workbook Exploration

On the basis of how frequently students tend to make errors when squaring a binomial, it's a good idea to have groups complete the following 15-minute exploration.

EXPLORATION *Simplifying squares of binomials*

1. Write $(x+3)^2$ without parentheses by first writing it as $(x+3)(x+3)$ and then multiplying pairs of terms. We say you have *simplified* $(x+3)^2$.
2. Simplify $(A+B)^2$.
3. In Problem 1 you simplified $(x+3)^2$ by first writing it as $(x+3)(x+3)$. Now simplify it by using the formula that you found in Problem 2. Compare your result with the result you found in Problem 1.
4. Use the formula that you found in Problem 2 to simplify $(3x+4y)^2$.
5. Use the formula that you found in Problem 5 to simplify $(5p-2w)^2$.

OBJECTIVE 4 COMMENTS

For Problem 11, some of my students distribute the coefficient -2 before simplifying $(x+3)^2$. When simplifying $(x+3)^2$ some students omit the middle term. After simplifying $(x+3)^2$ and then distributing the coefficient -2, some students forget to subtract the 4, the last term of $-2(x+3)^2 - 4$.

Workbook Exploration

The following 15-minute exploration affords groups yet more practice with squaring binomials.

EXPLORATION *Writing quadratic functions in standard form*

1. Three students all attempt to simplify $(x+3)^2$. Which of these students simplify the expression correctly? If any errors were made, describe them and where they occurred. Also verify your decision by comparing graphing calculator tables or graphs for $y=(x+3)^2$, $y=x^2+9$, and $y=x^2+6x+9$.

Student #1's work	Student #2's work	Student #3's work
$\begin{aligned}(x+3)^2 &= x^2+3^2\\ &= x^2+9\end{aligned}$	$\begin{aligned}(x+3)^2 &= (x+3)(x+3)\\ &= x^2+3x+3x+3^2\\ &= x^2+6x+9\end{aligned}$	$\begin{aligned}(x+3)^2 &= x^2+2(3x)+3^2\\ &= x^2+6x+9\end{aligned}$

2. Simplify. Use graphing calculator tables or graphs to verify your result.

a. $(x+2)^2$ **b.** $(x-4)^2$

3. Write the equation in standard form $f(x)=ax^2+bx+c$.

a. $f(x)=(x+3)^2$ **b.** $k(x)=(x-5)^2$
c. $g(x)=(x-7)^2+1$ **d.** $h(x)=2(x+4)^2+5$

OBJECTIVE 5 COMMENTS

My students do well with this skill. Nonetheless, it doesn't hurt to use arrows as shown in the solution of Example 5 on page 400.

Problems 15-24 address the contents of Sections 7.3, 7.4, and 7.5. These types of mixed sets of problems are good practice for exams.

Textbook and Workbook Exploration

The 15-minute "Product of binomial conjugates" exploration helps groups discover how to multiply binomial conjugates. Groups usually have an easy time with this exploration.

SECTION 7.6 LECTURE NOTES

Objectives

1. Apply properties of exponents.
2. Recognize whether a power expression is simplified.
3. Describe the meaning of the exponent zero.
4. Use combinations of properties of exponents to simplify power expressions.

Main point: Apply the properties of exponents to simplify power expressions.

OBJECTIVES 1 and 2

Recall: If n and m are counting numbers, then

$$x^m x^n = x^{m+n} \qquad \text{Product property for exponents (Section 7.4)}$$

$$(xy)^n = x^n y^n \qquad \text{Raising a product to a power (Section 7.5)}$$

Perform the indicated operations.

1. x^4x^8
2. $(xy)^6$
3. $(3x^3)(-4x^6)$
4. $(5a^4b^2)(3a^5b^4)$
5. $(-3tw)^4$

- A **power expression** is an expression that contains one or more powers.
- Some examples: $4x^6y^2(3x^2y)^2$, $\dfrac{25x^6y^3}{35x^3y}$, $\left(\dfrac{3x^2}{5y}\right)^3$.

Simplifying a Power Expression

A power expression is simplified if

1. It includes no parentheses.
2. In any monomial, each variable or constant appears as a base at most once. For example, for nonzero x, we write x^3x^5 as x^8.
3. Each numerical expression (such as 5^2) has been calculated, and each numerical fraction has been simplified.

$$\begin{aligned}
\frac{x^5}{x^3} &= \frac{x\cdot x\cdot x\cdot x\cdot x}{x\cdot x\cdot x} && \text{Write quotient without exponents.}\\
&= \frac{x\cdot x\cdot x}{x\cdot x\cdot x}\cdot\frac{x\cdot x}{1} && \frac{ac}{bd}=\frac{a}{b}\cdot\frac{c}{d}\\
&= 1\cdot x\cdot x && \text{Simplify: } \frac{x\cdot x\cdot x}{x\cdot x\cdot x}=1.\\
&= x^2 && \text{Simplify.}
\end{aligned}$$

This suggests we can subtract the exponents: $\dfrac{x^5}{x^3} = x^{5-3} = x^2$.

Quotient Property for Exponents

If m and n are counting numbers and x is nonzero, then

$$\frac{x^m}{x^n} = x^{m-n}$$

Simplify.

6. $\dfrac{6x^7}{9x^4}$ **7.** $\dfrac{4x^9y^5}{6x^7y^2}$ **8.** $\dfrac{20a^8b^5}{15a^6b}$

OBJECTIVE 3

Show that $2^4 = 16$, $2^3 = 8$, $2^2 = 4$, $2^1 = 2$ suggests that $2^0 = 1$.

Definition *Zero exponent*

For nonzero x, $x^0 = 1$

So, $7^0 = 1$, $(3xy^2)^0 = 1$, and $\left(\dfrac{7x}{y}\right)^0 = 1$.

OBJECTIVE 4

$$\begin{aligned} \left(\frac{x}{y}\right)^3 &= \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} && \text{Write power expression without exponents.} \\ &= \frac{x \cdot x \cdot x}{y \cdot y \cdot y} && \text{Multiply numerators; multiply denominators.} \\ &= \frac{x^3}{y^3} && \text{Simplify.} \end{aligned}$$

This suggests that we can distribute the exponent 3: $\left(\dfrac{x}{y}\right)^3 = \dfrac{x^3}{y^3}$

Raising a Quotient to a Power

If n is a counting number and y is nonzero, then

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

Simplify.

9. $\left(\dfrac{x}{y}\right)^5$ **10.** $\left(\dfrac{x}{4}\right)^3$

$(x^2)^4 = x^2 \cdot x^2 \cdot x^2 \cdot x^2$	Write expression without exponent 4.
$= x^{2+2+2+2}$	Add exponents: $x^m x^n = x^{m+n}$.
$= x^8$	Simplify.

This suggests we can multiply the exponents: $(x^2)^4 = x^{2\cdot 4} = x^8$.

Raising a Power to a Power

If m and n are counting numbers, then

$$(x^m)^n = x^{mn}$$

Simplify.

11. $(x^3)^5$

12. $(b^4)^7$

Sometimes we use more than one property of exponents to simplify an expression.

Simplify.

13. $(2x^2)^3$

14. $(3x^5y^2)^3$

15. $5x^6y(3x^3y^5)^2$

16. $(b^5c^2)^3(b^4c^8)^5$

17. $\dfrac{12x^3x^5}{8x^2}$

18. $\left(\dfrac{x^5}{2y}\right)^4$

19. $\left(\dfrac{4x^7}{3y^5w}\right)^3$

20. $\dfrac{(2ab^5)^4}{8a^3b^9}$

SHORT HW 9, 17, 21, 25, 29, 35, 37, 49, 53, 55, 65, 71, 77, 83, 88

MEDIUM HW 1, 9, 11, 17, 21, 25, 29, 31, 35, 37, 41, 43, 49, 53, 55, 65, 71, 77, 81, 83, 88, 93, 95, 97, 99, 101, 103

SECTION 7.6 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students learn some properties of exponents and simplify power expressions. They will use these skills many times in the rest of the course.

OBJECTIVES 1 and 2 COMMENTS

For Problem 3, I discuss why we multiply the coefficients but add the appropriate exponents. For Problem 5, I emphasize that we distribute the exponent 4 to the coefficient -3 as well as to the other factors x and y.

When discussing the boxed statement about "Simplifying a Power Expression," I refer to our work on Problems 1–5.

For Problem 6, to make sure students see how to apply the quotient property of exponents, I write $\dfrac{6x^7}{9x^4}$ as the product of two fractions:

$$
\begin{aligned}
\frac{6x^7}{9x^4} &= \frac{6}{9} \cdot \frac{x^7}{x^4} \\
&= \frac{2}{3} \cdot x^{7-4} \\
&= \frac{2}{3} \cdot x^3 \\
&= \frac{2x^3}{3}
\end{aligned}
$$

Then I discuss how we can simplify $\frac{6x^7}{9x^4}$ without writing it as the product of two fractions.

OBJECTIVE 3 COMMENTS

Many students are surprised to learn that it makes sense to define x^0 to be 1 (for nonzero values of x).

Workbook Exploration

The following 5-minute exploration is so quick, it's probably worth doing.

EXPLORATION *Zero exponent definition*

1. Complete the following table with ordered-pair solutions of $y = 2^x$.

x	2^x Before Simplifying	2^x After Simplifying
5	2^5	32
4		
3		
2		
1		

2. Look for a pattern in your completed table that suggests a value for 2^0. Use a calculator to verify your result.

OBJECTIVE 4 COMMENTS

I emphasize that for $x^m x^n$, we add the exponents and that for $(x^m)^n$, we multiply the exponents; students often confuse the product property and the property of raising a power to a power.

Many students are challenged by Problems 13–20. Typical errors include

- applying the properties in the wrong order, sometimes because students do not understand or pay attention to the roles of the parentheses.
- not completely simplifying expressions.
- not distributing exponents to coefficients.
- confusing the product property and the property of raising a power to a power.

Because of these difficulties, I go as quickly as possible over Problems 1–12 and the related concepts so that I have plenty of time to discuss Problems 13–20.

Textbook and Workbook Exploration

The 20-minute "Section Opener: Properties of exponents" exploration helps groups discover the property of raising a quotient to a power, the property of raising a power to a power, and the quotient property.

Textbook and Workbook Exploration

The 15-minute "Properties of exponents" exploration addresses common student errors.

SECTION 7.7 LECTURE NOTES

Objectives

1. Describe the meaning of a negative-integer exponent.
2. Simplify power expressions containing negative-integer exponents.
3. Work with models that have negative-integer exponents.
4. Use scientific notation.

Main point: Describe the meaning of a negative exponent and simplify power expressions.

OBJECTIVE 1

Here we write $\frac{x^2}{x^5}$ in two different forms:

$$\frac{x^2}{x^5} = x^{2-5} = x^{-3}$$
$$\frac{x^2}{x^5} = \frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^3}$$

Equating the two results, we have

$$x^{-3} = \frac{1}{x^3}$$

Definition *Negative-integer exponent*

If n is a counting number and x is nonzero, then $x^{-n} = \frac{1}{x^n}$.

OBJECTIVE 2

Part of simplifying a power expression includes writing it so that each exponent is positive.

Simplify.

1. 7^{-2}

2. b^{-4}

$$\begin{aligned}
\frac{1}{x^{-n}} &= 1 \div x^{-n} && \frac{a}{b} = a \div b \\
&= 1 \div \frac{1}{x^n} && \text{Write power so exponent is positive: } x^{-n} = \frac{1}{x^n} \\
&= 1 \cdot \frac{x^n}{1} && \text{Multiply by reciprocal of } \frac{1}{x^n}\text{, which is } \frac{x^n}{1}. \\
&= x^n && \text{Simplify.}
\end{aligned}$$

So, $\frac{1}{x^{-n}} = x^n$.

Negative-Integer Exponent in a Denominator

If x is nonzero and n is a counting number, then $\dfrac{1}{x^{-n}} = x^n$.

Simplify.

3. $\dfrac{1}{5^{-2}}$

4. $\dfrac{1}{x^{-8}}$

5. $\dfrac{a^{-3}}{b^6}$

6. $\dfrac{x^4}{y^{-9}}$

7. $\dfrac{5x^{-4}}{2y^{-8}}$

8. $\dfrac{6a^5b^{-3}}{12c^{-2}}$

Properties of Integer Exponents

If m and n are integers and x and y are nonzero, then

1. $x^m x^n = x^{m+n}$ Product property for exponents

2. $\dfrac{x^m}{x^n} = x^{m-n}$ Quotient property for exponents

3. $(xy)^n = x^n y^n$ Raising a product to a power

4. $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$ Raising a quotient to a power

5. $(x^m)^n = x^{mn}$ Raising a power to a power

Simplifying Power Expressions

A power expression is simplified if

1. It includes no parentheses.

2. In any monomial, each variable or constant appears as a base at most once.

3. Each numerical expression (such as 7^2) has been calculated, and each numerical fraction has been simplified.

4. Each exponent is positive.

Simplify.

9. $(x^6)^{-3}$

10. $(4x^3)(-5x^{-6})$

11. $\dfrac{b^3}{b^8}$

12. $\dfrac{7x^4}{3x^{-3}}$

13. $\dfrac{14b^4c^{-7}}{21b^{-2}c^{-3}}$

14. $(3x^{-4})^{-3}$

15. $\left(\dfrac{2x^5}{y^{-2}}\right)^{-4}$

16. $\dfrac{(5a^{-4}b)^{-2}}{(2cd^{-5})^{-3}}$

OBJECTIVE 3

17. The weight of an astronaut w (in pounds) when d thousand miles from the center of Earth is described by the equation $w = 3060d^{-2}$.

a. Simplify the right-hand side of the equation.

b. When at sea level, the astronaut is about 4 thousand miles from Earth's center. How much does the astronaut weigh at sea level?

c. Substitute 6 for d in the equation found in part (a), and solve for w. What does the result mean in this situation?

OBJECTIVE 4

Definition *Scientific notation*

A number is written in **scientific notation** if it has the form $N \times 10^k$, where k is an integer and the absolute value of N is between 1 and 10 or is equal to 1.

Examples: 6.74×10^7, 9.2×10^{-6}

Write the number in standard decimal notation.

18. 5.36×10^3 **19.** 8.192×10^6 **20.** 4.6×10^{-3} **21.** 2.99×10^{-5}

Converting from Scientific Notation to Standard Decimal Notation

To write the scientific notation $N \times 10^k$ in standard decimal notation, we move the decimal point of the number N as follows:

- If k is *positive*, we multiply N by 10 k times and hence move the decimal point k places to the *right*.
- If k is *negative*, we divide N by 10 k times and hence move the decimal point k places to the *left*.

Write the number in scientific notation.

22. The first evidence of life on Earth dates back to 3.6×10^9 years ago.

23. The wavelength of violet light is about 0.00000047 meter.

Converting from Standard Decimal Notation to Scientific Notation

To write a number in scientific notation, count the number k of places that the decimal point needs to be moved so that the absolute value of the new number N is between 1 and 10 or is equal to 1.

- If the decimal point is moved to the left, then the scientific notation is written as $N \times 10^k$.
- If the decimal point is moved to the right, then the scientific notation is written as $N \times 10^{-k}$.

- Most calculators represent 7.29×10^{26} as 7.29 E 26.
- Most calculators represent 5.96×10^{-23} as 5.96 E -23.

SHORT HW 1, 17, 27, 35, 49, 51, 55, 61, 63, 65, 69, 73, 83, 95, 99, 103

MEDIUM HW 1, 11, 17, 19, 27, 29, 31, 35, 43, 49, 51, 55, 61, 63, 65, 69, 73, 83, 95, 99, 103, 111, 113, 121, 123, 125

SECTION 7.7 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students learn the meaning of a negative exponent and simplify power expressions containing negative-integer exponents. Students also use scientific notation.

OBJECTIVE 1 COMMENTS

Most students have difficulty with negative exponents. The work showing why $x^{-3} = \dfrac{1}{x^3}$ doesn't help students' intuition much. The statement $x^{-n} = \dfrac{1}{x^n}$ seems mysterious to most students.

Background Information for the Exploration That Follows

In the following 15-minute exploration, groups discover the meaning of a negative-integer exponent. This may be the best way to help students gain some confidence with negative-integer exponents. Some groups may need a hint with Problem 6 of the exploration.

Group Exploration

Negative-integer exponent

In this exploration, you will explore the meaning of a negative-integer exponent.

1. Use your graphing calculator to compute 2^{-1}. Then press MATH **1** ENTER to convert your result into a fraction.

2. Use your graphing calculator to compute 3^{-1}. Then press MATH **1** ENTER to convert your result into a fraction.

3. For each part, use your graphing calculator to compute the expression and then convert the result into a fraction.

   a. 4^{-1} **b.** 5^{-1} **c.** 6^{-1}

4. Review your work in Problems 1–3. What pattern do you notice? If you do not see a pattern, then keep working with more powers of the form x^{-1} until you do.

5. Without using any type of calculator, write 13^{-1} as a fraction.

6. Write x^{-1} as a fraction.

7. For each part, use your graphing calculator to compute the expression and then convert the result into a fraction.

   a. 3^{-2} **b.** 4^{-2} **c.** 5^{-2}

8. Review your work in Problem 7. What pattern do you notice? If you do not see a pattern, then keep working with more powers of the form x^{-2} until you do.

9. Without using any type of calculator, write 9^{-2} as a fraction.

10. Write x^{-2} as a fraction.

11. Without using any type of calculator, write each expression as a fraction.

a. 8^{-1} **b.** 7^{-2} **c.** 4^{-3} **d.** 2^{-5}

12. Write each expression as a fraction.

a. x^{-3} **b.** x^{-4} **c.** x^{-5} **d.** x^{-6}

13. Write the expression x^{-n} as a fraction.

Workbook Exploration

The following 15-minute exploration is another option for groups to discover the meaning of a negative exponent.

EXPLORATION *Negative-integer exponent definition*

1. Complete the following table with ordered-pair solutions of $y = 2^x$. For results that are less than 1, use fractions, not decimal numbers.

x	2^x Before Simplifying	2^x After Simplifying
5	2^5	32
4		
3		
2		
1		
0		
-1		
-2		
-3		
-4		
-5		

2. Look for a pattern in your completed table that suggests a value for 2^{-6}. Write your result as a fraction. We say we have *simplified* 2^{-6}.

3. Simplify. Write your result as a fraction. If you have difficulty doing this, construct a table similar to the one you constructed in Problem 1.

a. 5^{-2} **b.** 3^{-4} **c.** 4^{-3} **d.** 7^{-1}

4. Simplify.

a. x^{-2} **b.** b^{-4} **c.** y^{-3} **d.** t^{-1}

5. Simplify b^{-n}, where n is a positive integer.

OBJECTIVE 2 COMMENTS

Most students have a lot of trouble with this objective, especially when they must use more than one property of exponents to simplify a power expression. I go as quickly as possible over Problems 1–8 and the related concepts so that students can have as much time as possible to work on Problems 9–16.

For Problem 7, many students write $\frac{5x^{-4}}{2y^{-8}} = \frac{2y^8}{5x^4}$; they think that the coefficient 5 is permanently joined to the power x^{-4} and that the coefficient 2 is permanently joined to the power y^{-8}. Students think this because of their work with adding monomials. For example, when we apply the commutative law to $5x^{-4} + 2y^{-8}$, the coefficients "stay" with their original powers: $2y^{-8} + 5x^{-4}$.

The following work helps correct students' misconceptions:

$$\begin{aligned}\frac{5x^{-4}}{2y^{-8}} &= \frac{5}{2} \cdot \frac{x^{-4}}{1} \cdot \frac{1}{y^{-8}} \\ &= \frac{5}{2} \cdot \frac{1}{x^4} \cdot \frac{y^8}{1} \\ &= \frac{5y^8}{2x^4}\end{aligned}$$

I emphasize that for a simplified power expression, each exponent is positive. Many students forget this on exams.

Textbook and Workbook Exploration

The 10-minute "Properties of exponents" exploration address several common student errors.

OBJECTIVE 3 COMMENTS

For parts (b) and (c) of Problem 17, I remind students that their results should include units.

OBJECTIVE 4 COMMENTS

Deemphasizing scientific notation is one way to handle getting through this long section.

Sometimes I motivate scientific notation by asking students to use their graphing calculators to compute 2^{50} or 2^{-50}. Other times, I motivate scientific notation by working with the national debt.

