Chapter 2

Section 2.1

2.1.**1** Not a linear transformation, since $y_2 = x_2 + 2$ is not linear in our sense.

2.1.**2** Linear, with matrix $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}$

2.1.**3** Not linear, since $y_2 = x_1 x_3$ is nonlinear.

2.1.**4** $A = \begin{bmatrix} 9 & 3 & -3 \\ 2 & -9 & 1 \\ 4 & -9 & -2 \\ 5 & 1 & 5 \end{bmatrix}$

2.1.**5** By Theorem 2.1.2, the three columns of the 2×3 matrix A are $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$, so that

$A = \begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}$.

2.1.**6** Note that $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so that T is indeed linear, with matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

2.1.**7** Note that $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = [\vec{v}_1 \ldots \vec{v}_m] \begin{bmatrix} x_1 \\ \cdots \\ x_m \end{bmatrix}$, so that T is indeed linear, with matrix $[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_m]$.

2.1.**8** Reducing the system $\begin{bmatrix} x_1 + 7x_2 & = y_1 \\ 3x_1 + 20x_2 & = y_2 \end{bmatrix}$, we obtain $\begin{bmatrix} x_1 & & = & -20y_1 & + & 7y_2 \\ & x_2 & = & 3y_1 & - & y_2 \end{bmatrix}$.

2.1.**9** We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\begin{bmatrix} 2x_1 & + & 3x_2 & = & y_1 & \\ 6x_1 & + & 9x_2 & = & & y_2 \end{bmatrix}$ we obtain $\begin{bmatrix} x_1 + 1.5x_2 & = 0.5y_1 \\ 0 & = -3y_1 + y_2 \end{bmatrix}$.

No unique solution (x_1, x_2) can be found for a given (y_1, y_2); the matrix is noninvertible.

2.1.**10** We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\begin{bmatrix} x_1 & + & 2x_2 & = & y_1 & \\ 4x_1 & + & 9x_2 & = & & y_2 \end{bmatrix}$ we find that $\begin{bmatrix} x_1 & & = & 9y_1 & + & 2y_2 \\ & x_2 & = & -4y_1 & + & y_2 \end{bmatrix}$ or $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

The inverse matrix is $\begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$.

2.1.**11** We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2. Reducing the system $\begin{bmatrix} x_1 & + & 2x_2 & = & y_1 & \\ 3x_1 & + & 9x_2 & = & & y_2 \end{bmatrix}$ we find that $\begin{bmatrix} x_1 & & = & 3y_1 - \frac{2}{3}y_2 \\ & x_2 & = & -y_1 + \frac{1}{3}y_2 \end{bmatrix}$. The inverse matrix is $\begin{bmatrix} 3 & -\frac{2}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$.

2.1.**12** Reducing the system $\begin{bmatrix} x_1 + kx_2 & = y_1 \\ x_2 & = y_2 \end{bmatrix}$ we find that $\begin{bmatrix} x_1 & & = & y_1 & - & ky_2 \\ & x_2 & = & & & y_2 \end{bmatrix}$. The inverse matrix is $\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$.

2.1.**13** a First suppose that $a \neq 0$. We have to attempt to solve the equation $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for x_1 and x_2.

$$\begin{bmatrix} ax_1 & + & bx_2 & = & y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} \div a \rightarrow \begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} -c(I) \rightarrow$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & (d - \frac{bc}{a})x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & (\frac{ad-bc}{a})x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix}$$

We can solve this system for x_1 and x_2 if (and only if) $ad - bc \neq 0$, as claimed.

If $a = 0$, then we have to consider the system

$$\begin{bmatrix} & & bx_2 & = & y_1 & \\ cx_1 & + & dx_2 & = & & y_2 \end{bmatrix} \text{swap} : I \leftrightarrow II \begin{bmatrix} cx_1 & + & dx_2 & = & & y_2 \\ & & bx_2 & = & y_1 & \end{bmatrix}$$

We can solve for x_1 and x_2 provided that both b and c are nonzero, that is if $bc \neq 0$. Since $a = 0$, this means that $ad - bc \neq 0$, as claimed.

b First suppose that $ad - bc \neq 0$ and $a \neq 0$. Let $D = ad - bc$ for simplicity. We continue our work in part (a):

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & \frac{D}{a}x_2 & = & -\frac{c}{a}y_1 & + & y_2 \end{bmatrix} \cdot \frac{a}{D} \rightarrow$$

$$\begin{bmatrix} x_1 & + & \frac{b}{a}x_2 & = & \frac{1}{a}y_1 & & \\ & & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix} -\frac{b}{a}(II) \rightarrow$$

$$\begin{bmatrix} x_1 & & = & (\frac{1}{a} + \frac{bc}{aD})y_1 & - & \frac{b}{D}y_2 \\ & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & & = & \frac{d}{D}y_1 & - & \frac{b}{D}y_2 \\ & x_2 & = & -\frac{c}{D}y_1 & + & \frac{a}{D}y_2 \end{bmatrix}$$

(Note that $\frac{1}{a} + \frac{bc}{aD} = \frac{D+bc}{aD} = \frac{ad}{aD} = \frac{d}{D}$.)

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, as claimed. If $ad - bc \neq 0$ and $a = 0$, then we have to solve the system

$$\begin{bmatrix} cx_1+ & dx_2 & = y_2 \\ & bx_2 & = y_1 \end{bmatrix} \begin{matrix} \div c \\ \div b \end{matrix}$$

$$\begin{bmatrix} x_1+ & \frac{d}{c}x_2 & = \frac{1}{c}y_2 \\ & x_2 & = \frac{1}{b}y_1 \end{bmatrix} -\frac{d}{c}(II)$$

$$\begin{bmatrix} x_1 & & = & -\frac{d}{bc}y_1 & +\frac{1}{c}y_2 \\ & x_2 & = & \frac{1}{b}y_1 & \end{bmatrix}$$

It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (recall that $a = 0$), as claimed.

2.1.**14** a By Exercise 13a, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ is invertible if (and only if) $2k - 15 \neq 0$, or $k \neq 7.5$.

b By Exercise 13b, $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15}\begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$.

If all entries of this inverse are integers, then $\frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15}$ is a (nonzero) integer n, so that $2k-15 = \frac{1}{n}$ or $k = 7.5 + \frac{1}{2n}$. Since $\frac{k}{2k-15} = kn = 7.5n + \frac{1}{2}$ is an integer as well, n must be odd.

We have shown: If all entries of the inverse are integers, then $k = 7.5 + \frac{1}{2n}$, where n is an odd integer. The converse is true as well: If k is chosen in this way, then the entries of $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$ will be integers.

2.1.**15** By Exercise 13a, the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible if (and only if) $a^2 + b^2 \neq 0$, which is the case unless $a = b = 0$. If $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is invertible, then its inverse is $\frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, by Exercise 13b.

2.1.**16** If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then $A\vec{x} = 3\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$, so that A represents a scaling by a factor of 3. Its inverse is a scaling by a factor of $\frac{1}{3}$: $A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. (See Figure 2.1.)

2.1.**17** If $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A\vec{x} = -\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$, so that A represents a reflection about the origin.

This transformation is its own inverse: $A^{-1} = A$. (See Figure 2.2.)

2.1.**18** Compare with Exercise 16: This matrix represents a scaling by the factor of $\frac{1}{2}$; the inverse is a scaling by 2. (See Figure 2.3.)

2.1.**19** If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, so that A represents the orthogonal projection onto the $\vec{e}_1$ axis. (See Figure 2.1.) This transformation is not invertible, since the equation $A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has infinitely many solutions $\vec{x}$. (See Figure 2.4.)

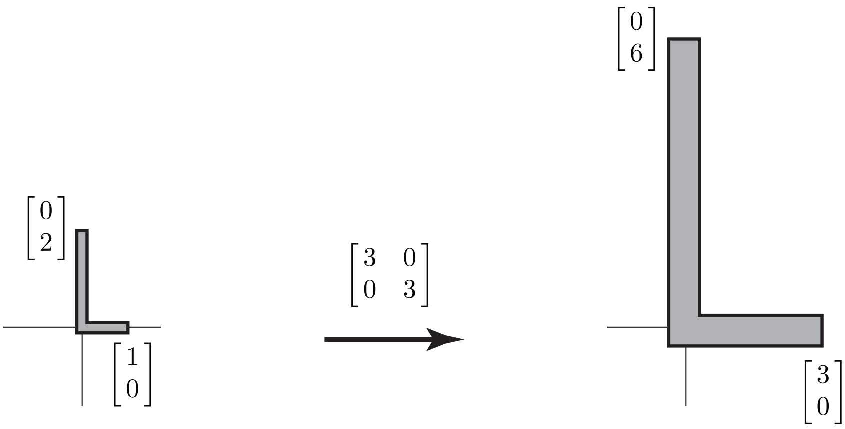

Figure 2.1: for Problem 2.1.16.

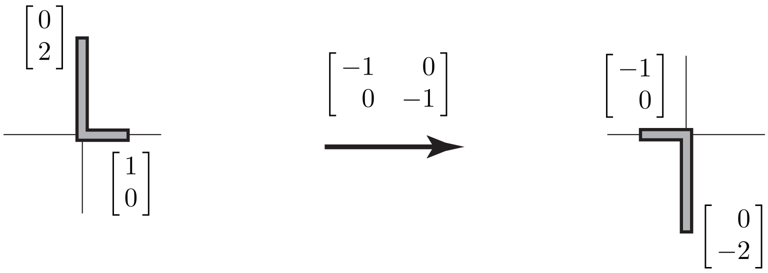

Figure 2.2: for Problem 2.1.17.

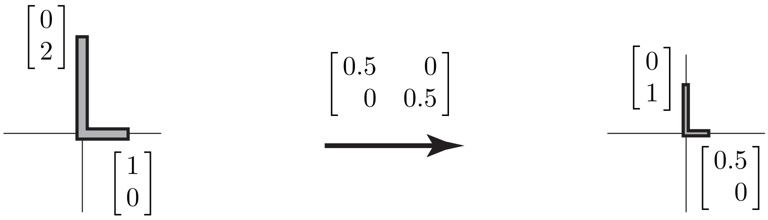

Figure 2.3: for Problem 2.1.18.

2.1.**20** If $A = \begin{bmatrix}0&1\\1&0\end{bmatrix}$, then $A\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\x_1\end{bmatrix}$, so that A represents the reflection about the line $x_2 = x_1$. This transformation is its own inverse: $A^{-1} = A$. (See Figure 2.5.)

2.1.**21** Compare with Example 5.

If $A = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$, then $A\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\-x_1\end{bmatrix}$. Note that the vectors $\vec{x}$ and $A\vec{x}$ are perpendicular and have the same

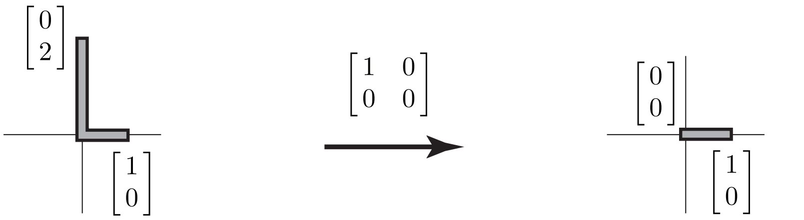

Figure 2.4: for Problem 2.1.19.

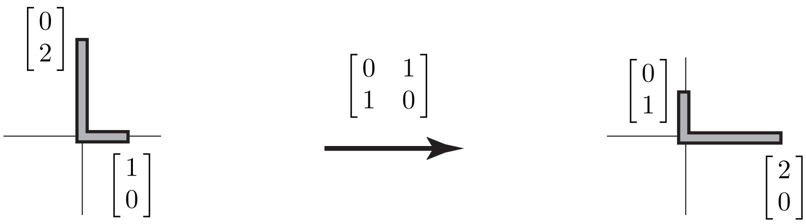

Figure 2.5: for Problem 2.1.20.

length. If $\vec{x}$ is in the first quadrant, then $A\vec{x}$ is in the fourth. Therefore, A represents the rotation through an angle of 90° in the clockwise direction. (See Figure 2.6.) The inverse $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents the rotation through 90° in the counterclockwise direction.

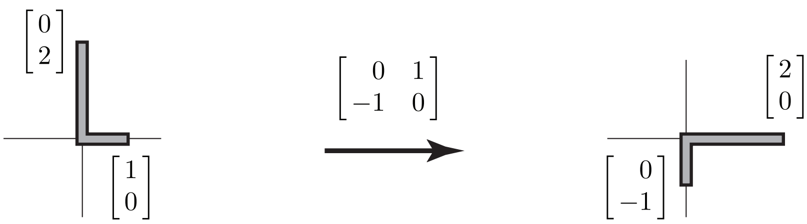

Figure 2.6: for Problem 2.1.21.

2.1.**22** If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$, so that A represents the reflection about the $\vec{e}_1$ axis. This transformation is its own inverse: $A^{-1} = A$. (See Figure 2.7.)

2.1.**23** Compare with Exercise 21.

Note that $A = 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, so that A represents a rotation through an angle of 90° in the clockwise direction, followed by a scaling by the factor of 2.

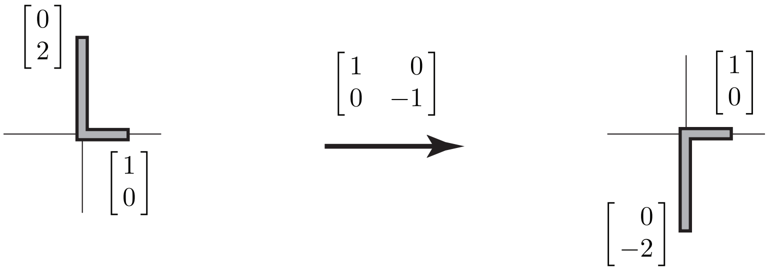

Figure 2.7: for Problem 2.1.22.

The inverse $A^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$ represents a rotation through an angle of 90° in the counterclockwise direction, followed by a scaling by the factor of $\frac{1}{2}$. (See Figure 2.8.)

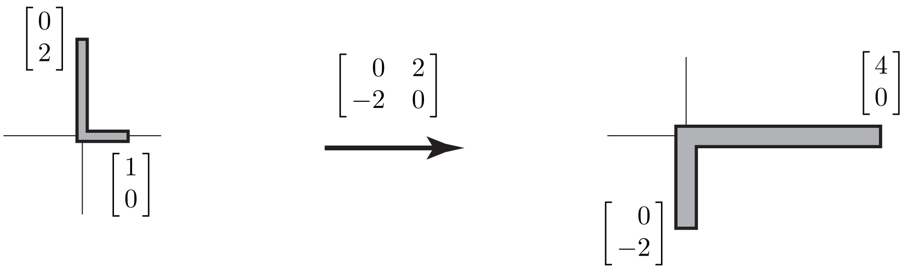

Figure 2.8: for Problem 2.1.23.

2.1.**24** Compare with Example 5. (See Figure 2.9.)

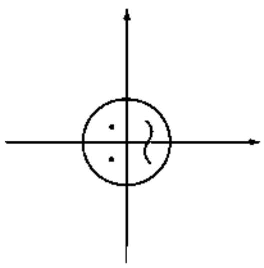

Figure 2.9: for Problem 2.1.24.

2.1.**25** The matrix represents a scaling by the factor of 2. (See Figure 2.10.)

2.1.**26** This matrix represents a reflection about the line $x_2 = x_1$. (See Figure 2.11.)

2.1.**27** This matrix represents a reflection about the $\vec{e}_1$ axis. (See Figure 2.12.)

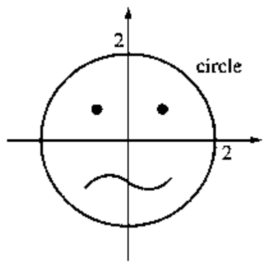

Figure 2.10: for Problem 2.1.25.

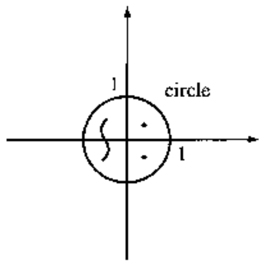

Figure 2.11: for Problem 2.1.26.

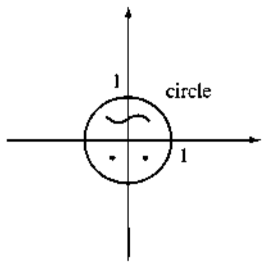

Figure 2.12: for Problem 2.1.27.

2.1.**28** If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix}$, so that the x_2 component is multiplied by 2, while the x_1 component remains unchanged. (See Figure 2.13.)

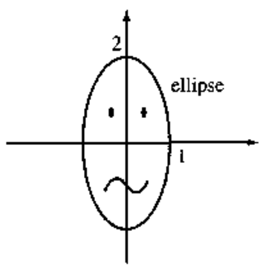

Figure 2.13: for Problem 2.1.28.

2.1.**29** This matrix represents a reflection about the origin. Compare with Exercise 17. (See Figure 2.14.)

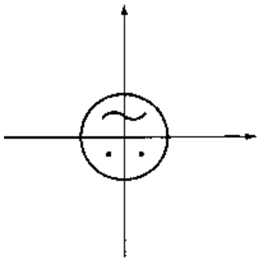

Figure 2.14: for Problem 2.1.29.

2.1.**30** If $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$, so that A represents the projection onto the $\vec{e}_2$ axis. (See Figure 2.15.)

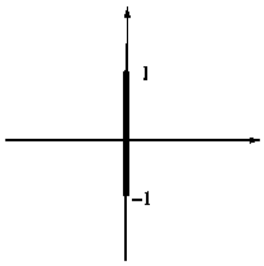

Figure 2.15: for Problem 2.1.30.

2.1.**31** The image must be reflected about the $\vec{e}_2$ axis, that is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ must be transformed into $\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$: This can be accomplished by means of the linear transformation $T(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}$.

2.1.**32** Using Theorem 2.1.2, we find $A = \begin{bmatrix} 3 & 0 & \cdot & 0 \\ 0 & 3 & \cdot & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}$. This matrix has 3's on the diagonal and 0's everywhere else.

2.1.**33** By Theorem 2.1.2, $A = \left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix} \; T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$. (See Figure 2.16.)

Therefore, $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

2.1.**34** As in Exercise 2.1.33, we find $T(\vec{e}_1)$ and $T(\vec{e}_2)$; then by Theorem 2.1.2, $A = [T(\vec{e}_1) \; T(\vec{e}_2)]$. (See Figure 2.17.)

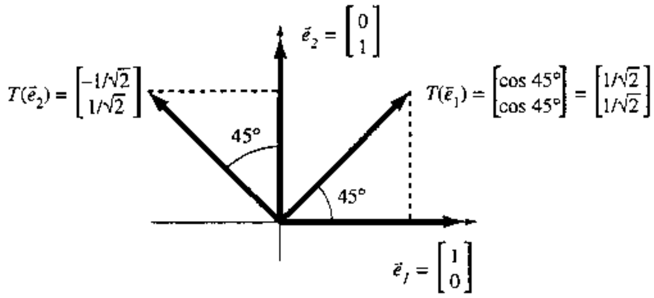

Figure 2.16: for Problem 2.1.33.

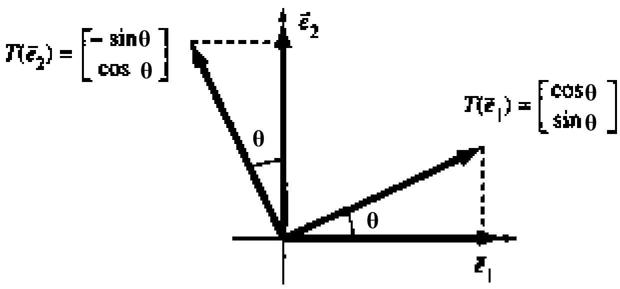

Figure 2.17: for Problem 2.1.34.

Therefore, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

2.1.**35** We want to find a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A \begin{bmatrix} 5 \\ 42 \end{bmatrix} = \begin{bmatrix} 89 \\ 52 \end{bmatrix}$ and $A \begin{bmatrix} 6 \\ 41 \end{bmatrix} = \begin{bmatrix} 88 \\ 53 \end{bmatrix}$. This amounts to solving the system $\begin{bmatrix} 5a + 42b & & = 89 \\ 6a + 41b & & = 88 \\ & 5c + 42d & = 52 \\ & 6c + 41d & = 53 \end{bmatrix}$.

(Here we really have two systems with two unknowns each.)

The unique solution is $a = 1,\ b = 2,\ c = 2$, and $d = 1$, so that $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

2.1.**36** First we draw $\vec{w}$ in terms of $\vec{v}_1$ and $\vec{v}_2$ so that $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some c_1 and c_2. Then, we scale the $\vec{v}_2$-component by 3, so our new vector equals $c_1\vec{v}_1 + 3c_2\vec{v}_2$.

2.1.**37** Since $\vec{x} = \vec{v} + k(\vec{w} - \vec{v})$, we have $T(\vec{x}) = T(\vec{v} + k(\vec{w} - \vec{v})) = T(\vec{v}) + k(T(\vec{w}) - T(\vec{v}))$, by Theorem 2.1.3

Since k is between 0 and 1, the tip of this vector $T(\vec{x})$ is on the line segment connecting the tips of $T(\vec{v})$ and $T(\vec{w})$. (See Figure 2.18.)

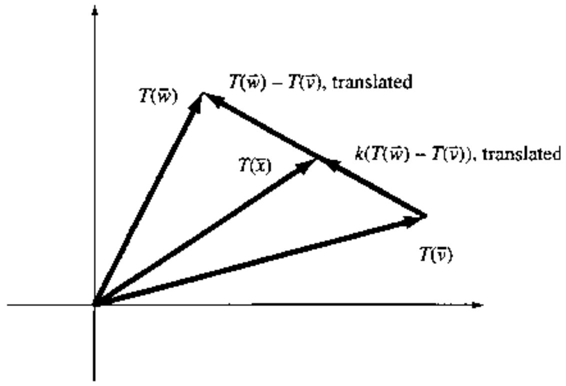

Figure 2.18: for Problem 2.1.37.

2.1.**38** $T\begin{bmatrix} 2 \\ -1 \end{bmatrix} = [\vec{v}_1 \quad \vec{v}_2]\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\vec{v}_1 - \vec{v}_2 = 2\vec{v}_1 + (-\vec{v}_2)$. (See Figure 2.19.)

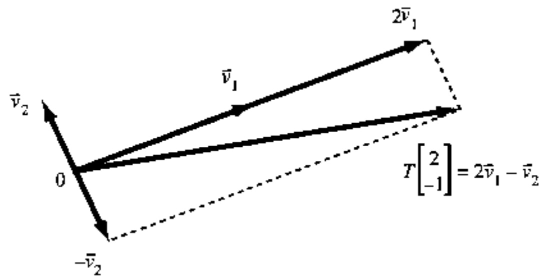

Figure 2.19: for Problem 2.1.38.

2.1.**39** By Theorem 2.1.2, we have $T\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_m) \end{bmatrix}\begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} = x_1T(\vec{e}_1) + \dots + x_mT(\vec{e}_m)$.

2.1.**40** These linear transformations are of the form $[y] = [a][x]$, or $y = ax$. The graph of such a function is a line through the origin.

2.1.**41** These linear transformations are of the form $[y] = [a\ b]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, or $y = ax_1 + bx_2$. The graph of such a function is a plane through the origin.

2.1.**42** a See Figure 2.20.

b The image of the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ is the origin, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

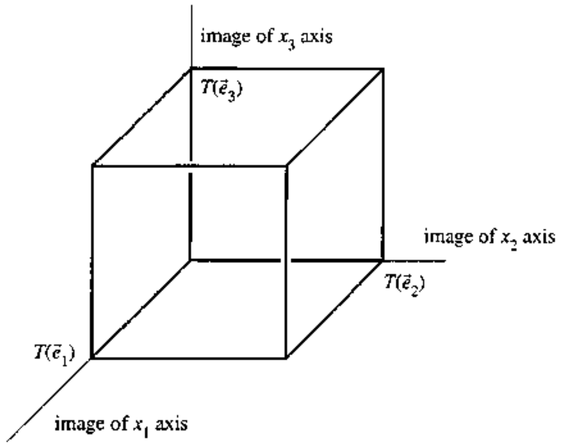

Figure 2.20: for Problem 2.1.42.

c Solve the equation $\begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $\begin{bmatrix} -\frac{1}{2}x_1 & + & x_2 & & = 0 \\ -\frac{1}{2}x_1 & + & & x_3 & = 0 \end{bmatrix}$. (See Figure 2.16.)

