2

Linear Transformations and Matrices

2.1 LINEAR TRANSFORMATIONS, NULL SPACES, AND RANGES

3. The nullity is 0, and the rank is 2. Thus T is one-to-one, but not onto.

6. The nullity is $n-1$, and the rank is 1. Thus T is not one-to-one unless $n=1$, and T is not onto unless $n=1$.

11. $\mathsf{T}(8,11) = (5,-3,16)$

18. $\mathsf{T}(a,b) = (b,0)$. $\mathsf{N}(\mathsf{T}) = \text{span}\{(1,0)\} = \mathsf{R}(\mathsf{T})$.

19. Let $\mathsf{V} = \mathsf{W} = R$, and define $\mathsf{T} = \mathsf{I}$ and $\mathsf{U} = 2\mathsf{I}$.

23. All of R^3 or a plane in R^3 through the origin

25. **(a)** $\mathsf{T}(a,b) = (0,b)$ **(b)** $\mathsf{T}(a,b) = (0,b-a)$

26. **(b)** $\mathsf{T}(a,b,c) = (0,0,c)$

28. **(b)** See (a) and (b) of Exercise 25.

32. **(c)** Let $\mathsf{V} = \mathsf{P}(F)$ and $\mathsf{W} = \text{span}(\{1\})$. Define T first on the standard basis of V by $\mathsf{T}(1) = \mathsf{T}(x) = \mathit{0}$, and $\mathsf{T}(x^k) = x^{k-1}$ for $k \geq 2$. Now extend T to a linear transformation from V to V. Then $\mathsf{N}(\mathsf{T}) = \text{span}(\{1,x\})$, and $\mathsf{R}(\mathsf{T}) = \text{span}(\{x^k\colon\ k \geq 1\})$. So $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{W}$, but $\mathsf{W} \neq \mathsf{N}(\mathsf{T})$.

2.2 THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

2. **(b)** $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$ **(e)** $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

5. **(c)** $(1\ 0\ 0\ 1)$ **(d)** $(1\ 2\ 4)$ **(f)** $\begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}$ **(g)** (a)

2.3 COMPOSITION OF LINEAR TRANSFORMATIONS AND MATRIX MULTIPLICATION

2. **(a)** $(AB)D = \begin{pmatrix} 29 \\ -26 \end{pmatrix}$

(b) $A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$, $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$, $CA = \begin{pmatrix} 20 \\ 26 \end{pmatrix}$

3. **(b)** $[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $[\mathsf{U}(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

4. **(b)** $\begin{pmatrix} -6 \\ 2 \\ 0 \\ 6 \end{pmatrix}$ **(d)** (12)

9. $\mathsf{T}(a_1, a_2) = (0, a_1 + a_2)$, $\mathsf{U}(a_1, a_2) = (0, a_1)$

$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

21. **(a)** $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{pmatrix}$ There are no cliques.

(b) $B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 3 \\ 3 & 0 & 3 & 2 \end{pmatrix}$

Persons 1, 3, and 4 belong to a clique.

24. $\dfrac{n^2 - n}{2}$

2.4 INVERTIBILITY AND ISOMORPHISMS

14. $\mathsf{T}\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c)$

19. **(a)** $[\mathsf{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2.5 THE CHANGE OF COORDINATE MATRIX

2. **(b)** $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

3. **(b)** $\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$ **(f)** $\begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}$

6. **(b)** $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad [\mathsf{L}_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

(d) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}, \quad [\mathsf{L}_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$

7. **(b)** $\mathsf{T}(x, y) = \frac{1}{m^2+1}(x + my, mx + m^2 y)$

2.6 DUAL SPACES

3. **(b)** $f_1(a + bx + cx^2) = a,\ f_2(a + bx + cx^2) = b,\ f_3(a + bx + cx^2) = c$

4. The basis for V is $\{(.4, -.3, -.1),\ (.6, .3, .1),\ (.2, .1, -.3)\}$

6. **(a)** $\mathsf{T}^t(f)(x, y) = 7x + 4y$ **(b)** $[\mathsf{T}^t]_{\beta^*} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

(c) $[\mathsf{T}]_\beta = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ and $([\mathsf{T}]_\beta)^t = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$

2.7 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

3. **(b)** $\{1, e^t, e^{-t}\}$ **(d)** $\{e^{-t}, te^{-t}\}$

4. **(b)** $\{e^t, te^t, t^2 e^t\}$

16. **(a)** $\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{l}}\,t\right) + c_2 \sin\left(\sqrt{\frac{g}{l}}\,t\right)$

(b) $\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}\,t\right)$

17. $y(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

18. **(a)** *Case 1:* $r^2 = 4km$. $y(t) = e^{-(r/2m)t}[c_1 + c_2 t]$

Case 2: $r^2 > 4km$. $y(t) = c_1 e^{at} + c_2 e^{bt}$, where

$$a = \frac{-r}{2m} + \frac{\sqrt{r^2 - 4mk}}{2m}, \quad b = \frac{-r}{2m} - \frac{\sqrt{r^2 - 4mk}}{2m}$$

Case 3: $r^2 < 4km$. $y(t) = e^{at}[c_1 \cos bt + c_2 \sin bt]$, where

$$a = \frac{-r}{2m}, \quad b = \frac{\sqrt{4mk - r^2}}{2m}$$

(b) Referring to the three cases listed in (a):

Case 1: $y(t) = v_0 t e^{-(r/2m)t}$

Case 2: $y(t) = \dfrac{v_0 m}{\sqrt{r^2 - 4mk}}[e^{at} - e^{bt}]$

Case 3: $y(t) = \dfrac{v_0}{b} e^{at} \sin bt$