For converting scientific notation to and from standard decimal notation, students are not sure whether to move the decimal point to the right or left. I encourage students to reflect on what makes sense, rather than to memorize rules. For example, for 3.56×10^{-8}, 3.56 is divided by 10^8 (a product of eight 10's), so we need to move the decimal point 8 places to the left.

SECTION 7.8 LECTURE NOTES

Objectives

1. Divide a polynomial by a monomial.
2. Use long division to divide a polynomial by a binomial.

Main point: Divide a polynomial by a monomial and perform long division.

OBJECTIVE 1 COMMENTS

Dividing by a Monomial

If A, B, and C are monomials and B is nonzero, then

$$\frac{A+C}{B} = \frac{A}{B} + \frac{C}{B}$$

Warning: When dividing a polynomial by a monomial, make sure you divide *every* term of the polynomial by the monomial.

Divide.

1. $\dfrac{8x^2+6x}{2x}$

2. $\dfrac{9x^3-15x^2}{-3x^2}$

3. $\dfrac{12p^4+5p^2-8p}{-4p^3}$

4. $\dfrac{24w^4-7w^2-18w}{6w^2}$

5. $\dfrac{25x^3y-35x^2y^2+10xy^4}{-5xy^2}$

6. $\dfrac{12a^4b^2-3a^3b^3+6ab^4}{3a^3b}$

OBJECTIVE 2 COMMENTS

Begin by reminding students how to perform long division with numbers. Also, review how to check the work.

Perform long division.

7. $\dfrac{2x^2+11x+15}{x+3}$ [Remainder is 0.]

8. $\dfrac{12x^2-17x+6}{3x-2}$ [Remainder is 0.]

9. $\dfrac{2x^2-7x-18}{x-5}$ [Remainder is not 0.]

10. $\dfrac{4x^3-8x^2+9x-7}{2x-3}$ [Remainder is not 0.]

11. $\dfrac{27x^3-8}{3x-2}$ [Remainder is 0.]

12. $\dfrac{6x^3-8x^2+9x-16}{2x^2+3}$ [Remainder is not 0.]

SHORT HW 3, 7, 9, 11, 17, 19, 25, 29, 33, 35, 43, 47, 49, 51, 55

MEDIUM HW 1, 3, 7, 9, 11, 13, 17, 19, 21, 25, 29, 33, 35, 37, 43, 45, 47, 49, 51, 55, 59, 61, 63, 67, 69

SECTION 7.8 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students divide a polynomial by a monomial and perform long division. The content of this section will not be needed in subsequent sections, except for the Related Review exercises in Section 10.2, where long division is compared to the technique of factoring the numerator and denominator and then using the property $\frac{AB}{AC} = \frac{B}{C}$, where $A \neq 0$.

OBJECTIVE 1 COMMENTS

Begin by reminding students that we add fractions with a common denominator by using the property $\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}$. So, it follows that we can divide a polynomial by a monomial by using the property $\frac{A+C}{B} = \frac{A}{B} + \frac{C}{B}$.

Textbook and Workbook Exploration

Although a few groups may have a bit of trouble with signs and/or exponents in Problems 3 and 4, most groups will have an easy time with the 10-minute exploration "Dividing by a monomial" exploration.

OBJECTIVE 2 COMMENTS

If you haven't taught this course before, you'll be surprised by how many of your students don't recall how to perform long division with numbers. Nevertheless, one quick example sets the stage well for performing long division with polynomials. Students often forget to change the signs before adding terms, so make sure to emphasize this. Although it isn't absolutely necessary to use expressions such as $0x^2$ for missing terms, it helps students line up like terms very well.

Many students tend to think that the answer is just the remainder, so I insist that they write their answer separate from their work.

CHAPTER 8 OVERVIEW

In Section 8.1, students factor trinomials of the form $x^2 + bx + c$ and factor differences of two squares. In Section 8.2, students factor out the GCF of polynomials and factor polynomials by grouping. They factor trinomials of the form $ax^2 + bx + c$ in Section 8.3. In Section 8.4, students factor sums and differences of cubes and learn a factoring strategy. In Section 8.5, students solve polynomial equations. They use quadratic models to make estimates and predictions in Section 8.6.

SECTION 8.1 LECTURE NOTES

Objectives

1. Explain why multiplying and *factoring* are reverse processes.
2. Factor a trinomial of the form $x^2 + bx + c$.
3. Describe the meaning of a *prime* polynomial.
4. Factor a difference of two squares.

Main point: Factor polynomials.

OBJECTIVE 1

Find the product $(x + 2)(x + 4)$.

- Writing $x^2 + 6x + 8$ as $(x + 2)(x + 4)$ is called *factoring*.
- We **factor** a polynomial by writing it as a product.
- We say $(x + 2)(x + 4)$ is a **factored polynomial** and that both $(x + 2)$ and $(x + 4)$ are factors of the polynomial.
- Multiplying and factoring are reverse processes.

OBJECTIVE 2

Find the product $(x + p)(x + q)$:

$$\begin{aligned}(x + p)(x + q) &= x^2 + qx + px + pq \\ &= x^2 + px + qx + pq \\ &= x^2 + (p + q)x + pq\end{aligned}$$

In the result, the coefficient of x is the sum of the last terms p and q and the constant term is the product of the last terms p and q.

1. Factor $x^2 + 9x + 20$.

Factoring $x^2 + bx + c$

To factor $x^2 + bx + c$, look for two integers p and q whose product is c and whose sum is b. That is, $pq = c$ and $p + q = b$. If such integers exist, the factored polynomial is

$$(x + p)(x + q)$$

Factor.

2. $x^2 + 11x + 28$
3. $x^2 - 7x + 12$
4. $b^2 - 6b + 9$
5. $x^2 - 5x - 14$
6. $x^2 + 3x - 18$
7. $w^2 - 5w - 24$
8. $p^2 + 7pq + 10q^2$
9. $x^2 - 4xy - 32y^2$

Factoring $x^2 + bx + c$ with c Positive

- To factor a trinomial of the form $x^2 + bx + c$ with a positive constant term c,
  - * If b is positive, look for *two positive* integers whose product is c and whose sum is b.
  - * If b is negative, look for *two negative* integers whose product is c and whose sum is b.

Factoring $x^2 + bx + c$ with c Negative

To factor a trinomial of the form $x^2 + bx + c$ with a negative constant term c, look for two integers with *different* signs whose product is c and whose sum is b.

OBJECTIVE 3

10. Factor $x^2 + 9x + 21$.

A polynomial that cannot be factored is called **prime**.

OBJECTIVE 4

Difference of Two Squares

$$A^2 - B^2 = (A + B)(A - B)$$

Factor.

11. $x^2 - 9$

12. $x^2 - 81$

13. $4r^2 - 25$

14. $36k^2 - 49$

15. $49x^2 - 64y^2$

16. $25p^2 - 16w^2$

A polynomial of the form $x^2 + k^2$, where $k \neq 0$, is prime. For example, $x^2 + 9$ is prime.

SHORT HW 1, 11, 17, 29, 35, 41, 45, 49, 57, 61, 63, 71, 75, 77, 87

MEDIUM HW 1, 11, 17, 29, 35, 41, 45, 49, 57, 59, 61, 63, 67, 71, 75, 77, 81, 83, 87, 89, 91, 97, 99, 101

SECTION 8.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students factor trinomials of the form $x^2 + bx + c$ and factor differences of squares in this section. They will use these skills many times in the course.

OBJECTIVE 1 COMMENTS

Most students tend to think of factoring as a procedure. Some students do not realize that we factor a polynomial by writing it as a product.

OBJECTIVE 2 COMMENTS

To factor a polynomial of the form $x^2 + bx + c$, I usually do not break this process into cases of signs of b and c.

It works fine to leave it at the general strategy of trying to find two numbers whose product is c and whose sum is b.

For Problem 8, I first factor $p^2 + 7p + 10$ and then insert the variable q in the appropriate places. I use the same tactic with Problem 9.

Some of my students need yet another reminder that they can verify their work using graphing calculator tables or graphs.

Workbook Exploration

Groups can discover the key idea behind factoring polynomials of the form $x^2 + bx + c$ by completing the following 15-minute exploration. Problem 4 gets across why the patterns discovered in this exploration only hold if the coefficient of x^2 is 1.

Group Exploration

Section Opener: Factoring trinomials

The product $(x + 4)(x + 8)$ is equivalent to $x^2 + 12x + 32$ (try it). Working backward, we write

$$x^2 + 12x + 32 = (x + 4)(x + 8)$$

and we call the process factoring. We say $(x+4)(x+8)$ is a factored polynomial. In general, a factored polynomial is a product of two or more polynomials.

1. Complete the following table.

Factored Polynomial	Last Terms	Product of Factored Polynomial	Coefficient of x	Constant Term
$(x + 3)(x + 4)$	3 and 4	$x^2 + 7x + 12$	7	12
$(x + 5)(x - 3)$	5 and -3	$x^2 + 2x - 15$	2	-15
$(x - 2)(x + 6)$				
$(x - 4)(x - 3)$				

2. For each row of the table you completed, what connection do you notice between the last terms of the factored polynomial and the coefficient of x of the product of the factored polynomial? Explain why this happens.

3. For each row of the table you completed, what connection do you notice between the last terms of the factored polynomial and the constant term of the product of the factored polynomial? Explain why this happens.

4. **a.** Do the observations you have made in this exploration apply to the polynomial $(2x + 3)(x + 4)$? If yes, show that they do. If no, explain why not in terms of how you find the product $(2x + 3)(x + 4)$.

b. Do the observations you have made in this exploration apply to the polynomial $(x + 5)(3x + 4)$? If yes, show that they do. If no, explain why not in terms of how you find the product $(x + 5)(3x + 4)$.

c. Discuss, in general, when your observations apply and when they do not.

OBJECTIVE 3 COMMENTS

For Problem 10, I ask my students to factor $x^2 + 9x + 21$ without any preamble about prime polynomials. Once they determine that we cannot factor the polynomial, I tell them that the polynomial is called *prime*. I then compare the meaning of a prime polynomial with the meaning of a prime number.

OBJECTIVE 4 COMMENTS

Some of my students think that a sum of squares can be factored. To dismantle that belief, I write $x^2 + 9$ on the board and ask how to factor it. Someone usually suggests $(x + 3)(x + 3)$ and we observed that the product is $x^2 + 6x + 9$, not $x^2 + 9$. Someone else usually suggests $(x + 3)(x - 3)$ and we observe that the product is $x^2 - 9$, not $x^2 + 9$. It is about this time that the class concludes that $x^2 + 9$ is prime.

Textbook and Workbook Exploration

Teams can discover how to factor differences of squares by completing the 20-minute "Factoring the difference of two squares" exploration.

SECTION 8.2 LECTURE NOTES

Objectives

1. Factor out the *greatest common factor (GCF)* of a polynomial.
2. Factor polynomials completely.
3. Factor out the opposite of the GCF of a polynomial.
4. Factor a polynomial by grouping.

Main point: Factor the *greatest common factor (GCF)* of a polynomial and factor a polynomial *by grouping*.

OBJECTIVE 1

Factor.

1. $5x + 10$ **2.** $12x^2 - 20x$ **3.** $15w^3 + 12w^2$

Definition *Greatest common factor*

The **greatest common factor (GCF)** of two or more terms is the monomial with the largest coefficient and the largest degree that is a factor of all the terms.

Factor.

4. $25p^3 + 15p^2$
5. $2x^3 - 6x^2 + 14x$
6. $8a^2b^2 - 12ab^3$
7. $6p^4q + 12p^3q - 9pq$

OBJECTIVE 2

If a result cannot be factored further, it is said to be **factored completely**.

8. $3x^2 - 12$
9. $5t^2 + 15t - 50$
10. $36x^3 - 16x$
11. $75x + 3x^3 - 30x^2$
12. $2a^4b - 2a^3b^2 - 12a^2b^3$

When factoring a polynomial, always factor it *completely*.

OBJECTIVE 3

How to Factor when the Leading Coefficient Is Negative

When the leading coefficient of a polynomial is negative, we first factor out the opposite of the GCF.

Factor.

13. $-2x^3 - 8x^2 + 24x$ **14.** $-5x^3 + 45x$ **15.** $-x^2 + 6x - 9$

OBJECTIVE 4

- First, factor $ax + 5x$: $ax + 5x = (a + 5)x$.
- Next, use the above work to suggest how to factor $a(b + 3) + 5(b + 3)$:

$$a(b + 3) + 5(b + 3) = (a + 5)(b + 3)$$

- Finally, factor $ab + 3a + 5b + 15$:

$$ab + 3a + 5b + 15 = a(b + 3) + 5(b + 3) = (a + 5)(b + 3)$$

16. Factor $5x^3 - 15x^2 + 2x - 6$.

Factoring by Grouping

For a polynomial with four terms, we **factor by grouping** (if it can be done) by

1. Factoring the first two terms and the last two terms.
2. Factoring out the binomial GCF.

Factor.

17. $12x^3 + 9x^2 - 8x - 6$

18. $20x^3 + 8x^2 - 45x - 18$

19. $3a - 3b - ax + xb$

20. $12x^3 - 21x^2 + 4x - 7$

SHORT HW 5, 13, 15, 25, 27, 31, 39, 41, 43, 47, 53, 59, 81, 83, 90

MEDIUM HW 1, 5, 13, 15, 17, 25, 27, 31, 35, 39, 41, 43, 47, 53, 59, 73–83 odd, 90, 95, 97, 101, 103

SECTION 8.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students factor out the GCF of a polynomial and factor a polynomial by grouping. Students will use this material many times in the rest of the course.

OBJECTIVE 1 COMMENTS

For Problem 2, I point out that if we only factor out 2, 4, or x, one of the factors of the result can still be (and should be) factored. This motivates factoring out the GCF $4x$, because by doing so, the result is factored completely.

Textbook and Workbook Exploration

In the 15-minute "Factoring out the greatest common factor" exploration, teams discover how to factor out the GCF of a polynomial.

OBJECTIVE 2 COMMENTS

I emphasize that when we factor a polynomial, we always factor it *completely*. Many of my students either do not know or forget to factor a polynomial completely.

OBJECTIVE 3 COMMENTS

When the leading coefficient of a polynomial is negative, many students fail to first factor out the opposite of the

GCF. Sorting this out with students now will pay dividends not only with the rest of Chapter 8, but also when working with rational expressions in Chapter 10.

For Problem 15, some students do not see that they can factor out -1.

OBJECTIVE 4 COMMENTS

For Problem 18, I remind my students that for the expression $-45x - 18$, we factor out the opposite of the GCF to get

$$4x^2(5x+2) - 9(5x+2)$$

before factoring out the common factor $(5x+2)$. Some students think that the expression

$$4x^2(5x+2) - 9(5x+2)$$

is factored completely. I remind these students that we factor a polynomial by writing it as a *product*. Finally, some students forget to consider whether the expression $(4x^2-9)(5x+2)$ is factored completely.

For Problem 20, it does not occur to many of my students to factor out a 1 from the polynomial $4x-7$ to get $3x^2(4x-7) + 1(4x-7)$ before then factoring out the common factor $(4x-7)$.

Workbook Exploration

Problems 1–3 of the following 15-minute exploration illustrate errors most students tend to make. It's well worth the time to have students work on this.

Group Exploration

Factoring out the GCF and factoring by grouping

For each of Problems 1–4, a student tries to factor a polynomial. If the result is correct, decide whether there is a more efficient way to factor the polynomial. If the result is incorrect, describe any errors and factor the polynomial correctly.

1. $3m^3 - 12m^2n - 36mn^2 = 3m(m^2 - 4mn - 12n^2)$

2. $4x^3y^2 + 8x^2y^3 - 12xy^4 = 2xy^2(2x^2 + 4xy - 6y^2)$

3. $8x^3 - 20x^2 - 18x + 45 = 4x^2(2x-5) - 9(2x-5)$

4. $$\begin{aligned} 4x^2 - 16 &= (2x+4)(2x-4) \\ &= 2(x+2)2(x-2) \\ &= 4(x+2)(x-2) \end{aligned}$$

SECTION 8.3 LECTURE NOTES

Objectives

1. Factor a trinomial by trial and error.
2. Factor out the GCF and then factor by trial and error.
3. Rule out possibilities when factoring by trial and error.
4. Factor a trinomial by grouping.

Main point: Factor trinomials of the form $ax^2 + bx + c$, where $a \neq 1$.

OBJECTIVE 1

When factoring by trial and error, it is extremely helpful to find the product of two binomials in one step.

Factor by trial and error.

1. $3x^2 + 17x + 10$

2. $2x^2 - 5x - 12$

3. $4w^2 - 16w + 15$

4. $9p^2 + 12pq + 4q^2$

Factoring $ax^2 + bx + c$ by Trial and Error

To **factor a trinomial** of the form $ax^2 + bx + c$ **by trial and error,** if the trinomial can be factored as a product of two binomials, then the product of the coefficients of the first terms of the binomials is equal to a and the product of the last terms of the binomials is equal to c. To find the correct factored expression, multiply the possible products and identify the one for which the coefficient of x is b.

OBJECTIVE 2

Recall:

- When factoring a polynomial, if the GCF is not 1, then we first factor out the GCF (or its opposite) and continue factoring if possible.
- Always factor a polynomial *completely*.

Factor.

5. $4x^2 + 18x - 36$

6. $60x^4 - 110x^3 - 350x^2$

OBJECTIVE 3

Factor by trial and error. Rule out possible factorizations to help speed up the process.

7. $6x^2 - 5x - 4$

8. $8a^3b - 49a^2b^2 + 6ab^3$

OBJECTIVE 4

Factoring $ax^2 + bx + c$ by Grouping

To **factor a trinomial** of the form $ax^2 + bx + c$ **by grouping** (if it can be done),

1. Find pairs of numbers whose product is ac.
2. Determine which of the pairs of numbers from step 1 has the sum b. Call this pair of numbers m and n.
3. Write the bx term as $mx + nx$:

$$ax^2 + bx + c = ax^2 + mx + nx + c$$

4. Factor $ax^2 + mx + nx + c$ by grouping.

Another name for this technique is the ***ac*** **method.**

Factor the trinomial by grouping.

9. $3x^2 + 17x + 10$
10. $2x^2 + 11x + 15$
11. $2x^2 - 5x - 12$
12. $4w^2 - 16w + 15$
13. $3x^2 - 10xy - 8y^2$
14. $9p^2 + 12pq + 4q^2$
15. $12k^3 - 28k^2 + 8k$
16. $20t^3w - 38t^2w + 12tw$

SHORT HW 1, 7, 13, 27, 29, 43, 51, 59, 63, 65, 67, 69, 71, 75, 79

MEDIUM HW 1, 7, 13, 27, 29, 37, 43, 45, 51, 59, 63, 65, 67, 69, 71, 73, 75, 79, 81, 85, 87, 95, 97

SECTION 8.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students factor trinomials of the form ax^2+bx+c, where $a \neq 1$, by trial and error and by grouping. Students will use these skills throughout most of the rest of the text.

OBJECTIVE 1 COMMENTS

I remind my students that for trinomials of the form $ax^2 + bx + c$, where $a = 1$, we can use the method discussed in Section 8.1. I emphasize that for trinomials of the form $ax^2 + bx + c$, where $a \neq 1$, we must use one of the two methods discussed in this section.

Due to time constraints, when factoring trinomials of the form $ax^2 + bx + c$, you will likely have to choose between showing students how to factor by trial and error or to factor by grouping. One advantage of using trial and error is that it requires students to find lots of products, and so students will become proficient at finding products. Another advantage is that the procedure naturally reinforces the concept that factoring is the reverse process of expanding, and vice versa. However, most students seem to be more successful at using grouping than using trial and error.

Some students' greatest challenge with using trial and error is that they lack the organizational skills to methodically exhaust all possible factorizations. Students tend to try a few possibilities and then stop or retry possibilities, sometimes without realizing that they're revisiting a possible factorization. So, any type of organizational structure you can offer to students will likely greatly improve their success in using this method.

OBJECTIVE 2 COMMENTS

The two most prevalent issues with factoring are that students forget to factor out the GCF first and they forget to factor completely. Probably the best way to convince students to factor out the GCF first is to have them factor a polynomial such as $6x^2 + 30x + 36$, which is much easier to factor if the GCF is factored out first.

OBJECTIVE 3 COMMENTS

I don't spend much time discussing how to rule out possibilities while factoring by trial and error, because I suspect that few of my students will remember to use such a strategy in the midst of hunting down a factorization. However, a brief mention probably helps my stronger students.