The solutions are of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t \end{bmatrix}$, where t is an arbitrary real number. For example, for $t = \frac{1}{2}$, we find the point $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ considered in part b.These points are on the line through the origin and the observer's eye.

2.1.**43** a $T(\vec{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 3x_2 + 4x_3 = [2\ 3\ 4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The transformation is indeed linear, with matrix [2 3 4].

b If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then T is linear with matrix $[v_1\ v_2\ v_3]$, as in part (a).

c Let $[a\ b\ c]$ be the matrix of T. Then $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [a\ b\ c] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = ax_1 + bx_2 + cx_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ does the job.

2.1.**44** $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so that T is linear, with matrix $\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$.

2.1.**45** Yes, $\vec{z} = L(T(\vec{x}))$ is also linear, which we will verify using Theorem 2.1.3. Part a holds, since $L(T(\vec{v}+\vec{w})) = L(T(\vec{v}) + T(\vec{w})) = L(T(\vec{v})) + L(T(\vec{w}))$, and part b also works, because $L(T(k\vec{v})) = L(kT(\vec{v})) = kL(T(\vec{v}))$.

2.1.**46** $T\begin{bmatrix}1\\0\end{bmatrix} = B\left(A\begin{bmatrix}1\\0\end{bmatrix}\right) = B\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}pa+qc\\ra+sc\end{bmatrix}$

$T\begin{bmatrix}0\\1\end{bmatrix} = B\left(A\begin{bmatrix}0\\1\end{bmatrix}\right) = B\begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}pb+qd\\rb+sd\end{bmatrix}$

So, $T\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\left(T\begin{bmatrix}1\\0\end{bmatrix}\right) + x_2\left(T\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}b\\d\end{bmatrix} = \begin{bmatrix}pb+qd\\rb+sd\end{bmatrix}$

2.1.**47** Write $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2 : \vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$. (See Figure 2.21.)

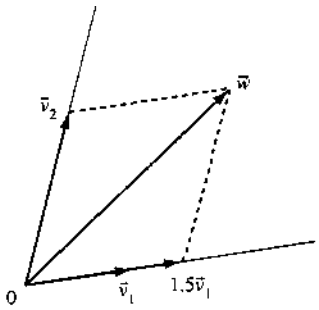

Figure 2.21: for Problem 2.1.47.

Measurements show that we have *roughly* $\vec{w} = 1.5\vec{v}_1 + \vec{v}_2$.

Therefore, by linearity, $T(\vec{w}) = T(1.5\vec{v}_1 + \vec{v}_2) = 1.5T(\vec{v}_1) + T(\vec{v}_2)$. (See Figure 2.22.)

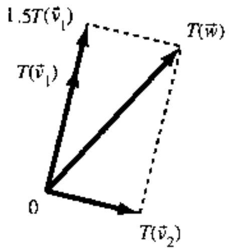

Figure 2.22: for Problem 2.1.47.

2.1.**48** Let $\vec{x}$ be some vector in $\mathbb{R}^2$. Since $\vec{v}_1$ and $\vec{v}_2$ are not parallel, we can write $\vec{x}$ in terms of components of $\vec{v}_1$ and $\vec{v}_2$. So, let c_1 and c_2 be scalars such that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$. Then, by Theorem 2.1.3, $T(\vec{x}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) = c_1L(\vec{v}_1) + c_2L(\vec{v}_2) = L(c_1\vec{v}_1 + c_2\vec{v}_2) = L(\vec{x})$. So $T(\vec{x}) = L(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^2$.

2.1.**49** Denote the components of $\vec{x}$ with x_j and the entries of A with a_{ij}. We are told that $\sum_{j=1}^n x_j = 1$ and $\sum_{i=1}^n a_{ij} = 1$ for all $j = 1, ..., n$. Now the i^{th} component of $A\vec{x}$ is $\sum_{j=1}^n a_{ij}x_j$, so that the sum of all components of $A\vec{x}$ is $\sum_{i=1}^n \sum_{j=1}^n a_{ij}x_j = \sum_{j=1}^n \sum_{i=1}^n a_{ij}x_j = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\right) x_j = \sum_{j=1}^n x_j = 1$, as claimed.

Also, the components of $A\vec{x}$ are nonnegative since all the scalars a_{ij} and x_j are nonnegative. Therefore, $A\vec{x}$ is a distribution vector.

2.1.**50** Proceeding as in Exercise 51, we find

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 1 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix} \text{ and } \vec{x}_{equ} = \frac{1}{11}\begin{bmatrix} 4 \\ 4 \\ 2 \\ 1 \end{bmatrix}.$$

Pages 1 and 2 have the highest naive PageRank.

2.1.**51** a. We can construct the transition matrix A column by column, as discussed in Example 9:

$$A = \begin{bmatrix} 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 1/3 & 0 \end{bmatrix}.$$

For example, the first column represents the fact that half of the surfers from page 1 take the link to page 2, while the other half go to page 3.

b. To find the equilibrium vector, we need to solve the system $A\vec{x} = \vec{x} = I_4\vec{x}$ or $(A - I_4)\vec{x} = \vec{0}$. We use technology to find

$$\text{rref}(A - I_4) = \begin{bmatrix} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions are of the form $\vec{x} = \begin{bmatrix} t \\ 4t \\ 3t \\ 5t \end{bmatrix}$, where t is arbitrary. The distribution vector among these solutions must satisfy the condition $t + 4t + 3t + 5t = 13t = 1$, or $t = \frac{1}{13}$. Thus $\vec{x}_{equ} = \frac{1}{13}\begin{bmatrix} 1 \\ 4 \\ 3 \\ 5 \end{bmatrix}$.

c. Page 4 has the highest naive PageRank.

2.1.**52** Proceeding as in Exercise 51, we find

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \text{ and } \vec{x}_{equ} = \frac{1}{5}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Pages 1 and 2 have the highest naive PageRank.

2.1.**53** a. Constructing the matrix B column by column, as explained for the second column, we find

$$B = \begin{bmatrix} 0.05 & 0.45 & 0.05 & 0.05 \\ 0.45 & 0.05 & 0.05 & 0.85 \\ 0.45 & 0.45 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.85 & 0.05 \end{bmatrix}$$

b. The matrix $0.05E$ accounts for the jumpers, since 5% of the surfers from a given page jump to any other page (or stay put). The matrix $0.8A$ accounts for the 80% of the surfers who follow links.

c. To find the equilibrium vector, we need to solve the system $B\vec{x} = \vec{x} = I_4\vec{x}$ or $(B - I_4)\vec{x} = \vec{0}$. We use technology to find

$$\text{rref}(B - I_4) = \begin{bmatrix} 1 & 0 & 0 & -5/7 \\ 0 & 1 & 0 & -9/7 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solutions are of the form $\vec{x} = \begin{bmatrix} 5t \\ 9t \\ 7t \\ 7t \end{bmatrix}$, where t is arbitrary. Now $\vec{x}$ is a distribution vector when $t = \frac{1}{28}$. Thus $\vec{x}_{equ} = \frac{1}{28}\begin{bmatrix} 5 \\ 9 \\ 7 \\ 7 \end{bmatrix}$. Page 2 has the highest PageRank.

2.1.**54** a. Here we consider the same mini-Web as in Exercise 50. Using the formula for B from Exercise 53b , we find

$$B = \begin{bmatrix} 0.05 & 0.85 & 0.05 & 0.05 \\ 0.45 & 0.05 & 0.45 & 0.85 \\ 0.45 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.45 & 0.05 \end{bmatrix}.$$

b. Proceeding as in Exercise 53, we find $\vec{x}_{equ} = \frac{1}{1124}\begin{bmatrix} 377 \\ 401 \\ 207 \\ 139 \end{bmatrix}$.

c. Page 2 has the highest PageRank.

2.1.**55** Here we consider the same mini-Web as in Exercise 51. Proceeding as in Exercise 53, we find

$B = \begin{bmatrix} 0.05 & 0.05 & 19/60 & 0.05 \\ 0.45 & 0.05 & 19/60 & 0.45 \\ 0.45 & 0.05 & 0.05 & 0.45 \\ 0.05 & 0.85 & 19/60 & 0.05 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{2860}\begin{bmatrix} 323 \\ 855 \\ 675 \\ 1007 \end{bmatrix}$.

Page 4 has the highest PageRank.

2.1.**56** Here we consider the same mini-Web as in Exercise 52. Proceeding as in Exercise 53, we find

$B = \frac{1}{15}\begin{bmatrix} 1 & 13 & 1 \\ 7 & 1 & 13 \\ 7 & 1 & 1 \end{bmatrix}$ and $\vec{x}_{equ} = \frac{1}{159}\begin{bmatrix} 61 \\ 63 \\ 35 \end{bmatrix}$.

Page 2 has the highest PageRank

2.1.**57** a Let x_1 be the number of 2 Franc coins, and x_2 be the number of 5 Franc coins. Then $\begin{bmatrix} 2x_1 & +5x_2 & = & 144 \\ x_1 & +x_2 & = & 51 \end{bmatrix}$.

From this we easily find our solution vector to be $\begin{bmatrix} 37 \\ 14 \end{bmatrix}$.

b $\begin{bmatrix} \text{total value of coins} \\ \text{total number of coins} \end{bmatrix} = \begin{bmatrix} 2x_1 & +5x_2 \\ x_1 & +x_2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

So, $A = \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}$.

c By Exercise 13, matrix A is invertible (since $ad - bc = -3 \neq 0$), and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix}$.

Then $-\frac{1}{3} \begin{bmatrix} 1 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 144 \\ 51 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 144 & -5(51) \\ -144 & +2(51) \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -111 \\ -42 \end{bmatrix} = \begin{bmatrix} 37 \\ 14 \end{bmatrix}$, which was the vector we found in part a.

2.1.**58** a Let $\begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{mass of the platinum alloy} \\ \text{mass of the silver alloy} \end{bmatrix}$. Using the definition density = mass/volume, or volume = mass/density, we can set up the system:

$\begin{bmatrix} p & +s & = & 5,000 \\ \frac{p}{20} & +\frac{s}{10} & = & 370 \end{bmatrix}$, with the solution $p = 2,600$ and $s = 2,400$. We see that the platinum alloy makes up only 52 percent of the crown; this gold smith is a crook!

b We seek the matrix A such that $A \begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} p+s \\ \frac{p}{20} + \frac{s}{10} \end{bmatrix}$. Thus $A = \begin{bmatrix} 1 & 1 \\ \frac{1}{20} & \frac{1}{10} \end{bmatrix}$.

c Yes. By Exercise 13, $A^{-1} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}$. Applied to the case considered in part a, we find that $\begin{bmatrix} p \\ s \end{bmatrix} = A^{-1} \begin{bmatrix} \text{total mass} \\ \text{total volume} \end{bmatrix} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix} \begin{bmatrix} 5,000 \\ 370 \end{bmatrix} = \begin{bmatrix} 2,600 \\ 2,400 \end{bmatrix}$, confirming our answer in part a.

2.1.**59** a $\begin{bmatrix} C \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}(F-32) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9}F - \frac{160}{9} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix}$.

So $A = \begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix}$.

b Using Exercise 13, we find $\frac{5}{9}(1) - (-\frac{160}{9})0 = \frac{5}{9} \neq 0$, so A is invertible.

$A^{-1} = \frac{9}{5} \begin{bmatrix} 1 & \frac{160}{9} \\ 0 & \frac{5}{9} \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & 32 \\ 0 & 1 \end{bmatrix}$. So, $F = \frac{9}{5}C + 32$.

2.1.**60** a $A\vec{x} = \begin{bmatrix} 300 \\ 2,400 \end{bmatrix}$, meaning that the total value of our money is C\$300, or, equivalently, $ZAR2400$.

b From Exercise 13, we test the value $ad - bc$ and find it to be zero. Thus A is not invertible. To determine when A is consistent, we begin to compute rref $\begin{bmatrix} A \vdots \vec{b} \end{bmatrix}$:

$$\left[\begin{array}{cccc} 1 & \frac{1}{8} & \vdots & b_1 \\ 8 & 1 & \vdots & b_2 \end{array}\right] -8I \rightarrow \left[\begin{array}{cccc} 1 & \frac{1}{8} & \vdots & b_1 \\ 0 & 0 & \vdots & b_2 - 8b_1 \end{array}\right].$$

Thus, the system is consistent only when $b_2 = 8b_1$. This makes sense, since b_2 is the total value of our money in terms of Rand, while b_1 is the value in terms of Canadian dollars. Consider the example in part a. If the system $A\vec{x} = \vec{b}$ is consistent, then there will be infinitely many solutions $\vec{x}$, representing various compositions of our portfolio in terms of Rand and Canadian dollars, all representing the same total value.

2.1.**61** All four entries along the diagonal must be 1: they represent the process of converting a currency to itself. We also know that $a_{ij} = a_{ji}^{-1}$ for all i and j because converting currency i to currency j is the inverse of converting currency j to currency i. This gives us three more entries, $A = \begin{bmatrix} 1 & 4/5 & * & 5/4 \\ 5/4 & 1 & * & * \\ * & * & 1 & 10 \\ 4/5 & * & 1/10 & 1 \end{bmatrix}$. Next let's find the entry a_{31}, giving the value of one Euro expressed in Yuan. Now $E1 = £(4/5)$ and $£1 = ¥10$ so that $E1 = ¥10(4/5) = ¥8$. We have found that $a_{31} = a_{34}a_{41} = 8$. Similarly we have $a_{ij} = a_{ik}a_{kj}$ for all indices $i, j, k = 1, 2, 3, 4$. This gives $a_{24} = a_{21}a_{14} = 25/16$ and $a_{23} = a_{24}a_{43} = 5/32$. Using the fact that $a_{ij} = a_{ji}^{-1}$, we can complete the matrix:

$$A = \begin{bmatrix} 1 & 4/5 & 1/8 & 5/4 \\ 5/4 & 1 & 5/32 & 25/16 \\ 8 & 32/5 & 1 & 10 \\ 4/5 & 16/25 & 1/10 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.8 & 0.125 & 1.25 \\ 1.25 & 1 & 0.15625 & 1.5625 \\ 8 & 6.4 & 1 & 10 \\ 0.8 & 0.64 & 0.1 & 1 \end{bmatrix}$$

2.1.**62** a 1: this represents converting a currency to itself.

b a_{ij} is the reciprocal of a_{ji}, meaning that $a_{ij}a_{ji} = 1$. This represents converting on currency to another, then converting it back.

c Note that a_{ik} is the conversion factor from currency k to currency i meaning that

(1 unit of currency k) = (a_{ik} units of currency i)

Likewise,

(1 unit of currency j) = (a_{kj} units of currency k).

It follows that

(1 unit of currency j) = ($a_{kj}a_{ik}$ units of currency i) = (a_{ij} units of currency i), so that $a_{ik}a_{kj} = a_{ij}$.

d The rank of A is only 1, because every row is simply a scalar multiple of the top row. More precisely, since $a_{ij} = a_{i1}a_{1j}$, by part c, the i^{th} row is a_{i1} times the top row. When we compute the rref, every row but the top will be removed in the first step. Thus, rref(A) is a matrix with the top row of A and zeroes for all other entries.

2.1.**63** a We express the leading variables x_1, x_3, x_4 in terms of the free variables x_2, x_5:

$$\begin{aligned} x_1 &= x_2 - 4x_5 \\ x_3 &= x_5 \\ x_4 &= 2x_5 \end{aligned}$$

Written in vector form,

$$\begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}, \text{ with } B = \begin{bmatrix} 1 & -4 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$$

2.1.**64** a The given system reduces to

$$\begin{array}{llll} x_1 & +2x_2 & & +3x_4 = 0 \\ & & x_3 & +4x_4 = 0 \end{array} \quad \text{or} \quad \begin{array}{lll} x_1 & = & -2x_2 \; -3x_4 \\ x_3 & = & \quad\quad\;\; -4x_4 \end{array}$$

Written in vector form,

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}, \text{ with } B = \begin{bmatrix} -2 & -3 \\ 0 & -4 \end{bmatrix}$$

Section 2.2

2.2.**1** The standard L is transformed into a distorted L whose foot is the vector $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Meanwhile, the back becomes the vector $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

2.2.**2** By Theorem 2.2.3, this matrix is $\begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

2.2.**3** If $\vec{x}$ is in the unit square in $\mathbb{R}^2$, then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ with $0 \leq x_1,\ x_2 \leq 1$, so that

$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1T(\vec{e}_1) + x_2T(\vec{e}_2)$.

The image of the unit square is a parallelogram in $\mathbb{R}^3$; two of its sides are $T(\vec{e}_1)$ and $T(\vec{e}_2)$, and the origin is one of its vertices. (See Figure 2.23.)

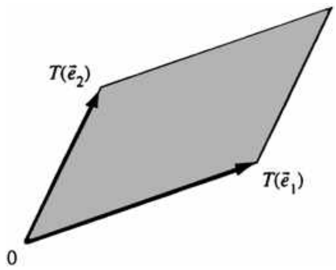

Figure 2.23: for Problem 2.2.3.

2.2.**4** By Theorem 2.2.4, this is a rotation combined with a scaling. The transformation rotates 45 degrees counter-clockwise, and has a scaling factor of $\sqrt{2}$.

2.2.**5** Note that $\cos(\theta) = -0.8$, so that $\theta = \arccos(-0.8) \approx 2.498$.

2.2.**6** By Theorem 2.2.1, $\text{proj}_L \begin{bmatrix}1\\1\\1\end{bmatrix} = \left(\vec{u}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\right)\vec{u}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix}2\\1\\2\end{bmatrix}$:

$\vec{u} = \frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$, so that $\text{proj}_L \begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{5}{3}\cdot\frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}\frac{10}{9}\\ \frac{5}{9}\\ \frac{10}{9}\end{bmatrix}$.

2.2.**7** According to the discussion in the text, $\text{ref}_L \begin{bmatrix}1\\1\\1\end{bmatrix} = 2\left(\vec{u}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\right)\vec{u} - \begin{bmatrix}1\\1\\1\end{bmatrix}$, where $\vec{u}$ is a unit vector on L. To get $\vec{u}$, we normalize $\begin{bmatrix}2\\1\\2\end{bmatrix}$: $\vec{u} = \frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$, so that $\text{ref}_L \begin{bmatrix}1\\1\\1\end{bmatrix} = 2(\frac{5}{3})\frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{11}{9}\\ \frac{1}{9}\\ \frac{11}{9}\end{bmatrix}$.

2.2.**8** From Definition 2.2.2, we can see that this is a reflection about the line $x_1 = -x_2$.

2.2.**9** By Theorem 2.2.5, this is a vertical shear.

2.2.**10** By Theorem 2.2.1, $\text{proj}_L\vec{x} = (\vec{u}\cdot\vec{x})\vec{u}$, where $\vec{u}$ is a unit vector on L. We can choose $\vec{u} = \frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}0.8\\0.6\end{bmatrix}$. Then $\text{proj}_L\begin{bmatrix}x_1\\x_2\end{bmatrix} = \left(\begin{bmatrix}0.8\\0.6\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\begin{bmatrix}0.8\\0.6\end{bmatrix} = (0.8x_1+0.6x_2)\begin{bmatrix}0.8\\0.6\end{bmatrix} = \begin{bmatrix}0.64x_1+0.48x_2\\0.48x_1+0.36x_2\end{bmatrix} = \begin{bmatrix}0.64 & 0.48\\0.48 & 0.36\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$.
The matrix is $A = \begin{bmatrix}0.64 & 0.48\\0.48 & 0.36\end{bmatrix}$.

2.2.**11** In Exercise 10 we found the matrix $A = \begin{bmatrix}0.64 & 0.48\\0.48 & 0.36\end{bmatrix}$ of the projection onto the line L. By Theorem 2.2.2, $\text{ref}_L\vec{x} = 2(\text{proj}_L\vec{x}) - \vec{x} = 2A\vec{x} - \vec{x} = (2A - I_2)\vec{x}$, so that the matrix of the reflection is $2A - I_2 = \begin{bmatrix}0.28 & 0.96\\0.96 & -0.28\end{bmatrix}$.

2.2.**12** a. If $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ relative to the line L of reflection, then $A\vec{x} = \vec{x}^{\parallel} - \vec{x}^{\perp}$ and $A(A\vec{x}) = \vec{x}^{\parallel} - (-\vec{x}^{\perp}) = \vec{x}^{\parallel} + \vec{x}^{\perp} = \vec{x}$. In summary, $A(A\vec{x}) = \vec{x}$.

b. $A\vec{v} = A(\vec{x} + A\vec{x}) = A\vec{x} + A(A\vec{x}) \underbrace{=}_{Step 3} A\vec{x} + \vec{x} = \vec{v}$. In step 3 we use part a.

c. $A\vec{w} = A(\vec{x} - A\vec{x}) = A\vec{x} - A(A\vec{x}) \underbrace{=}_{Step 3} A\vec{x} - \vec{x} = -\vec{w}$. Again, in step 3 we use part a.

d. $\vec{v}\cdot\vec{w} = (\vec{x} + A\vec{x})\cdot(\vec{x} - A\vec{x}) = \vec{x}\cdot\vec{x} - (A\vec{x})\cdot(A\vec{x}) = \|\vec{x}\|^2 - \|A\vec{x}\|^2 = 0$, since a reflection preserves length. Thus $\vec{v}$ is perpendicular to $\vec{w}$.

e. $\vec{v} = \vec{x} + A\vec{x} = \vec{x} + \text{ref}_L\vec{x} = 2\text{proj}_L\vec{x}$, by Definition 2.2.2, so that $\vec{v}$ is parallel to L.

2.2.**13** By Theorem 2.2.2,

$$\text{ref}_L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2(u_1x_1 + u_2x_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2u_1^2 - 1)x_1 + 2u_1u_2x_2 \\ 2u_1u_2x_1 + (2u_2^2 - 1)x_2 \end{bmatrix}.$$

The matrix is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$. Note that the sum of the diagonal entries is $a + d = 2(u_1^2 + u_2^2) - 2 = 0$, since $\vec{u}$ is a unit vector. It follows that $d = -a$. Since $c = b$, A is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Also, $a^2 + b^2 = (2u_1^2 - 1)^2 + 4u_1^2u_2^2 = 4u_1^4 - 4u_1^2 + 1 + 4u_1^2(1 - u_1^2) = 1$, as claimed.

2.2.**14** a Proceeding as on Page 61/62 in the text, we find that A is the matrix whose ijth entry is u_iu_j:

$$A = \begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2^2 & u_2u_3 \\ u_nu_1 & u_nu_2 & u_3^2 \end{bmatrix}$$

b The sum of the diagonal entries is $u_1^2 + u_2^2 + u_3^2 = 1$, since $\vec{u}$ is a unit vector.

2.2.**15** According to the discussion on Page 60 in the text, $\text{ref}_L(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$

$$= 2(x_1u_1 + x_2u_2 + x_3u_3)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1u_1^2 & +2x_2u_2u_1 & +2x_3u_3u_1 & -x_1 \\ 2x_1u_1u_2 & +2x_2u_2^2 & +2x_3u_3u_2 & -x_2 \\ 2x_1u_1u_3 & +2x_2u_2u_3 & +2x_3u_3^2 & -x_3 \end{bmatrix} = \begin{bmatrix} (2u_1^2 - 1)x_1 & +2u_2u_1x_2 & +2u_1u_3x_3 \\ 2u_1u_2x_1 & +(2u_2^2 - 1)x_2 & +2u_2u_3x_3 \\ 2u_1u_3x_1 & +2u_2u_3x_2 & +(2u_3^2 - 1)x_3 \end{bmatrix}.$$

So $A = \begin{bmatrix} (2u_1^2 - 1) & 2u_2u_1 & 2u_1u_3 \\ 2u_1u_2 & (2u_2^2 - 1) & 2u_2u_3 \\ 2u_1u_3 & 2u_2u_3 & (2u_3^2 - 1) \end{bmatrix}$.