OBJECTIVE 4 COMMENTS

For factoring trinomials by grouping, I do not try to explain why the four-step algorithm works, because I do not think many of my students would follow such an explanation. Nonetheless, students are quite successful at using this approach, provided that the coefficients of the polynomial are relatively small.

SECTION 8.4 LECTURE NOTES

Objectives

1. Factor a sum or difference of two cubes.
2. Apply a factoring strategy.

Main point: Factor a sum or difference of two cubes and learn a factoring strategy.

OBJECTIVE 1

Show that $(A+B)(A^2-AB+B^2)=A^3+B^3$.

Sum or Difference of Two Cubes

$A^3+B^3=(A+B)(A^2-AB+B^2)$ Sum of two cubes
$A^3-B^3=(A-B)(A^2+AB+B^2)$ Difference of two cubes

Remind students of the following cubes:

$$2^3=8 \qquad 3^3=27 \qquad 4^3=64 \qquad 5^3=125 \qquad 10^3=1000$$

Factor.

1. x^3+27
2. x^3-8
3. $8x^3-125$
4. $64w^3+27$
5. $27p^3-q^3$
6. $125t^3+8w^3$
7. $4x^3+32$
8. $16x^3-54$

OBJECTIVE 2

Five-Step Factoring Strategy

The five steps that follow can be used to factor many polynomials. Steps 2–4 can be applied to the entire polynomial or to a factor of the polynomial.

1. If the leading coefficient is positive and the GCF is not 1, we factor out the GCF. If the leading coefficient is negative, we factor out the opposite of the GCF.
2. For a binomial, try using one of the properties for the difference of two squares, the sum of two cubes, or the difference of two cubes.
3. For a trinomial of the form ax^2+bx+c,
   - **a.** If $a=1$, try to find two integers whose product is c and whose sum is b.
   - **b.** If $a\neq 1$, try to factor by trial and error or by grouping.
4. For an expression with four terms, try factoring by grouping.
5. Continue applying steps 2–4 until the polynomial is factored completely.

Factor.

9. $3w^4 - 27w^3 + 60w^2$
10. $x^3 - 27$
11. $4m^2 - 16m + 15$
12. $80x^2 - 45$
13. $3x^3 - 2x^2 - 27x + 18$
14. $40 - x^2 - 3x$
15. $6a^2b - 32ab + 10b$
16. $x^2 - 2x - 48$
17. $25p^2 + 9$
18. $8a^3 + 125b^3$

SHORT HW 1, 5, 11, 13, 19, 29, 39, 45, 47, 51, 65, 69, 73, 81, 88

MEDIUM HW 1, 5, 9, 11, 13, 19, 29, 31, 33, 39, 45, 47, 51, 57, 61, 65, 69, 71, 73, 81, 88, 89, 95

SECTION 8.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students factor sums of cubes and differences of cubes in this section. Students also learn a factoring strategy. They will use these skills throughout most of the rest of the text.

OBJECTIVE 1 COMMENTS

It will greatly benefit students to memorize the cubes of 2, 3, 4, 5, and 10. These are the only cubes that students will need to know to complete the exercises in Homework 8.4.

For Problem 1, I write the formula $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$ above the equation $x^3 + 27 = (x + 3)(x^2 - x \cdot 3 + 3^2)$ and draw arrows between the terms (see Example 1 on page 455 of the textbook for a similar use of arrows). I use a similar approach for Problems 2–8 until my students get the idea.

When factoring the sum of two cubes, many of my students want to write the incorrect statement $A^3 + B^3 = (A + B)(A^2 - 2AB + B^2)$. They also tend to write a similar incorrect statement for the difference of two cubes. Some students also have trouble with the signs of the coefficients.

Workbook Exploration

Students have such trouble with factoring sums and differences of cubes, the following exploration might help them keep the big picture in mind. Keep close tabs on groups for Problem 1; if their result is wrong, the point of Problem 2 might be lost on them.

Group Exploration

Section Opener: Factoring a sum and a difference of two cubes

1. Find the product $(A + B)(A^2 - AB + B^2)$.
2. Refer to the result you found in Problem 1 to help you factor $x^3 + y^3$.
3. What is the cube root of 8? Use your result and the result you found in Problem 1 to help you factor $w^3 + 8$.
4. Find the product $(A - B)(A^2 + AB + B^2)$.
5. Refer to the result you found in Problem 4 to help you factor $x^3 - y^3$.
6. What is the cube root of 27? Use your result and the result you found in Problem 4 to help you factor $p^3 - 27$.

OBJECTIVE 2 COMMENTS

Once my students begin to get comfortable with all the factoring techniques discussed in Chapter 8, they still have trouble recognizing which technique is appropriate for a particular polynomial, and they will often fail to factor the polynomial completely. Spending some time discussing the five-step factoring strategy is a good idea. For each of the Problems 9–18, I walk students through the strategy, asking students lots of questions as I do so. Or, I have teams of students work on the problems and then go over a few of them, referring to the strategy as I do so.

Workbook Exploration

The following 20-minute exploration would be nice way to wrap up class.

Group Exploration

Developing a factoring strategy

In this exploration, you will summarize what you have learned about factoring.

1. When factoring a polynomial, what should you try to do first? Give an example.
2. Describe various techniques that you can use to factor a polynomial with the given number of terms. For each technique, give an example of factoring a polynomial.

   a. two terms **b.** three terms **c.** four terms

3. Explain how you know when you are done factoring a polynomial.

SECTION 8.5 LECTURE NOTES

Objectives

1. Apply the *zero factor property*.
2. Use factoring to solve *quadratic equations in one variable*.
3. Use graphing to solve a polynomial equation in one variable.
4. Combine like terms to help solve quadratic equations in one variable.
5. Solve quadratic equations in one variable that contain fractions.
6. Find any x-intercept(s) of a parabola.
7. Use factoring to solve *cubic equations in one variable*.
8. Compare solving polynomial equations with factoring polynomials.

Main point: Use factoring to solve polynomial equations.

OBJECTIVE 1

A **quadratic equation in one variable** is an equation that can be put into the form $ax^2 + bx + c = 0$, where a, b, and c are constants and $a \neq 0$.

Zero Factor Property

Let A and B be real numbers:

If $AB = 0$, then $A = 0$ or $B = 0$.

OBJECTIVE 2

Solve.

1. $(x-3)(x+5) = 0$

2. $x^2 + 3x - 28 = 0$

3. $p^2 - 7p + 10 = 0$

4. $18x^2 - 8 = 0$

5. $9x^2 = 12x - 4$

6. $5w^2 = 30w + 80$

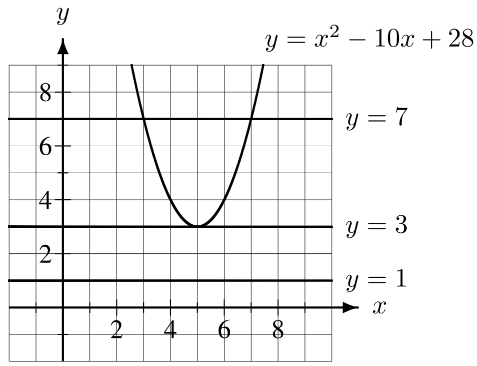
OBJECTIVE 3

The graphs of $y = x^2 - 10x + 28$, $y = 7$, $y = 3$, and $y = 1$ are shown below. Use these graphs to solve the given equation.

7. $x^2 - 10x + 28 = 7$

8. $x^2 - 10x + 28 = 3$

9. $x^2 - 10x + 28 = 1$



10. Use "intersect" to solve $x^2 - 5x + 1 = x - 3$. Round the solution(s) to the second decimal place.

Our results in Problems 7–9 suggest the following property.

Number of Solutions of a Quadratic Equation in One Variable

A quadratic equation in one variable has either two real-number solutions, one real-number solution, or no real-number solutions.

OBJECTIVE 4

Solve.

11. $x^2 - 5x = 3x - 16$

12. $x^2 - 6x = 35 - 4x$

13. $(p - 4)(p - 3) = 2$

OBJECTIVE 5

Solve.

14. $\frac{1}{6}x^2 - \frac{1}{3}x = \frac{4}{3}$

15. $\frac{3}{2}x^2 = \frac{1}{2} - \frac{1}{4}x$

OBJECTIVE 6

Find all x-intercepts.

16. $y = x^2 + 4x - 45$

17. $y = x^2 - 10x + 25$

OBJECTIVE 7

A **cubic equation in one variable** is an equation that can be put into the form $ax^3 + bx^2 + cx + d = 0$, where a, b, c, and d are constants and $a \neq 0$.

Solve.

18. $5x^3 - 30x^2 + 40x = 0$

19. $k^3 + 2k^2 = 4k + 8$

Solving Quadratic and Cubic Equations in One Variable

If an equation can be solved by factoring, we solve it by the following steps:

1. Write the equation so that one side of the equation is equal to zero.
2. Factor the nonzero side of the equation.
3. Apply the zero factor property.
4. Solve each equation that results from using the zero factor property.

OBJECTIVE 8

When we solve any equation, our objective is to find any *numbers* that satisfy the equation. When we factor a polynomial, our objective is to write the polynomial as a product of polynomials, which is an *expression.*

20. Solve $x^2 + 5x + 6 = 0$.

21. Factor $x^2 + 5x + 6$.

SHORT HW 3, 11, 15, 23, 29, 33, 41, 47, 57, 63, 69, 79, 83, 87, 94

MEDIUM HW 3, 7, 9, 11, 15, 17, 19, 23, 29, 33, 41, 47, 49, 53, 57, 59, 63, 69, 75, 79, 83, 85, 87, 91, 93

SECTION 8.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students use factoring to solve polynomial equations in this section. They will use this skill for most of the rest of the course. Students also find x-intercepts of parabolas and use graphing to solve quadratic equations in one variable.

OBJECTIVE 1 COMMENTS

I ask my students, "If $AB = 0$, what do we know about A or B?" Students always know the answer.

OBJECTIVE 2 COMMENTS

Most students who are proficient at factoring do well with Problems 1–6.

For Problems 5 and 6, I say that in order to apply the zero factor property, one side of the equation must be 0.

For Problem 6, some students are baffled when they arrive at the equation $5(w - 8)(w + 2) = 0$. I ask my students "If $5AB = 0$, then what do we know about A or B?" Students then see that either $w - 8$ or $w + 2$ must be 0.

Textbook and Workbook Exploration

The 15-minute "Zero factor property" exploration is a nice way for teams to discover how to use factoring to help solve polynomial equations.

OBJECTIVE 3 COMMENTS

Although this concept was addressed in Sections 4.3 and 4.4, students will need a refresher. For Problem 10, I point out that we could not use factoring to solve $x^2 - 5x + 1 = x - 3$.

Textbook and Workbook Exploration

The 20 minute "Solving systems of quadratic equations" exploration first reviews the concepts related to using graphing to solve an equation in one variable (see Problems 1–5 of the exploration). If your students remember these concepts, you could have them just do Problem 6; you could also assign Problem 5 if you want your students to have a little more practice.

Problems 7–9 are a great way to get across how many real-number solutions are possible for a quadratic equation in one variable.

OBJECTIVE 4 COMMENTS

For Problem 13, I remind my students that one side of the equation must be zero in order to apply the zero factor property. Some students think that the solutions are 3 and 4.

OBJECTIVE 5 COMMENTS

For Problems 14 and 15, some students need to be reminded to clear each equation of fractions by multiplying by the LCD.

OBJECTIVE 6 COMMENTS

For Problems 16 and 17, I remind students that we find the x-intercepts by substituting 0 for y.

Workbook Exploration

The following 20-minute exploration is a great way to review finding intercepts (and solving related equations) for linear and quadratic equations in two variables. At various times in the semester, it's good practice to pick a theme such as solving equations and review that theme for all types discussed so far.

Group Exploration

Finding intercepts

Find all intercepts of the graph of the function.

1. $f(x) = 4(2)^x$ **2.** $f(x) = x^2 - 5x - 36$ **3.** $f(x) = -3x + 7$

4. $f(x) = 3x^2 + 4x - 4$ **5.** $f(x) = \log_3(x)$ **6.** $f(x) = 4x^3 - 12x^2 - 9x + 27$

OBJECTIVE 7 COMMENTS

Again, students who are proficient at factoring do well with Problems 18 and 19.

OBJECTIVE 8 COMMENTS

Some students confuse solving polynomial equations with factoring polynomials on exams. For example, students will swap the instructions for Problems 20 and 21. Warning students about this issue now will improve their work on polynomial expressions and equations in Chapter 8 as well as their work on rational expressions and equations in Chapter 10.

SECTION 8.6 LECTURE NOTES

Objectives

1. Use an equation of a quadratic model to make estimates and predictions.
2. Model projectile motion.
3. Model the area of rectangular objects.

Main point: Use a quadratic model to make estimates and predictions about an authentic situation. Also solve area problems.

OBJECTIVE 1

1. The global average smartphone prices are shown in the following table for various years.

Year	Global Average Smartphone Price (dollars)
2012	378
2013	332
2014	309
2015	302
2016	306
2017	324

Source: *GfK*

Let p be the global average smartphone price (in dollars) at t years since 2012.

a. Use a graphing calculator to draw a scatterplot of the data. Can the data be described better by a linear model or a quadratic model? Explain.

b. Use a graphing calculator to draw the graph of the model $p = 8t^2 - 48t + 370$ and, in the same viewing window, the scatterplot of the data. Does the model fit the data well?

c. Predict the global average smartphone price in 2022.

d. Estimate when the global average smartphone price will be \$586.

2. A company's annual profit can be modeled by $p = t^2 - 8t + 20$, where p is the annual profit (in millions of dollars) at t years since 2010. Estimate when the annual profit was \$8 million and predict when the annual profit will be \$8 million.

OBJECTIVE 2

3. A batter hits a baseball into the air. The height h (in feet) of the baseball after t seconds is given by

$$h = -16t^2 + 80t + 4$$

a. When is the baseball at a height of 68 feet? Explain why there are two such times.

b. When is the baseball at a height of 104 feet? Explain why there is one such time.

c. When is the baseball at a height of 110 feet?

OBJECTIVE 3

4. A rectangular hallway floor has an area of 64 square feet. If the length is 12 feet more than the width, find the dimensions of the floor.

SHORT HW 1, 3, 9, 13, 15, 19, 21, 23

MEDIUM HW 1, 3, 5, 9, 13, 15, 19–33 odds

SECTION 8.6 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use quadratic models to make estimates and predictions. They also solve area problems.

OBJECTIVE 1 COMMENTS

Although using factoring to make estimates or predictions when working with a quadratic model is contrived, Problem 1 and similar exercises of Homework 8.6 are a good way to get students thinking about aspects of quadratic modeling such as constructing scatterplots, evaluating the fit of a model, and deciding for which variable to substitute a value.

Students have a hard time with part (d) of Problem 1, although after eventually factoring out the GCF, the remaining factoring is easy. Likewise, students have trouble making predictions for the explanatory variable via factoring In Exercises 2–8 of Homework 8.6. In general, the difficulty stems from not getting the equation equal to 0, not factoring out the GCF, and getting overwhelmed due to the complexity of the problem.

In particular, students have the most trouble with part (b) of Exercise 7 and part (d) of Exercise 8 due to the fractional coefficients of the models.

Textbook and Workbook Exploration

The 25-minute “Using a quadratic model to make a prediction” exploration can give groups much-needed practice with quadratic modeling. You’ll have to show them how to use “maximum” on a graphing calculator in Problem 3.

OBJECTIVES 2 and 3 COMMENTS

Problem 3 is a nice opportunity for students to reflect on the connection between the results of the symbolic work and the graph of the model. For part (c), we can deduce that the baseball never reaches a height of 110 feet, because the maximum height is 104 feet [see part (b)].

CHAPTER 9 OVERVIEW

In Section 9.1, students use the product property for square roots to simplify radicals. In Section 9.2, students use the quotient property for square roots and the technique of rationalizing denominators to simplify radical expressions. Students use the square root property to solve quadratic equations in Section 9.3. They also use the Pythagorean theorem to find the length of a side of a right triangle in Section 9.3. In Section 9.4, students solve quadratic equations by completing the square. In Section 9.5, students use the quadratic formula to solve quadratic equations. They use quadratic models to make estimates and predictions in Section 9.6.

SECTION 9.1 LECTURE NOTES

Objectives

1. Describe the meaning of *square root* and *principal square root*.
2. Approximate a principal square root.
3. Apply the *product property for square roots*.
4. Use the product property for square roots to simplify radicals.
5. Simplify a square root whose radicand contains x^2.

Main point: Use the product property for square roots to simplify radicals.

OBJECTIVE 1

- A **square root** of a number a is the number we square to get a.
- The square roots of 25 are -5 and 5, because $(-5)^2 = 25$ and $5^2 = 25$.

Definition *Principal square root*

If a is a nonnegative number, then $\sqrt{a}$ is the nonnegative number we square to get a. We call $\sqrt{a}$ the **principal square root** of a.

- For example $\sqrt{49} = 7$, because $7^2 = 49$.
- The symbol "$\sqrt{}$" is called a **radical sign**.
- An expression under a radical sign is called a **radicand**.
- For $\sqrt{4x-7}$, the radicand is $4x-7$.
- A radical sign together with a radicand is called a **radical**.
- Here are some more radicals: $\sqrt{3}$, $\sqrt{x}$, $\sqrt{2x+7}$.
- An expression that contains a radical is called a **radical expression**.
- Here are some radical expressions: $\sqrt{7}$, $\sqrt{x}$, $\sqrt{3x+1}$, $2\sqrt{x+2}-9x$, $(4\sqrt{x}+3)(\sqrt{x}-5)$
- A square root of a negative number is not a real number.

Find the square root.

1. $\sqrt{64}$ **2.** $\sqrt{-64}$ **3.** $-\sqrt{64}$ **4.** $-\sqrt{-64}$

- A **perfect square** is a number whose principal square root is rational.
- Students should memorize the integer perfect squares from 0 to 225.

OBJECTIVE 2

State whether the square root is rational or irrational. If the square root is rational, find the (exact) value. If the square root is irrational, estimate its value by rounding to the second decimal place.

5. $\sqrt{23}$ **6.** $\sqrt{196}$

OBJECTIVE 3

Show students that $\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25}$. This equation suggests the following property.

Product Property for Square Roots

For nonnegative numbers a and b,

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

For example, $\sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6$.

OBJECTIVE 4

A square root is **simplified** when the radicand does not have any perfect-square factors other than 1.

Simplify.

7. $\sqrt{18}$ **8.** $-\sqrt{20}$ **9.** $\sqrt{50t}$ **10.** $3\sqrt{28x}$ **11.** $5\sqrt{27x}$

Simplifying a Square Root with a Small Radicand

To simplify a square root in which it is easy to determine the largest perfect-square factor of the radicand,

1. Write the radicand as the product of the *largest* perfect-square factor and another number.
2. Apply the product property for square roots.

Simplify.

12. $\sqrt{150}$ **13.** $\sqrt{360}$

Simplifying a Square Root with a Large Radicand

To simplify a square root in which it is difficult to determine the largest perfect-square factor of the radicand,

1. Find the prime factorization of the radicand.
2. Look for pairs of identical factors of the radicand, and rearrange the factors to highlight them.
3. Use the product property for square roots.

OBJECTIVE 5

Simplifying $\sqrt{x^2}$

If x is nonnegative, then

$$\sqrt{x^2} = x$$

Simplify.

14. $\sqrt{19x^2}$ **15.** $3\sqrt{8x^2}$ **16.** $\sqrt{45a^2b}$ **17.** $\sqrt{75t^2w^2}$ **18.** $4\sqrt{90xy^2}$

SHORT HW 7, 13, 17, 23, 25, 37, 39, 43, 45, 51, 55, 57, 59, 61, 67

MEDIUM HW 7, 11, 13, 15, 17, 23, 25, 31, 37, 39, 43, 45, 51, 55, 57, 59, 61, 67, 69, 71, 75, 77

SECTION 9.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use the product property for square roots to simplify radicals. Students will use this material in Chapters 9 and 11.

OBJECTIVE 1 COMMENTS

Most students are surprised to learn that a positive number has two square roots.

I tell my students that it is important to know the meanings of *radical*, *radicand*, and *radical sign*, because I will use these terminologies frequently in Chapters 9 and 11. I also make a big deal about memorizing the integer perfect squares from 0 to 225.

OBJECTIVE 2 COMMENTS

Students have an easy time with this objective.

Workbook Exploration

Students tend to not understand that a square root such as $\sqrt{13}$ is simply a (real) number. The following 15-minute exploration may help such students.