2.2.**16** a See Figure 2.24.

b By Theorem 2.1.2, the matrix of T is $[T(\vec{e}_1) \quad T(\vec{e}_2)]$.

$T(\vec{e}_2)$ is the unit vector in the fourth quadrant perpendicular to $T(\vec{e}_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$, so that

$T(\vec{e}_2) = \begin{bmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{bmatrix}$. The matrix of T is therefore $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Alternatively, we can use the result of Exercise 13, with $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ to find the matrix

$$\begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}.$$

You can use trigonometric identities to show that the two results agree. (See Figure 2.25.)

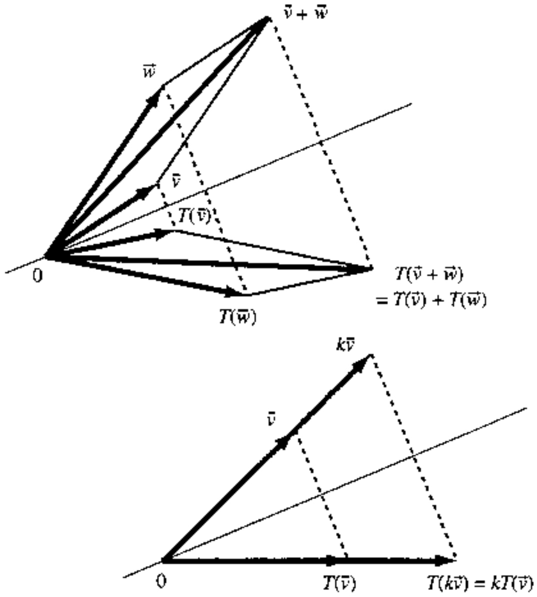

Figure 2.24: for Problem 2.2.16a.

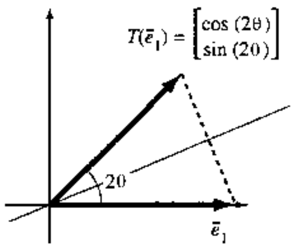

Figure 2.25: for Problem 2.2.16b.

2.2.**17** We want, $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 & +bv_2 \\ bv_1 & -av_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

Now, $(a-1)v_1 + bv_2 = 0$ and $bv_1 - (a+1)v_2$, which is a system with solutions of the form $\begin{bmatrix} bt \\ (1-a)t \end{bmatrix}$, where t is an arbitrary constant.

Let's choose $t = 1$, making $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix}$.

Similarly, we want $A\vec{w} = -\vec{w}$. We perform a computation as above to reveal $\vec{w} = \begin{bmatrix} a-1 \\ b \end{bmatrix}$ as a possible choice. A quick check of $\vec{v}\cdot\vec{w} = 0$ reveals that they are indeed perpendicular.

Now, any vector $\vec{x}$ in $\mathbb{R}$ can be written in terms of components with respect to $L = \text{span}(\vec{v})$ as $\vec{x} = \vec{x}^{||} + \vec{x}^{\perp} = c\vec{v} + d\vec{w}$. Then, $T(\vec{x}) = A\vec{x} = A(c\vec{v} + d\vec{w}) = A(c\vec{v}) + A(d\vec{w}) = cA\vec{v} + dA\vec{w} = c\vec{v} - d\vec{w} = \vec{x}^{||} - \vec{x}^{\perp} = \text{ref}_L(\vec{x})$, by Definition 2.2.2.

(The vectors $\vec{v}$ and $\vec{w}$ constructed above are both zero in the special case that $a = 1$ and $b = 0$. In that case, we can let $\vec{v} = \vec{e}_1$ and $\vec{w} = \vec{e}_2$ instead.)

2.2.**18** From Exercise 17, we know that the reflection is about the line parallel to $\vec{v} = \begin{bmatrix} b \\ 1-a \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \end{bmatrix} = 0.4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So, every point on this line can be described as $\begin{bmatrix} x \\ y \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So, $y = k = \frac{1}{2}x$, and $y = \frac{1}{2}x$ is the line we are looking for.

2.2.**19** $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{0}$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2.2.**20** $T(\vec{e}_1) = \vec{e}_1$, $T(\vec{e}_2) = -\vec{e}_2$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.2.**21** $T(\vec{e}_1) = \vec{e}_2$, $T(\vec{e}_2) = -\vec{e}_1$, and $T(\vec{e}_3) = \vec{e}_3$, so that the matrix is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (See Figure 2.26.)

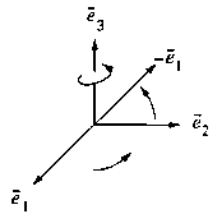

Figure 2.26: for Problem 2.2.21.

2.2.**22** Sketch the $\vec{e}_1 - \vec{e}_3$ plane, as viewed from the positive $\vec{e}_2$ axis.

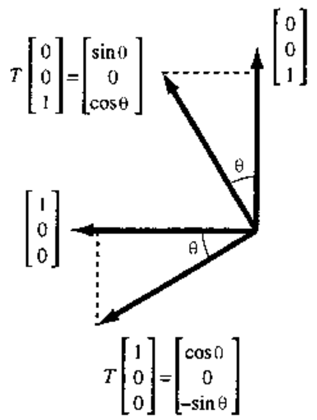

Figure 2.27: for Problem 2.2.22.

Since $T(\vec{e}_2) = \vec{e}_2$, the matrix is $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$. (See Figure 2.27.)

2.2.**23** $T(\vec{e}_1) = \vec{e}_3$, $T(\vec{e}_2) = \vec{e}_2$, and $T(\vec{e}_3) = \vec{e}_1$, so that the matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. (See Figure 2.28.)

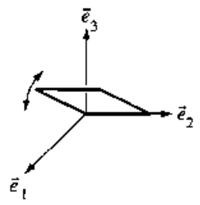

Figure 2.28: for Problem 2.2.23.

2.2.**24** a $A = [\vec{v} \quad \vec{w}]$, so $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}$ and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{w}$. Since A preserves length, both $\vec{v}$ and $\vec{w}$ must be unit vectors. Furthermore, since A preserves angles and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are clearly perpendicular, $\vec{v}$ and $\vec{w}$ must also be perpendicular.

b Since $\vec{w}$ is a unit vector perpendicular to $\vec{v}$, it can be obtained by rotating $\vec{v}$ through 90 degrees, either in the counterclockwise or in the clockwise direction. Using the corresponding rotation matrices, we see that $\vec{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{v} = \begin{bmatrix} -b \\ a \end{bmatrix}$ or $\vec{w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{v} = \begin{bmatrix} b \\ -a \end{bmatrix}$.

c Following part b, A is either of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, representing a rotation, or $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, representing a reflection.

2.2.**25** The matrix $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a horizontal shear, and its inverse $A^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ represents such a shear as well, but "the other way."

2.2.**26** a $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$. So $k = 4$ and $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.

b This is the orthogonal projection onto the horizontal axis, with matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

c $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -5b \\ 5a \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. So $a = \frac{4}{5}, b = -\frac{3}{5}$, and $C = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$. Note that $a^2 + b^2 = 1$, as required for a rotation matrix.

d Since the x_1 term is being modified, this must be a horizontal shear.

Then $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3k \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$. So $k = 2$ and $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

e $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7a+b \\ 7b-a \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$. So $a = -\frac{4}{5}, b = \frac{3}{5}$, and $E = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$. Note that $a^2 + b^2 = 1$, as required for a reflection matrix.

2.2.**27** Matrix B clearly represents a scaling.

Matrix C represents a projection, by Definition 2.2.1, with $u_1 = 0.6$ and $u_2 = 0.8$.

Matrix E represents a shear, by Theorem 2.2.5.

Matrix A represents a reflection, by Definition 2.2.2.

Matrix D represents a rotation, by Definition 2.2.3.

2.2.**28** a D is a scaling, being of the form $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.

b E is the shear, since it is the only matrix which has the proper form (Theorem 2.2.5).

c C is the rotation, since it fits Theorem 2.2.3.

d A is the projection, following the form given in Definition 2.2.1.

e F is the reflection, using Definition 2.2.2.

2.2.**29** To check that L is linear, we verify the two parts of Theorem 2.1.3:

a) Use the hint and apply L to both sides of the equation $\vec{x} + \vec{y} = T(L(\vec{x}) + L(\vec{y}))$:

$L(\vec{x} + \vec{y}) = L(T(L(\vec{x}) + L(\vec{y}))) = L(\vec{x}) + L(\vec{y})$ as claimed.

$$b) L(k\vec{x}) = \underset{\substack{\uparrow \\ \vec{x} = T(L(\vec{x}))}}{L(kT(L(\vec{x})))} = \underset{\substack{\uparrow \\ T \text{ is linear}}}{L(T(kL(\vec{x})))} = kL(\vec{x}), \text{ as claimed}$$

2.2.**30** Write $A = [\vec{v}_1 \quad \vec{v}_2]$; then $A\vec{x} = [\vec{v}_1 \quad \vec{v}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2$. We must choose $\vec{v}_1$ and $\vec{v}_2$ in such a way that $x_1\vec{v}_1 + x_2\vec{v}_2$ is a scalar multiple of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, for all x_1 and x_2. This is the case if (and only if) both $\vec{v}_1$ and $\vec{v}_2$ are scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For example, choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so that $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$.

2.2.**31** Write $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$; then $A\vec{x} = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$.

We must choose $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ in such a way that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ is perpendicular to $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ for all x_1, x_2, and x_3. This is the case if (and only if) all the vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are perpendicular to $\vec{w}$, that is, if $\vec{v}_1 \cdot \vec{w} = \vec{v}_2 \cdot \vec{w} = \vec{v}_3 \cdot \vec{w} = 0$.

For example, we can choose $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \vec{v}_3 = \vec{0}$, so that $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2.2.**32** a See Figure 2.29.

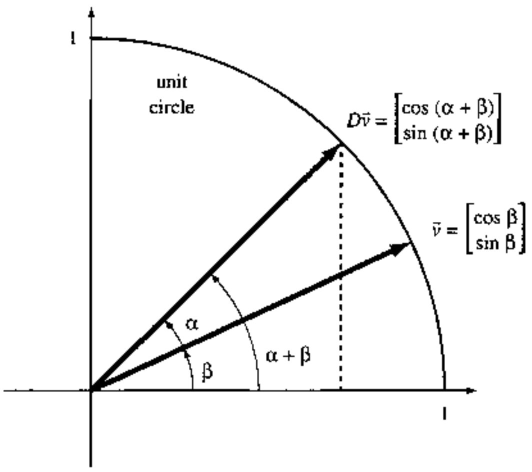

Figure 2.29: for Problem 2.2.32a.

b Compute $D\vec{v} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix}$.

Comparing this result with our finding in part (a), we get the addition theorems

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

2.2.**33** Geometrically, we can find the representation $\vec{v} = \vec{v}_1 + \vec{v}_2$ by means of a parallelogram, as shown in Figure 2.30.

To show the existence and uniqueness of this representation algebraically, choose a nonzero vector $\vec{w}_1$ in L_1 and a nonzero $\vec{w}_2$ in L_2. Then the system $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{0}$ or $[\vec{w}_1\ \vec{w}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$ has only the solution $x_1 = x_2 = 0$ (if $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{0}$ then $x_1\vec{w}_1 = -x_2\vec{w}_2$ is both in L_1 and in L_2, so that it must be the zero vector).

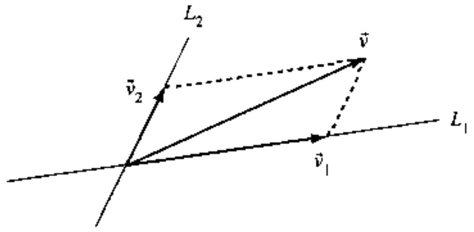

Figure 2.30: for Problem 2.2.33.

Therefore, the system $x_1\vec{w}_1 + x_2\vec{w}_2 = \vec{v}$ or $[\vec{w}_1\ \vec{w}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{v}$ has a unique solution x_1, x_2 for all $\vec{v}$ in $\mathbb{R}^2$ (by Theorem 1.3.4). Now set $\vec{v}_1 = x_1\vec{w}_1$ and $\vec{v}_2 = x_2\vec{w}_2$ to obtain the desired representation $\vec{v} = \vec{v}_1 + \vec{v}_2$. (Compare with Exercise 1.3.57.)

To show that the transformation $T(\vec{v}) = \vec{v}_1$ is linear, we will verify the two parts of Theorem 2.1.3.

Let $\underset{\text{in } L_1}{\vec{v} = \vec{v}_1} + \underset{\text{in } L_2}{\vec{v}_2}$, $\underset{\text{in } L_1}{\vec{w} = \vec{w}_1} + \underset{\text{in } L_2}{\vec{w}_2}$, so that $\vec{v} + \vec{w} = \underset{\text{in } L_1}{(\vec{v}_1 + \vec{w}_1)} + \underset{\text{in } L_2}{(\vec{v}_2 + \vec{w}_2)}$ and $k\vec{v} = \underset{\text{in } L_1}{k\vec{v}_1} + \underset{\text{in } L_2}{k\vec{v}_2}$.

a. $T(\vec{v} + \vec{w}) = \vec{v}_1 + \vec{w}_1 = T(\vec{v}) + T(\vec{w})$, and

b. $T(k\vec{v}) = k\vec{v}_1 = kT(\vec{v})$, as claimed.

2.2.**34** Keep in mind that the columns of the matrix of a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$.

If T is the orthogonal projection onto a line L, then $T(\vec{x})$ will be on L for all $\vec{x}$ in $\mathbb{R}^3$; in particular, the three columns of the matrix of T will be on L, and therefore pairwise parallel. This is the case only for matrix B: B represents an orthogonal projection onto a line.

A reflection transforms orthogonal vectors into orthogonal vectors; therefore, the three columns of its matrix must be pairwise orthogonal. This is the case only for matrix E: E represents the reflection about a line.

2.2.**35** If the vectors $\vec{v}_1$ and $\vec{v}_2$ are defined as shown in Figure 2.27, then the parallelogram P consists of all vectors of the form $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, where $0 \le c_1,\ c_2 \le 1$.

The image of P consists of all vectors of the form $T(\vec{v}) = T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$.

These vectors form the parallelogram shown in Figure 2.31 on the right.

2.2.**36** If the vectors $\vec{v}_0, \vec{v}_1$, and $\vec{v}_2$ are defined as shown in Figure 2.28, then the parallelogram P consists of all vectors $\vec{v}$ of the form $\vec{v} = \vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2$, where $0 \le c_1, c_2 \le 1$.

The image of P consists of all vectors of the form $T(\vec{v}) = T(\vec{v}_0 + c_1\vec{v}_1 + c_2\vec{v}_2) = T(\vec{v}_0) + c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$.

These vectors form the parallelogram shown in Figure 2.32 on the right.

2.2.**37** a By Definition 2.2.1, a projection has a matrix of the form $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector.

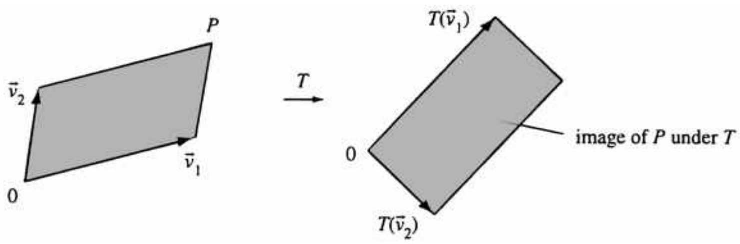

Figure 2.31: for Problem 2.2.35.

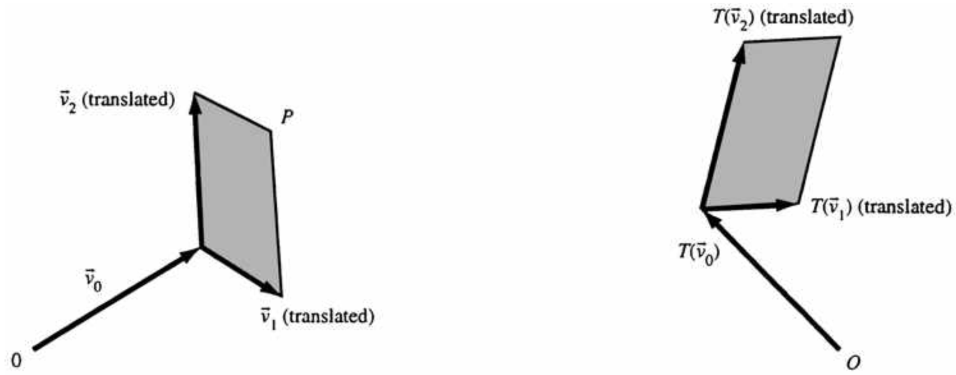

Figure 2.32: for Problem 2.2.36.

So the trace is $u_1^2 + u_2^2 = 1$.

b By Definition 2.2.2, reflection matrices look like $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, so the trace is $a - a = 0$.

c According to Theorem 2.2.3, a rotation matrix has the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, so the trace is $\cos\theta + \cos\theta = 2\cos\theta$ for some θ. Thus, the trace is in the interval $[-2, 2]$.

d By Theorem 2.2.5, the matrix of a shear appears as either $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, depending on whether it represents a vertical or horizontal shear. In both cases, however, the trace is $1 + 1 = 2$.

2.2.**38** a $A = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, so $\det(A) = u_1^2u_2^2 - u_1u_2u_1u_2 = 0$.

b $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, so $\det(A) = -a^2 - b^2 = -(a^2 + b^2) = -1$.

c $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, so $\det(A) = a^2 - (-b^2) = a^2 + b^2 = 1$.

d $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, both of which have determinant equal to $1^2 - 0 = 1$.

2.2.**39** a Note that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. The matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ represents an orthogonal projection (Definition 2.2.1), with $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$. So, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ represents a projection combined with a scaling by a factor of 2.

b This looks similar to a shear, with the one zero off the diagonal. Since the two diagonal entries are identical, we can write $\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$, showing that this matrix represents a vertical shear combined with a scaling by a factor of 3.

c We are asked to write $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = k \begin{bmatrix} \frac{3}{k} & \frac{4}{k} \\ \frac{4}{k} & -\frac{3}{k} \end{bmatrix}$, with our scaling factor k yet to be determined. This matrix, $\begin{bmatrix} \frac{3}{k} & \frac{4}{k} \\ \frac{4}{k} & -\frac{3}{k} \end{bmatrix}$ has the form of a reflection matrix $\left(\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\right)$. This form further requires that $1 = a^2 + b^2 = (\frac{3}{k})^2 + (\frac{4}{k})^2$, or $k = 5$. Thus, the matrix represents a reflection combined with a scaling by a factor of 5.

2.2.**40** $\vec{x} = \text{proj}_P\vec{x} + \text{proj}_Q\vec{x}$, as illustrated in Figure 2.33.

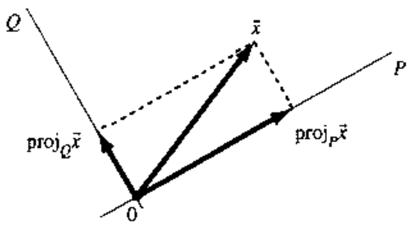

Figure 2.33: for Problem 2.2.40.

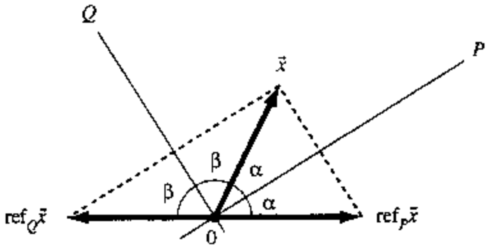

Figure 2.34: for Problem 2.2.41.

2.2.**41** $\text{ref}_Q\vec{x} = -\text{ref}_P\vec{x}$ since $\text{ref}_Q\vec{x}$, $\text{ref}_P\vec{x}$, and $\vec{x}$ all have the same length, and $\text{ref}_Q\vec{x}$ and $\text{ref}_P\vec{x}$ enclose an angle of $2\alpha + 2\beta = 2(\alpha + \beta) = \pi$. (See Figure 2.34.)

2.2.**42** $T(\vec{x}) = T(T(\vec{x}))$ since $T(\vec{x})$ is on L hence the projection of $T(\vec{x})$ onto L is $T(\vec{x})$ itself.

2.2.43 Since $\vec{y} = A\vec{x}$ is obtained from $\vec{x}$ by a rotation through θ in the counterclockwise direction, $\vec{x}$ is obtained from $\vec{y}$ by a rotation through θ in the *clockwise* direction, that is, a rotation through $-\theta$. (See Figure 2.35.)

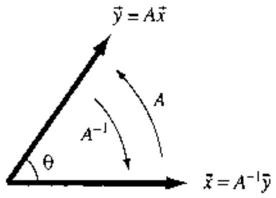

Figure 2.35: for Problem 2.2.43.

Therefore, the matrix of the inverse transformation is $A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. You can use the formula in Exercise 2.1.13b to check this result.

2.2.44 By Exercise 1.1.13b, $A^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

If A represents a rotation through θ followed by a scaling by r, then A^{-1} represents a rotation through $-\theta$ followed by a scaling by $\frac{1}{r}$. (See Figure 2.36.)

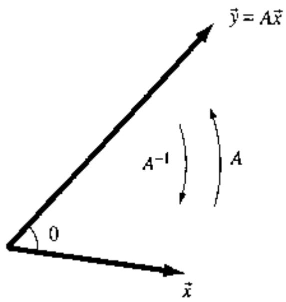

Figure 2.36: for Problem 2.2.44.

2.2.45 By Exercise 2.1.13, $A^{-1} = \frac{1}{-a^2-b^2}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \frac{1}{-(a^2+b^2)}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = -1\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$.

So $A^{-1} = A$, which makes sense. Reflecting a vector twice about the same line will return it to its original state.

2.2.46 We want to write $A = k\begin{bmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{b}{k} & -\frac{a}{k} \end{bmatrix}$, where the matrix $B = \begin{bmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{b}{k} & -\frac{a}{k} \end{bmatrix}$ represents a reflection. It is required that $(\frac{a}{k})^2 + (\frac{b}{k})^2 = 1$, meaning that $a^2 + b^2 = k^2$, or, $k = \sqrt{a^2+b^2}$. Now $A^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \frac{1}{k^2}A = \frac{1}{k}B$, for the reflection matrix B and the scaling factor k introduced above. In summary: If A represents a reflection combined with a scaling by k, then A^{-1} represents the same reflection combined with a scaling by $\frac{1}{k}$.

2.2.47 a. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$f(t) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}\right) \cdot \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\right) = \begin{bmatrix} a\cos t + b\sin t \\ c\cos t + d\sin t \end{bmatrix} \cdot \begin{bmatrix} -a\sin t + b\cos t \\ -c\sin t + d\cos t \end{bmatrix}$$

$$= (a\cos t + b\sin t)(-a\sin t + b\cos t) + (c\cos t + d\sin t)(-c\sin t + d\cos t),$$

a continuous function.

b. $f(0) = \left(T\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ and $f\left(\frac{\pi}{2}\right) = \left(T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = -\left(T\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -f(0)$

c. If $f(0) = f\left(\frac{\pi}{2}\right) = 0$, then we can let $c = 0$.

If $f(0)$ and $f\left(\frac{\pi}{2}\right)$ are both nonzero, with opposite signs (by part b), then the intermediate value theorem (with $L = 0$) guarantees that there exists a c between 0 and $\frac{\pi}{2}$ with $f(c) = 0$. See Figure 2.37.

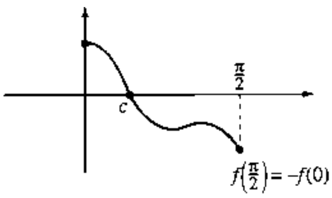

Figure 2.37: for Problem 2.2.47c.

2.2.**48** Since rotations preserve angles, any two perpendicular unit vectors $\vec{v}_1$ and $\vec{v}_2$ will do the job.