Group Exploration

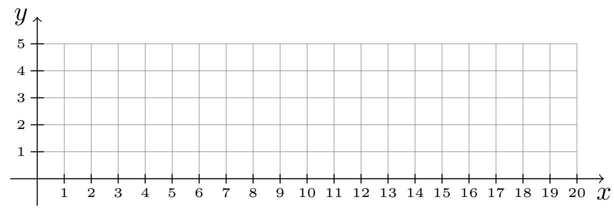
Section Opener: Estimating square roots

If a is a nonnegative number, then $\sqrt{a}$ is the nonnegative number we square to get a. We call $\sqrt{a}$ the principal square root of a. So, $\sqrt{9} = 3$ because $3^2 = 9$.

1. Complete the following table.

x	$\sqrt{x}$
0	
1	
4	
9	
16	

2. Graph $y = \sqrt{x}$.



3. Draw arrows on the coordinate system in Problem 2 to show how to find $\sqrt{4}$.

4. Draw arrows on the coordinate system in Problem 2 to help find the two consecutive integers nearest $\sqrt{7}$.

5. Draw arrows on the coordinate system in Problem 2 to help you estimate $\sqrt{13}$. Round your result to the first decimal place.

OBJECTIVE 3 COMMENTS

I tell my students that we will use the product property for square roots frequently in Chapters 9 and 10.

Textbook and Workbook Exploration

The 10-minute "Product property for square roots" exploration leads groups to discover the product property for square roots and to observe that the statement $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ is not true in general. This is a good exploration to assign, because so many students think that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ is true.

OBJECTIVE 4 COMMENTS

Students have difficulty simplifying square roots when the radicand is a relatively large number. If a student has difficulty finding the prime factorization of such a number, I encourage them to simply divide the radicand by the perfect squares in order: 4, 9, 16, ... I caution them that they need to find the largest perfect-square factor, or after simplifying (perhaps not completely), they need to see if the result's radicand still has a perfect-square factor.

OBJECTIVE 5 COMMENTS

I use several different ways to explain that $\sqrt{x^2} = x$, where x is nonnegative:

- I explain that to find $\sqrt{x^2}$, we want an expression that when squared gives x^2. I then write $?^2 = x^2$ and ask what "?" must be.

- I write the following:

$$\sqrt{3^2} = \sqrt{9} = 3$$
$$\sqrt{5^2} = \sqrt{25} = 5$$
$$\sqrt{6^2} = \sqrt{36} = 6$$

  This pattern suggests that for nonnegative values of x, $\sqrt{x^2}$ is equal to the base of x^2, or x.

- I show that squaring a number and finding the principal square root of a number are processes that "undo" each other. So, $\sqrt{x^2} = x$, where x is nonnegative.

SECTION 9.2 LECTURE NOTES

Objectives

1. Apply the *quotient property for square roots*.
2. Use the quotient property for square roots to simplify a radical.
3. *Rationalize the denominator* of a radical expression.
4. Simplify a radical quotient.
5. Summarize how to simplify a radical expression.

Main point: Use the *quotient property for square roots* and the technique of *rationalizing denominators* to simplify radical expressions.

OBJECTIVE 1

Here we compute the square root of a quotient and a quotient of square roots:

Square root of a quotient: $\sqrt{\dfrac{9}{25}} = \dfrac{3}{5}$ because $\left(\dfrac{3}{5}\right)^2 = \dfrac{9}{25}$

Quotient of square roots: $\dfrac{\sqrt{9}}{\sqrt{25}} = \dfrac{3}{5}$

Because the two results are equal, we can write

$$\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}}$$

This equation suggests the **quotient property for square roots**.

Quotient Property for Square Roots

For a nonnegative number a and a positive number b,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

OBJECTIVE 2

If a radical has a fractional radicand, we **simplify the radical** by writing it as an expression whose radicand is not a fraction.

Simplify. Assume that all variables are positive.

1. $\sqrt{\dfrac{25}{81}}$ **2.** $\sqrt{\dfrac{7}{w^2}}$ **3.** $\sqrt{\dfrac{12}{25}}$ **4.** $\sqrt{\dfrac{5a^2b}{49}}$

OBJECTIVE 3

We simplify an expression of the form $\dfrac{p}{\sqrt{q}}$ by leaving no denominator as a radical expression. We call this process **rationalizing the denominator**.

5. Simplify $\dfrac{2}{\sqrt{5}}$.

To rationalize the denominator of a fraction of the form $\dfrac{p}{\sqrt{q}}$, where q is positive, we multiply the fraction by 1 in the form $\dfrac{\sqrt{q}}{\sqrt{q}}$.

Simplify.

6. $\dfrac{7}{\sqrt{x}}$ **7.** $\sqrt{\dfrac{x}{5}}$ **8.** $\sqrt{\dfrac{5x}{18}}$ **9.** $\sqrt{\dfrac{7w^2}{20}}$ **10.** $\sqrt{\dfrac{3a^2b}{7}}$

OBJECTIVE 4

Simplify the *radical quotient*.

11. $\dfrac{8+6\sqrt{7}}{10}$

12. $\dfrac{35+15\sqrt{3}}{45}$

13. $\dfrac{14-\sqrt{12}}{8}$

14. $\dfrac{6-\sqrt{18}}{12}$

OBJECTIVE 5

Simplifying a Radical Expression

To simplify a radical expression,

1. Use the quotient property for square roots so that no radicand is a fraction.
2. Use the product property for square roots so that no radicands have perfect-square factors other than 1.
3. Rationalize denominators so that no denominator is a radical expression.
4. Use the property $\dfrac{a}{a} = 1$, where a is nonzero, to simplify.
5. Continue applying steps 1–4 until the radical expression is simplified completely.

SHORT HW 1, 11, 17, 21, 23, 31, 37, 41, 43, 45, 51, 55, 57, 59, 64

MEDIUM HW 1, 7, 11, 15, 17, 21, 23, 31, 37, 41, 43, 45, 47, 51, 55, 57, 59, 61, 64, 65, 67, 73, 75, 77

SECTION 9.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use the quotient property for square roots and the process of rationalizing denominators to simplify radical expressions. Students will use this material when using the quadratic formula in Sections 9.5 and 9.6.

OBJECTIVE 1 COMMENTS

I compare the quotient property for square roots with the product property for square roots.

OBJECTIVE 2 COMMENTS

For Problem 2, I usually have to remind students why $\sqrt{w^2} = w$, where w is nonnegative. For Problem 3, after writing $\sqrt{\frac{12}{25}} = \frac{\sqrt{12}}{\sqrt{25}}$, some students need to be reminded to simplify $\sqrt{12}$.

Textbook and Workbook Exploration

The 15-minute "Quotient property for square roots" exploration helps groups discover the quotient property and how to use it to simplify radicals. For part (b) of Problem 5 of the exploration, some groups may need to be reminded that $\sqrt{x^2} = x$, where x is nonnegative.

OBJECTIVE 3 COMMENTS

When rationalizing the denominator of an expression, I emphasize that we can only multiply the expression by a strategic form $\frac{a}{a}$, where $a \neq 0$, of the number 1.

For Problem 8, after applying the quotient property, many students multiply the expression $\frac{\sqrt{5x}}{\sqrt{18}}$ by $\frac{\sqrt{18}}{\sqrt{18}}$, when it is probably better to simplify $\frac{\sqrt{5x}}{\sqrt{18}}$ to $\frac{\sqrt{5x}}{3\sqrt{2}}$ and then multiply this result by $\frac{\sqrt{2}}{\sqrt{2}}$.

OBJECTIVES 4 and 5 COMMENTS

I go over Problems 1–10 and related concepts as quickly as possible so I have plenty of time to discuss Problems 11–14, because these problems involve an important skill when using the quadratic formula.

For Problem 11, I first remind my students that we can simplify $\frac{8}{10}$ because we can write

$$\begin{aligned}\frac{8}{10} &= \frac{2 \cdot 4}{2 \cdot 5}\\ &= \frac{2}{2} \cdot \frac{4}{5}\\ &= 1 \cdot \frac{4}{5}\\ &= \frac{4}{5}\end{aligned}$$

Next, I compare factoring $8 + 6\sqrt{7}$ with factoring $8 + 6x$. Then, I simplify $\frac{8 + 6\sqrt{7}}{10}$:

$$\begin{aligned}\frac{8 + 6\sqrt{7}}{10} &= \frac{2(4 + 3\sqrt{7})}{2(5)}\\ &= \frac{2}{2} \cdot \frac{4 + 3\sqrt{7}}{5}\\ &= 1 \cdot \frac{4 + 3\sqrt{7}}{5}\\ &= \frac{4 + 3\sqrt{7}}{5}\end{aligned}$$

Finally, I discuss how to simplify the radical quotient in fewer steps.

SECTION 9.3 LECTURE NOTES

Objectives

1. Use the *square root property* to solve a quadratic equation of the form $x^2 = k$.
2. Use the square root property to solve a quadratic equation of the form $(x+p)^2 = k$.
3. Describe the *Pythagorean theorem*.
4. Use the Pythagorean theorem to find the length of a side of a right triangle.
5. Use the Pythagorean theorem to model a situation.

Main point: Use the square root property to solve quadratic equations and use the Pythagorean theorem to find the length of a side of a right triangle.

OBJECTIVE 1

- Use factoring to show that the solutions of $x^2 = 16$ are -4 and 4.
- We can use the notation ± 4 to stand for the numbers -4 and 4.
- Discuss why the solutions of $x^2 = 25$ are $\pm\sqrt{25}$ (or ± 5).

Square Root Property

Let k be a nonnegative constant. Then $x^2 = k$ is equivalent to

$$x = \pm\sqrt{k}$$

Solve.

1. $x^2 = 9$ **2.** $x^2 = 7$

Warning:

- The equation $x^2 = 9$ has *two* solutions, -3 and 3.
- Recall from Section 9.1 that $\sqrt{9}$ is the *one* number 3.

Solve.

3. $x^2 = 12$ **5.** $r^2 - 50 = 0$ **7.** $3t^2 - 2 = 5$
4. $x^2 = -25$ **6.** $8x^2 - 5 = 0$

To solve an equation of the form $ax^2 + c = k$, we isolate x^2 to one side of the equation before using the square root property.

OBJECTIVE 2

Solve.

8. $(x+5)^2 = 3$ **9.** $(x-7)^2 = 27$ **10.** $(t+5)^2 = -64$ **11.** $(p-7)^2 = 0$

OBJECTIVE 3

- An angle of $90°$ is called a *right angle*.
- If one angle of a triangle measures $90°$, the triangle is a **right triangle**.
- The side opposite the right angle is the triangle's longest side. We call that side the **hypotenuse**, and we call the two shorter sides the **legs**. [Draw a sketch.]

Pythagorean theorem

If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then

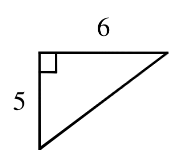
$$a^2 + b^2 = c^2$$

[Draw a sketch.]

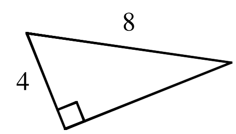
OBJECTIVE 4

12. The lengths of two sides of a right triangle are given. Find the length of the third side.

a.



b.



Let a and b represent the lengths of the legs of a right triangle, and let c represent the length of the hypotenuse. Values for two of the three lengths are given. Find the third length.

13. $a = 4$ and $b = 10$

14. $b = 6$ and $c = 12$

OBJECTIVE 5

15. A student drives 9 miles north from home and then 5 miles east to get to school. What would be the length of the trip if it was possible to drive along a straight line from home to school?

16. The size of a rectangular television screen is usually described as the length of a diagonal. If a 20-inch screen has a height of 12 inches, what is the screen's width?

SHORT HW 1, 11, 17, 21, 27, 29, 37, 43, 49, 55, 67, 77, 85, 89, 93

MEDIUM HW 1, 11, 15, 17, 21, 27, 29, 35, 37, 41, 43, 47, 49, 55, 67, 75, 77, 81, 85, 87, 89, 93, 95, 97

SECTION 9.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use the square root property to solve quadratic equations and use the Pythagorean theorem to find the length of a side of a right triangle.

OBJECTIVE 1 COMMENTS

After solving an equation such as $x^2 = 9$, some students start thinking that $\sqrt{9} = \pm 3$.

Most students have a tough time with Problems 6 and 7. It is partly because of the large number of steps required to solve the equations. In the long run, students have trouble recognizing that the square root property can be used to solve such equations. I emphasize that the square root property can be used to solve quadratic equations that can be put into the form $ax^2 + bx + c = 0$, where $b = 0$, or the form $(x + p)^2 = k$.

Workbook Exploration

This 15-minute exploration can be used to introduce using the square root property to solve quadratic equations.

Group Exploration

Section Opener: Using the square root property to solve an equation

1. Use factoring to solve $x^2 = 9$.
2. Describe how to find the solutions of $x^2 = 9$ in one step.
3. Use factoring to solve $x^2 = 25$.
4. Describe how to find the solutions of $x^2 = 25$ in one step.
5. Find the solutions of $x^2 = 7$ in one step.
6. Find the solutions of $x^2 = k$ in one step, where k is a nonnegative constant. This is called the **square root property**.
7. Use the square root property to solve $2x^2 = 10$.

OBJECTIVE 2 COMMENTS

Some students have trouble with solving equations of the form $(x+b)^2 = k$ such as $(x+5)^2 = 3$ (see Problem 8). To solve $(x + 5)^2 = 3$, I write the equation on the board. Then I cover the expression $x + 5$ with my hand and say the equation is now "hand" squared is equal to 3. Next, I ask students what the value of "hand" must be. Many students see that "hand" must be $\pm\sqrt{3}$. Therefore, $x + 5 = \pm\sqrt{3}$.

Many of my students forget to write "$\pm$." I remind my students several times to include this notation while discussing this section and Section 9.4.

Problems 8–10 prepare students to solve quadratic equations by completing the square.

OBJECTIVE 3 COMMENTS

When introducing the Pythagorean theorem, I emphasize that the right-hand side of the equation $a^2 + b^2 = c^2$ is the square of the length of the hypotenuse. I point out that it doesn't matter which leg has length a and which leg has length b, because $a^2 + b^2 = b^2 + a^2$ by the commutative law.

Textbook and Workbook Exploration

After providing a brief introduction to the Pythagorean theorem, you could assign the fun 20-minute "Pythagorean theorem and its converse" exploration as a class activity. I emphasize that the value of c represents the length of the hypotenuse.

OBJECTIVE 4 COMMENTS

For Problems 12–14, I emphasize that students should write the length of the requested side in radical form (not decimal form) and that the radical should be simplified. I point out that the results of these problems should be positive, because lengths are positive values.

OBJECTIVE 5 COMMENTS

For Problems 15 and 16, I encourage students to draw sketches. I emphasize that students should include units for their results.

SECTION 9.4 LECTURE NOTES

Objectives

1. Describe the relationship between b and c for a perfect-square trinomial of the form $x^2 + bx + c$.
2. Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.
3. Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.
4. Explain why any quadratic equation can be solved by completing the square.

Main point: Solve quadratic equations by completing the square.

OBJECTIVE 1

For the right-hand side of each of the following equations, show that if we divide the coefficient of x by 2 and square the result, we get the constant term.

- $(x + 3)^2 = x^2 + 6x + 9$
- $(x - 5)^2 = x^2 - 10x + 25$
- $(x + k)^2 = x^2 + (2k)x + k^2$

Perfect-Square Trinomial Property

For a perfect-square trinomial of the form $x^2 + bx + c$, dividing b by 2 and then squaring the result gives c:

$$x^2 + bx + c$$

$$\left(\frac{b}{2}\right)^2 = c$$

Note that this property is for a perfect-square trinomial $ax^2 + bx + c$, where $a = 1$.

Find the value of c such that the expression is a perfect-square trinomial. Then factor the perfect-square trinomial.

1. $x^2 + 12x + c$ **2.** $x^2 + 18x + c$ **3.** $x^2 - 8x + c$ **4.** $x^2 - 14x + c$

OBJECTIVE 2

Solve.

5. $x^2 + 8x = -2$ [Radicand does not require simplifying.]

6. $x^2 - 16x = -14$ [Radicand requires simplifying.]

7. $p^2 + 4p - 41 = 0$ [Radicand requires simplifying.]

8. $x^2 - 10x + 30 = 0$ [No real-number solutions]

OBJECTIVE 3

- The perfect-square trinomial property given for $ax^2 + bx + c$, where $a = 1$, does not extend to trinomials with $a \neq 1$.
- For example, consider simplifying $(2x + 3)^2$:

$$(2x + 3)^2 = 4x^2 + 12x + 9$$

Dividing $b = 12$ by 2 and then squaring the result does *not* give 9:

$$\left(\frac{12}{2}\right)^2 = 6^2 = 36 \neq 9$$

Solve.

9. $3x^2 + 30x = 12$ [Radicand does not require simplifying.]

10. $2x^2 - 12x = 6$ [Radicand requires simplifying.]

11. $5t^2 + 40t - 20 = 0$ [Radicand requires simplifying.]

12. $2x^2 - 24x + 100 = 0$ [No real-number solutions]

Warning: To solve an equation of the form $ax^2 + bx = k$ with $a \neq 1$, we must divide both sides of the equation by a before completing the square for the left side of the equation.

Solving a Quadratic Equation by Completing the Square

To solve a quadratic equation by **completing the square,**

1. Write the equation in the form $ax^2 + bx = k$, where a, b, and k are constants.

2. If $a \neq 1$, divide both sides of the equation by a.

3. Complete the square for the expression on the left side of the equation.

4. Solve the equation by using the square root property.

OBJECTIVE 4

Any quadratic equation can be solved by completing the square.

Solve by the method of your choice.

13. $x^2 - 50 = 0$ [Square root property]

14. $x^2 - 8x - 12 = 0$ [Complete the square: Radicand requires simplifying.]

15. $x^2 = -3x + 40$ [Factor.]

16. $(r - 5)^2 = 60$ [Square root property]

17. $2x^2 + 28x = -2$ [Complete the square: Radicand requires simplifying.]

18. $3m^2 + 10m - 8 = 0$ [Factor.]

SHORT HW 3, 7, 13, 17, 21, 25, 27, 39, 41, 43, 51, 59, 63, 65, 70

MEDIUM HW 3, 7, 13, 17, 21, 25, 27, 39, 41, 43, 51–81 odds

SECTION 9.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students solve quadratic equations by completing the square. The problems are on the easier side, leaving work with fractions for intermediate algebra. This technique will not be needed in subsequent sections, except for the derivation of the quadratic formula in Section 9.5.

OBJECTIVE 1 COMMENTS

Most students have an easy time with this objective.

Workbook Exploration

The following 15-minute exploration includes enough detail that groups have a reasonable chance of learning the main point of completing the square; it is doubtful that many students would pick this up from a lecture.

Group Exploration

Section Opener: The connection between b and c of a perfect square trinomial $x^2 + bx + c$

1. Simplify $(x + 5)^2$.
2. For $(x + 5)^2 = x^2 + 10x + 25$, explain why the coefficient of x, 10, turns out to be twice the second term of $x + 5$, 5.
3. For $(x + 5)^2 = x^2 + 10x + 25$, explain why the constant term, 25, turns out to be the square of the second term of $x + 5$, 5.
4. For the perfect square trinomial $x^2 + 10x + 25$, show that if you divide the coefficient of x, 10, by 2 and then square the result, you will get the constant term, 25. Why does this make sense? [**Hint:** See your explanations in Problems 2 and 3.]
5. Simplify $(x + k)^2$.
6. Let k be a nonzero constant. For $(x + k)^2 = x^2 + (2k)x + k^2$, explain why the coefficient of x, $2k$, turns out to be twice the second term of $x + k$, k.
7. Let k be a nonzero constant. For $(x + k)^2 = x^2 + (2k)x + k^2$, explain why the constant term, k^2, turns out to be the square of the second term of $x + k$, k.
8. For the perfect square trinomial $x^2 + (2k)x + k^2$, show that if you divide the coefficient of x, $2k$, by 2 and then square the result, you will get the constant term, k^2. Why does this make sense? [**Hint:** See your explanations in Problems 6 and 7.]
9. Find the value of c for which the polynomial $x^2 + 8x + c$ is a perfect-square trinomial. Then factor the perfect-square trinomial.

OBJECTIVE 2 COMMENTS

For Problem 5, I remind students that if we add 16 to the left side of the equation, we must add 16 to the right side as well (addition property of equality). Some students forget to do this. For the step where the expression $x^2+8x+16$ is factored as $(x+4)^2$, many students wonder "where did the $8x$ term (the middle term of $x^2+8x+16$) go?" It helps to show students that $(x + 4)^2 = x^2 + 8x + 16$ by multiplying pairs of terms (and then combining like terms).