2.2.**49** a. A straightforward computation gives $f(t) = 15\ \cos(t)\ \sin(t)$.

b. The equation $f(t) = 0$ has the solutions $c = 0$ and $c = \frac{\pi}{2}$.

c. Using $c = 0$ we find $\vec{v}_1 = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -\sin c \\ \cos c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

2.2.**50** a. $f(t) = \cos(t)\sin(t) + \cos^2(t) - \sin^2(t) = \frac{1}{2}\sin(2t) + \cos(2t)$

b. The equation $f(t) = 0$ has the solution $c = \frac{\pi - \arctan 2}{2} \approx 1.017222 \approx 1.02$

c. $\vec{v}_1 = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix} \approx \begin{bmatrix} 0.526 \\ 0.851 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -\sin c \\ \cos c \end{bmatrix} \approx \begin{bmatrix} -0.851 \\ 0.526 \end{bmatrix}$

2.2.**51** a. $f(t) = 4\left(\cos^2(t) - \sin^2(t)\right)$

b. The equation $f(t) = 0$ has the solution $c = \frac{\pi}{4}$.

c. $\vec{v}_1 = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -\sin c \\ \cos c \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2.2.**52** a. $f(t) = 15\left(\sin^2(t) - \cos^2(t)\right)$

b. The equation $f(t) = 0$ has the solution $c = \frac{\pi}{4}$. See Figure 2.38.

c. $\vec{v}_1 = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -\sin c \\ \cos c \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ See Figure 2.39.

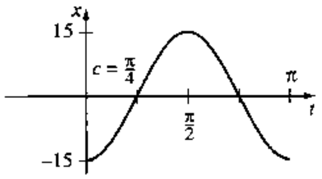

Figure 2.38: for Problem 2.2.52.

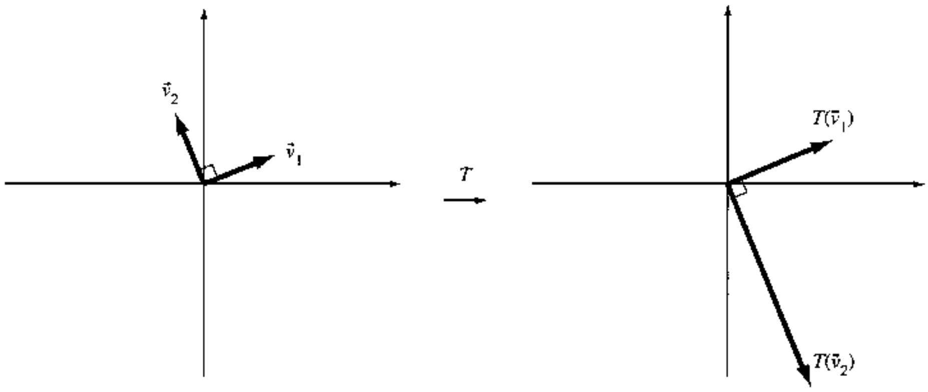

Figure 2.39: for Problem 2.2.52.

2.2.**53** If $\vec{x} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ then $T(\vec{x}) = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} 5\cos(t) \\ 2\sin(t) \end{bmatrix} = \cos(t) \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \sin(t) \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

These vectors form an ellipse; consider the characterization of an ellipse given in the footnote on Page 75, with $\vec{w}_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. (See Figure 2.40.)

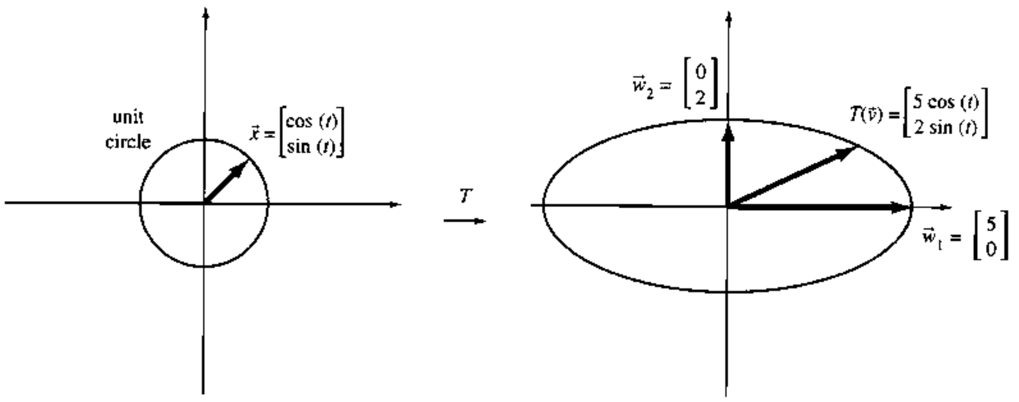

Figure 2.40: for Problem 2.2.53.

2.2.**54** Use the hint: Since the vectors on the unit circle are of the form $\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2$, the image of the unit circle consists of the vectors of the form $T(\vec{v}) = T(\cos(t)\vec{v}_1 + \sin(t)\vec{v}_2) = \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2)$.

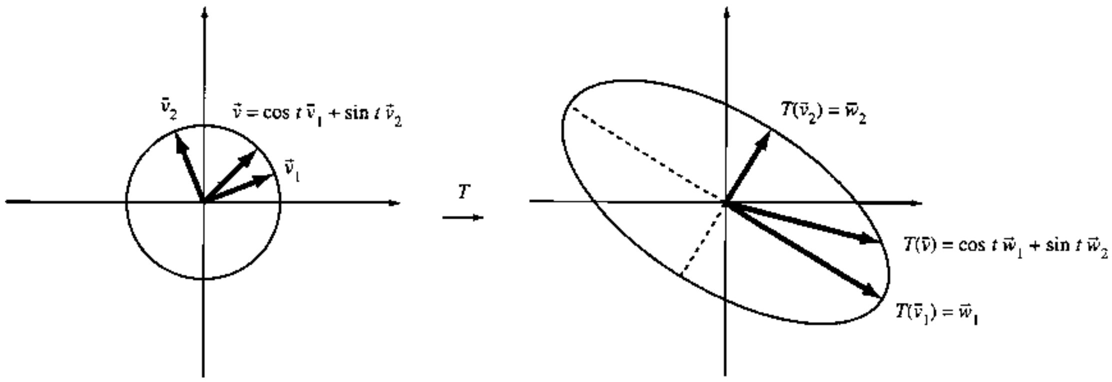

Figure 2.41: for Problem 2.2.54.

These vectors form an ellipse: Consider the characterization of an ellipse given in the footnote, with $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$. The key point is that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular. See Figure 2.41.

2.2.**55** Consider the linear transformation T with matrix $A = [\vec{w}_1 \quad \vec{w}_2]$, that is,

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\vec{w}_1 \quad \vec{w}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{w}_1 + x_2\vec{w}_2.$$

The curve C is the image of the unit circle under the transformation T: if $\vec{v} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ is on the unit circle, then $T(\vec{v}) = \cos(t)\vec{w}_1 + \sin(t)\vec{w}_2$ is on the curve C. Therefore, C is an ellipse, by Exercise 54. (See Figure 2.42.)

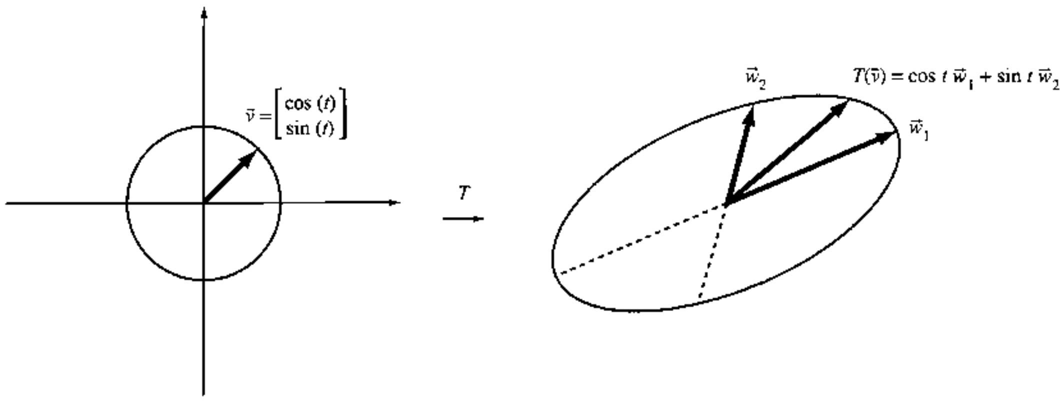

Figure 2.42: for Problem 2.2.55.

2.2.**56** By definition, the vectors $\vec{v}$ on an ellipse E are of the form $\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2$, for some perpendicular vectors $\vec{v}_1$ and $\vec{v}_2$. Then the vectors on the image C of E are of the form $T(\vec{v}) = \cos(t)T(\vec{v}_1) + \sin(t)T(\vec{v}_2)$. These vectors form an ellipse, by Exercise 55 (with $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$). See Figure 2.43.

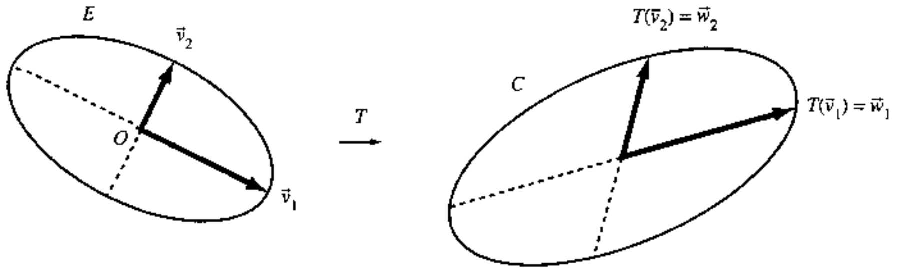

Figure 2.43: for Problem 2.2.56.

Section 2.3

2.3.**1** $\begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$

2.3.**2** $\begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 7 & 4 \end{bmatrix}$

2.3.**3** Undefined

2.3.**4** $\begin{bmatrix} 4 & 4 \\ -8 & -8 \end{bmatrix}$

2.3.**5** $\begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix}$

2.3.**6** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.3.**7** $\begin{bmatrix} -1 & 1 & 0 \\ 5 & 3 & 4 \\ -6 & -2 & -4 \end{bmatrix}$

2.3.**8** $\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$

2.3.**9** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.3.**10** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

2.3.**11** $[10]$

2.3.12 $[0\ 1]$

2.3.13 $[h]$

2.3.14 $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $BC = [14\ \ 8\ \ 2]$, $BD = [6]$, $C^2 = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$, $CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$, $DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$,

$DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$, $EB = [5\ 10\ 15]$, $E^2 = [25]$

2.3.15 $\left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[3] & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}[4] \\ \hline [1\ \ 3]\begin{bmatrix} 1 \\ 2 \end{bmatrix} + [4][3] & [1\ \ 3]\begin{bmatrix} 0 \\ 0 \end{bmatrix} + [4][4] \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [19] & [16] \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 19 & 16 \end{bmatrix}$

2.3.16 $\left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{array}\right] = \left[\begin{array}{cc} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{array}\right] = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$

2.3.17 We must find all S such that $SA = AS$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So $\begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$, meaning that $b = 2b$ and $c = 2c$, so b and c must be zero.

We see that all diagonal matrices (those of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$) commute with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

2.3.18 Following the form of Exercise 17, we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So, $\begin{bmatrix} 2a - 3b & 3a + 2b \\ 2c - 3d & 3c + 2d \end{bmatrix} = \begin{bmatrix} 2a + 3c & 2b + 3d \\ -3a + 2c & -3b + 2d \end{bmatrix}$, revealing that $a = d$ (since $3a + 2b = 2b + 3d$) and $-b = c$ (since $2a + 3c = 2a - 3b$).

Thus B is any matrix of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

2.3.19 Again, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Thus, $\begin{bmatrix} 2b & -2a \\ 2d & -2c \end{bmatrix} = \begin{bmatrix} -2c & -2d \\ 2a & 2b \end{bmatrix}$, meaning that $c = -b$ and $d = a$.

We see that all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ commute with $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

2.3.**20** As in Exercise 2.3.17, we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So, $\begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$, revealing that $c = 0$ (since $a + 2c = a$) and $a = d$ (since $b + 2d = 2a + b$).

Thus B is any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

2.3.**21** Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Thus, $\begin{bmatrix} a+2b & 2a-b \\ c+2d & 2c-d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 2a-c & 2b-d \end{bmatrix}$. So $a + 2b = a + 2c$, or $c = b$, and $2a - b = b + 2d$, revealing $d = a - b$. (The other two equations are redundant.)

All matrices of the form $\begin{bmatrix} a & b \\ b & a-b \end{bmatrix}$ commute with $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

2.3.**22** As in Exercise 17, we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

So, $\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$, revealing that $a = d$ (since $a + b = b + d$) and $b = c$ (since $a + c = a + b$).

Thus B is any matrix of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

2.3.**23** We want $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $\begin{bmatrix} a+2b & 3a+6b \\ c+2d & 3c+6d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ 2a+6c & 2b+6d \end{bmatrix}$. So $a + 2b = a + 3c$, or $c = \frac{2}{3}b$, and $3a + 6b = b + 3d$, revealing $d = a + \frac{5}{3}b$. The other two equations are redundant.

Thus all matrices of the form $\begin{bmatrix} a & b \\ \frac{2}{3}b & a+\frac{5}{3}b \end{bmatrix}$ commute with $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

2.3.**24** Following the form of Exercise 2.3.17, we let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Then we want $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

So, $\begin{bmatrix} 2a & 2b & 3c \\ 2d & 2e & 3f \\ 2g & 2h & 3i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 3g & 3h & 3i \end{bmatrix}$. Thus c, f, g and h must be zero, leaving B to be any matrix of the form

$\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & i \end{bmatrix}$.

2.3.**25** Now we want $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$,

or, $\begin{bmatrix} 2a & 3b & 2c \\ 2d & 3e & 2f \\ 2g & 3h & 2i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 2g & 2h & 2i \end{bmatrix}$. So, $3b = 2b$, $2d = 3d$, $3f = 2f$ and $3h = 2h$, meaning that b, d, f and h must all be zero.

Thus all matrices of the form $\begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & i \end{bmatrix}$ commute with $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

2.3.**26** Following the form of Exercise 2.3.17, we let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Now we want $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

So, $\begin{bmatrix} 2a & 3b & 4c \\ 2d & 3e & 4f \\ 2g & 3h & 4i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{bmatrix}$, which forces b, c, d, f, g and h to be zero. a, e and i, however, can be chosen freely.

Thus B is any matrix of the form $\begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix}$.

2.3.**27** We will prove that $A(C + D) = AC + AD$, repeatedly using Theorem 1.3.10a: $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$.

Write $B = [\vec{v}_1 \ \dots \ \vec{v}_m]$ and $C = [\vec{w}_1 \ \dots \ \vec{w}_m]$. Then

$A(C + D) = A[\vec{v}_1 + \vec{w}_1 \cdots \vec{v}_m + \vec{w}_m] = [A\vec{v}_1 + A\vec{w}_1 \ \cdots \ A\vec{v}_m + A\vec{w}_m]$, and

$AC + AD = A[\vec{v}_1 \ \cdots \ \vec{v}_m] + A[\vec{w}_1 \ \cdots \ \vec{w}_m] = [A\vec{v}_1 + A\vec{w}_1 \ \cdots A\vec{v}_m + A\vec{w}_m]$.

The results agree.

2.3.**28** The ijth entries of the three matrices are

$$\sum_{h=1}^{p}(ka_{ih})b_{hj}, \ \sum_{h=1}^{p} a_{ih}(kb_{hj}), \text{ and } k\left(\sum_{h=1}^{p} a_{ih}b_{hj}\right)$$

.

The three results agree.

2.3.**29** a $D_\alpha D_\beta$ and $D_\beta D_\alpha$ are the same transformation, namely, a rotation through $\alpha+\beta$.

b $D_\alpha D_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$D_\beta D_\alpha$ yields the same answer.

2.3.**30** a See Figure 2.44.

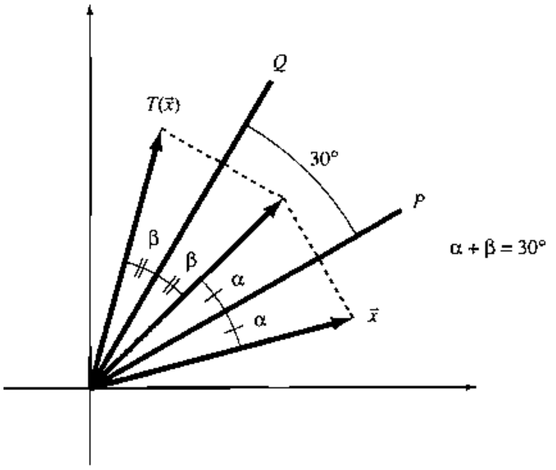

Figure 2.44: for Problem 2.4.30.

The vectors $\vec{x}$ and $T(\vec{x})$ have the same length (since reflections leave the length unchanged), and they enclose an angle of $2(\alpha+\beta) = 2\cdot 30^\circ = 60^\circ$

b Based on the answer in part (a), we conclude that T is a rotation through 60°.

c The matrix of T is $\begin{bmatrix} \cos(60^\circ) & -\sin(60^\circ) \\ \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

2.3.**31** Write A in terms of its rows: $A = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \cdots \\ \vec{w}_n \end{bmatrix}$ (suppose A is $n\times m$).

We can think of this as a partition into n

$1 \times m$ matrices. Now $AB = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \cdots \\ \vec{w}_n \end{bmatrix} B = \begin{bmatrix} \vec{w}_1 B \\ \vec{w}_2 B \\ \cdots \\ \vec{w}_n B \end{bmatrix}$ (a product of partitioned matrices).

We see that the ith row of AB is the product of the ith row of A and the matrix B.

2.3.**32** Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we want $X \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} X$, or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, meaning that $b = c = 0$. Also, we want $X \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X$, or $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, or $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$ so $a = d$. Thus, $X = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI_2$ must be a multiple of the identity matrix. (X will then commute with any 2×2 matrix M, since $XM = aM = MX$.)

2.3.**33** $A^2 = I_2$, $A^3 = A, A^4 = I_2$. The power A^n alternates between $A = -I_2$ and I_2. The matrix A describes a reflection about the origin. Alternatively one can say A represents a rotation by $180° = \pi$. Since A^2 is the identity, A^{1000} is the identity and $A^{1001} = A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.3.**34** $A^2 = I_2$, $A^3 = A, A^4 = I_2$. The power A^n alternates between A and I_2. The matrix A describes a reflection about the x axis. Because A^2 is the identity, A^{1000} is the identity and $A^{1001} = A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.3.**35** $A^2 = I_2$, $A^3 = A, A^4 = I_2$. The power A^n alternates between A and I_2. The matrix A describes a reflection about the diagonal $x = y$. Because A^2 is the identity, A^{1000} is the identity and $A^{1001} = A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

2.3.**36** $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$. The power A^n represents a horizontal shear along the x-axis. The shear strength increases linearly in n. We have $A^{1001} = \begin{bmatrix} 1 & 1001 \\ 0 & 1 \end{bmatrix}$.

2.3.**37** $A^2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ and $A^4 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$. The power A^n represents a vertical shear along the y axis. The shear magnitude increases linearly in n. We have $A^{1001} = \begin{bmatrix} 1 & 0 \\ -1001 & 1 \end{bmatrix}$.

2.3.**38** $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^3 = -A$, $A^4 = I_2$. The matrix A represents the rotation through $\pi/2$ in the counterclockwise direction. Since A^4 is the identity matrix, we know that A^{1000} is the identity matrix and $A^{1001} = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

2.3.**39** $A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A^3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, $A^4 = -I_2$. The matrix A describes a rotation by $\pi/4$ in the clockwise direction. Because A^8 is the identity matrix, we know that A^{1000} is the identity matrix and $A^{1001} = A = (1/\sqrt{2}) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

2.3.40 $A^2 = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$, $A^3 = I_2$, $A^4 = A$. The matrix A describes a rotation by $120° = 2\pi/3$ in the counterclockwise direction. Because A^3 is the identity matrix, we know that A^{999} is the identity matrix and $A^{1001} = A^2 = A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$.

2.3.41 $A^2 = I_2$, $A^3 = A$, $A^4 = I_2$. The power A^n alternates between I_2 for even n and A for odd n. Therefore $A^{1001} = A$. The matrix represents a reflection about a line.

2.3.42 $A^n = A$. The matrix A represents a projection on the line $x = y$ spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We have $A^{1001} = A = (1/2)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

2.3.43 An example is $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, representing the reflection about the horizontal axis.

2.3.44 A rotation by $\pi/2$ given by the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

2.3.45 For example, $A = (1/2)\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, the rotation through $2\pi/3$. See Problem 2.3.40.

2.3.46 For example, $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the orthogonal projection onto the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2.3.47 For example, $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the orthogonal projection onto the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2.3.48 For example, the shear $A = \begin{bmatrix} 1 & 1/10 \\ 0 & 1 \end{bmatrix}$.

2.3.49 $AF = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represents the reflection about the x-axis, while $FA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ represents the reflection about the y-axis. (See Figure 2.45.)

2.3.50 $CG = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents a reflection about the line $x = y$, while $GC = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ represents a reflection about the line $x = -y$. (See Figure 2.46.)

2.3.51 $FJ = JF = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ both represent a rotation through $3\pi/4$ combined with a scaling by $\sqrt{2}$. (See Figure 2.47.)

2.3.52 $JH = HJ = \begin{bmatrix} 0.2 & -1.4 \\ 1.4 & 0.2 \end{bmatrix}$. Since H represents a rotation and J represents a rotation through $\pi/4$ combined with a scaling by $\sqrt{2}$, the products in either order will be the same, representing a rotation combined with a scaling by $\sqrt{2}$. (See Figure 2.48.)

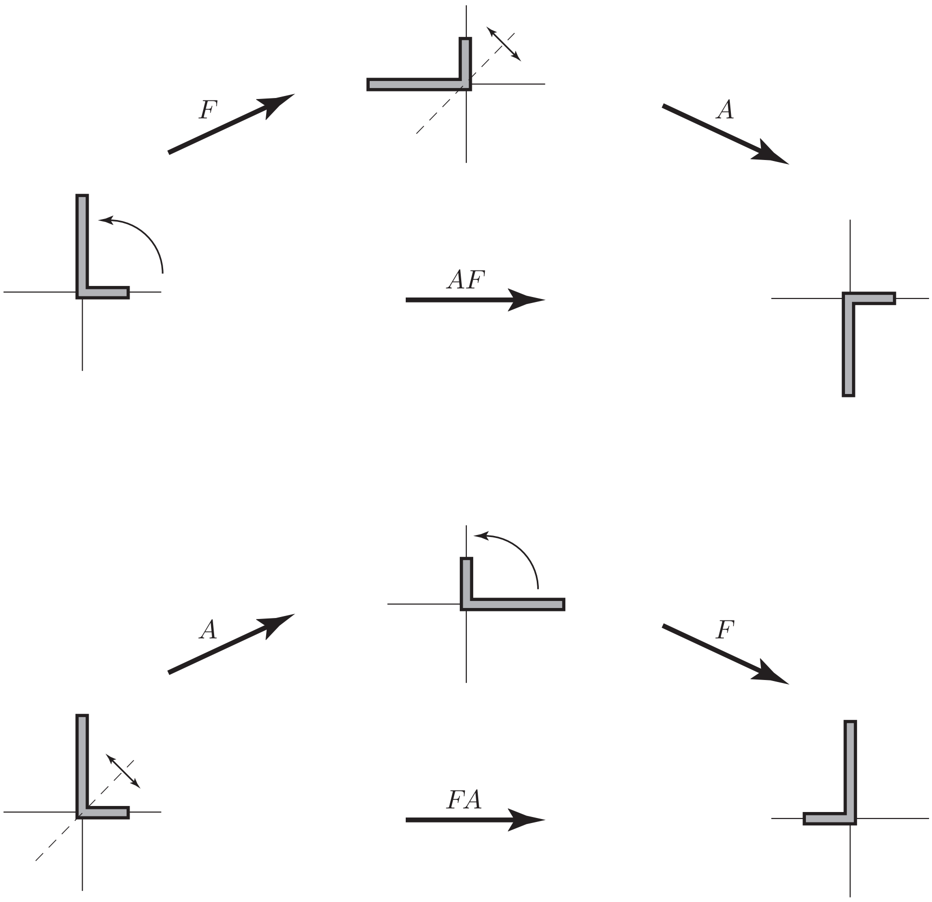

Figure 2.45: for Problem 2.3.49.