I remind students to write "±," as appropriate.

OBJECTIVE 3 COMMENTS

Most students need several reminders that the perfect square trinomial property given for $ax^2 + bx + c$, where $a = 1$, does not extend to trinomials with $a \neq 1$.

Textbook and Workbook Exploration

The 15 minute "Identifying errors in solving by completing the square" exploration addresses two common student errors.

OBJECTIVE 4 COMMENTS

I tell students that although any quadratic equation can be solved by completing the square, we will discuss an easier method in Section 9.5 that can also be used to solve any quadratic equation. I say this method involves the *quadratic formula*, which we will derive by completing the square.

Problems 13–18 serve as a summary of solving equations by factoring, the square root property, and completing the square. I make sure I have time to do at least some of these problems, because most students have trouble remembering all the techniques and knowing which technique to use for various problems on exams.

SECTION 9.5 LECTURE NOTES

Objectives

1. Solve quadratic equations by using the *quadratic formula*.
2. Find approximate solutions of a quadratic equation.
3. Find approximate x-intercepts of a parabola.
4. Determine whether to solve a quadratic equation by factoring, by using the square root property, by completing the square, or by using the quadratic formula.

Main point: Use the quadratic formula to solve quadratic equations.

OBJECTIVE 1

Solve the general quadratic equation $ax^2 + bx + c = 0$ by completing the square.

Quadratic Formula

The solutions of a quadratic equation $ax^2 + bx + c = 0$ are given by the **quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Warning: For the fraction in the quadratic formula, note that $-b$ is part of the numerator.

1. Solve $x^2 + 2x - 24 = 0$ by using the quadratic formula and then by factoring. Compare your results.

Solve.

2. $2x^2 - 7x - 3 = 0$ [Radical does not require simplifying.]
3. $3x^2 - 4x = 5$ [Radical requires simplifying.]
4. $5t(3t - 2) = -1$ [Radical requires simplifying.]
5. $\frac{3}{8}x^2 = \frac{3}{4} + \frac{1}{8}x$ [Radical does not require simplifying.]
6. $(x - 5)(x - 2) = -3$ [No real-number solutions.]

OBJECTIVE 2

7. Find approximate solutions of the equation $2.7x(1.4x + 2.7) = 3.9x + 8.2$. Round any results to the second decimal place.

OBJECTIVE 3

8. Find approximate x-intercepts of the parabola $y = -2.5x^2 + 13.2x + 70.9$. Round the coordinates of the results to the second decimal place.

OBJECTIVE 4

Method:	**When to Use:**
Factoring	For equations that can easily be put into the form $ax^2 + bx + c = 0$ and where $ax^2 + bx + c$ can easily be factored.
Square root property	For equations that can easily be put into the form $x^2 = k$ or $(x + p)^2 = k$.
Completing the square	Use when the directions require it.
Quadratic formula	For all equations except those that can easily be solved by factoring or by the square root property.

Solve.

9. $(t + 4)^2 = 27$ [Square root property: radical requires simplifying.]

10. $x^2 = 4(x + 8)$ [Factor.]

11. $2m^2 = 8m - 3$ [Quadratic formula: radical requires simplifying.]

12. $7x^2 - 5 = 3$ [Square root property: radical requires simplifying.]

13. Solve the equation $3x^2 - 6x - 21 = 0$ by completing the square.

SHORT HW 7, 9, 11, 13, 21, 25, 29, 39, 43, 45, 67, 73, 75, 79, 86

MEDIUM HW 1, 7, 9, 11, 13, 21, 25, 29, 39, 43, 45, 53, 55, 57, 59, 65–75 odd, 79, 81, 86, 87, 89, 91

SECTION 9.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use the quadratic formula to solve quadratic equations. They will use this skill in Section 9.6.

OBJECTIVE 1 COMMENTS

Before deriving the quadratic formula, I tell my students that the derivation is in their textbooks and I encourage students to watch carefully and not take notes. That way, they can focus on the concepts and I can get through the derivation more quickly, because I won't have to wait for them to copy it from the board.

For Problem 3, a few students may think that $a = 3$, $b = -4$, and $c = 5$ (when really, $c = -5$). I emphasize that we must write the equation so that one side is 0 before determining the values of a, b, and c. Also for Problem 3, many students do not simplify the radical of their results. I revisit this topic a couple of times before the next exam.

OBJECTIVE 2 COMMENTS

For Problem 7, after I have applied the quadratic formula, I show students how to use their calculators to compute the two approximate solutions. First, I have students calculate the square root of the discriminant. Next, I have them calculate the numerators (both the sum and the difference). Then, I have them calculate the entire fractions. Students do pretty well by this stepwise calculator-entry approach. I warn them that they had better practice such problems until they can do one in about five minutes. Without such a warning, some students can take as long as 20 minutes for one problem! Problem 7 is good preparation for Section 9.6, where students will use quadratic models to make estimates and predictions.

OBJECTIVE 3 COMMENTS

Problem 8 is also good preparation for Section 9.6.

Workbook Exploration

The following 20-minute exploration has students find intercepts by using numerical, graphical, and symbolic methods. Students need lots of practice with making such connections.

EXPLORATION *Finding intercepts by tables, graphs, and the quadratic formula*

1. Use a graphing calculator to help you complete the following table for the function $f(x) = 2x^2 - 5x - 2$. Then use the table to estimate the x-intercepts of the graph of f.

x	$f(x)$	x	$f(x)$
-4		1	
-3		2	
-2		3	
-1		4	
0		5	

2. Use a graphing calculator to graph $f(x) = 2x^2 - 5x - 2$. Copy the screen. Use your graph to estimate the x-intercepts of the graph of $f(x) = 2x^2 - 5x - 2$.

3. Use the quadratic formula to find the x-intercepts of the graph of $f(x) = 2x^2 - 5x - 2$. Round your results to the second decimal place.

4. Are the results you found in Problems 1, 2, and 3 equal? Are they approximately equal? Which method is easiest to use? Which method gives the most precise results?

5. Would your responses to the questions in Problem 4 be the same for all quadratic functions? [**Hint:** Carefully consider all sorts of quadratic functions.]

OBJECTIVE 4 COMMENTS

Determining which method to use to solve a quadratic equation is a great challenge for most students. Without much discussion, most students will rarely think to solve such an equation by using the square root property. Some students will use the quadratic formula to solve an equation which can easily be solved by factoring. Some students will do a poor job of solving an equation by completing the square when they could have used the quadratic formula (and hopefully have done a better job).

I try to train my students to first inspect an equation and see if the square root property can easily be used. If not, I suggest that students check whether the equation can easily be solved by factoring. If not, then I strongly recommend that students use the quadratic formula. I say flat out that they should only use completing the square when the directions require it. And I promise that there will be such directions for one problem on the upcoming exam.

Textbook and Workbook Exploration

The 20-minute "Comparing methods of solving quadratic equations" exploration encourages students to consider the big picture of solving quadratic equations. It sets the stage for the upcoming exam where students will have to grapple with which technique is most appropriate to use in solving an equation.

SECTION 9.6 LECTURE NOTES

Objective

1. Use quadratic models to make estimates and predictions.

Main point: Use quadratic models to make estimates and predictions.

OBJECTIVE 1

We can use the quadratic formula to make estimates and predictions for the explanatory variable for any situation that is described well by a quadratic model.

1. The U.S. revenues (in billions of dollars) from herbal supplements are shown in the following table for various years.

Year	U.S. Revenue from Herbal Supplements (billions of dollars)
2010	5.0
2011	5.3
2012	5.6
2013	6.0
2014	6.4
2015	6.9
2016	7.5

Source: *Nutrition Business Journal*

Let r be the U.S. annual revenue (in billions of dollars) from herbal supplements at t years since 2010. A model is $r = 0.03t^2 + 0.23t + 5.02$.

a. Use a graphing calculator to draw the graph of the model and, in the same viewing window, the scatterplot of the data. Does the model fit the data well?

b. What is the r-intercept of the model? What does it mean in this situation?

c. Use the model to predict the revenue in 2021.

d. Use the model to estimate when the annual revenue will be \$12 billion.

- To make a prediction about the response variable of a quadratic model, we substitute a value for the explanatory variable and then solve for the response variable.
- To make a prediction about the explanatory variable, we substitute a value for the response variable and then solve for the explanatory variable, usually by using the quadratic formula.

2. The numbers of Proctor & Gamble employees are shown in the following table for various years.

Year	Number of Proctor & Gable Employees (thousands)
2009	132
2011	129
2013	121
2015	110
2016	105

Source: *Proctor & Gamble*

Let n be the number (in thousands) of employees at t years since 2000. A model is $n = -0.4t^2 + 6t + 111$.

a. Use a graphing calculator to draw the graph of the model and, in the same viewing window, the scatterplot of the data. Does the model fit the data well?

b. Use "maximum" on a graphing calculator to estimate when the number of employees was the greatest. According to the model, what was that number of employees?

c. Predict the number of employees in 2022.

d. Predict when there will be 60 thousand employees.

SHORT HW 1, 5, 9, 15, 17, 19

MEDIUM HW 1, 5, 7, 9, 13–33 odds [Warning: 13 is long yet relevant.]

SECTION 9.6 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use quadratic models to make estimates and predictions.

OBJECTIVE 1 COMMENTS

For part (a) of Problem 1, I begin by using a graphing calculator to draw a scatterplot of the data and ask my students whether we should use a line, parabola, or neither to model the data. Then we check how well the quadratic model fits the data.

For part (d) of Problem 1, I remind my students how to use a calculator after having applied the quadratic formula. Once students have mastered this skill, I encourage them to continue practicing until they can do such a computation quickly.

I have groups of students complete Problem 2. Most groups do well with this application, because it is so similar to Problem 1. For part (b), some groups may need a reminder on how to use "maximum" on a graphing calculator.

CHAPTER 10 OVERVIEW

In Section 10.1, students find excluded values of rational expressions and simplify rational expressions. In Section 10.2, students multiply and divide rational expressions and convert units of quantities. Students add and subtract rational expressions in Section 10.3 and Section 10.4, respectively. In Section 10.5, students solve rational equations and use a rational model to make estimates and predictions. In Section 10.6, students use proportions to make estimates and find the length of a side of a similar triangle. Students use direct variation and inverse variation equations to make estimates in Section 10.7. In Section 10.8, students simplify complex rational expressions.

SECTION 10.1 LECTURE NOTES

Objectives

1. Describe the meaning of *rational expression*.
2. Find all *excluded values* of a rational expression.
3. Simplify a rational expression.
4. Describe a rational equation in two variables.
5. Make estimates and predictions with a *rational model*.

Main point: Find all *excluded values* of a rational expression and simplify rational expressions.

OBJECTIVE 1

- If P and Q are polynomials with Q nonzero, we call the ratio $\frac{P}{Q}$ a **rational expression.**
- Some rational expressions: $\frac{2}{x}, \frac{x-6}{x+2}, \frac{5x^2-36}{x^2-3x+8}$

Definition *Excluded value*

A number is an **excluded value** of a rational expression if substituting the number into the expression leads to a division by 0.

OBJECTIVE 2

Find all excluded values of the given expression.

1. $\frac{8}{x}$
2. $\frac{7}{x-3}$

Warning: For $\frac{7}{x-3}$, 0 is *not* an excluded value.

Find all excluded values of the given expression.

3. $\frac{x-2}{5}$
4. $\frac{x+3}{5x+4}$
5. $\frac{2}{t^2+4t-12}$
6. $\frac{5x-2}{4x^2-49}$
7. $\frac{x-5}{3x^2-2x-8}$
8. $\frac{w-3}{4w^3-8w^2-9w+18}$

Finding Excluded Values

To find any excluded values of a rational expression $\frac{P}{Q}$,

1. Set the denominator Q equal to 0.
2. Solve the equation $Q = 0$.

OBJECTIVE 3

- A rational expression is in **lowest terms** if the numerator and the denominator have no common factors other than 1 or -1.
- We **simplify a rational expression** by writing it in lowest terms.

Simplifying a Rational Expression

To simplify a rational expression,

1. Factor the numerator and the denominator.
2. Use the property
$$\frac{AB}{AC} = \frac{A}{A} \cdot \frac{B}{C} = 1 \cdot \frac{B}{C} = \frac{B}{C}$$
where A and C are nonzero, so that the expression is in lowest terms.

Simplify.

9. $\dfrac{25x^2}{35x^5}$

10. $\dfrac{k^2-9}{k^2-5k+6}$

11. $\dfrac{-15x+6}{25x^2-4}$

12. $\dfrac{a^2-3ab+2b^2}{a^2-4b^2}$

13. $\dfrac{6w^2-7w-5}{2w^3+w^2-50w-25}$

14. $\dfrac{x^3+8}{x^2+4x+4}$

OBJECTIVE 4

A **rational equation in two variables** is an equation in two variables in which each side can be written as a rational expression.

OBJECTIVE 5

15. Some students have a party and share equally in the expense, which is \$100 for both soft drinks and snacks. Let p be the per-person cost (in dollars per person) if n students go to the party.
    a. Use a table to help find an equation for n and p. Show the arithmetic to help you see a pattern.
    b. Perform a unit analysis of the equation you found in part (a).
    c. Substitute 40 for n in the equation and solve for p. What does the result mean in this situation?
    d. For positive values of n, what happens to the value of p as the value of n is increased? What does that trend mean in this situation?

16. The numbers of U.S. banks and their total deposits are shown in the following table for various years.

Year	Number of Banks (thousands)	Total Deposits (trillions of dollars)
2010	9.8	7.7
2011	9.8	8.2
2012	9.7	8.9
2013	9.6	9.4
2014	9.4	10.1
2015	9.3	10.7

Source: *Federal Deposit Insurance Corp.*

Let A be the average annual total deposits (in billions of dollars) per bank at t years since 2010. A reasonable model is $A = \dfrac{0.61t + 7.65}{-0.11t + 9.87}$.

a. Use the model to estimate the average total deposits per bank in 2012.

b. Now find the *actual* average total deposits per bank in 2012.

c. Is the result you found in part (a) an underestimate or an overestimate? Explain.

d. Predict the average total deposits per bank in 2021.

SHORT HW 9, 15, 21, 23, 27, 33, 49, 63, 65, 71, 73, 75, 79, 81, 89

MEDIUM HW 1, 3, 9, 15, 21, 23, 27, 33, 39, 49, 57, 63, 65, 71, 73, 75, 79, 81, 89, 91, 93, 95

SECTION 10.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students find excluded values of rational expressions and simplify rational expressions. Students will use these skills throughout Chapter 10.

OBJECTIVE 1 COMMENTS

I point out that it makes sense that the word “rational” contains the word “ratio,” because a rational expression is the ratio of two polynomials.

OBJECTIVE 2 COMMENTS

To begin, I remind my students that division by 0 is undefined. Then we determine the excluded values of $\frac{8}{x}$ (Problem 1). For Problem 2, students can determine the excluded values by inspection. For each of Problems 4–8, I set the denominator equal to zero to find the excluded values.

The skill of finding excluded values will come in handy when students solve rational equations in Section 10.5.

OBJECTIVE 3 COMMENTS

I remind my students several times throughout this chapter that the property

$$\frac{AB}{AC} = \frac{B}{C}, \quad \text{where } A \text{ and } C \text{ are nonzero}$$

can only be used if A is a factor both of the (entire) numerator and of the (entire) denominator. Many students have difficulty with this concept. For example, it is a common student error to think that

$$\frac{4x^2 - 7}{3x} = \frac{4x - 7}{3}$$

Some of my students have difficulties with Problem 9, because they have forgotten the properties of exponents discussed in Sections 7.6 and 7.7.

Some students have trouble with Problem 13, because they’ve forgotten how to factor by grouping.

Some students have trouble with Problem 14, because they’ve forgotten how to factor a sum of two cubes.

I encourage my students to verify their work with graphing calculator tables and graphs. I discuss how to use parentheses correctly for their equation entries. One way to avoid using as many parentheses is to build equations using Y_n references as described in Section A.22.

Workbook Exploration

The following 15-minute exploration has groups consider whether they can use the property $\frac{a}{a} = 1$, where $a \neq 0$.

During the debrief make sure students get the main point: we can use the property for common factors of the entire numerator and the entire denominator.

Group Exploration
Section Opener: Simplifying rational expressions

We can write $\frac{2x}{5x} = \frac{2}{5}$ because x is a common factor of the numerator and the denominator. For each problem, use this concept to help you determine whether the work is correct. If you are unsure, it may help to decide whether you can show work similar to the following:

$$\frac{2x}{5x} = \frac{2}{5} \cdot \frac{x}{x} = \frac{2}{5} \cdot 1 = \frac{2}{5}$$

Explain.

1. $\frac{4x}{9x} = \frac{4}{9}$

2. $\frac{4+x}{9x} = \frac{4+1}{9} = \frac{5}{9}$

3. $\frac{3x \cdot 5}{7x} = \frac{3 \cdot 5}{7} = \frac{15}{7}$

4. $\frac{3x+5}{7x} = \frac{3+5}{7} = \frac{8}{7}$

5. $\frac{8+x^5}{6+x^2} = \frac{4+x^3}{3}$

6. $\frac{8x^5}{6x^2} = \frac{4x^3}{3}$

7. $\frac{(x+2)(x-4)}{5(x-4)} = \frac{x+2}{5}$

8. $\frac{(x+2)(x-4)+1}{5(x-4)} = \frac{(x+2)+1}{5} = \frac{x+3}{5}$

Textbook and Workbook Exploration
The 15-minute "Simplifying rational expressions" addresses many common student errors of simplifying rational expressions. I highly recommend either assigning this exploration or discussing these types of issues with your students.

OBJECTIVES 4 and 5 COMMENTS
For Problem 16, the rational model can be found by finding models for the number of banks and the total money deposited and then finding the appropriate quotient function of the two models. The textbook does not discus this process, leaving it to be discussed in intermediate algebra.

For part (a) of Problem 16, students have an easy time using the model to make the estimate, but some students are not sure how to do part (b), where they must use the data values in the table to find the actual value. Problem 15 should suggest how to find the actual value.

Workbook Exploration
In the following 15-minute exploration, groups may feel a bit intimidated by the complexity of the rational model, but the material is certainly within their grasp.

Group Exploration
Section Opener: Using a rational model to make predictions

The average amounts employees pay for single-person coverage of health insurance and the average total costs (paid by employees and employers) are shown in the following table for various years.

	Average Cost (dollars)	
Year	Employee	Total
2006	627	4242
2009	779	4824
2012	951	5615
2015	1071	6251
2016	1129	6435

Source: *Kaiser/HRET Survey of Employer-Sponsored Health Benefits, 1999–2016*

Let P be the percentage of the average total annual cost of health insurance paid by employees at t years since 2000. A reasonable model is

$$P = \frac{4900t + 33,261}{224t + 2872}$$

1. Use the model to estimate the percentage of the average total annual cost of health insurance paid by employees in 2016.

2. Find the *actual* percentage of the average total annual cost of health insurance paid by employees in 2016.

3. Is the result you found in Problem 1 an underestimate or an overestimate? Explain.

4. Use the model to predict the percentage of the average total annual cost of health insurance paid by employees in 2021.

SECTION 10.2 LECTURE NOTES

Objectives

1. Multiply rational expressions.
2. Divide rational expressions.
3. Convert units of quantities.

Main point: Multiply and divide rational expressions.

OBJECTIVE 1

Recall how to multiply two fractions such as $\frac{2}{5}$ and $\frac{4}{3}$:

$$\frac{2}{5} \cdot \frac{4}{3} = \frac{2 \cdot 4}{5 \cdot 3} = \frac{8}{15}$$

Multiplying Rational Expressions

If $\frac{A}{B}$ and $\frac{C}{D}$ are rational expressions and B and D are nonzero, then

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

Find the product.

1. $\frac{2x-7}{x^3} \cdot \frac{x^4}{3x+5}$

2. $\frac{5p-15}{7} \cdot \frac{21p}{2p-6}$

How to Multiply Rational Expressions

To multiply two rational expressions,

1. Factor the numerators and the denominators.
2. Multiply by using the property $\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$, where B and D are nonzero.
3. Simplify the result.

Find the product.

3. $\frac{x^2+6x+9}{20x-15} \cdot \frac{8x^2-6x}{x^2-9}$

4. $\frac{t^2-3t-10}{2t^2-9t+4} \cdot \frac{6t^2-3t}{t^2-4}$

OBJECTIVE 2

Recall how to divide two fractions such as $\frac{5}{3}$ and $\frac{7}{2}$:

$$\frac{5}{3} \div \frac{7}{2} = \frac{5}{3} \cdot \frac{2}{7} = \frac{10}{21}$$

Dividing Rational Expressions

If $\dfrac{A}{B}$ and $\dfrac{C}{D}$ are rational expressions and B, C, and D are nonzero, then

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

Find the quotient.

5. $\dfrac{6x^3}{5} \div \dfrac{4x^5}{7}$

6. $\dfrac{x-4}{x^2-2x-35} \div \dfrac{3x-12}{x^2-25}$

How to Divide Rational Expressions

To divide two rational expressions,

1. Write the quotient as a product by using the property $\dfrac{A}{B} \div \dfrac{C}{D} = \dfrac{A}{B} \cdot \dfrac{D}{C}$, where B, C, and D are nonzero.
2. Find the product.
3. Simplify.

For Problems 7 and 8, find the quotient.

7. $\dfrac{2x^2-9x+7}{x^2+4x-12} \div \dfrac{4x^2-49}{-2x^2-12x}$.

8. $\dfrac{b^2+12b+36}{b-4} \div (b^2+3b-18)$

OBJECTIVE 3

Make the indicated unit conversions. Round approximate results to two decimal places.

9. The Sears Tower in Chicago is 1450 feet tall. What is its height in yards?

10. Red meat production in Iowa is 590 million pounds per year. What is Iowa's meat production in ounces per day?

11. A driver travels at 50 miles per hour. What is the driver's speed in meters per second?

Converting Units

To convert the units of a quantity,

1. Write the quantity in the original units.
2. Multiply by fractions equal to 1 so that the units you want to eliminate appear in one numerator and one denominator.

SHORT HW 5, 13, 37, 43, 45, 49, 51, 53, 57, 59, 61, 63, 69, 77, 85

MEDIUM HW 1, 5, 13, 37, 43, 45, 49, 51, 53, 57, 59, 61, 63, 69, 77, 79, 85, 87, 89, 91, 93, 95

SECTION 10.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students multiply and divide rational expressions. These skills will be needed in Chapter 10.

OBJECTIVES 1 and 2 COMMENTS

Most students have an easy time with this section.

For Problems 5–8, some students forget to take the reciprocal of the appropriate rational expression. Some students make mistakes when simplifying, usually because they try to simplify before taking the reciprocal.

For Problem 7, I remind students that if the leading coefficient of a polynomial is negative, then we factor out the opposite of the GCF.

Background Information for the Exploration That Follows

Students can discover how to multiply rational expressions in the following 10-minute exploration.

EXPLORATION *Multiplying rational expressions*

Find the indicated product. Use a graphing calculator table to verify your result.

1. $\frac{1}{2} \cdot \frac{4}{5}$

2. $\frac{3}{x} \cdot \frac{2}{x}$

3. $\frac{x}{4} \cdot \frac{5}{x+3}$

4. $\frac{x-3}{5} \cdot \frac{x+2}{x}$

5. $\frac{t^2+5t+6}{t-5} \cdot \frac{t+1}{t^2+7t+10}$

6. $\frac{w^2-25}{w^2-w-12} \cdot \frac{(w-4)^2}{4w+20}$

Background Information for the Exploration That Follows

Students can discover how to divide rational expressions in the following 10-minute exploration. If they have not divided fractions in a long time, they may need to be reminded that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, where b, c, and d are nonzero.

EXPLORATION *Dividing rational expressions*

Find the indicated quotient. Use a graphing calculator table to verify your result.

1. $\frac{3}{2} \div \frac{7}{5}$

2. $\frac{x}{5} \div \frac{x}{3}$

3. $\frac{p-2}{3} \div \frac{p+3}{x}$

4. $\frac{x^2+7x+12}{x-2} \div \frac{x+3}{x^2-7x+10}$

5. $\frac{y^2-4}{y^7} \div \frac{3y+6}{4y^3}$

OBJECTIVE 3 COMMENTS

Some equivalent units are shown in the margin on page 546 of the textbook.

SECTION 10.3 LECTURE NOTES

Objectives

1. Add rational expressions that have a common denominator.
2. Add rational expressions that have different denominators.

Main point: Add rational expressions.

OBJECTIVE 1

Recall how to add two fractions with a common denominator such as $\frac{3}{11}$ and $\frac{6}{11}$:

$$\frac{3}{11} + \frac{6}{11} = \frac{9}{11}$$

Adding Rational Expressions That Have a Common Denominator

If $\frac{A}{B}$ and $\frac{C}{B}$ are rational expressions, where B is nonzero, then $\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}$.

Find the sum.

1. $\dfrac{5x+2}{x} + \dfrac{4}{x}$
2. $\dfrac{x^2}{x^2+7x+12} + \dfrac{5x+6}{x^2+7x+12}$
3. $\dfrac{c^2-5c}{c^2-16} + \dfrac{3c-24}{c^2-16}$

OBJECTIVE 2

Suppose that two brothers, John and Paul, own the following numbers and types of musical instruments:

John	Paul
3 guitars	1 guitar
1 bass guitar	2 bass guitars
1 sitar	1 sitar

- The brothers will not give their instruments to each other, but each brother wants to own the same numbers and types of musical instruments that the other brother has. Which instruments do the brothers want?
- Compare this situation with finding the LCD for $\frac{3}{40} + \frac{7}{50}$.

Find the sum.

4. $\dfrac{3}{8} + \dfrac{5}{4x}$
5. $\dfrac{3}{10x} + \dfrac{4}{15x^3}$
6. $\dfrac{2}{w-5} + \dfrac{7}{w+3}$
7. $\dfrac{4}{3x-6} + \dfrac{5}{6}$

How to Add Two Rational Expressions That Have Different Denominators

To add two rational expressions that have different denominators,

1. Factor the denominators of the expressions if possible. Determine which factors are missing.
2. Use the property $\frac{A}{A} = 1$, where A is nonzero, to introduce missing factors.
3. Add the expressions by using the property $\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}$, where B is nonzero.
4. Simplify.

Find the sum.

8. $\frac{2x}{x^2-4x+3} + \frac{4}{x^2-9}$

9. $\frac{5}{x^2-8x+16} + \frac{3x}{x^2+4x-32}$

10. $\frac{x-1}{x-4} + \frac{x-3}{x-2}$

11. $\frac{7}{y^2+y-6} + \frac{y-1}{y+3}$

12. $\frac{3x}{2x^2+x-3} + \frac{2}{4x^2-9}$

13. $\frac{2x}{x+y} + \frac{3y}{x^2+3xy+2y^2}$

14. $\frac{2}{3x^2-6x+12} + \frac{5x}{x^3+8}$

SHORT HW 3, 13, 15, 21, 25, 39, 45, 51, 53, 63, 67, 69, 73, 77, 83

MEDIUM HW 3, 7, 13, 15, 21, 25, 33, 39, 45, 51, 53, 57, 59, 63, 67, 69, 71, 73, 75, 77, 83, 89, 91, 93

SECTION 10.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students add rational expressions. Students will use these concepts in Chapter 10.

OBJECTIVE 1 COMMENTS

For Problem 2, I remind students to simplify their results.

Background Information for the Exploration That Follows

The following exploration serves as a nice introduction to adding rational expressions with common denominators.

Group Exploration

Adding rational expressions

Perform each addition. Simplify the result.

1. $\frac{2}{7} + \frac{3}{7}$

2. $\frac{4}{x} + \frac{5}{x}$

3. $\frac{3}{x+2} + \frac{6}{x+2}$

4. $\frac{2x+5}{3x} + \frac{4x+1}{3x}$

5. $\dfrac{x^2}{x+3}+\dfrac{5x+6}{x+3}$

6. $\dfrac{x^2+x}{x^2-4}+\dfrac{2x-10}{x^2-4}$

OBJECTIVE 2 COMMENTS

I find that the John-Paul analogy (see the lecture notes or the textbook) is extremely effective in getting across the main idea of finding the LCD. The analogy has students focus on the key element of the procedure, which is to determine which factors are missing in each denominator.

Many of my students struggle with Problem 13, because the factoring is fairly challenging.

I remind my students that they should simplify their results.

Background Information for the Exploration That Follows

Students can discover how to add rational expressions by completing the following 25-minute exploration.

EXPLORATION *Adding rational expressions*

We can perform the addition for $\dfrac{5}{2}+\dfrac{3}{7}$ by the following steps:

$$\begin{aligned}\frac{5}{2}+\frac{3}{7}&=\frac{5}{2}\cdot 1+\frac{3}{7}\cdot 1\\&=\frac{5}{2}\cdot\frac{7}{7}+\frac{3}{7}\cdot\frac{2}{2}\\&=\frac{35}{14}+\frac{6}{14}\\&=\frac{35+6}{14}\\&=\frac{41}{14}\end{aligned}$$

Find the sum.

1. $\dfrac{3}{5}+\dfrac{7}{2}$

2. $\dfrac{x}{2}+\dfrac{x}{3}$

3. $\dfrac{x-2}{4x}+\dfrac{x+3}{6x}$

4. $\dfrac{5}{t-3}+\dfrac{2}{t+4}$

5. $\dfrac{p}{(p+3)(p-5)}+\dfrac{4}{(p-2)(p-5)}$

6. $\dfrac{3}{x^2+3x+2}+\dfrac{2}{x^2+7x+6}$

7. $\dfrac{x+1}{x^2-16}+\dfrac{x-2}{2x-8}$

Textbook and Workbook Exploration

For the 10-minute "Performing operations with rational expressions" exploration groups consider errors or inefficient algorithms typically performed by students.

SECTION 10.4 LECTURE NOTES

Objectives

1. Subtract rational expressions that have a common denominator.
2. Subtract rational expressions that have different denominators.

Main point: Subtract rational expressions.

OBJECTIVE 1

Recall how to subtract two fractions with a common denominator such as $\frac{6}{7}$ and $\frac{4}{7}$:

$$\frac{6}{7} - \frac{4}{7} = \frac{2}{7}$$

Subtracting Rational Expressions That Have a Common Denominator

If $\frac{A}{B}$ and $\frac{C}{B}$ are rational expressions, where B is nonzero, then

$$\frac{A}{B} - \frac{C}{B} = \frac{A-C}{B}$$

Find the difference.

1. $\frac{9x}{x-3} - \frac{5x}{x-3}$

2. $\frac{x^2}{x^2-9} - \frac{4x+21}{x^2-9}$

3. $\frac{5a^2-3a-10}{a^2+10a+25} - \frac{4a^2-2a+20}{a^2+10a+25}$

Warning: When subtracting rational expressions, be sure to subtract the *entire* numerator.

OBJECTIVE 2

Find the difference.

4. $\frac{3}{5x} - \frac{7}{x}$

5. $\frac{7}{4x^3} - \frac{5}{6x}$

6. $\frac{2x}{x-4} - \frac{3}{x-5}$

7. $\frac{3}{r^2-36} - \frac{r}{r^2-8r+12}$

How to Subtract Rational Expressions That Have Different Denominators

To subtract two rational expressions that have different denominators,

1. Factor the denominators of the expressions if possible. Determine which factors are missing.
2. Use the property $\frac{A}{A} = 1$, where A is nonzero, to introduce missing factors.
3. Subtract the expressions by using the property $\frac{A}{B} - \frac{C}{B} = \frac{A - C}{B}$, where B is nonzero.
4. Simplify.

Find the difference.

8. $\frac{5}{x^2 - 2x - 24} - \frac{3}{x^2 - 6x}$
9. $\frac{2b}{b^2 + 3b - 40} - \frac{7}{b^2 + 16b + 64}$
10. $\frac{x-4}{x+5} - 3$
11. $\frac{x-2}{x+3} - \frac{x+5}{x-4}$
12. $\frac{5x}{x-y} - \frac{2y}{x^2 - y^2}$
13. $\frac{4}{t^3 - 2t^2 - t + 2} - \frac{2t}{t^2 - 1}$

SHORT HW 3, 7, 13, 21, 27, 35, 43, 47, 51, 55, 59, 63, 67, 69, 73

MEDIUM HW 3, 7, 13, 15, 21, 27, 35, 43, 47, 49, 51, 53, 55, 59, 63, 65, 67, 69, 73, 79, 81, 83, 85, 87

SECTION 10.4 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students subtract rational expressions.

OBJECTIVE 1 COMMENTS

For Problem 2, most of my students fail to subtract the *entire* numerator $4x + 21$. I revisit this topic many times before we reach the next exam.

OBJECTIVE 2 COMMENTS

Due to having worked with finding common denominators in Section 10.3, students do quite well with this material. The primary task is to train students to subtract the entire numerator.

Some students have trouble with Problem 13 due to the more challenging factoring.

Background Information for the Exploration That Follows

The following 15-minute exploration has groups discover how to subtract two rational expressions. It helps to check with groups after they complete Problem 2 to make sure they've got the idea before they begin working on Problem 3.

EXPLORATION *Subtracting rational expressions*

1. Find the difference. Simplify the result.

   a. $\frac{7}{5} - \frac{4}{5}$

   b. $\frac{x^2 - 2x}{x-5} - \frac{15}{x-5}$

c. $\dfrac{4}{2x} - \dfrac{7}{x^2}$

d. $\dfrac{5}{x+3} - \dfrac{2}{x^2-9}$

2. For finding the difference $\dfrac{x^2}{x+1} - \dfrac{3x+7}{x+1}$, which of the methods that follow yields the correct answer? Use graphing calculator tables to help you decide.

Method A

$$\begin{aligned} \frac{x^2}{x+1} - \frac{3x+7}{x+1} &= \frac{x^2-(3x+7)}{x+1} \\ &= \frac{x^2-3x-7}{x+1} \end{aligned}$$

Method B

$$\frac{x^2}{x+1} - \frac{3x+7}{x+1} = \frac{x^2-3x+7}{x+1}$$

3. Find the difference. Simplify the result.

a. $\dfrac{x^2}{x-5} - \dfrac{4x-3}{x-5}$

b. $\dfrac{3x}{x^2+6x+8} - \dfrac{5}{x^2+7x+12}$

Textbook and Workbook Exploration

The 15 minute "Performing operations with rational expressions" exploration addresses some common student errors.

SECTION 10.5 LECTURE NOTES

Objectives

1. Solve *rational equations in one variable*.
2. Compare solving rational equations with simplifying rational expressions.
3. Use a rational model to make estimates and predictions.

Main point: Solve rational equations.

OBJECTIVE 1

- A **rational equation in one variable** is an equation in one variable in which each side can be written as a rational expression.
- With rational equations, it is possible to take the usual steps for solving equations, yet arrive at x values that are excluded values for one or more of the fractions in the equation. These values of x are *not* solutions. We call these values **extraneous solutions**.

Solve.

1. $\frac{8}{x} - 3 = \frac{4}{x} - 1$ [one solution]
2. $\frac{x}{3} - \frac{5}{6x} = \frac{1}{2}$ [two solutions]
3. $2 - \frac{1}{t-2} = \frac{t-3}{t-2}$ [extraneous solution]

Because multiplying both sides of a rational equation by the LCD and then simplifying both sides may introduce extraneous solutions, we must always check that any proposed solution is not an excluded value.

Solve.

4. $\frac{x-2}{x+5} = \frac{x+3}{x-4}$ [one solution]
5. $\frac{w}{w+2} + \frac{7}{w-5} = \frac{14}{w^2 - 3w - 10}$ [one extraneous solution, one solution]

Solving a Rational Equation in One Variable

To solve a rational equation in one variable,

1. Factor the denominator(s) if possible.
2. Identify any excluded values.
3. Find the LCD of all the fractions.
4. Multiply both sides of the equation by the LCD, which gives a simpler equation to solve.
5. Solve the simpler equation.
6. Discard any proposed solutions that are excluded values.

Solve.

6. $\dfrac{6}{p^2-9}-\dfrac{1}{p+3}=\dfrac{1}{p-3}$ [one extraneous solution]

7. $\dfrac{x}{x-2}-\dfrac{x-1}{x+2}=\dfrac{3}{x^2-4}$ [two solutions]

OBJECTIVE 2

Solving a Rational Equation versus Simplifying a Rational Expression

To solve a rational equation, clear the fractions in it by multiplying both sides of the equation by the LCD. To simplify a rational expression, do *not* multiply it by the LCD. The only multiplication permissible is multiplication by 1, usually in the form $\dfrac{A}{A}$, where A is a nonzero polynomial.

Solve or simplify, whichever is appropriate.

8. $\dfrac{5}{x}+\dfrac{3}{x-2}-\dfrac{7}{x}$

9. $\dfrac{5}{x}+\dfrac{3}{x-2}=\dfrac{7}{x}$

10. $\dfrac{4}{x-6}+\dfrac{3x}{x-2}=\dfrac{-19}{x^2-8x+12}$

11. $\dfrac{4}{x-6}+\dfrac{3x}{x-2}$

The result of solving a rational equation is the empty set or a set of one or more numbers. The result of simplifying a rational expression is an expression.

OBJECTIVE 3

12. In Problem 16 of the lecture notes of Section 10.1, we found the model

$$A=\frac{0.61t+7.65}{-0.11t+9.87}$$

where A is the average total deposits (in billions of dollars) per bank at t years since 2010 (see the following table).

Year	Number of Banks (thousands)	Total Deposits (trillions of dollars)
2010	9.8	7.7
2011	9.8	8.2
2012	9.7	8.9
2013	9.6	9.4
2014	9.4	10.1
2015	9.3	10.7

Source: *Federal Deposit Insurance Corp.*

Predict when the average annual total deposits per bank will be \$1.7 billion.

SHORT HW 1, 7, 13, 19, 21, 27, 31, 35, 41, 43, 47, 49, 55, 59, 65

MEDIUM HW 1, 7, 13, 19, 21, 23, 27, 31, 35, 41, 43, 47, 49, 53, 55, 57, 59, 61,65, 69, 71, 73

SECTION 10.5 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students solve rational equations in this section. They use a rational model to make predictions for the explanatory variable.

OBJECTIVE 1 COMMENTS

After solving Problem 3, I provide an explanation similar to the one on the middle of page 565 of the textbook to show where we introduced the extraneous solution, 2. Almost all students fail to check for extraneous solutions. I tell my students to write "extraneous solutions" at the top of their exam to remind them to check for extraneous solutions.

For Problem 5, after multiplying both sides of the original equation by $(w+2)(w-5)$, some of my students make the following error on the left-hand side:

$$(w+2)(w-5)\left(\frac{w}{w+2}+\frac{7}{w-5}\right)=w+7$$

I remind my students that they cannot "cancel" across an addition symbol.

For Problems 6 and 7, many students have difficulties with the signs due to the difference on the left-hand side of each equation.

OBJECTIVE 2 COMMENTS

My students often confuse simplifying expressions with solving equations. This issue has been addressed in various Related Review exercises and Expressions, Equations, and Graphs exercises in preceding sections. However, most of my students still need this issue to be discussed in this section. Without frequent warnings, many students introduce multiplications of rational expressions by expressions not equivalent to 1. Many students multiply both sides of an equation by an expression equivalent to 1, rather than simply multiplying both sides by the LCD. Exercises 64 and 65 and Related Review Exercises 67–74 of Homework 10.5 attend to this issue.

Workbook Exploration

The following 15-minute exploration addresses some student misconceptions about simplifying rational expressions and solving rational equations.

EXPLORATION *Simplifying versus solving*

1. Two students tried to solve $4=\dfrac{5}{x}+\dfrac{3}{x}$. Did one, both, or neither of these students solve the equation correctly? Explain.

Student A's Work

$$\begin{aligned}
4&=\frac{5}{x}+\frac{3}{x}\\
4x&=x\left(\frac{5}{x}+\frac{3}{x}\right)\\
4x&=x\cdot\frac{5}{x}+x\cdot\frac{3}{x}\\
4x&=5+3\\
4x&=8\\
x&=2
\end{aligned}$$

Student B's Work

$$\begin{aligned}
4&=\frac{5}{x}+\frac{3}{x}\\
&=\frac{5}{x}+\frac{3}{x}-4\\
&=\frac{8}{x}-4\\
&=\frac{8}{x}-4\cdot\frac{x}{x}\\
&=\frac{8}{x}-\frac{4x}{x}\\
&=\frac{-4x+8}{x}
\end{aligned}$$

2. Three students tried to simplify $4 + \frac{5}{x} + \frac{3}{x}$. Which students, if any, simplified the expression correctly? Explain.