2.3.**53** $CD = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents the rotation through $\pi/2$, while $DC = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ represents the rotation through $-\pi/2$. (See Figure 2.49.)

2.3.**54** $BE = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix}$ represents the rotation through the angle $\theta = \arccos(-0.6) \approx 2.21$, while $EB = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$ represents the rotation through $-\theta$. (See Figure 2.50.)

2.3.**55** We need to solve the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which amounts to solving the system $a + 2c = 0, 2a + 4c = 0, b + 2d = 0$ and $2b + 4d = 0$. The solutions are of the form $a = -2c$ and $b = -2d$. Thus $X = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix}$, where c, d are arbitrary constants.

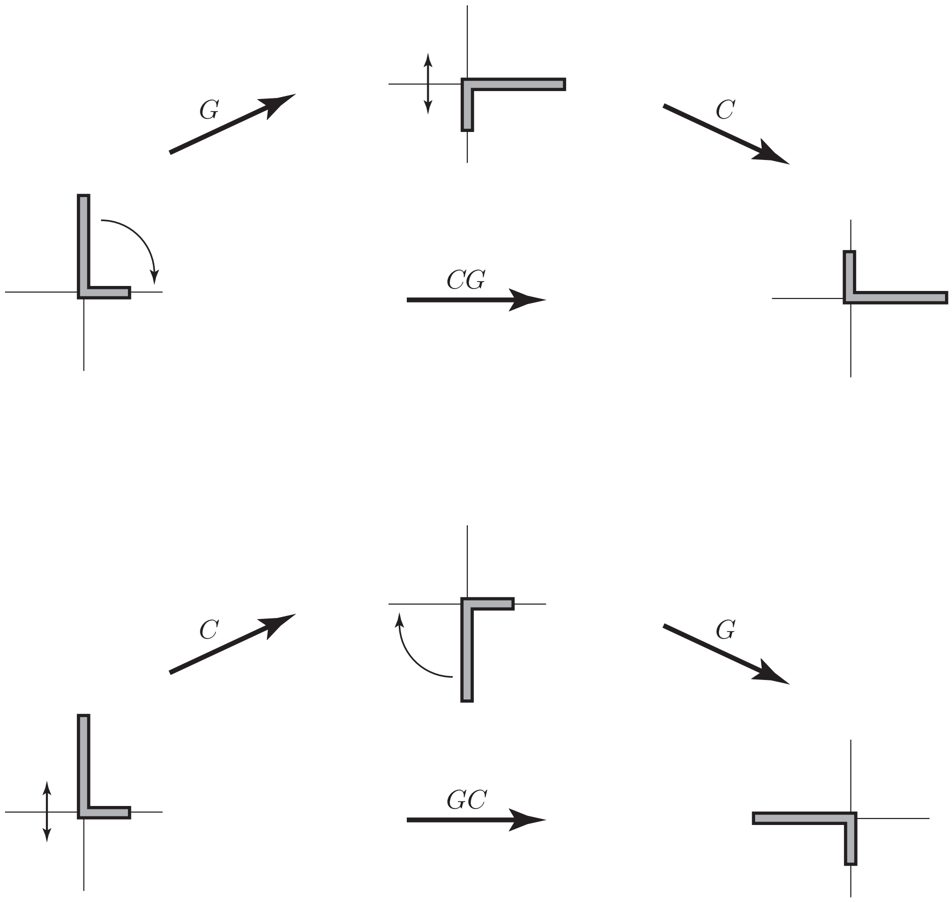

Figure 2.46: for Problem 2.3.50.

2.3.**56** Proceeding as in Exercise 55, we find $X = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

2.3.**57** We need to solve the matrix equation

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which amounts to solving the system $a + 2c = 1$, $3a + 5c = 0, b + 2d = 0$ and $3b + 5d = 1$. The solution is $X = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

2.3.**58** Proceeding as in Exercise 55, we find $X = \begin{bmatrix} -2b & b \\ -2d & d \end{bmatrix}$, where b and d are arbitrary.

2.3.**59** The matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has no solutions, since we have the inconsistent equations $2a + 4b = 1$ and $a + 2b = 0$.

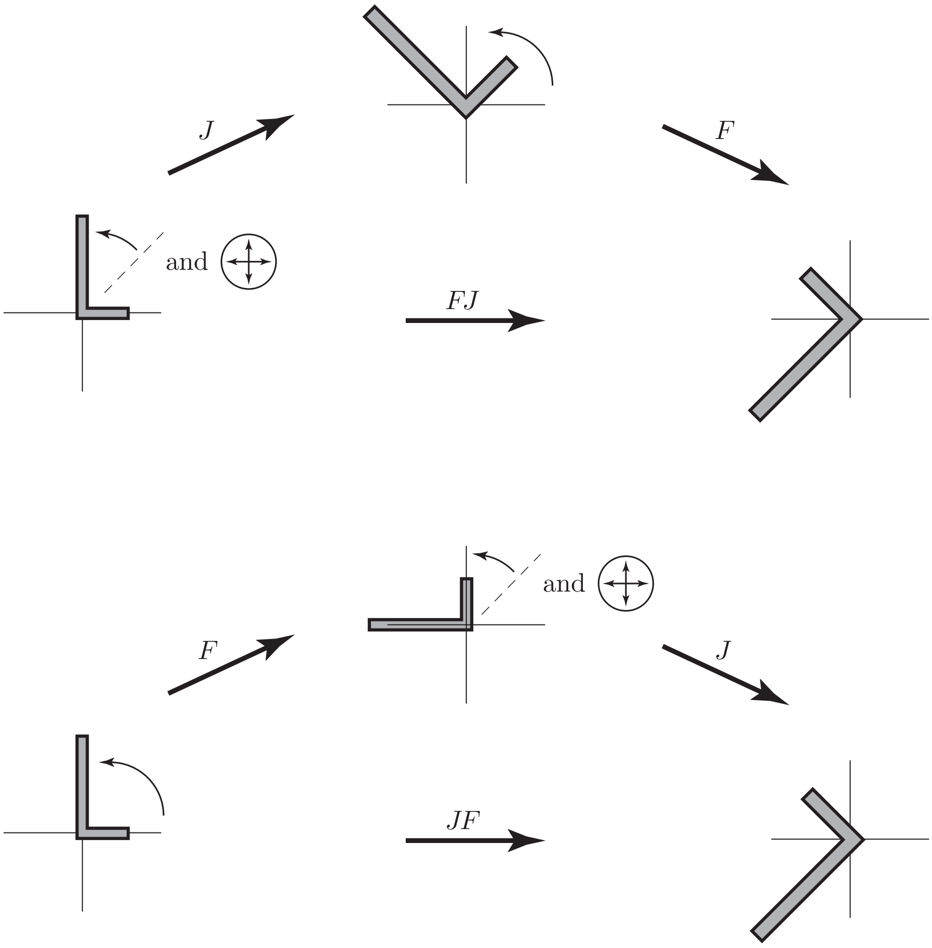

Figure 2.47: for Problem 2.3.51.

2.3.**60** Proceeding as in Exercise 59, we find that this equation has no solutions.

2.3.**61** We need to solve the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which amounts to solving the system $a + 2c + 3e = 0, c + 2e = 0, b + 2d + 3f = 0$ and $d + 2f = 1$. The solutions are of the form $X = \begin{bmatrix} e+1 & f-2 \\ -2e & 1-2f \\ e & f \end{bmatrix}$, where e, f are arbitrary constants.

2.3.**62** The matrix equation

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

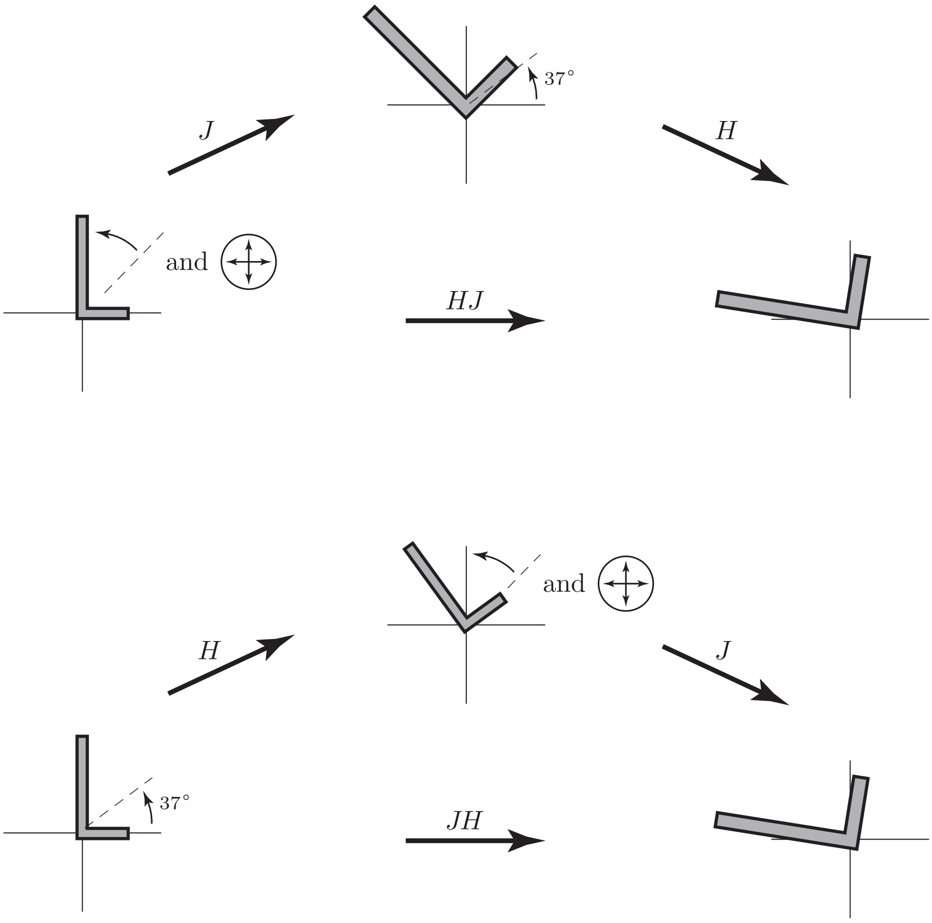

Figure 2.48: for Problem 2.3.52.

has no solutions, since we have the inconsistent equations $a = 1, 2a + d = 0$ and $3a + 2d = 0$.

2.3.**63** The matrix equation

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has no solutions, since we have the inconsistent equations $a + 4d = 1, 2a + 5d = 0$, and $3a + 6d = 0$.

2.3.**64** Proceeding as in Exercise 61, we find $X = \begin{bmatrix} e - 5/3 & f + 2/3 \\ -2e + 4/3 & -2f - 1/3 \\ e & f \end{bmatrix}$, where e, f are arbitrary constants.

2.3.**65** With $X = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, we have to solve $X^2 = \begin{bmatrix} a^2 & ab + bc \\ 0 & c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This means $a = 0, c = 0$ and b can be arbitrary. The general solution is $X = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$.

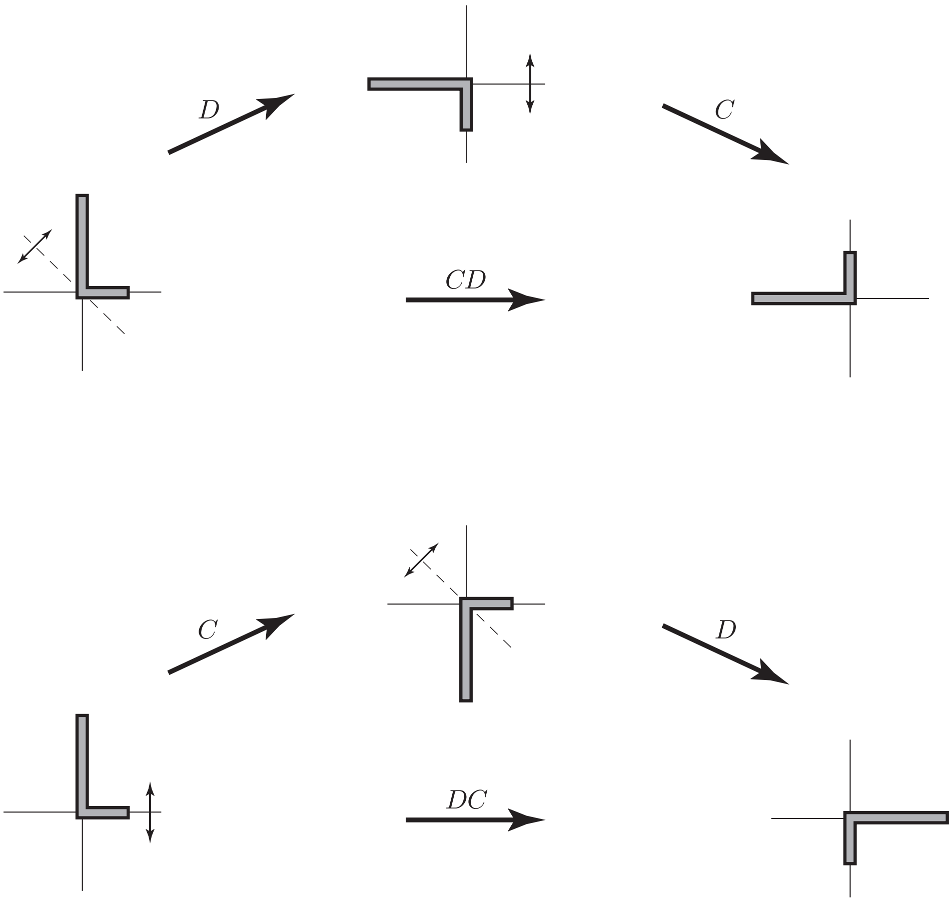

Figure 2.49: for Problem 2.3.53.

2.3.**66** If $X = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ then the diagonal entries of X^3 will be a^3, c^3, and f^3. Since we want $X^3 = 0$, we must have $a = c = f = 0$. If $X = \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{bmatrix}$, then a direct computation shows that $X^3 = 0$. Thus the solutions are of the form $X = \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & e & 0 \end{bmatrix}$, where b, d, e are arbitrary.

2.3.**67** a. $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} a+b+c & d+e+f & g+h+i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, by Definition 2.1.4.

b. For an $n \times n$ matrix A with nonnegative entries, the following statements are equivalent:

$$\begin{aligned} &\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \\ &\Leftrightarrow \textstyle\sum_{i=1}^{n} 1 a_{ij} = 1 \quad \text{for all } j = 1, \ldots, n \\ &\Leftrightarrow A \text{ is a transition matrix} \end{aligned}$$

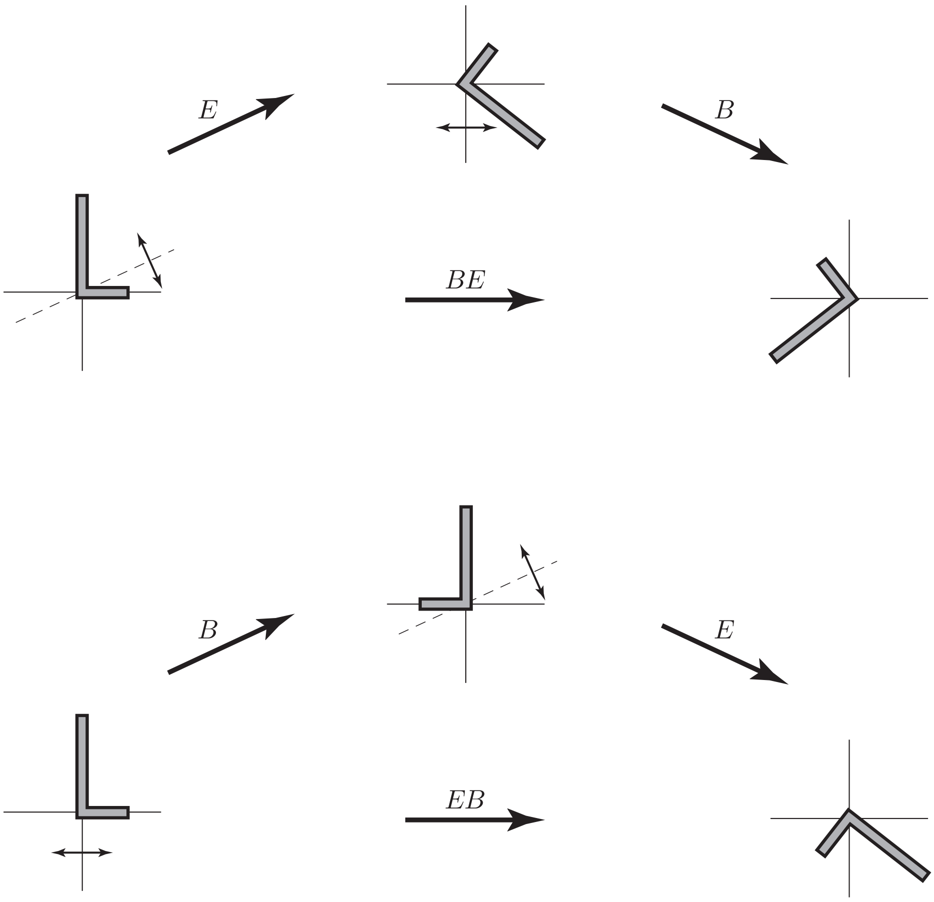

Figure 2.50: for Problem 2.3.54.

2.3.**68** From Exercise 67b we know that $\begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix}$. Now $\begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix} AB = (\begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix} A)B = \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 1 & ... & 1 \end{bmatrix}$, so that AB is a transition matrix, again by Exercise 67b (note that all entries of AB will be nonnegative).

2.3.**69** a. It means that 25% of the surfers who are on page 1 initially will find themselves on page 3 after following two links.

b. The ijth entry of A^2 is 0 if it is impossible to get from page j to page i by following two consecutive links. This means that there is no path of length 2 in the graph of the mini-Web from point j to point i.

2.3.**70** a.

$$A^3 = \begin{bmatrix} 0 & 1/8 & 1/2 & 0 \\ 5/8 & 1/2 & 0 & 1/4 \\ 1/8 & 1/8 & 1/2 & 1/4 \\ 1/4 & 1/4 & 0 & 1/2 \end{bmatrix}$$

b. It means that 25% of the surfers who are on page 1 initially will find themselves on page 4 after following three links.

c. The ijth entry of A^3 is 0 if it is impossible to get from page j to page i by following three consecutive links. This means that there is no path of length 3 in the graph of the mini-Web from point j to point i.

d. There are two paths, $1 \to 2 \to 1 \to 2$ and $1 \to 3 \to 4 \to 2$. Of the surfers who are on page 1 initially, $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = 12.5\%$ will follow the path $1 \to 2 \to 1 \to 2$, while $\frac{1}{2} \times 1 \times 1 = \frac{1}{2} = 50\%$ will follow the path $1 \to 3 \to 4 \to 2$.

2.3.**71** We compute

$$A^4 = \begin{bmatrix} 5/16 & 1/4 & 0 & 1/8 \\ 1/4 & 5/16 & 1/4 & 1/2 \\ 5/16 & 1/16 & 1/4 & 1/8 \\ 1/8 & 1/8 & 1/2 & 1/4 \end{bmatrix}.$$

We see that it is impossible to get from page 3 to page 1 by following four consecutive links.

2.3.**72** We compute

$$A^5 = \begin{bmatrix} 1/8 & 5/32 & 1/8 & 1/4 \\ 9/32 & 1/4 & 1/2 & 5/16 \\ 9/32 & 9/32 & 1/8 & 5/16 \\ 5/16 & 5/16 & 1/4 & 1/8 \end{bmatrix}.$$

Considering the matrix A^4 we found in Exercise 71, we see that 5 is the smallest positive integer m such that all entries of A^m are positive. A surfer can get from any page j to any page i by following five consecutive links. Equivalently, there is a path of length five in the mini-Web from any point j to any point i.

2.3.**73** $$\lim_{m\to\infty}(A^m\vec{x}) = \left(\lim_{m\to\infty} A^m\right)\vec{x} = \begin{bmatrix} \vec{x}_{equ} & \vec{x}_{equ} & \dots & \vec{x}_{equ} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = (x_1 + x_2 + \dots + x_n)\vec{x}_{equ} = \vec{x}_{equ}.$$

Note that $x_1 + x_2 + \dots + x_n = 1$ since $\vec{x}$ is a transition vector.

2.3.**74** The transition matrix AB is not necessarily positive. Consider the case where A has a row of zeros, for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, with $AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
The matrix BA, on the other hand, must be positive. Each entry of BA is the dot product $\vec{w} \cdot \vec{v}$ of a row $\vec{w}$ of B with a column $\vec{v}$ of A. All components of $\vec{w}$ are positive, all components of $\vec{v}$ are nonnegative, and at least one component v_i of $\vec{v}$ is positive (since $\vec{v}$ is a distribution vector). Thus $\vec{w} \cdot \vec{v} \geq w_i v_i > 0$, showing that all entries of BA are positive as claimed.

2.3.**75** Each entry of $A^{m+1} = A^m A$ is the dot product $\vec{w} \cdot \vec{v}$of a row $\vec{w}$of A^m with a column $\vec{v}$of A. All components of $\vec{w}$ are positive (since A^m is positive) , all components of $\vec{v}$ are nonnegative (since A is a transition matrix), and at least one component $v_i$of $\vec{v}$ is positive (since $\vec{v}$ is a distribution vector). Thus $\vec{w} \cdot \vec{v} \geq w_i v_i > 0$, showing that all entries of A^{m+1} are positive as claimed.

2.3.**76** $$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 0 \end{bmatrix} \text{ and } A^{20} \approx \begin{bmatrix} 0.2002 & 0.2002 & 0.2002 & 0.1992 \\ 0.4004 & 0.3994 & 0.4004 & 0.4004 \\ 0.1992 & 0.2002 & 0.1992 & 0.2012 \\ 0.2002 & 0.2002 & 0.2002 & 0.1992 \end{bmatrix}$$

suggest $\vec{x}_{equ} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.2 \\ 0.2 \end{bmatrix}$. We can verify that $A \begin{bmatrix} 0.2 \\ 0.4 \\ 0.2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.2 \\ 0.2 \end{bmatrix}$.

Page 2 has the highest naive PageRank.

2.3.**77** $A^{10} \approx \begin{bmatrix} 0.5003 & 0.4985 & 0.5000 \\ 0.0996 & 0.1026 & 0.0999 \\ 0.4002 & 0.3989 & 0.4001 \end{bmatrix}$ suggests $\vec{x}_{equ} = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.4 \end{bmatrix}$.

We can verify that $A \begin{bmatrix} 0.5 \\ 0.1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.4 \end{bmatrix}$.

2.3.**78** $B^{15} \approx \begin{bmatrix} 0.1785 & 0.1785 & 0.1786 & 0.1787 \\ 0.3214 & 0.3214 & 0.3216 & 0.3213 \\ 0.2500 & 0.2500 & 0.2499 & 0.2501 \\ 0.2501 & 0.2501 & 0.2499 & 0.2499 \end{bmatrix} \approx \frac{1}{28} \begin{bmatrix} 5 & 5 & 5 & 5 \\ 9 & 9 & 9 & 9 \\ 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \end{bmatrix}$ suggests $\vec{x}_{equ} = \begin{bmatrix} 5/28 \\ 9/28 \\ 7/28 \\ 7/28 \end{bmatrix} = \begin{bmatrix} 5/28 \\ 9/28 \\ 1/4 \\ 1/4 \end{bmatrix}$.

We can verify that $B \begin{bmatrix} 5/28 \\ 9/28 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 5/28 \\ 9/28 \\ 1/4 \\ 1/4 \end{bmatrix}$.

2.3.**79** An extreme example is the identity matrix I_n, where $I_n\vec{x} = \vec{x}$ for all distribution vectors $\vec{x}$.

2.3.**80** One example is the reflection matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, where $A^m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for even m and $A^m = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for odd m.

2.3.**81** If $A\vec{v} = 5\vec{v}$, then $A^2\vec{v} = A(A\vec{v}) = A(5\vec{v}) = 5A\vec{v} = 5^2\vec{v}$. We can show by induction on m that $A^m\vec{v} = 5^m\vec{v}$. Indeed, $A^{m+1}\vec{v} = A\left(A^m\vec{v}\right) \underbrace{=}_{step 2} A\left(5^m\vec{v}\right) = 5^m A\vec{v} = 5^m 5\vec{v} = 5^{m+1}\vec{v}$. In step 2 we have used the induction hypothesis.

2.3.**82** a. $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b. $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

c. Using Exercise 81, we find that $A^m \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A^m \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{10^m} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for all positive integers m.

Now $A^m\vec{x} = A^m \left(\frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \frac{1}{3} A^m \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{3} A^m \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2}{3 \cdot 10^m} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

d. $\lim_{m \to \infty} (A^m\vec{x}) = \lim_{m \to \infty} \left(\frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{2 \cdot 10^m} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. We can verify that $A \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$, so that $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ is indeed the equilibrium distribution for A.

2.3.**83** Pick a positive number m such that A^m is a positive transition matrix; note that $A^m\vec{x} = \vec{x}$. The equation $A^m\vec{x} = \vec{x}$ implies that the jth component x_j of $\vec{x}$ is the dot product $\vec{w} \cdot \vec{x}$, where $\vec{w}$ is the jth row of A^m. All components of $\vec{w}$ are positive, all components of $\vec{x}$ are nonnegative, and at least one component $x_i$of $\vec{x}$ is positive (since $\vec{x}$ is a distribution vector). Thus $x_j = \vec{w} \cdot \vec{x} \geq w_i x_i > 0$, showing that all components of $\vec{x}$ are positive as claimed.

2.3.**84** Let $\vec{v}_1, \ldots, \vec{v}_n$ be the columns of the matrix X. Solving the matrix equation $AX = I_n$ amounts to solving the linear systems $A\vec{v}_i = \vec{e}_i$ for $i = 1, \ldots, n$. Since A is a $n \times m$ matrix of rank n, all these systems are consistent, so that the matrix equation $AX = I_n$ does have at least one solution. If $n < m$, then each of the systems $A\vec{v}_i = \vec{e}_i$ has infinitely many solutions, so that the matrix equation $AX = I_n$ has infinitely many solutions as well. See the examples in Exercises 2.3.57, 2.3.61 and 2.3.64.

2.3.**85** Let $\vec{v}_1, \ldots, \vec{v}_n$ be the columns of the matrix X. Solving the matrix equation $AX = I_n$ amounts to solving the linear systems $A\vec{v}_i = \vec{e}_i$ for $i = 1, \ldots, n$. Since A is an $n \times n$ matrix of rank n, all these systems have a unique solution, by Theorem 1.3.4, so that the matrix equation $AX = I_n$ has a unique solution as well.

Section 2.4

2.4.**1** rref $\begin{bmatrix} 2 & 3 & \vdots & 1 & 0 \\ 5 & 8 & \vdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 8 & -3 \\ 0 & 1 & \vdots & -5 & 2 \end{bmatrix}$, so that $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$.

2.4.**2** rref $\begin{bmatrix} 1 & 1 & \vdots & 1 & 0 \\ 1 & 1 & \vdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \vdots & 0 & 1 \\ 0 & 0 & \vdots & 1 & -1 \end{bmatrix}$, so that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ fails to be invertible.

2.4.**3** rref $\begin{bmatrix} 0 & 2 & \vdots & 1 & 0 \\ 1 & 1 & \vdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & -\frac{1}{2} & 1 \\ 0 & 1 & \vdots & \frac{1}{2} & 0 \end{bmatrix}$, so that $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

2.4.**4** Use Theorem 2.4.5; the inverse is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

2.4.**5** rref $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Theorem 2.4.3.

2.4.**6** Use Theorem 2.4.5; the inverse is $\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}$.

2.4.**7** rref $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Theorem 2.4.3.

2.4.**8** Use Theorem 2.4.5; the inverse is $\begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$.

2.4.9 rref $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that the matrix fails to be invertible, by Theorem 2.4.3.

2.4.10 Use Theorem 2.4.5; the inverse is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

2.4.11 Use Theorem 2.4.5; the inverse is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.4.12 Use Theorem 2.4.5; the inverse is $\begin{bmatrix} 3 & -5 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$.

2.4.13 Use Theorem 2.4.5; the inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$.

2.4.14 Use Theorem 2.4.5; the inverse is $\begin{bmatrix} 5 & -20 & -2 & -7 \\ 0 & -1 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix}$

2.4.15 Use Theorem 2.4.5; the inverse is $\begin{bmatrix} -6 & 9 & -5 & 1 \\ 9 & -1 & -5 & 2 \\ -5 & -5 & 9 & -3 \\ 1 & 2 & -3 & 1 \end{bmatrix}$

2.4.16 Solving for x_1 and x_2 in terms of y_1 and y_2 we find that

$$\begin{aligned} x_1 &= -8y_1 + 5y_2 \\ x_2 &= 5y_1 - 3y_2 \end{aligned}$$

2.4.17 We make an attempt to solve for x_1 and x_2 in terms of y_1 and y_2:

$$\begin{bmatrix} x_1 + 2x_2 & = & y_1 & \\ 4x_1 + 8x_2 & = & & y_2 \end{bmatrix} \xrightarrow[-4(I)]{} \begin{bmatrix} x_1 + 2x_2 & = y_1 \\ 0 & = -4y_1 + y_2 \end{bmatrix}.$$

This system has no solutions (x_1, x_2) for some (y_1, y_2), and infinitely many solutions for others; the transformation fails to be invertible.

2.4.18 Solving for x_1, x_2, and x_3 in terms of y_1, y_2, and y_3 we find that

$$\begin{aligned} x_1 &= y_3 \\ x_2 &= y_1 \\ x_3 &= y_2 \end{aligned}$$

2.4.**19** Solving for x_1, x_2, and x_3 in terms of y_1, y_2, and y_3, we find that

$$x_1 = 3y_1 - \frac{5}{2}y_2 + \frac{1}{2}y_3$$
$$x_2 = -3y_1 + 4y_2 - y_3$$
$$x_3 = y_1 - \frac{3}{2}y_2 + \frac{1}{2}y_3$$

2.4.**20** Solving for x_1, x_2, and x_3 in terms of y_1, y_2, and y_3 we find that

$$x_1 = -8y_1 - 15y_2 + 12y_3$$
$$x_2 = 4y_1 + 6y_2 - 5y_3$$
$$x_3 = -y_1 - y_2 + y_3$$

2.4.**21** $f(x) = x^2$ fails to be invertible, since the equation $f(x) = x^2 = 1$ has two solutions, $x = \pm 1$.

2.4.**22** $f(x) = 2^x$ fails to be invertible, since the equation $f(x) = 2^x = 0$ has no solution x.

2.4.**23** Note that $f'(x) = 3x^2 + 1$ is always positive; this implies that the function $f(x) = x^3 + x$ is increasing throughout. Therefore, the equation $f(x) = b$ has *at most* one solution x for all b. (See Figure 2.51.)

Now observe that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$; this implies that the equation $f(x) = b$ has at least one solution x for a given b (for a careful proof, use the intermediate value theorem; compare with Exercise 2.2.47c).

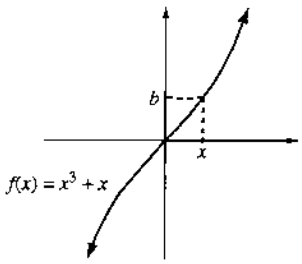

Figure 2.51: for Problem 2.3.23.

2.4.**24** We can write $f(x) = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$.

The equation $f(x) = 0$ has three solutions, $x = 0, 1, -1$, so that $f(x)$ fails to be invertible.

2.4.**25** Invertible, with inverse $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt[3]{y_1} \\ y_2 \end{bmatrix}$

2.4.**26** Invertible, with inverse $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt[3]{y_2 - y_1} \\ y_1 \end{bmatrix}$

2.4.**27** This transformation fails to be invertible, since the equation $\begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solution.

2.4.**28** We are asked to find the inverse of the matrix $A = \begin{bmatrix} 22 & 13 & 8 & 3 \\ -16 & -3 & -2 & -2 \\ 8 & 9 & 7 & 2 \\ 5 & 4 & 3 & 1 \end{bmatrix}$.

We find that $A^{-1} = \begin{bmatrix} 1 & -2 & 9 & -25 \\ -2 & 5 & -22 & 60 \\ 4 & -9 & 41 & -112 \\ -9 & 17 & 80 & 222 \end{bmatrix}$.

T^{-1} is the transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ with matrix A^{-1}.

2.4.**29** Use Theorem 2.4.3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix} \begin{matrix} \\ -I \\ -I \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 3 & k^2-1 \end{bmatrix} \begin{matrix} -II \\ \\ -3(II) \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 2-k \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-3k+2 \end{bmatrix}$$

The matrix is invertible if (and only if) $k^2 - 3k + 2 = (k-2)(k-1) \neq 0$, in which case we can further reduce it to I_3. Therefore, the matrix is invertible if $k \neq 1$ and $k \neq 2$.

2.4.**30** Use Theorem 2.4.3:

$$\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix} \xrightarrow{I \leftrightarrow II} \begin{bmatrix} -1 & 0 & c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \xrightarrow{\div(-1)} \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \\ \\ +b(I)+c(II) \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix fails to be invertible, regarless of the values of b and c.

2.4.**31** Use Theorem 2.4.3; first assume that $a \neq 0$.

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \text{swap :} \\ I \leftrightarrow II \end{matrix} \rightarrow \begin{bmatrix} -a & 0 & c \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \div(-a) \\ \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \\ \\ +b(I) \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & a & b \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} \div a \rightarrow \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & 1 & \frac{b}{a} \\ 0 & -c & -\frac{bc}{a} \end{bmatrix} \begin{matrix} \\ \\ +c(II) \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{c}{a} \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & 0 \end{bmatrix}$$

Now consider the case when $a = 0$:

$$\begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ -b & -c & 0 \end{bmatrix} \begin{matrix} \text{swap :} \\ I \leftrightarrow III \end{matrix} \rightarrow \begin{bmatrix} -b & -c & 0 \\ 0 & 0 & c \\ 0 & 0 & b \end{bmatrix}$$: The second entry on the diagonal of rref will be 0.

It follows that the matrix $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ fails to be invertible, regardless of the values of a, b, and c.

2.4.**32** Use Theorem 2.4.9.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix such that $ad - bc = 1$ and $A^{-1} = A$, then

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so that $b = 0,\ c = 0$, and $a = d$.

The condition $ad - bc = a^2 = 1$ now implies that $a = d = 1$ or $a = d = -1$.

This leaves only two matrices A, namely, I_2 and $-I_2$. Check that these two matrices do indeed satisfy the given requirements.

2.4.**33** Use Theorem 2.4.9.

The requirement $A^{-1} = A$ means that $-\frac{1}{a^2+b^2}\begin{bmatrix} -a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. This is the case if (and only if) $a^2 + b^2 = 1$.

2.4.**34** a By Theorem 2.4.3, A is invertible if (and only if) a, b, and c are all nonzero. In this case, $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$.

b In general, a diagonal matrix is invertible if (and only if) all of its diagonal entries are nonzero.

2.4.**35** a A is invertible if (and only if) all its diagonal entries, a, d, and f, are nonzero.

b As in part (a): if all the diagonal entries are nonzero.

c Yes, A^{-1} will be upper triangular as well; as you construct rref$[A \vdots I_n]$, you will perform only the following row operations:

- divide rows by scalars
- subtract a multiple of the jth row from the ith row, where $j > i$.

Applying these operations to I_n, you end up with an upper triangular matrix.

d As in part (b): if all diagonal entries are nonzero.

2.4.**36** If a matrix A can be transformed into B by elementary row operations, then A is invertible if (and only if) B is invertible. The claim now follows from Exercise 35, where we show that a triangular matrix is invertible if (and only if) its diagonal entries are nonzero.

2.4.**37** Make an attempt to solve the linear equation $\vec{y} = (cA)\vec{x} = c(A\vec{x})$ for $\vec{x}$:

$A\vec{x} = \frac{1}{c}\vec{y}$, so that $\vec{x} = A^{-1}\left(\frac{1}{c}\vec{y}\right) = \left(\frac{1}{c}A^{-1}\right)\vec{y}$.

This shows that cA is indeed invertible, with $(cA)^{-1} = \frac{1}{c}A^{-1}$.

2.4.**38** Use Theorem 2.4.9; $A^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix} (= A)$.

2.4.**39** Suppose the ijth entry of M is k, and all other entries are as in the identity matrix. Then we can find rref$[M \vdots I_n]$ by subtracting k times the jth row from the ith row. Therefore, M is indeed invertible, and M^{-1} differs from the identity matrix only at the ijth entry; that entry is $-k$. (See Figure 2.52.)

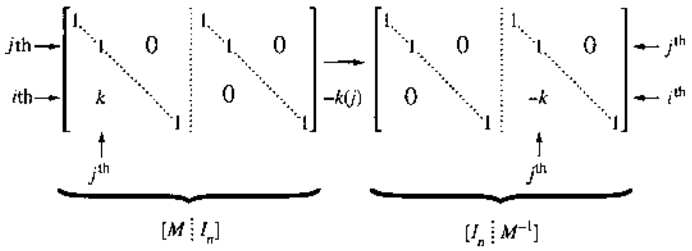

Figure 2.52: for Problem 2.3.39.

2.4.**40** If you apply an elementary row operation to a matrix with two equal columns, then the resulting matrix will also have two equal columns. Therefore, rref(A) has two equal columns, so that rref$(A) \neq I_n$. Now use Theorem 2.4.3.

2.4.**41** a Invertible: the transformation is its own inverse.

b Not invertible: the equation $T(\vec{x}) = \vec{b}$ has infinitely many solutions if $\vec{b}$ is on the plane, and none otherwise.

c Invertible: The inverse is a scaling by $\frac{1}{5}$ (that is, a contraction by 5). If $\vec{y} = 5\vec{x}$, then $\vec{x} = \frac{1}{5}\vec{y}$.

d Invertible: The inverse is a rotation about the same axis through the same angle in the opposite direction.

2.4.**42** Permutation matrices are invertible since they row reduce to I_n in an obvious way, just by row swaps. The inverse of a permutation matrix A is also a permutation matrix since rref$[A \vdots I_n] = [I_n \vdots A^{-1}]$ is obtained from $[A \vdots I_n]$ by a sequence of row swaps.

2.4.**43** We make an attempt to solve the equation $\vec{y} = A(B\vec{x})$ for $\vec{x}$:

$B\vec{x} = A^{-1}\vec{y}$, so that $\vec{x} = B^{-1}(A^{-1}\vec{y})$.

2.4.**44** a rref$(M_4) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so that rank$(M_4) = 2$.

b To simplify the notation, we introduce the row vectors $\vec{v} = [1\ 1\ \ldots\ 1]$ and $\vec{w} = [0\ n\ 2n\ \ldots\ (n-1)n]$ with n components.

Then we can write M_n in terms of its rows as $M_n = \begin{bmatrix} \vec{v} + \vec{w} \\ 2\vec{v} + \vec{w} \\ \cdots \\ n\vec{v} + \vec{w} \end{bmatrix} \begin{matrix} \\ -2(I) \\ \cdots \\ -n(I) \end{matrix}$.

Applying the Gauss-Jordan algorithm to the first column we get $\begin{bmatrix} \vec{v}+\vec{w} \\ -\vec{w} \\ -2\vec{w} \\ \dots \\ -(n-1)\vec{w} \end{bmatrix}$.

All the rows below the second are scalar multiples of the second; therefore, $\text{rank}(M_n) = 2$.

c By part (b), the matrix M_n is invertible only if $n = 2$.

2.4.**45** a Each of the three row divisions requires three multiplicative operations, and each of the six row subtractions requires three multiplicative operations as well; altogether, we have $3 \cdot 3 + 6 \cdot 3 = 9 \cdot 3 = 3^3 = 27$ operations.

b Suppose we have already taken care of the first m columns: $[A \vdots I_n]$ has been reduced the matrix in Figure 2.53.

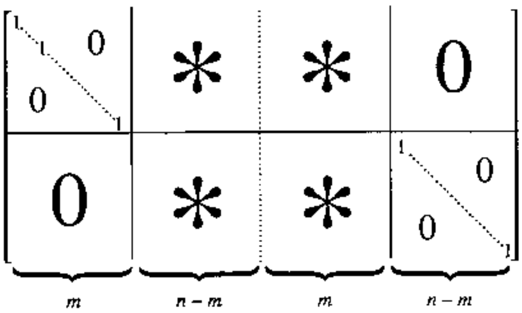

Figure 2.53: for Problem 2.3.45b.

Here, the stars represent arbitrary entries.

Suppose the $(m+1)$th entry on the diagonal is k. Dividing the $(m+1)$th row by k requires n operations: $n-m-1$ to the left of the dotted line (not counting the computation $\frac{k}{k} = 1$), and $m+1$ to the right of the dotted line (including $\frac{1}{k}$). Now the matrix has the form shown in Figure 2.54.

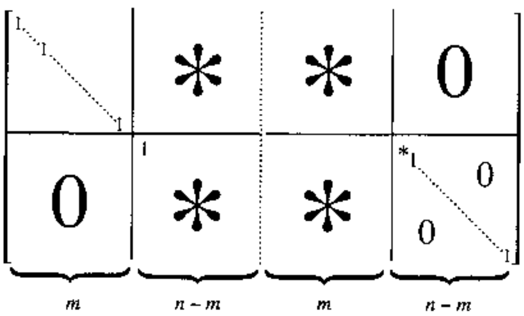

Figure 2.54: for Problem 2.4.45b.

Eliminating each of the other $n-1$ components of the $(m+1)$th column now requires n multiplicative operations ($n-m-1$ to the left of the dotted line, and $m+1$ to the right). Altogether, it requires $n + (n-1)n = n^2$ operations to process the mth column. To process all n columns requires $n \cdot n^2 = n^3$ operations.

c The inversion of a 12×12 matrix requires $12^3 = 4^3 3^3 = 64 \cdot 3^3$ operations, that is, 64 times as much as the inversion of a 3×3 matrix. If the inversion of a 3×3 matrix takes one second, then the inversion of a 12×12 matrix takes 64 seconds.

2.4.**46** Computing $A^{-1}\vec{b}$ requires $n^3 + n^2$ operations: First, we need n^3 operations to find A^{-1} (see Exercise 45b) and then n^2 operations to compute $A^{-1}\vec{b}$ (n multiplications for each component).

How many operations are required to perform Gauss-Jordan eliminations on $[A \vdots \vec{b}]$? Let us count these operations "column by column." If m columns of the coefficient matrix are left, then processing the next column requires nm operations (compare with Exercise 45b). To process all the columns requires

$n \cdot n + n(n-1) + \cdots + n \cdot 2 + n \cdot 1 = n(n + n - 1 + \cdots + 2 + 1) = n\frac{n(n+1)}{2} = \frac{n^3+n^2}{2}$ operations.

only half of what was required to compute $A^{-1}\vec{b}$.

We mention in passing that one can reduce the number of operations further (by about 50% for large matrices) by performing the steps of the row reduction in a different order.

2.4.**47** Let $f(x) = x^2$; the equation $f(x) = 0$ has the unique solution $x = 0$.

2.4.**48** Consider the linear system $A\vec{x} = \vec{0}$. The equation $A\vec{x} = \vec{0}$ implies that $BA\vec{x} = \vec{0}$, so $\vec{x} = \vec{0}$ since $BA = I_m$. Thus the system $A\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$. This implies $m \leq n$, by Theorem 1.3,3. Likewise the linear system $B\vec{y} = \vec{0}$ has the unique solution $\vec{y} = \vec{0}$, implying that $n \leq m$. It follows that $n = m$, as claimed.

2.4.**49** a $A = \begin{bmatrix} 0.293 & 0 & 0 \\ 0.014 & 0.207 & 0.017 \\ 0.044 & 0.01 & 0.216 \end{bmatrix}$, $I_3 - A = \begin{bmatrix} 0.707 & 0 & 0 \\ -0.014 & 0.793 & -0.017 \\ -0.044 & -0.01 & 0.784 \end{bmatrix}$

$(I_3 - A)^{-1} = \begin{bmatrix} 1.41 & 0 & 0 \\ 0.0267 & 1.26 & 0.0274 \\ 0.0797 & 0.0161 & 1.28 \end{bmatrix}$

b We have $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so that $\vec{x} = (I_3 - A)^{-1}\vec{e}_1 =$ first column of $(I_3 - A)^{-1} \approx \begin{bmatrix} 1.41 \\ 0.0267 \\ 0.0797 \end{bmatrix}$.

c As illustrated in part (b), the ith column of $(I_3 - A)^{-1}$ gives the output vector required to satisfy a consumer demand of 1 unit on industry i, in the absence of any other consumer demands. In particular, the ith diagonal entry of $(I_3 - A)^{-1}$ gives the output of industry i required to satisfy this demand. Since industry i has to satisfy the consumer demand of 1 as well as the interindustry demand, its total output will be at least 1.

d Suppose the consumer demand increases from $\vec{b}$ to $\vec{b} + \vec{e}_2$ (that is, the demand on manufacturing increases by one unit). Then the output must change from $(I_3 - A)^{-1}\vec{b}$ to

$(I_3 - A)^{-1}(\vec{v} + \vec{e}_2) = (I_3 - A)^{-1}\vec{b} + (I_3 - A)^{-1}\vec{e}_2 = (I_3 - A)^{-1}\vec{b}+$ (second column of $(I_3 - A)^{-1}$).

The components of the second column of $(I_3 - A)^{-1}$ tells us by how much each industry has to increase its output.

e The ijth entry of $(I_n - A)^{-1}$ gives the required increase of the output x_i of industry i to satisfy an increase of the consumer demand b_j on industry j by one unit. In the language of multivariable calculus, this quantity is $\frac{\partial x_i}{\partial b_j}$.

2.4.**50** Recall that $1 + k + k^2 + \cdots = \frac{1}{1-k}$.

The top left entry of $I_3 - A$ is $I - k$, and the top left entry of $(I_3 - A)^{-1}$ will therefore be $\frac{1}{1-k}$, as claimed:

$$\begin{bmatrix} 1-k & 0 & 0 & \vdots & 1 & 0 & 0 \\ * & * & * & \vdots & 0 & 1 & 0 \\ * & * & * & \vdots & 0 & 0 & 1 \end{bmatrix} \overset{\div(1-k)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{1}{1-k} & 0 & 0 \\ * & * & * & \vdots & 0 & 1 & 0 \\ * & * & * & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$\rightarrow \ldots$ (first row will remain unchanged).

In terms of economics, we can explain this fact as follows: The top left entry of $(I_3 - A)^{-1}$ is the output of industry 1 (Agriculture) required to satisfy a consumer demand of 1 unit on industry 1. Producting this one unit to satisfy the consumer demand will generate an extra demand of $k = 0.293$ units on industry 1. Producting these k units in turn will generate an extra demand of $k \cdot k = k^2$ units, and so forth. We are faced with an infinite series of (ever smaller) demands, $1 + k + k^2 + \cdots$.

2.4.**51** a Since rank$(A) < n$, the matrix $E =$rref(A) will not have a leading one in the last row, and all entries in the last row of E will be zero.

Let $\vec{c} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. Then the last equation of the system $E\vec{x} = \vec{c}$ reads $0 = 1$, so this system is inconsistent.

Now, we can "rebuild" $\vec{b}$ from $\vec{c}$ by performing the reverse row-operations in the opposite order on $\begin{bmatrix} E \vdots \vec{c} \end{bmatrix}$ until we reach $\begin{bmatrix} A \vdots \vec{b} \end{bmatrix}$. Since $E\vec{x} = \vec{c}$ is inconsistent, $A\vec{x} = \vec{b}$ is inconsistent as well.

b Since rank$(A) \leq \min(n, m)$, and $m < n$, rank$(A) < n$ also. Thus, by part a, there is a $\vec{b}$ such that $A\vec{x} = \vec{b}$ is inconsistent.

2.4.**52** Let $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\begin{bmatrix} A \vdots \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & \vdots & 0 \\ 0 & 2 & 4 & \vdots & 0 \\ 0 & 3 & 6 & \vdots & 1 \\ 1 & 4 & 8 & \vdots & 0 \end{bmatrix}$. We find that rref $\begin{bmatrix} A \vdots \vec{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$, which has an inconsistency in the third row.