Student C's Work

$$\begin{aligned} 4 + \frac{5}{x} + \frac{3}{x} &= x\left(4 + \frac{5}{x} + \frac{3}{x}\right) \\ &= 4x + x \cdot \frac{5}{x} + x \cdot \frac{3}{x} \\ &= 4x + 5 + 3 \\ &= 4x + 8 \end{aligned}$$

Student D's Work

$$\begin{aligned} 4 + \frac{5}{x} + \frac{3}{x} &= 4 \cdot \frac{x}{x} + \frac{5}{x} + \frac{3}{x} \\ &= \frac{4x}{x} + \frac{8}{x} \\ &= \frac{4x+8}{x} \end{aligned}$$

Student E's Work

$$\begin{aligned} 4 + \frac{5}{x} + \frac{3}{x} &= 0 \\ x\left(4 + \frac{5}{x} + \frac{3}{x}\right) &= x \cdot 0 \\ 4x + 5 + 3 &= 0 \\ 4x &= -8 \\ x &= -2 \end{aligned}$$

3. **a.** What is the difference in your goals in solving a rational equation versus simplifying a rational expression?

b. Explain how that difference in goals relates to the techniques you use to solve an equation versus simplify an expression.

OBJECTIVE 3 COMMENTS

Problem 12 is at the same level of difficulty as Exercises 55–57 in Homework 10.5. Note that Exercise 58 is significantly harder, because it requires using the quadratic formula.

Workbook Exploration

Having groups work on the following 10-minute exploration will prepare them well for similar exercises in the homework.

Group Exploration

Using a rational model to make predictions

The average amounts employees pay for single-person coverage of health insurance and the average total costs (paid by employees and employers) are shown in the following table for various years.

	Average Cost (dollars)	
Year	Employee	Total
2006	627	4242
2009	779	4824
2012	951	5615
2015	1071	6251
2016	1129	6435

Source: *Kaiser/HRET Survey of Employer-Sponsored Health Benefits, 1999–2016*

Let P be the percentage of the average total annual cost of health insurance paid by employees at t years since 2000. A reasonable model is

$$P = \frac{4900t + 33,261}{224t + 2872}$$

Predict when 18% of the average total annual cost of health insurance will be paid by employees.

SECTION 10.6 LECTURE NOTES

Objectives

1. Use *proportions* to make estimates.
2. Find the length of a side of a *similar triangle*.

Main point: Use proportions to make estimates. Work with similar triangles.

OBJECTIVE 1

- A **proportion** is a statement of the equality of two ratios.
- Here is a proportion: $\frac{6}{8} = \frac{3}{4}$.

If the ratio of two related quantities is constant, we say that the two quantities are **proportional**.

1. A person pays $297 for 6 months' use of cable service. How much will the person pay for 8 months' use of cable service?
2. The weight of an object on Earth and the weight of the same object on the Moon are proportional. An astronaut who weighs 160 pounds on Earth weighs 26.5 pounds on the Moon. What is the weight of a person on Earth if she weighs 24.0 pounds on the Moon?
3. A family spends $416 on groceries in a 4-month period.
   - **a.** Estimate the family's total money for groceries during a 6-month period.
   - **b.** Discuss any assumptions that you made in part (a).
4. In a poll of 1000 adults, 770 adults said that engineers are largely responsible for Americans having a high standard of living (Source: *Harris Interactive, Inc.*).
   - **a.** Estimate how many of 12,500 students at a college would say that engineers are largely responsible for Americans having a high standard of living.
   - **b.** Discuss any assumptions that you made in part (a). Describe a possible scenario in which the result in part (a) is an underestimate.

OBJECTIVE 2

- Two triangles are called **similar triangles** if their corresponding angles are equal in measure.
- The two triangles in Fig. 2.9 are similar triangles because the measures of angles A and X are equal, the measures of angles B and Y are equal, and the measures of angles C and Z are equal.

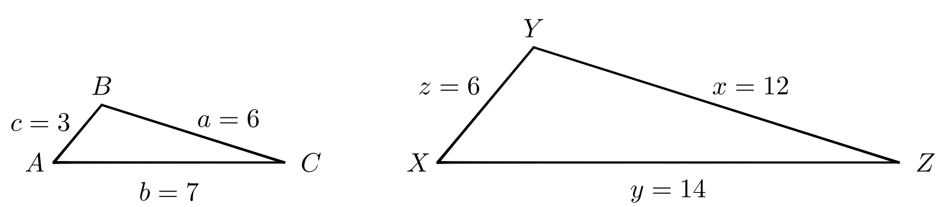


Figure 2.9: Two similar triangles

- Here we find the ratio of the lengths of corresponding sides of the two triangles in Fig. 2.9:

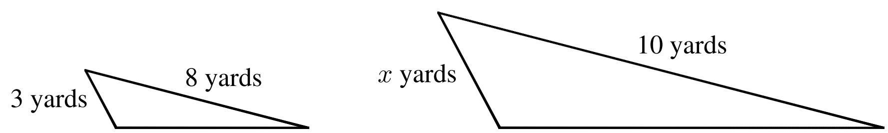
$$\frac{c}{z} = \frac{3}{6} = \frac{1}{2}, \qquad \frac{a}{x} = \frac{6}{12} = \frac{1}{2}, \qquad \frac{b}{y} = \frac{7}{14} = \frac{1}{2}$$

  Notice that all three ratios are equal.

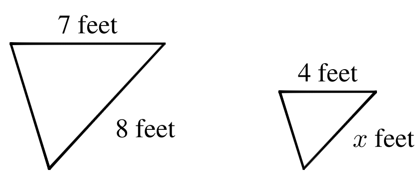
Side Lengths of Similar Triangles
The lengths of the corresponding sides of two similar triangles are proportional.

Similar triangles have the same shape but not necessarily the same size.

5. The two triangles shown in the following figure are similar. Find the length of the side labeled "x yards" on the larger triangle. Round your result to the second decimal place.



6. The two triangles shown in the following figure are similar. Find the length of the side labeled "x feet" on the smaller triangle. Round your result to the second decimal place.



SHORT HW 1, 3, 7, 13, 15, 17, 19, 21, 23, 27, 33

MEDIUM HW 1, 3, 7, 11–23 odds, 27, 29, 31, 32, 33–39 odds

SECTION 10.6 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students use proportions to make estimates about authentic situations and find the length of a side of a similar triangle.

OBJECTIVE 1 COMMENTS

Students have an easy time working with proportions. To begin, I quickly remind students of the meaning of a ratio. When setting up proportions, I emphasize that it's important to be consistent in describing quantities in the numerators and the denominators. I remind students that they should describe their result using units and a complete sentence.

Part (b) of Problem 3 and part (b) of Problem 4 are great opportunities for critical thinking. Most algebra textbooks do not address the dangers of generalizing from a sample. This is unfortunate, because without such a discussion, work with proportions can be misleading.

Textbook and Workbook Exploration

The 15-minute "Proportions" exploration is a nice way to introduce the concept of proportion.

OBJECTIVE 2 COMMENTS

Again, I emphasize the importance of being consistent. This time, students must be consistent in describing the lengths of the sides of the triangles in the numerators and the denominators of a proportion.

SECTION 10.7 LECTURE NOTES

Objectives

1. Describe the meaning of *direct variation* and *inverse variation*.
2. In direct variation and inverse variation, describe how a change in the value of the explanatory variable affects the value of the response variable.
3. Use a single point to find a direct variation equation or an inverse variation equation.
4. Use direct variation models and inverse variation models to make estimates.
5. Use a table of data to find a direct variation equation or an inverse variation equation.

Main point: Work with *direct variation equations* and *inverse variation equations*.

OBJECTIVE 1

1. A person drives d miles at 60 mph for t hours. Find an equation for t and d.

Definition *Direct variation*

If $y = kx$ for some constant k, we say that $\boldsymbol{y}$ **varies directly as** $\boldsymbol{x}$ or that $\boldsymbol{y}$ **is proportional to** $\boldsymbol{x}$. We call k the **variation constant** or the **constant of proportionality**. The equation $y = kx$ is called a **direct variation equation**.

OBJECTIVES 2 and 3

Graph $y = \frac{1}{2}x$, $y = x$, and $y = 2x$ to motivate the following property.

Changes in Values of Variables for Direct Variation

Assume that y varies directly as x for some positive variation constant k.

- If the value of x increases, then the value of y increases.
- If the value of x decreases, then the value of y decreases.

2. The variable y varies directly as x with positive variation constant k.

   a. What happens to the value of y as the value of x increases?

   b. If $y = 5$ when $x = 3$, find an equation for x and y.

3. The variable p varies directly as r. If $p = 8$ when $r = 3$, find the value of r when $p = 12$.

OBJECTIVE 4

4. The number of inches of water varies directly as the number of inches of snow. If 15 inches of snow will melt to 1.68 inches of water, to how many inches of water will 22 inches of snow melt?

Finding and Using a Direct Variation Model

Assume that a quantity p varies directly as a quantity t. To make estimates about an authentic situation,

1. Substitute the values of a data point into the equation $p = kt$ and then solve for k.
2. Substitute the value of k into the equation $p = kt$.
3. Use the equation found in step 2 to make estimates of quantity t or quantity p.

Give examples of equations such as $y = kx^2$ in which the response variable varies directly as an *expression*.

OBJECTIVE 1 (revisited)

Definition *Inverse variation*

If $y = \frac{k}{x}$ for some constant k, we say that $\boldsymbol{y}$ **varies inversely as** $\boldsymbol{x}$ or that $\boldsymbol{y}$ **is inversely proportional to** $\boldsymbol{x}$. We call k the **variation constant** or the **constant of proportionality**. The equation $y = \frac{k}{x}$ is called an **inverse variation equation**.

OBJECTIVES 2 and 3 (revisited)

Graph $y = \frac{1}{x}$, $y = \frac{2}{x}$, and $y = \frac{4}{x}$ to motivate the following property.

Changes in Values of Variables for Inverse Variation

Assume that y varies inversely as x for some positive variation constant k. For positive values of x,

- If the value of x increases, then the value of y decreases.
- If the value of x decreases, then the value of y increases.

5. The variable y varies inversely as x with positive variation constant k.

a. What happens to the value of y as the value of x increases?

b. If $y = 6$ when $x = 2$, find an equation for x and y.

6. The variable w varies inversely as t. If $w = 8$ when $t = 5$, find the value of t when $w = 4$.

OBJECTIVE 4 (revisited)

7. When a stone is tied to a string and whirled in a circle at constant speed, the tension in the string varies inversely as the radius of the circle. If the radius is 50 centimeters, the tension is 70 newtons. Find the radius if the tension is 90 newtons.

Finding and Using an Inverse Variation Model

Assume that a quantity p varies inversely as a quantity t. To make estimates about an authentic situation,

1. Substitute the values of a data point into the equation $p = \dfrac{k}{t}$ and then solve for k.
2. Substitute the value of k into the equation $p = \dfrac{k}{t}$.
3. Use the equation found in step 2 to make estimates of quantity t or quantity p.

Give examples of equations such as $y = \dfrac{5}{x^2}$ in which the response variable varies inversely as an *expression*.

8. The frequency F (in hertz) of a tuning fork varies inversely as the square of the length L (in cm) of the prongs. The frequency is 32 hertz when the prong length is 10 cm. Find the frequency if the prong length is 8 cm.

OBJECTIVE 5

Discuss Example 8 on pages 584 and 585.

SHORT HW 1, 15, 17, 25, 27, 31, 33, 39, 43, 47, 49, 51, 53, 55, 60

MEDIUM HW 1, 7, 11, 15, 17, 23–43 odds, 47, 49, 51, 53, 55, 60, 61, 63

SECTION 10.7 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

Students learn about direct and inverse variation in this section. They use variation models to make estimates.

OBJECTIVE 1 COMMENTS

When discussing direct variation, I emphasize that an equation of the form $y = kx$ is a linear equation in two variables. Hence, most of the concepts associated with direct variation are review material.

I make sure that my students understand the difference between two variables that can be described by a direct variation equation and two variables that can be described by using an increasing curve. Most people (for example, in the media) state that two quantities vary directly when what they mean is that the quantities can be modeled by an increasing curve.

OBJECTIVES 2 and 3 COMMENTS

For the equation $y = kx$, I emphasize how the value of y changes as the value of x increases or decreases.

OBJECTIVE 4 COMMENTS

When working with direct and inverse variation applications, some students do not read carefully to determine whether the situation should be modeled using a direct variation or an inverse variation equation.

OBJECTIVE 1 (REVISITED) COMMENTS

I spend more time discussing inverse variation than direct variation, because students are less familiar with graphs and tables of equations of the form $y = \dfrac{k}{x^p}$, where k and p are constants.

OBJECTIVE 2 and 3 (REVISITED) COMMENTS

For the equation $y = \dfrac{k}{x}$, I emphasize how the value of y changes as the value of x increases or decreases.

Textbook and Workbook Exploration

The 15-minute “Inverse variation” exploration has groups consider asymptotic behavior for a model of the form $y = \dfrac{k}{x}$.

OBJECTIVE 4 (REVISITED) COMMENTS

For Problem 7, I emphasize that the phrase “varies inversely” is important. Many students who do not read the problem carefully treat such a problem as a direct variation application.

For Problem 8, I emphasize the word “square.” Many students who do not read the problem carefully write the equation $F = \dfrac{k}{L}$.

OBJECTIVE 5 COMMENTS

Example 8 on pages 584 and 585 and Exercises 53–56 in Homework 10.7 provide a more detailed account about how the concepts in this section can be applied to authentic situations.

Textbook and Workbook Exploration

The 15-minute “Comparing methods of finding a direct variation equation” exploration serves as a nice summary of three techniques we have used to find linear models in the course.

SECTION 10.8 LECTURE NOTES

Objective

1. Simplify *complex rational expressions*.

Main point: Simplify complex rational expressions.

OBJECTIVE 1

- A **complex rational expression** is a rational expression whose numerator or denominator (or both) is a rational expression.
- Here are some examples:

$$\frac{\frac{5}{x-2}}{\frac{x}{x^2-49}} \qquad \frac{\frac{3}{a}+\frac{1}{a^3}}{\frac{2}{5}-\frac{2}{a}} \qquad \frac{\frac{x+5}{x-3}}{x^2-2x-8}$$

- Here we find the values of two numerical complex rational expressions:

$$\frac{2}{\frac{2}{2}} = \frac{2}{1} = 2 \qquad \frac{\frac{2}{2}}{2} = \frac{1}{2}$$

- From these two examples, we see that it is important to keep track of the main fraction bar (the longest one) of the complex fraction.
- We **simplify a complex rational expression** by writing it as a rational expression $\frac{P}{Q}$, with $\frac{P}{Q}$ in lowest terms.

Method 1: An expression of the form $\frac{R}{S}$ can be written in the form $R \div S$. Use this fact to simplify the following complex rational expression:

$$\frac{\frac{2}{9}}{\frac{5}{7}}$$

Simplify by method 1.

1. $\dfrac{\frac{9}{x^5}}{\frac{6}{x^2}}$ **2.** $\dfrac{\frac{18x}{x^2+7x+10}}{\frac{12}{x^2-25}}$ **3.** $\dfrac{\frac{4}{a^2}-\frac{1}{a}}{\frac{5}{a^3}+\frac{3}{a}}$

Using Method 1 to Simplify a Complex Rational Expression

To simplify a complex rational expression by method 1,

1. Write both the numerator and the denominator as fractions.
2. To write the complex rational expression as the quotient of two rational expressions, use the property

$$\frac{\frac{A}{B}}{\frac{C}{D}} = \frac{A}{B} \div \frac{C}{D}, \text{ where } B, C, \text{ and } D \text{ are nonzero}$$

3. Divide the rational expressions.

Method 2: First find the LCD of all the fractions in the numerator and the denominator. Then multiply by 1 in the form $\frac{\text{LCD}}{\text{LCD}}$.

Simplify by method 2.

4. $\dfrac{\frac{6}{x^3}}{\frac{4}{x^5}}$ **5.** $\dfrac{\frac{15x}{x^2 - 4x - 21}}{\frac{35}{x^2 - 49}}$ **6.** $\dfrac{\frac{5}{x^3} - \frac{7}{2x}}{\frac{3}{x^2} - \frac{5}{4x}}$

Using Method 2 to Simplify a Complex Rational Expression

To simplify a rational expression by method 2,

1. Find the LCD of all the fractions appearing in the numerator and denominator.
2. Multiply by 1 in the form $\frac{\text{LCD}}{\text{LCD}}$.
3. Simplify the numerator and the denominator to polynomials.
4. Simplify the rational expression.

7. Use the method of your choice to simplify $\dfrac{3 + \frac{4}{x^3}}{2 - \frac{2}{x^2}}$.

SHORT HW 1, 7, 11, 15, 17, 19, 21, 25, 27, 29, 35, 37, 39, 43

MEDIUM HW 1, 7, 11, 15, 17, 19, 21, 25, 27, 29, 35–59 odds

SECTION 10.8 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students simplify complex rational expressions using one of two methods called "method 1" (writing such expressions as a quotient of two rational expressions) and "method 2" (multiplying such an expression by $\frac{\text{LCD}}{\text{LCD}}$). For instructors who would like to decrease the amount of symbol manipulation in the course, this may be a good section to skip.

OBJECTIVE 1 COMMENTS

Many reviewers of the text swear by method 2. Most of my students prefer method 1. The section has been written so that students can compare method 1 and method 2 and decide for themselves which method they think is better for a particular expression. In my course, I discuss method 1 and leave it up to my students to read about and use method 2, if they so choose.

When using method 1, some students use pairs of equality symbols aligned vertically. They seem to treat a complex rational expression as if it were *two* separate expressions. I am not sure what such students are thinking, but the error of using two equality symbols is definitely worth discussing!

When I use method 2 (for a higher level course), I emphasize that the expression

$$\frac{\frac{\text{LCD}}{1}}{\frac{\text{LCD}}{1}}$$

is equal to 1. I remind my students that multiplying an expression by 1 in this form will give an equivalent expression.

For Problem 3, many of my students perform the following incorrect work:

$$\frac{\frac{4}{a^2}-\frac{1}{a}}{\frac{5}{a^3}+\frac{3}{a}} = \left(\frac{4}{a^2}-\frac{1}{a}\right)\cdot\left(\frac{a^3}{5}+\frac{a}{3}\right)$$

To make matters worse, some of my students omit using any parentheses.

For Problem 4, it is a common student error to multiply the original expression by $\frac{\frac{x^3}{1}}{\frac{x^5}{1}}$. Some students make similar mistakes for Problems 5 and 6.

Background Information for the Exploration That Follows

The following exploration guides groups in using method 1 to simplify complex rational expressions.

EXPLORATION *Simplifying complex rational expressions*

1. A fraction that has a fraction appearing in its numerator or denominator is called a **complex rational expression**. Steps used to simplify a complex rational expression are provided here:

$$\begin{aligned}\frac{\frac{a}{b}}{\frac{c}{d}} &= \frac{a}{b}\div\frac{c}{d}\\ &= \frac{a}{b}\cdot\frac{d}{c}\\ &= \frac{ad}{bc}\end{aligned}$$

Simplify:

$$\frac{\frac{2}{x}}{\frac{3}{x}}$$

2. A more complicated complex rational expression is simplified below. To begin, the sum in the numerator is written as one fraction and the difference in the denominator is written as one fraction.

$$\begin{aligned}
\frac{4+\frac{3}{a}}{\frac{2}{b}-3} &= \frac{4\cdot\frac{a}{a}+\frac{3}{a}}{\frac{2}{b}-3\cdot\frac{b}{b}} \\
&= \frac{\frac{4a}{a}+\frac{3}{a}}{\frac{2}{b}-\frac{3b}{b}} \\
&= \frac{\frac{4a+3}{a}}{\frac{2-3b}{b}} \\
&= \frac{4a+3}{a} \div \frac{2-3b}{b} \\
&= \frac{4a+3}{a} \cdot \frac{b}{2-3b} \\
&= \frac{4ab+3b}{2a-3ab}
\end{aligned}$$

Simplify.

a. $\dfrac{\frac{3}{x}+5}{\frac{2}{x}+4}$ **b.** $\dfrac{\frac{2}{a}-\frac{3}{b}}{\frac{5}{a}+\frac{4}{b}}$ **c.** $\dfrac{\frac{5}{4x}+\frac{2}{x^3}}{\frac{4}{3x}-\frac{1}{x^2}}$

CHAPTER 11 OVERVIEW

In Section 11.1, students add and subtract radical expressions. In Section 11.2, students multiply radical expressions. Students solve square root equations and use square root models to make predictions in Section 11.3.