2.4.**53** a $A - \lambda I_2 = \begin{bmatrix} 3-\lambda & 1 \\ 3 & 5-\lambda \end{bmatrix}$.

This fails to be invertible when $(3 - \lambda)(5 - \lambda) - 3 = 0$,

or $15 - 8\lambda + \lambda^2 - 3 = 0$,

or $12 - 8\lambda + \lambda^2 = 0$

or $(6 - \lambda)(2 - \lambda) = 0$. So $\lambda = 6$ or $\lambda = 2$.

b For $\lambda = 6$, $A - \lambda I_2 = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$.

The system $(A - 6I_2)\vec{x} = \vec{0}$ has the solutions $\begin{bmatrix} t \\ 3t \end{bmatrix}$, where t is an arbitrary constant. Pick $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, for example.

For $\lambda = 2$, $A - \lambda I_2 = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$.

The system $(A - 2I_2)\vec{x} = \vec{0}$ has the solutions $\begin{bmatrix} t \\ -t \end{bmatrix}$, where t is an arbitrary constant. Pick $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, for example.

c For $\lambda = 6$, $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda = 2$, $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2.4.**54** $A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 10 \\ -3 & 12-\lambda \end{bmatrix}$. This fails to be invertible when $\det(A - \lambda I_2) = 0$,

so $0 = (1-\lambda)(12-\lambda) + 30 = 12 - 13\lambda + \lambda^2 + 30 = \lambda^2 - 13\lambda + 42 = (\lambda - 6)(\lambda - 7)$. In order for this to be zero, λ must be 6 or 7.

If $\lambda = 6$, then $A - 6I_2 = \begin{bmatrix} -5 & 10 \\ -3 & 6 \end{bmatrix}$. We solve the system $(A - 6I_2)\,\vec{x} = \vec{0}$ and find that the solutions are of the form $\vec{x} = \begin{bmatrix} 2t \\ t \end{bmatrix}$. For example, when $t = 1$, we find $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

If $\lambda = 7$, then $A - 7I_2 = \begin{bmatrix} -6 & 10 \\ -3 & 5 \end{bmatrix}$. Here we solve the system $(A - 7I_2)\,\vec{x} = \vec{0}$, this time finding that our solutions are of the form $\vec{x} = \begin{bmatrix} 5t \\ 3t \end{bmatrix}$. For example, for $t = 1$, we find $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

2.4.**55** The determinant of A is equal to 4 and $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$. The linear transformation defined by A is a scaling by a factor 2 and A^{-1} defines a scaling by 1/2. The determinant of A is the area of the square spanned by $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. The angle θ from $\vec{v}$ to $\vec{w}$ is $\pi/2$. (See Figure 2.55.)

2.4.**56** The determinant of A is 1. The matrix is invertible with inverse $A^{-1} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$. The linear transformation defined by A is a rotation by angle α in the counterclockwise direction. The inverse represents a rotation by the angle α in the clockwise direction. The determinant of A is the area of the unit square spanned by $\vec{v} = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{bmatrix}$. The angle θ from $\vec{v}$ to $\vec{w}$ is $\pi/2$. (See Figure 2.56.)

2.4.**57** The determinant of A is -1. Matrix A is invertible, with $A^{-1} = A$. Matrices A and A^{-1} define reflection about the line spanned by the $\vec{v} = \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix}$. The absolute value of the determinant of A is the area of the

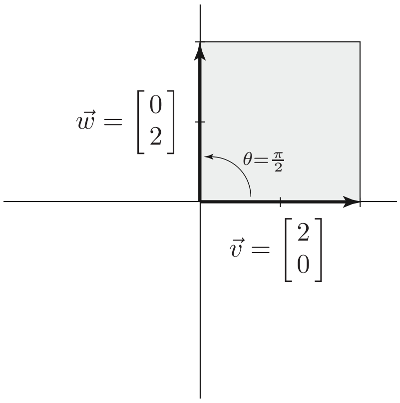

Figure 2.55: for Problem 2.4.55.

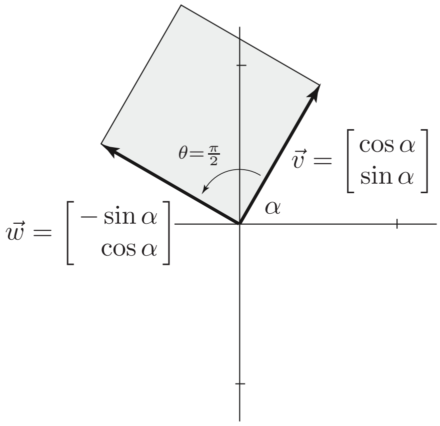

Figure 2.56: for Problem 2.4.56.

unit square spanned by $\vec{v} = \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{bmatrix}$. The angle θ from $\vec{v}$ to $\vec{w}$ is $-\pi/2$. (See Figure 2.57.)

2.4.**58** The determinant of A is 9. The matrix is invertible with inverse $A^{-1} = \begin{bmatrix} -3^{-1} & 0 \\ 0 & -3^{-1} \end{bmatrix}$. The linear transformation defined by A is a reflection about the origin combined with a scaling by a factor 3. The inverse defines a reflection about the origin combined with a scaling by a factor 1/3. The determinant is the area of the square spanned by $\vec{v} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$. The angle θ from $\vec{v}$ to $\vec{w}$ is $\pi/2$. (See Figure 2.58.)

2.4.**59** The determinant of A is 1. The matrix A is invertible with inverse $A^{-1} = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$. The matrix A represents the rotation through the angle $\alpha = \arccos(0.6)$. Its inverse represents a rotation by the same angle in the clockwise direction. The determinant of A is the area of the unit square spanned by $\vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$ and

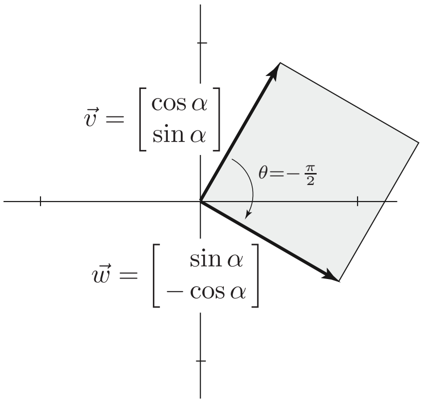

Figure 2.57: for Problem 2.4.57.

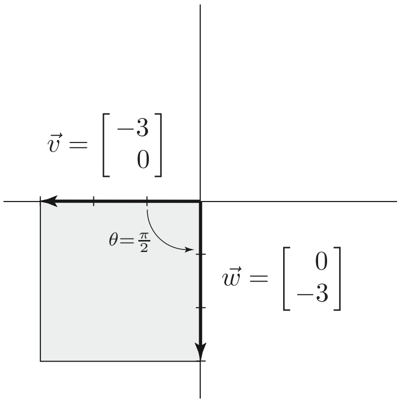

Figure 2.58: for Problem 2.4.58.

$\vec{w} = \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$. The angle θ from $\vec{v}$ to $\vec{w}$ is $\pi/2$. (See Figure 2.59.)

2.4.**60** The determinant of A is -1. The matrix A is invertible with inverse $A^{-1} = A$. Matrices A and A^{-1} define the reflection about the line spanned by $\vec{v} = \begin{bmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{bmatrix}$, where $\alpha = \arccos(-0.8)$. The absolute value of the determinant of A is the area of the unit square spanned by $\vec{v} = \begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$. The angle θ from v to w is $-\pi/2$. (See Figure 2.60.)

2.4.**61** The determinant of A is 2 and $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The matrix A represents a rotation through the angle $-\pi/4$ combined with scaling by $\sqrt{2}$. describes a rotation through $\pi/4$ and scaling by $1/\sqrt{2}$. The determinant of A is the area of the square spanned by $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with side length $\sqrt{2}$. The angle θ from $\vec{v}$ to $\vec{w}$ is $\pi/2$. (See Figure 2.61.)

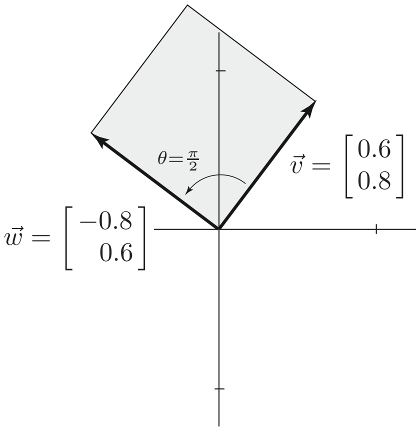

Figure 2.59: for Problem 2.4.59.

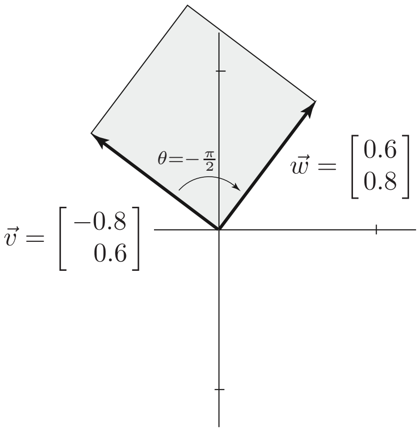

Figure 2.60: for Problem 2.4.60.

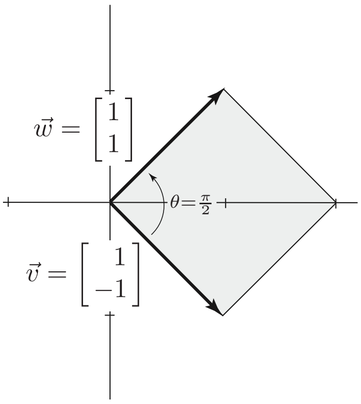

Figure 2.61: for Problem 2.4.61.

2.4.**62** The determinant of A is 1 and $A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Both A and A^{-1} represent horizontal shears. The determinant of A is the area of the parallelogram spanned by $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The angle from $\vec{v}$ to $\vec{w}$ is $3\pi/4$. (See Figure 2.62.)

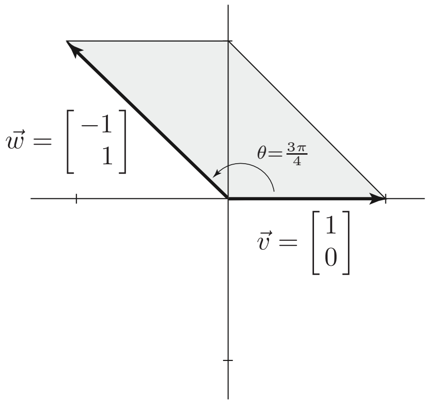

Figure 2.62: for Problem 2.4.62.

2.4.**63** The determinant of A is -25 and $A^{-1} = (1/25)\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} = \frac{1}{25}A$. The matrix A represents a reflection about a line combined with a scaling by 5 whilc A^{-1} represents a reflection about the same line combined with a scaling by 1/5. The absolute value of the determinant of A is the area of the square spanned by $\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ with side length 5. The angle from $\vec{v}$ to $\vec{w}$ is $-\pi/2$. (See Figure 2.63.)

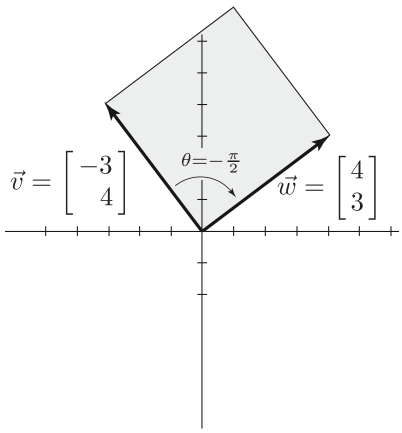

Figure 2.63: for Problem 2.4.63.

2.4.**64** The determinant of A is 25. The matrix A is a rotation dilation matrix with scaling factor 5 and rotation by an angle arccos(0.6) in the clockwise direction. The inverse $A^{-1} = (1/25)\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ is a rotation dilation too with a scaling factor 1/5 and rotation angle arccos(0.6). The determinant of A is the area of the parallelogram spanned by $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ with side length 5. The angle from $\vec{v}$ to $\vec{w}$ is $\pi/2$. (See Figure 2.64.)

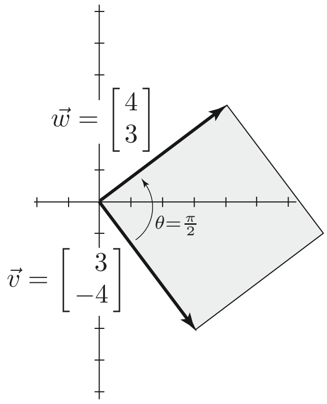

Figure 2.64: for Problem 2.4.64.

2.4.**65** The determinant of A is 1 and $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Both A and A^{-1} represent vertical shears. The determinant of A is the area of the parallelogram spanned by $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The angle from $\vec{v}$ to $\vec{w}$ is $\pi/4$. (See Figure 2.65.)

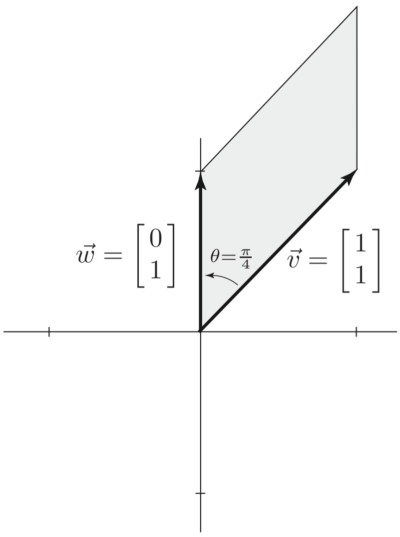

Figure 2.65: for Problem 2.4.65.

2.4.**66** We can write $AB(AB)^{-1} = A(B(AB)^{-1}) = I_n$ and $(AB)^{-1}AB = ((AB)^{-1}A)B = I_n$.

By Theorem 2.4.8, A and B are invertible.

2.4.**67** Not necessarily true; $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$ if $AB \neq BA$.

2.4.**68** Not necessarily true; $(A-B)(A+B) = A^2 + AB - BA - B^2 \neq A^2 - B^2$ if $AB \neq BA$.

2.4.**69** Not necessarily true; consider the case $A = I_n$ and $B = -I_n$.

2.4.**70** True; apply Theorem 2.4.7 to $B = A$.

2.4.**71** True; $ABB^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$.

2.4.**72** Not necessarily true; the equation $ABA^{-1} = B$ is equivalent to $AB = BA$ (multiply by A from the right), which is not true in general.

2.4.**73** True; $(ABA^{-1})^3 = ABA^{-1}ABA^{-1}ABA^{-1} = AB^3A^{-1}$.

2.4.**74** True; $(I_n + A)(I_n + A^{-1}) = I_n^2 + A + A^{-1} + AA^{-1} = 2I_n + A + A^{-1}$.

2.4.**75** True; $(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A$ (use Theorem 2.4.7).

2.4.**76** We want A such that $A\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, so that $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}$.

2.4.**77** We want A such that $A\vec{v}_i = \vec{w}_i$, for $i = 1, 2, \ldots, m$, or $A[\vec{v}_1\ \vec{v}_2\ \ldots\ \vec{v}_m] = [\vec{w}_1\ \vec{w}_2\ \ldots\ \vec{w}_m]$, or $AS = B$.

Multiplying by S^{-1} from the right we find the unique solution $A = BS^{-1}$.

2.4.**78** Use the result of Exercise 2.4.77, with $S = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$;

$A = BS^{-1} = \begin{bmatrix} 33 & -13 \\ 21 & -8 \\ 9 & -3 \end{bmatrix}$

2.4.**79** Use the result of Exercise 2.4.77, with $S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 3 \\ 2 & 6 \end{bmatrix}$;

$A = BS^{-1} = \frac{1}{5}\begin{bmatrix} 9 & 3 \\ -2 & 16 \end{bmatrix}$.

2.4.**80** $P_0 \xrightarrow{T} P_1,\ P_1 \xrightarrow{T} P_3,\ P_2 \xrightarrow{T} P_2,\ P_3 \xrightarrow{T} P_0$

$P_0 \xrightarrow{L} P_0,\ P_1 \xrightarrow{L} P_2,\ P_2 \xrightarrow{L} P_1,\ P_3 \xrightarrow{L} P_3$

a. T^{-1} is the rotation about the axis through 0 and P_2 that transforms P_3 into P_1.

b. $L^{-1} = L$

c. $T^2 = T^{-1}$ (See part (a).)

d. $P_0 \xrightarrow{T \circ L} P_1$ $\quad P_0 \xrightarrow{L \circ T} P_2$ $\quad$ The transformations $T \circ L$ and $L \circ T$ are not the same.

$P_1 \longrightarrow P_2 \quad P_1 \longrightarrow P_3$
$P_2 \longrightarrow P_3 \quad P_2 \longrightarrow P_1$
$P_3 \longrightarrow P_0 \quad P_3 \longrightarrow P_0$

e.

$P_0 \xrightarrow{L \circ T \circ L} P_2$
$P_1 \longrightarrow P_1$
$P_2 \longrightarrow P_3$
$P_3 \longrightarrow P_0$

This is the rotation about the axis through 0 and P_1 that sends P_0 to P_2.

2.4.**81** Let A be the matrix of T and C the matrix of L. We want that $AP_0 = P_1$, $AP_1 = P_3$, and $AP_2 = P_2$. We can use the result of Exercise 77, with $S = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$.

Then $A = BS^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$.

Using an analogous approach, we find that $C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.4.**82** a $EA = \begin{bmatrix} a & b & c \\ d-3a & e-3b & f-3c \\ g & h & k \end{bmatrix}$

The matrix EA is obtained from A by an elementary row operation: subtract three times the first row from the second.

b $EA = \begin{bmatrix} a & b & c \\ \frac{1}{4}d & \frac{1}{4}e & \frac{1}{4}f \\ g & h & k \end{bmatrix}$

The matrix EA is obtained from A by dividing the second row of A by 4 (an elementary row operation).

c If we set $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & k \\ d & e & f \end{bmatrix}$, as desired.

d An elementary $n \times n$ matrix E has the same form as I_n except that either

- $e_{ij} = k(\neq 0)$ for some $i \neq j$ [as in part (a)], or
- $e_{ii} = k(\neq 0, 1)$ for some i [as in part (b)], or
- $e_{ij} = e_{ji} = 1$, $e_{ii} = e_{jj} = 0$ for some $i \neq j$ [as in part (c)].

2.4.**83** Let E be an elementary $n \times n$ matrix (obtained from I_n by a certain elementary row operation), and let F be the elementary matrix obtained from I_n by the reversed row operation. Our work in Exercise 2.4.82 [parts (a) through (c)] shows that $EF = I_n$, so that E is indeed invertible, and $E^{-1} = F$ is an elementary matrix as well.

2.4.**84** a The matrix rref(A) is obtained from A by performing a sequence of p elementary row operations. By Exercise 2.4.82 [parts (a) through (c)] each of these operations can be represented by the left multiplication with an elementary matrix, so that rref$(A) = E_1E_2\dots E_pA$.

b $A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ swap rows 1 and 2, represented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \div 2$, represented by $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} -3(II)$, represented by $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$\downarrow$

rref$(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Therefore, rref$(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = E_1E_2E_3A$.

2.4.**85** a Let $S = E_1E_2\dots E_p$ in Exercise 2.4.84a.

By Exercise 2.4.83, the elementary matrices E_i are invertible: now use Theorem 2.4.7 repeatedly to see that S is invertible.

b $A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \div 2$, represented by $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} -4(I)$, represented by $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

rref$(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Therefore, rref$(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = E_1E_2A = SA$, where

$S = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -2 & 1 \end{bmatrix}$.

(There are other correct answers.)

2.4.**86** a By Exercise 2.4.84a, $I_n = \text{rref}(A) = E_1E_2\dots E_pA$, for some elementary matrices $E_1, \dots, E_p$. By Exercise 2.4.83, the E_i are invertible and their inverses are elementary as well. Therefore,

$A = (E_1E_2 \dots E_p)^{-1} = E_p^{-1} \dots E_2^{-1}E_1^{-1}$ expresses A as a product of elementary matrices.

b We can use out work in Exercise 2.4.84 b:

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = \left(\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

2.4.**87** $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represents a horizontal shear, $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ represents a vertical shear,

$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ represents a "scaling in $\vec{e}_1$ direction" (leaving the $\vec{e}_2$ component unchanged),

$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ represents a "scaling in $\vec{e}_2$ direction" (leaving the $\vec{e}_1$ component unchanged), and

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents the reflection about the line spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2.4.**88** Performing a sequence of p elementary row operations on a matrix A amounts to multiplying A with $E_1E_2 \dots E_p$ from the left, where the E_i are elementary matrices. If $I_n = E_1E_2 \dots E_pA$, then $E_1E_2 \dots E_p = A^{-1}$, so that

a. $E_1E_2 \dots E_pAB = B$, and

b. $E_1E_2 \dots E_pI_n = A^{-1}$.

2.4.**89** Let A and B be two lower triangular $n \times n$ matrices. We need to show that the ijth entry of AB is 0 whenever $i < j$.

This entry is the dot product of the ith row of A and the jth column of B,

$$[a_{i1}\ a_{i2} \dots a_{ii}\ 0 \dots 0] \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{jj} \\ \vdots \\ b_{nj} \end{bmatrix}, \text{ which is indeed 0 if } i < j.$$

2.4.**90** a $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix} \begin{matrix} \\ -2I \\ -2I \end{matrix}$, represented by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{matrix} \\ \\ +II \end{matrix}$ represented by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, so that

$$\underset{\substack{\uparrow \\ U}}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}} = \underset{\substack{\uparrow \\ E_3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ E_2}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ E_1}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ A}}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}}$$

b $A = (E_3E_2E_1)^{-1}U = E_1^{-1}E_2^{-1}E_3^{-1}U = \underset{\substack{\uparrow \\ M_1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ M_2}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ M_3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ U}}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}}$

c Let $L = M_1M_2M_3$ in part (b); we compute $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$.

Then $\underset{\substack{\uparrow \\ A}}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}} = \underset{\substack{\uparrow \\ L}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ U}}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}}$

d We can use the matrix L we found in part (c), but U needs to be modified. Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(Take the diagonal entries of the matrix U in part (c)).

Then $\underset{\substack{\uparrow \\ A}}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 2 & 2 & 4 \end{bmatrix}} = \underset{\substack{\uparrow \\ L}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}} \underset{\substack{\uparrow \\ D}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}} \underset{\substack{\uparrow \\ U}}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}}$.

2.4.**91** a Write the system $L\vec{y} = \vec{b}$ in components:

$\begin{bmatrix} y_1 & & & & & & & = -3 \\ -3y_1 & + & y_2 & & & & & = 14 \\ y_1 & + & 2y_2 & + & y_3 & & & = 9 \\ -y_1 & + & 8y_2 & - & 5y_3 & + & y_4 & = 33 \end{bmatrix}$, so that $y_1 = -3$, $y_2 = 14 + 3y_1 = 5$,

$y_3 = 9 - y_1 - 2y_2 = 2$, and $y_4 = 33 + y_1 - 8y_2 + 5y_3 = 0$:

$$\vec{y} = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 0 \end{bmatrix}.$$

b Proceeding as in part (a) we find that $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$.

2.4.**92** We try to find matrices $L = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ and $U = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$ such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} ad & ae \\ bd & be + cf \end{bmatrix}.$$

Note that the equations $ad = 0,\ ae = 1$, and $bd = 1$ cannot be solved simultaneously: If $ad = 0$ then a or d is 0 so that ae or bd is zero.

Therefore, the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does not have an LU factorization.

2.4.**93** a Write $L = \begin{bmatrix} L^{(m)} & 0 \\ L_3 & L_4 \end{bmatrix}$ and $U = \begin{bmatrix} U^{(m)} & U_2 \\ 0 & U_4 \end{bmatrix}$.