SECTION 11.1 LECTURE NOTES

Objectives

1. Add and subtract radical expressions.

Main point: Add and subtract radical expressions.

OBJECTIVE 1

- Square root radicals that have the same radicand are called **like radicals**.
- We add the like radicals $2\sqrt{x}$ and $4\sqrt{x}$ as follows:

$$2\sqrt{x}+4\sqrt{x}=(2+4)\sqrt{x}=6\sqrt{x}$$

- To add or subtract like radicals, we use the distributive law.
- When we add or subtract like radicals, we say we *combine like radicals*.

Perform the indicated operations.

1. $4\sqrt{3}+6\sqrt{3}$

2. $3\sqrt{x}+\sqrt{x}$

3. $8\sqrt{5x}-6\sqrt{5x}$

4. $4\sqrt{2}+7\sqrt{5}$

5. $3w\sqrt{7}-2w\sqrt{7}$

6. $2\sqrt{x}-4\sqrt{3}-7\sqrt{x}$

Sometimes simplifying radicals will allow us to combine like radicals.

Perform the indicated operations.

7. $\sqrt{18}+6\sqrt{2}$

8. $2\sqrt{45}-3\sqrt{20}$

9. $5\sqrt{16t}-4\sqrt{49t}+2\sqrt{25t}$

10. $3\sqrt{12x}+6\sqrt{8}-5\sqrt{27x}$

Recall from Section 9.1 that if x is nonnegative, then $\sqrt{x^2}=x$.

Perform the indicated operations. Assume that all variables are nonnegative.

11. $2\sqrt{5x^2}+x\sqrt{5}$

12. $\sqrt{32x^2}-\sqrt{50x^2}$

13. $4w\sqrt{63}+6\sqrt{28w^2}-2\sqrt{300w}$

14. $3\sqrt{80t^2}+5\sqrt{24t}-2t\sqrt{45}$

SHORT HW 1, 5, 13, 25, 29, 41, 43, 47, 51, 55, 57, 59, 63

MEDIUM HW 1, 5, 9, 13, 17, 21, 25, 29, 35, 41, 43, 47, 51, 55, 57, 59, 61, 63, 65, 67, 73, 75, 77

SECTION 11.1 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students add and subtract radical expressions.

OBJECTIVE 1 COMMENTS

I compare the process of combining like radicals to the process of combining like terms.

Most students add and subtract radical expressions well. Students' greatest challenge is to identify perfect-square factors of radicands and to remember that $\sqrt{x^2} = x$, where x is nonnegative.

Textbook and Workbook Exploration

In the 10-minute "Combining like radicals" exploration, groups discover how to combine like radicals. Some groups may need help with Problem 5 of the exploration.

SECTION 11.2 LECTURE NOTES

Objectives

1. Multiply two radical expressions.
2. Find the product of two *radical conjugates*.
3. Simplify the square of a radical expression that has two terms.

Main point: Multiply radical expressions and simplify the square of a radical expression with two terms.

OBJECTIVE 1

- Recall from Section 9.1 the product property for square roots: $\sqrt{ab} = \sqrt{a}\sqrt{b}$, where a and b are nonnegative.
- So, we have $\sqrt{2}\sqrt{7} = \sqrt{2 \cdot 7} = \sqrt{14}$.
- For x nonnegative: $(\sqrt{x})^2 = \sqrt{x}\sqrt{x} = \sqrt{x^2} = x$.

Power Property for Square Roots

If x is nonnegative, then

$$(\sqrt{x})^2 = x$$

Find the product.

1. $\sqrt{3} \cdot \sqrt{5}$ **2.** $7\sqrt{2} \cdot \sqrt{6}$ **3.** $8\sqrt{x} \cdot 5\sqrt{x}$ **4.** $\sqrt{7t} \cdot \sqrt{2t}$

Find the product.

5. $3(\sqrt{2} - \sqrt{7})$ **6.** $\sqrt{x}(8 - \sqrt{x})$ **7.** $4\sqrt{5}(2\sqrt{5} + 6\sqrt{3})$

Find the product.

8. $(4 + \sqrt{5})(2 + \sqrt{5})$ **9.** $(\sqrt{w} - 7)(\sqrt{w} + 4)$

Product of Two Radical Expressions, Each with Two Terms

To find the product of two radical expressions in which each factor has two terms,

1. Multiply each term in the first radical expression by each term in the second radical expression.
2. Combine like radicals.

Find the product.

10. $(6 - 3\sqrt{5})(3 + 2\sqrt{5})$ **11.** $(2 - 7\sqrt{3})(5 - 4\sqrt{3})$

OBJECTIVE 2

- Recall from Section 7.5 that we call binomials such as $4x + 7$ and $4x - 7$ conjugates of each other.
- We call the radical expressions $4 + 7\sqrt{x}$ and $4 - 7\sqrt{x}$ radical conjugates of each other.
- To find the **radical conjugate** of a radical expression with two terms, we change the addition symbol to a subtraction symbol or vice versa.

Product of Two Radical Conjugates

To find the product of two radical conjugates, use the property

$$(A + B)(A - B) = A^2 - B^2$$

Find the product.

12. $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

13. $(t + \sqrt{7})(t - \sqrt{7})$

14. $(3\sqrt{2} - 4\sqrt{5})(3\sqrt{2} + 4\sqrt{5})$

15. $(2\sqrt{a} + 7\sqrt{b})(2\sqrt{a} - 7\sqrt{b})$

OBJECTIVE 3

Recall from Section 7.5:

$$(A + B)^2 = A^2 + 2AB + B^2 \qquad \text{Square of a sum}$$
$$(A - B)^2 = A^2 - 2AB + B^2 \qquad \text{Square of a difference}$$

Simplify.

16. $(x + \sqrt{5})^2$

17. $(\sqrt{p} - \sqrt{7})^2$

Simplifying the Square of a Radical Expression

To simplify the square of a radical expression that has two terms,

- Use $C^2 = CC$ to write the square as a product of two identical expressions, and then multiply each term in the first radical expression by each term in the second radical expression,

or

- Use the square-of-a-sum property $(A+B)^2 = A^2+2AB+B^2$ or the square-of-a-difference property $(A - B)^2 = A^2 - 2AB + B^2$.

Simplify.

18. $(5 + 3\sqrt{b})^2$

19. $(3\sqrt{2} - 5\sqrt{3})^2$

SHORT HW 3, 9, 15, 21, 25, 27, 35, 37, 43, 45, 47, 53, 55, 61, 62

MEDIUM HW 1, 3, 9, 15, 17, 21, 25, 27, 35, 37, 43, 45, 47, 53, 55, 57, 61, 62, 63, 65, 71, 73, 75

SECTION 11.2 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students multiply radical expressions. They simplify the square of a radical expression that has two terms. Students will use these skills in Section 11.3.

OBJECTIVE 1 COMMENTS

For Problem 2, some students wonder if the 7 should be distributed to the factors $\sqrt{2}$ and $\sqrt{6}$, as if they were finding the product $7(\sqrt{2}+\sqrt{6})$. To address this, I compare the distributive law with the associative law for multiplication. I also point out that for $7(2 \cdot 6)$, we would not distribute the 7 to both 2 and to 6.

For Problem 3, some students wonder if the numbers "outside" the radical signs should be multiplied with the variables "inside" the radical signs. Similar issues exist for Problem 5. Usually referring to the product property for square roots helps clear up this confusion.

OBJECTIVE 2 COMMENTS

For Problem 14, some students have trouble simplifying $(3\sqrt{2})^2$. Usually it's because they skip writing the key step $(3\sqrt{2})^2 = 3^2(\sqrt{2})^2$. This issue also applies for simplifying $(4\sqrt{5})^2$. Weaker students are sometimes better off multiplying each term in the first radical expression, $(3\sqrt{2} - 4\sqrt{5})$, times each term in the second radical expression, $(3\sqrt{2} + 4\sqrt{5})$.

Textbook and Workbook Exploration

The 15-minute "Multiplying two radical expressions" exploration will likely only work well with very strong groups. Usually students have a tough time with such skills even after having seen some examples.

Textbook and Workbook Exploration

In the 15 minute "Rationalizing the denominator" exploration, groups discover how to rationalize the denominator in which the denominator contains a sum or difference of two radicals. Some groups may need a hint for Problem 4 of the exploration.

OBJECTIVE 3 COMMENTS

For Problems 16–19, I remind students to remember the middle term $2AB$ of the right-hand side of the equation $(A+B)^2 = A^2 + 2AB + B^2$. I make a similar comment about $(A-B)^2 = A^2 - 2AB + B^2$. Students will need to keep these ideas in mind when solving square root equations in Section 11.3.

For Problem 16, aside from the most frequent error of writing $(x+\sqrt{5})^2$ as x^2+5, a common error is to write $(x+\sqrt{5})^2$ as $x^2+x\sqrt{10}+5$. I find that weaker students have better success writing $(x+\sqrt{5})^2$ as $(x+\sqrt{5})(x+\sqrt{5})$ and then multiplying pairs of terms than by using the property $(A+B)^2 = A^2 + 2AB + B^2$.

Workbook Exploration

Although students are not led to any discoveries in the following 15-minute exploration, it's still a good idea to have groups work on it because students struggle a great deal with this skill.

Group Exploration

Simplifying the square of a radical expression that has two terms

Simplify.

1. $(x+5)^2$

2. $(x+\sqrt{5})^2$

3. $(x-3)^2$

4. $(x-\sqrt{3})^2$

5. $(\sqrt{w}+\sqrt{2})^2$

6. $(\sqrt{5}-\sqrt{7})^2$

SECTION 11.3 LECTURE NOTES

Objectives

1. Solve a *square root equation* in one variable.
2. Graph a square root equation in two variables.
3. Use a *square root model* to make predictions.

Main point: Solve square root equations in one variable and use a square root model to make predictions.

OBJECTIVE 1

A **square root equation** is an equation that contains at least one square root expression. Here are some examples:

$$\sqrt{x} = 4 \qquad 2\sqrt{x} - 7 = 5 \qquad \sqrt{x^2 + 3x - 2} = x \qquad \sqrt{x+5} + x = 3$$

1. Solve $\sqrt{x} = 3$ by squaring both sides of the equation.

Squaring Property of Equality

If A and B are expressions, then all solutions of the equation $A = B$ are *among* the solutions of the equation $A^2 = B^2$. That is, the solutions of an equation are among the solutions of the equation obtained by squaring both sides.

- If a proposed solution of any type of equation is *not* a solution, we call it an **extraneous solution.**
- Show that squaring both sides of the equation $x = 5$ introduces the extraneous solution -5.

Checking Proposed Solutions

Because squaring both sides of a square root equation may introduce extraneous solutions, it is essential to check that each proposed solution satisfies the original equation.

Solve.

2. $\sqrt{5x - 1} = 3$ [eventually linear: one solution]
3. $\sqrt{x + 7} = -4$ [eventually linear: one extraneous solution]
4. $\sqrt{7w - 2} = \sqrt{4w + 10}$ [eventually linear: one solution]
5. $3\sqrt{x} + 7 = 13$ [eventually linear: one solution]

To solve a square root equation, we isolate a square root term to one side of the equation before squaring both sides.

Solving a Square Root Equation in One Variable

We solve a square root equation in one variable by the following steps:

1. Isolate a square root term to one side of the equation.
2. Square both sides.
3. Solve the new equation.
4. Check that each proposed solution satisfies the original equation.

6. Solve $\sqrt{x-3} = x-3$ [factoring: two solutions]

- Warning: To solve $\sqrt{x-3} = x-3$, it is incorrect to square each term: $\left(\sqrt{x-3}\right)^2 = x^2 - 3^2$.
- It is correct to square both sides of $\sqrt{x-3} = x-3$. When simplifying $(x-3)^2$, remember the middle term $-6x$ of $x^2 - 6x + 9$.
- If an equation contains two or more square root terms, we may need to use the squaring property of equality twice.

Solve.

7. $\sqrt{x+5} + 1 = x$ [factoring: one extraneous solution, one solution]

8. $\sqrt{w+2} = \sqrt{w} + 1$ [linear: one solution]

9. $\sqrt{x-4} + 2 = \sqrt{x+2}$ [linear: one solution]

10. Use "intersect" on a graphing calculator to solve $\sqrt{x+4} = x^2 - 2x - 6$.

OBJECTIVE 2

11. Graph $y = \sqrt{x}$.

OBJECTIVE 3

12. Men's average expenditures on Valentine's Day gifts are shown in the following table for various years.

Year	Men's Average Expenditure on Valentine's Day (dollars)
2010	135
2011	159
2012	169
2013	176
2014	173
2015	191
2016	196

Source: *National Retail Federation*

Let A be men's average expenditure (in dollars) on Valentine's Day gifts at t years since 2010. A reasonable model is $A = 23.5\sqrt{t} + 135$.

a. Use a graphing calculator to draw the graph of the model and, in the same viewing window, the scatterplot of the data. Does the model fit the data well?

b. Predict men's average expenditure on Valentine's Day gifts in 2022.

c. Predict when men's average expenditure on Valentine's Day gifts will be $210.

SHORT HW 7, 9, 11, 13, 21, 25, 29, 33, 39, 41, 43, 49, 51, 59, 64 **MEDIUM HW** 1, 7, 9, 11, 13, 21, 25, 29, 33, 35, 37, 39, 41, 43, 45, 49, 51, 55, 59, 64, 71–79 odd

SECTION 11.3 DETAILED COMMENTS AND EXPLORATIONS

Main points/connection to other parts of the text

In this section, students solve square root equations.

OBJECTIVE 1 COMMENTS

The key concept in this section is the power property for square roots $(\sqrt{x})^2 = x$, where x is nonnegative. I remind students why this property is true.

I emphasize that we must check for extraneous solutions.

For each of Problems 5 and 7, I emphasize that we want to isolate the radical to one side of the equation. For Problems 8 and 9, I point out that it is helpful that one of the radicals is isolated to one side of the equation. I observe that for these two problems, each time we square both sides, we reduce the number of radicals in the equation by one radical.

I remind students several times that $(A+B)^2$ is not equivalent to A^2+B^2 and that $(A-B)^2$ is not equivalent to A^2-B^2 or A^2+B^2. To solve $\sqrt{x-3} = x-3$ (Problem 6), some students will correctly write

$$(\sqrt{x-3})^2 = (x-3)^2$$

as their first step and then incorrectly write

$$(\sqrt{x-3})^2 = x^2 - 3^2$$

Other students will not show the above first step, but will square each term of the right side of the equation:

$$(\sqrt{x-3})^2 = x^2 - 3^2$$

For the first type of error, I remind students how to simplify $(A-B)^2$. For the second type of error, I demonstrate that squaring each term of the true statement $3 = 5-2$ yields $3^2 = 5^2 - 2^2$, a false statement.

Background Information for the Exploration That Follows

When I use the following 15-minute exploration, I first show my students how to solve an equation such as $\sqrt{x} = 5$ by squaring both sides. Then I assign the exploration.

EXPLORATION *Solving square root equations*

In this exploration, you will solve radical equations. For Problems 1–4, check that each result satisfies the original equation.

1. Solve $\sqrt{x} = 3$.
2. Solve the equation $\sqrt{x} + 2 = 7$. [**Hint**: How do you solve $x+2=7$?]
3. Solve $6\sqrt{x} = 24$. [**Hint**: How do you solve $6x = 24$?]

4. Solve $3\sqrt{x}+4=10$. [**Hint**: How do you solve $3x+4=10$?]

5. **a.** A student tries to solve the equation $\sqrt{x}=-2$:

$$\begin{aligned}\sqrt{x} &= -2\\ (\sqrt{x})^2 &= (-2)^2\\ x &= 4\end{aligned}$$

Check whether 4 satisfies the equation $\sqrt{x}=-2$. (As was the case with rational equations, the result $x=4$ is called an *extraneous solution*.) What is the solution of the equation $\sqrt{x}=-2$?

b. Do you think checking your results using substitution is an optional or "mandatory" step when solving square root equations? Explain.

c. Solve the equation $\sqrt{x}=-3$.

d. If k is a negative constant, explain why the solution to the equation $\sqrt{x}=k$ is the empty set.

e. Solve the equation $2\sqrt{x}+9=1$.

For some groups, the hint in Problem 3 of the exploration is not helpful enough. For those groups, I suggest that they write the equation in the form $\sqrt{x}=k$.

Workbook Exploration

Although the following 30-minute exploration takes a lot of time for groups to complete, they will cover quite a lot of ground.

Group Exploration

Section Opener: Solving radical equations

1. Solve $\sqrt{x}=4$ by finding a number whose square root is 4.

2. Solve $\sqrt{x}=4$ by squaring both sides of the equation. Compare your result to the result you found in Problem 1.

3. Solve $\sqrt{x-3}=2$ by squaring both sides of the equation.

4. **a.** A student tries to solve the equation $\sqrt{x}=-2$:

$$\begin{aligned}\sqrt{x} &= -2\\ (\sqrt{x})^2 &= (-2)^2\\ x &= 4\end{aligned}$$

Check whether 4 satisfies the equation $\sqrt{x}=-2$. (As was the case with rational equations, the result $x=4$ is called an *extraneous solution*.) What is the solution of the equation $\sqrt{x}=-2$?

b. Do you think checking your results using substitution is an optional or "mandatory" step when solving radical equations? Explain.

c. Solve the equation $\sqrt{x}=-3$.

5. Three students tried to solve $\sqrt{x}+1=3$. Which students, if any, solved the equation correctly? Describe any errors and where they occurred.

Student 1's Work

$$\begin{aligned} \sqrt{x}+1&=3\\ (\sqrt{x}+1)^2&=3^2\\ (\sqrt{x})^2+1^2&=9\\ x+1&=9\\ x&=8 \end{aligned}$$

Student 2's Work

$$\begin{aligned} \sqrt{x}+1&=3\\ \sqrt{x}&=3-1\\ \sqrt{x}&=2\\ (\sqrt{x})^2&=2^2\\ x&=4 \end{aligned}$$

Student 3's Work

$$\begin{aligned} \sqrt{x}+1&=3\\ (\sqrt{x}+1)^2&=3^2\\ (\sqrt{x})^2+2\sqrt{x}+1^2&=9\\ x+2\sqrt{x}+1&=9\\ 2\sqrt{x}&=8-x\\ (2\sqrt{x})^2&=(8-x)^2\\ 4x&=64-16x+x^2\\ 0&=x^2-20x+64\\ 0&=(x-4)(x-16)\\ x=4 \quad \text{or} \quad x&=16 \end{aligned}$$

6. Solve.

   a. $2\sqrt{x}+9=1$ b. $\sqrt{x+5}-2=4$ c. $\sqrt{x+3}-x=3$

Textbook and Workbook Exploration
The 10-minute "Solving square root equations" exploration addresses some common student errors.

Workbook Exploration
Students can investigate how extraneous solutions are introduced in the 15-minute "Extraneous solutions" exploration.

EXPLORATION *Extraneous solutions*

1. Solve the equation $\sqrt{3x-2}+2=x$. Record each step of your work carefully.
2. In Problem 1, you found that 1 is an extraneous solution and that 6 is the only solution. Now substitute 1 for x in each step you recorded in Problem 1. Which of the equations are satisfied by 1?
3. What does it mean to say that 1 is an extraneous solution? Why do we sometimes get extraneous solutions when we solve square root equations but not when we solve linear or quadratic equations?

OBJECTIVE 2 COMMENTS
For Problem 11, I point out that it's convenient to substitute integer perfect-square values for x to find the corresponding values of y.

OBJECTIVE 3 COMMENTS
Students have an easy time with Problem 12.

Workbook Exploration
The following 15-minute exploration directs students to find a square root model and then use the model to make estimates.

Group Exploration
Using a square root model to make predictions

The number of shopping malls are shown in the following table for various years.

Year	Number of Malls
1977	576
1987	874
1997	1043
2007	1165
2017	1211

Source: *CoStar Group*

Let n be the number of shopping malls at t years since 1977.

1. Find an equation of a square root model.
2. What is the n-intercept? What does it mean in this situation?
3. Predict the number of malls in 2020.
4. Predict the average number of malls per state in 2022.
5. Predict when there will be 1275 malls.
6. As t increases, does the graph get steeper, have the same steepness, or get less steep? What does that mean in this situation?