Then $A = LU = \begin{bmatrix} L^{(m)}U^{(m)} & L^{(m)}U_2 \\ L_3U^{(m)} & L_3U_2 + L_4U_4 \end{bmatrix}$, so that $A^{(m)} = L^{(m)}U^{(m)}$, as claimed.

b By Exercise 2.4.66, the matrices L and U are both invertible. By Exercise 2.4.35, the diagonal entries of L and U are all nonzero. For any m, the matrices $L^{(m)}$ and $U^{(m)}$ are triangular, with nonzero diagonal entries, so that they are invertible. By Theorem 2.4.7, the matrix $A^{(m)} = L^{(m)}U^{(m)}$ is invertible as well.

c Using the hint, we write $A = \begin{bmatrix} A^{(n-1)} & \vec{v} \\ \vec{w} & k \end{bmatrix} = \begin{bmatrix} L' & 0 \\ \vec{x} & t \end{bmatrix} \begin{bmatrix} U' & \vec{y} \\ 0 & s \end{bmatrix}$.

We are looking for a column vector $\vec{y}$, a row vector $\vec{x}$, and scalars t and s satisfying these equations. The following equations need to be satisfied: $\vec{v} = L'\vec{y},\ \vec{w} = \vec{x}U'$, and $k = \vec{x}\vec{y} + ts$.

We find that $\vec{y} = (L')^{-1}\vec{v},\ \vec{x} = \vec{w}(U')^{-1}$, and $ts = k - \vec{w}(U')^{-1}(L')^{-1}\vec{v}$.

We can choose, for example, $s = 1$ and $t = k - \vec{w}(U')^{-1}(L')^{-1}\vec{v}$, proving that A does indeed have an LU factorization.

Alternatively, one can show that if all principal submatrices are invertible then no row swaps are required in the Gauss-Jordan Algorithm. In this case, we can find an LU-factorization as outlined in Exercise 2.4.90.

2.4.**94** a If $A = LU$ is an LU factorization, then the diagonal entries of L and U are nonzero (compare with Exercise 2.4.93). Let D_1 and D_2 be the diagonal matrices whose diagonal entries are the same as those of L and U, respectively.

Then $A = (LD_1^{-1})(D_1D_2)(D_2^{-1}U)$ is the desired factorization

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{new } L & D & \text{new } U \end{matrix}$$

(verify that LD_1^{-1} and $D_2^{-1}U$ are of the required form).

b If $A = L_1D_1U_1 = L_2D_2U_2$ and A is invertible, then $L_1, D_1, U_1, L_2, D_2, U_2$ are all invertible, so that we can multiply the above equation by $D_2^{-1}L_2^{-1}$ from the left and by U_1^{-1} from the right:

$D_2^{-1}L_2^{-1}L_1D_1 = U_2U_1^{-1}$.

Since products and inverses of upper triangular matrices are upper triangular (and likewise for lower triangular matrices), the matrix $D_2^{-1}L_2^{-1}L_1D_1 = U_2U_1^{-1}$ is both upper and lower triangular, that is, it is diagonal. Since the diagonal entries of U_2 and U_1 are all 1, so are the diagonal entries of $U_2U_1^{-1}$, that is $U_2U_1^{-1} = I_n$, and thus $U_2 = U_1$.

Now $L_1D_1 = L_2D_2$, so that $L_2^{-1}L_1 = D_2D_1^{-1}$ is diagonal. As above, we have in fact $L_2^{-1}L_1 = I_n$ and therefore $L_2 = L_1$.

2.4.**95** Suppose A_{11} is a $p \times p$ matrix and A_{22} is a $q \times q$ matrix. For B to be the inverse of A we must have $AB = I_{p+q}$. Let us partition B the same way as A:

$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where B_{11} is $p \times p$ and B_{22} is $q \times q$.

Then $AB = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$ means that

$A_{11}B_{11} = I_p,\ A_{22}B_{22} = I_q,\ A_{11}B_{12} = 0,\ A_{22}B_{21} = 0.$

This implies that A_{11} and A_{22} are invertible, and $B_{11} = A_{11}^{-1},\ B_{22} = A_{22}^{-1}$.

This in turn implies that $B_{12} = 0$ and $B_{21} = 0$.

We summarize: A is invertible if (and only if) both A_{11} and A_{22} are invertible; in this case

$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$.

2.4.**96** This exercise is very similar to Example 7 in the text. We outline the solution:

$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$ means that

$A_{11}B_{11} = I_q,\ A_{11}B_{12} = 0,\ A_{21}B_{11} + A_{22}B_{21} = 0,\ A_{21}B_{12} + A_{22}B_{22} = I_q.$

This implies that A_{11} is invertible, and $B_{11} = A_{11}^{-1}$. Multiplying the second equation with A_{11}^{-1}, we conclude that $B_{12} = 0$. Then the last equation simplifies to $A_{22}B_{22} = I_q$, so that $B_{22} = A_{22}^{-1}$.

Finally, $B_{21} = -A_{22}^{-1}A_{21}B_{11} = -A_{22}^{-1}A_{21}A_{11}^{-1}$.

We summarize: A is invertible if (and only if) both A_{11} and A_{22} are invertible. In this case,

$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}$.

2.4.**97** Suppose A_{11} is a $p \times p$ matrix. Since A_{11} is invertible, $\text{rref}(A) = \begin{bmatrix} I_p & A_{12} & * \\ 0 & 0 & \text{rref}(A_{23}) \end{bmatrix}$, so that

$\text{rank}(A) = p + \text{rank}(A_{23}) = \text{rank}(A_{11}) + \text{rank}(A_{23})$.

2.4.**98** Try to find a matrix $B = \begin{bmatrix} X & \vec{x} \\ \vec{y} & t \end{bmatrix}$ (where X is $n \times n$) such that

$AB = \begin{bmatrix} I_n & \vec{v} \\ \vec{w} & 1 \end{bmatrix} \begin{bmatrix} X & \vec{x} \\ \vec{y} & t \end{bmatrix} = \begin{bmatrix} X + \vec{v}\vec{y} & \vec{x} + t\vec{v} \\ \vec{w}X + \vec{y} & \vec{w}\vec{x} + t \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$.

We want $X + \vec{v}\vec{y} = I_n, \vec{x} + t\vec{v} = \vec{0},\ \vec{w}X + \vec{y} = \vec{0}$, and $\vec{w}\vec{x} + t = 1$.

Substituting $\vec{x} = -t\vec{v}$ into the last equation we find $-t\vec{w}\vec{v} + t = 1$ or $t(1 - \vec{w}\vec{v}) = 1$.

This equation can be solved only if $\vec{w}\vec{v} \neq 1$, in which case $t = \frac{1}{1-\vec{w}\vec{v}}$. Now substituting $X = I_n - \vec{v}\vec{y}$ into the third equation, we find $\vec{w} - \vec{w}\vec{v}\vec{y} + \vec{y} = \vec{0}$ or $\vec{y} = -\frac{1}{1-\vec{w}\vec{v}}\vec{w} = -t\vec{w}$.

We summarize: A is invertible if (and only if) $\vec{w}\vec{v} \neq 1$. In this case, $A^{-1} = \begin{bmatrix} I_n + t\vec{v}\vec{w} & -t\vec{v} \\ -t\vec{w} & t \end{bmatrix}$, where $t = \frac{1}{1-\vec{w}\vec{v}}$.

The same result can be found (perhaps more easily) by working with $\text{rref}[A \vdots I_{n+1}]$, rather than partitioned matrices.

2.4.**99** Multiplying both sides with A^{-1} we find that $A = I_n$: The identity matrix is the only invertible matrix with this property.

2.4.**100** Suppose the entries of A are all a, where $a \neq 0$. Then the entries of A^2 are all na^2. The equation $na^2 = a$ is satisfied if $a = \frac{1}{n}$. Thus the solution is $A = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ & & \ddots & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$.

2.4.**101** The ijth entry of AB is

$$\sum_{k=1}^{n} a_{ik}b_{kj}.$$

Then

$$\sum_{k=1}^{n} a_{ik}b_{kj} \leq \sum_{k=1}^{n} sb_{kj} = s\left(\sum_{k=1}^{n} b_{kj}\right) \leq sr.$$

↑ since $a_{ik} \leq s$ ↑ this is $\leq r$, as it is the j^{th} column sum of B.

2.4.**102** a We proceed by induction on m. Since the column sums of A are $\leq r$, the entries of $A^1 = A$ are also $\leq r^1 = r$, so that the claim holds for $m = 1$. Suppose the claim holds for some fixed m. Now write $A^{m+1} = A^m A$; since the entries of A^m are $\leq r^m$ and the column sums of A are $\leq r$, we can conclude that the entries of A^{m+1} are $\leq r^m r = r^{m+1}$, by Exercise 101.

b For a fixed i and j, let b_m be the ijth entry of A^m. In part (a) we have seen that $0 \leq b_m \leq r^m$.

Note that $\lim_{m\to\infty} r^m = 0$ (since $r < 1$), so that $\lim_{m\to\infty} b_m = 0$ as well (this follows from what some calculus texts call the "squeeze theorem").

c For a fixed i and j, let c_m be the ijth entry of the matrix $I_n + A + A^2 + \cdots + A^m$. By part (a),

$c_m \leq 1 + r + r^2 + \cdots + r^m < \frac{1}{1-r}$.

Since the c_m form an increasing bounded sequence, $\lim_{m\to\infty} c_m$ exists (this is a fundamental fact of calculus).

d $(I_n - A)(I_n + A + A^2 + \cdots + A^m) = I_n + A + A^2 + \cdots A^m - A - A^2 - \cdots - A^m - A^{m+1}$
$= I_n - A^{m+1}$

Now let m go to infinity; use parts (b) and (c). $(I_n - A)(I_n + A + A^2 + \cdots + A^m + \cdots) = I_n$, so that

$(I_n - A)^{-1} = I_n + A + A^2 + \cdots + A^m + \cdots$.

2.4.**103** a The components of the jth column of the technology matrix A give the demands industry J_j makes on the other industries, per unit output of J_j. The fact that the jth column sum is less than 1 means that industry J_j *adds value* to the products it produces.

b A productive economy can satisfy any consumer demand $\vec{b}$, since the equation

$(I_n - A)\vec{x} = \vec{b}$ can be solved for the output vector $\vec{x} : \vec{x} = (I_n - A)^{-1}\vec{b}$ (compare with Exercise 2.4.49).

c The output $\vec{x}$ required to satisfy a consumer demand $\vec{b}$ is

$\vec{x} = (I_n - A)^{-1}\vec{b} = (I_n + A + A^2 + \cdots + A^m + \cdots)\,\vec{b} = \vec{b} + A\vec{b} + A^2\vec{b} + \cdots + A^m\vec{b} + \cdots$.

To interpret the terms in this series, keep in mind that whatever output $\vec{v}$ the industries produce generates an interindustry demand of $A\vec{v}$.

The industries first need to satisfy the consumer demand, $\vec{b}$. Producing the output $\vec{b}$ will generate an interindustry demand, $A\vec{b}$. Producing $A\vec{b}$ in turn generates an extra interindustry demand, $A(A\vec{b}) = A^2\vec{b}$, and so forth.

For a simple example, see Exercise 2.4.50; also read the discussion of "chains of interindustry demands" in the footnote to Exercise 2.4.49.

2.4.**104** a We write our three equations below:

$$\begin{aligned} I &= \tfrac{1}{3}R + \tfrac{1}{3}G + \tfrac{1}{3}B \\ L &= R - G \\ S &= -\tfrac{1}{2}R - \tfrac{1}{2}G + B \end{aligned}, \text{ so that the matrix is } P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

b $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ is transformed into $\begin{bmatrix} R \\ G \\ 0 \end{bmatrix}$, with matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

c This matrix is $PA = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$ (we apply first A, then P.)

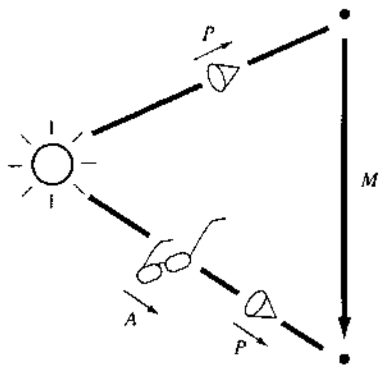

Figure 2.66: for Problem 2.4.104d.

d See Figure 2.66. A "diagram chase" shows that $M = PAP^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{9} \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{3} \end{bmatrix}$.

2.4.**105** a $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Matrix A^{-1} transforms a wife's clan into her husband's clan, and B^{-1} transforms a child's clan into the mother's clan.

b B^2 transforms a women's clan into the clan of a child of her daughter.

c AB transforms a woman's clan into the clan of her daughter-in-law (her son's wife), while BA transforms a man's clan into the clan of his children. The two transformations are different. (See Figure 2.67.)

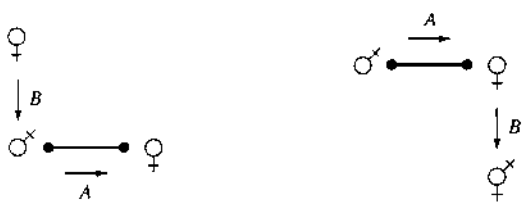

Figure 2.67: for Problem 2.4.105c.

d The matrices for the four given diagrams (in the same order) are $BB^{-1} = I_3$,

$$BAB^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ B(BA)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ BA(BA)^{-1} = I_3.$$

e Yes; since $BAB^{-1} = A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, in the second case in part (d) the cousin belongs to Bueya's husband's clan.

2.4.**106** a We need 8 multiplications: 2 to compute each of the four entries of the product.

b We need n multiplications to compute each of the mp entries of the product, mnp multiplications altogether.

2.4.**107** $g(f(x)) = x$, for all x, so that $g \circ f$ is the identity, but $f(g(x)) = \begin{cases} x & \text{if } x \text{ is even} \\ x+1 & \text{if } x \text{ is odd} \end{cases}$.

2.4.**108** a The formula $\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1-Rk & L+R-kLR \\ -k & 1-kL \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}$ is given, which implies that

$y = (1-Rk)x + (L+R-kLR)m.$

In order for y to be independent of x it is required that $1 - Rk = 0$, or $k = \frac{1}{R} = 40$ (diopters).

$\frac{1}{k}$ then equals R, which is the distance between the plane of the lens and the plane on which parallel incoming rays focus at a point; thus the term "focal length" for $\frac{1}{k}$.

b Now we want y to be independent of the slope m (it must depend on x alone). In view of the formula above, this is the case if $L + R - kLR = 0$, or $k = \dfrac{L+R}{LR} = \dfrac{1}{R} + \dfrac{1}{L} = 40 + \dfrac{10}{3} \approx 43.3$ (diopters).

c Here the transformation is

$$\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1-k_1D & D \\ k_1k_2D - k_1 - k_2 & 1-k_2D \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

We want the slope n of the outgoing rays to depend on the slope m of the incoming rays alone, and not on x; this forces $k_1k_2D - k_1 - k_2 = 0$, or, $D = \frac{k_1+k_2}{k_1k_2} = \frac{1}{k_1} + \frac{1}{k_2}$, the sum of the focal lengths of the two lenses. See Figure 2.68.

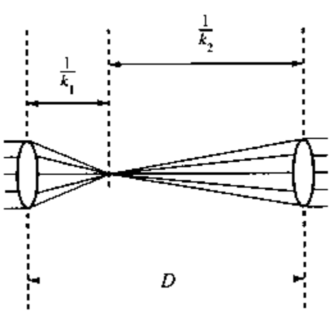

Figure 2.68: for Problem 2.4.108c.

True or False

Ch 2.TF.**1** T, by Theorem 2.2.4.

Ch 2.TF.**2** T, by Theorem 2.4.6.

Ch 2.TF.**3** T; The matrix is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Ch 2.TF.**4** F; The columns of a rotation matrix are unit vectors; see Theorem 2.2.3.

Ch 2.TF.**5** T, by Theorem 2.4.3.

Ch 2.TF.**6** T; Let $A = B$ in Theorem 2.4.7.

Ch 2.TF.**7** F, by Theorem 2.3.3.

Ch 2.TF.**8** T, by Theorem 2.4.8.

Ch 2.TF.**9** F; Matrix AB will be 3×5, by Definition 2.3.1b.

Ch 2.TF.**10** F; Note that $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A linear transformation transforms $\vec{0}$ into $\vec{0}$.

Ch 2.TF.**11** T; The equation $\det(A) = k^2 - 6k + 10 = 0$ has no real solution.

Ch 2.TF.**12** T; The matrix fails to be invertible for $k = 5$ and $k = -1$, since the determinant $\det A = k^2 - 4k - 5 = (k-5)(k+1)$ is 0 for these values of k.

Ch 2.TF.**13** F; Note that $\det(A) = (k-2)^2 + 9$ is always positive, so that A is invertible for all values of k.

Ch 2.TF.**14** F We can show by induction on m that the matrix A^m is of the form $A^m = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$ for all m, so that A^m fails to be positive. Indeed, $A^{m+1} = A^m A = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$.

Ch 2.TF.**15** F; Consider $A = I_2$ (or any other invertible 2×2 matrix).

Ch 2.TF.**16** T; Note that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1}$ is the unique solution.

Ch 2.TF.**17** F, by Theorem 2.4.9. Note that the determinant is 0.

Ch 2.TF.**18** T, by Theorem 2.4.3.

Ch 2.TF.**19** T; The shear matrix $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ works.

Ch 2.TF.**20** T; Simplify to see that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4y \\ -12x \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -12 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Ch 2.TF.**21** F; If matrix A has two identical rows, then so does AB, for any matrix B. Thus AB cannot be I_n, so that A fails to be invertible.

Ch 2.TF.**22** T, by Theorem 2.4.8. Note that $A^{-1} = A$ in this case.

Ch 2.TF.**23** F; For any 2×2 matrix A, the two columns of $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ will be identical.

Ch 2.TF.**24** T; One solution is $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Ch 2.TF.**25** F; A reflection matrix is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Here, $a^2 + b^2 = 1 + 1 = 2$.

Ch 2.TF.**26** T Let B be the matrix whose columns are all $\vec{x}_{equ}$, the equilibrium vector of A.

Ch 2.TF.**27** T; The product is $\det(A)I_2$.

Ch 2.TF.**28** T; Writing an upper triangular matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and solving the equation $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ we find that $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, where b is any nonzero constant.

Ch 2.TF.**29** T; Note that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a rotation through $\pi/2$. Thus $n = 4$ (or any multiple of 4) works.

Ch 2.TF.**30** F; If a matrix A is invertible, then so is A^{-1}. But $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ fails to be invertible.

Ch 2.TF.**31** T For example, $A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Ch 2.TF.**32** F Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, with $AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Ch 2.TF.**33** F; Consider matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, for example.

Ch 2.TF.**34** T; Apply Theorem 2.4.8 to the equation $(A^2)^{-1}AA = I_n$, with $B = (A^2)^{-1}A$.

Ch 2.TF.**35** F; Consider the matrix A that represents a rotation through the angle $2\pi/17$.

Ch 2.TF.**36** F; Consider the reflection matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Ch 2.TF.**37** T; We have $(5A)^{-1} = \frac{1}{5}A^{-1}$.

Ch 2.TF.**38** T; The equation $A\vec{e}_i = B\vec{e}_i$ means that the ith columns of A and B are identical. This observation applies to all the columns.

Ch 2.TF.**39** T; Note that $A^2B = AAB = ABA = BAA = BA^2$.

Ch 2.TF.**40** T; Multiply both sides of the equation $A^2 = A$ with A^{-1}.

Ch 2.TF.**41** T See Exercise 2.3.75

Ch 2.TF.**42** F Consider $A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$, with $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$.

Ch 2.TF.**43** F; Consider $A = I_2$ and $B = -I_2$.

Ch 2.TF.**44** T; Since $A\vec{x}$ is on the line onto which we project, the vector $A\vec{x}$ remains unchanged when we project again: $A(A\vec{x}) = A\vec{x}$, or $A^2\vec{x} = A\vec{x}$, for all $\vec{x}$. Thus $A^2 = A$.

Ch 2.TF.**45** T; If you reflect twice in a row (about the same line), you will get the original vector back: $A(A\vec{x}) = \vec{x}$, or, $A^2\vec{x} = \vec{x} = I_2\vec{x}$. Thus $A^2 = I_2$ and $A^{-1} = A$.

Ch 2.TF.**46** F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example.

Ch 2.TF.**47** T; Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example.

Ch 2.TF.**48** F; By Theorem 1.3.3, there is a nonzero vector $\vec{x}$ such that $B\vec{x} = \vec{0}$, so that $AB\vec{x} = \vec{0}$ as well. But $I_3\vec{x} = \vec{x} \neq \vec{0}$, so that $AB \neq I_3$.

Ch 2.TF.**49** T; We can rewrite the given equation as $A^2 + 3A = -4I_3$ and $-\frac{1}{4}(A + 3I_3)A = I_3$. By Theorem 2.4.8, the matrix A is invertible, with $A^{-1} = -\frac{1}{4}(A + 3I_3)$.

Ch 2.TF.**50** T; Note that $(I_n + A)(I_n - A) = I_n^2 - A^2 = I_n$, so that $(I_n + A)^{-1} = I_n - A$.

Ch 2.TF.**51** F; A and C can be two matrices which fail to commute, and B could be I_n, which commutes with anything.

Ch 2.TF.**52** F; Consider $T(\vec{x}) = 2\vec{x}$, $\vec{v} = \vec{e}_1$, and $\vec{w} = \vec{e}_2$.

Ch 2.TF.**53** F; Since there are only eight entries that are not 1, there will be at least two rows that contain only ones. Having two identical rows, the matrix fails to be invertible.

Ch 2.TF.**54** F; Let $A = B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, for example.

Ch 2.TF.**55** F; We will show that $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S$ fails to be diagonal, for an arbitrary invertible matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now, $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$. Since c and d cannot both be zero (as S must be invertible), at least one of the off-diagonal entries ($-c^2$ and d^2) is nonzero, proving the claim.

Ch 2.TF.**56** T; Consider an $\vec{x}$ such that $A^2\vec{x} = \vec{b}$, and let $\vec{x}_0 = A\vec{x}$. Then $A\vec{x}_0 = A(A\vec{x}) = A^2\vec{x} = \vec{b}$, as required.

Ch 2.TF.**57** T; Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $A^{-1} = -A$, or $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. This holds if $ad - bc = 1$ and $d = -a$. These equations have many solutions: for example, $a = d = 0, b = 1, c = -1$. More generally, we can choose an arbitrary a and an arbitrary nonzero b. Then, $d = -a$ and $c = -\frac{1+a^2}{b}$.

Ch 2.TF.**58** F; Consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We make an attempt to solve the equation $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Now the equation $b(a+d) = 0$ implies that $b = 0$ or $d = -a$.

If $b = 0$, then the equation $d^2 + bc = -1$ cannot be solved.

If $d = -a$, then the two diagonal entries of A^2, a^2+bc and d^2+bc, will be equal, so that the equations $a^2+bc = 1$ and $d^2+bc = -1$ cannot be solved simultaneously.

In summary, the equation $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ cannot be solved.

Ch 2.TF.**59** T; Recall from Definition 2.2.1 that a projection matrix has the form $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector. Thus, $a^2+b^2+c^2+d^2 = u_1^4 + (u_1u_2)^2 + (u_1u_2)^2 + u_2^4 = u_1^4 + 2(u_1u_2)^2 + u_2^4 = (u_1^2+u_2^2)^2 = 1^2 = 1$.

Ch 2.TF.**60** T; We observe that the systems $AB\vec{x} = 0$ and $B\vec{x} = 0$ have the same solutions (multiply with A^{-1} and A, respectively, to obtain one system from the other). Then, by True or False Exercise 45 in Chapter 1, rref(AB) =rref(B).

Ch 2.TF.**61** T For example, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, with $A^m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for even m and $A^m = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for odd m.

Ch 2.TF.**62** T We need to show that the system $A\vec{x} = \vec{x}$ or $(A - I_n)\vec{x} = \vec{0}$ has a nonzero solution $\vec{x}$. This amounts to showing that rank$(A - I_n) < n$, or, equivalently, that rref$(A - I_n)$ has a row of zeros. By definition of a transition matrix, the sum of all the row vectors of A is $\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$, so that the sum of all the row vectors of $A - I_n$ is the zero row vector. If we add rows I through $(n-1)$ to the last row of $A - I_n$, we generate a row of zeros as required.