## Instructor's Solutions Manual

# Linear Algebra WITH ApPLICATIONS Tenth Edition 

Steven J. Leon
University of Massachusetts Dartmouth

# Lisette G. de Pillis 

Harvey Mudd College


The author and publisher of this book have used their best efforts in preparing this book. These efforts include the development, research, and testing of the theories and programs to determine their effectiveness. The author and publisher make no warranty of any kind, expressed or implied, with regard to these programs or the documentation contained in this book. The author and publisher shall not be liable in any event for incidental or consequential damages in connection with, or arising out of, the furnishing, performance, or use of these programs.

Reproduced by Pearson from electronic files supplied by the author.

Copyright © 2020, 2015, 2010 by Pearson Education, Inc. 221 River Street, Hoboken, NJ 07030. All rights reserved.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publisher. Printed in the United States of America.

## Contents

Preface ..... V
1 Matrices and Systems of Equations ..... 1
1 Systems of Linear Equations ..... 1
2 Row Echelon Form ..... 2
3 Matrix Arithmetic ..... 3
4 Matrix Algebra ..... 7
5 Elementary Matrices ..... 12
6 Partitioned Matrices ..... 18
MATLAB Exercises ..... 22
Chapter Test A ..... 24
Chapter Test B ..... 27
2 Determinants ..... 30
1 The Determinant of a Matrix ..... 30
2 Properties of Determinants ..... 33
3 Additional Topics and Applications ..... 36
MATLAB Exercises ..... 38
Chapter Test A ..... 38
Chapter Test B ..... 39
3 Vector Spaces ..... 42
1 Definition and Examples ..... 42
2 Subspaces ..... 46
3 Linear Independence ..... 52
4 Basis and Dimension ..... 55
5 Change of Basis ..... 57
6 Row Space and Column Space ..... 57
MATLAB Exercises ..... 65
Chapter Test A ..... 66
Chapter Test B ..... 68
4 Linear Transformations ..... 72
1 Definition and Examples ..... 72
2 Matrix Representations of Linear Transformations ..... 75
3 Similarity ..... 78
MATLAB Exercise ..... 79
Chapter Test A ..... 80
Chapter Test B ..... 81
5 Orthogonality ..... 84
$1 \quad$ The Scalar product in $\mathbb{R}^{n}$ ..... 84
2 Orthogonal Subspaces ..... 86
3 Least Squares Problems ..... 89
4 Inner Product Spaces ..... 93
5 Orthonormal Sets ..... 99
6 The Gram-Schmidt Process ..... 107
$7 \quad$ Orthogonal Polynomials ..... 109
MATLAB Exercises ..... 112
Chapter Test A ..... 113
Chapter Test B ..... 115
6 Eigenvalues ..... 119
1 Eigenvalues and Eigenvectors ..... 119
2 Systems of Linear Differential Equations ..... 124
3 Diagonalization ..... 125
4 Hermitian Matrices ..... 133
5 Singular Value Decomposition ..... 141
6 Quadratic Forms ..... 143
$7 \quad$ Positive Definite Matrices ..... 146
8 Nonnegative Matrices ..... 149
MATLAB Exercises ..... 152
Chapter Test A ..... 154
Chapter Test B ..... 156
7 Numerical Linear Algebra ..... 160
1 Floating-Point Numbers ..... 160
2 Gaussian Elimination ..... 161
3 Pivoting Strategies ..... 162
4 Matrix Norms and Condition Numbers ..... 163
5 Orthogonal Transformations ..... 174
6 The Eigenvalue Problem ..... 175
7 Least Squares Problems ..... 179
8 Iterative Methods ..... 182
MATLAB Exercises ..... 183
Chapter Test A ..... 185
Chapter Test B ..... 186

## Preface

This solutions manual is designed to accompany the tenth edition of Linear Algebra with Applications by Steven J. Leon and Lisette de Pillis. The manual contains the complete solutions to all of the nonroutine exercises and Chapter test questions in the first seven chapters the book. Each of those chapters also includes a set of MATLAB computer exercises. Most of the MATLAB computations are straightforward. and consequently the computational results are not included in this manual. However, the MATLAB Exercises also include questions related to the computations. The purpose of the questions is to emphasize the significance of the computations. This manual does provide the answers to most of these questions.

## Chapter 1

# Matrices and 

 Systems of Equations
## 1 SYSTEMS OF LINEAR EQUATIONS

2. (d)
$\left(\begin{array}{rrrrr}1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 & 1 \\ 0 & 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2\end{array}\right)$
3. (a) $3 x_{1}+2 x_{2}=8$
$x_{1}+5 x_{2}=7$
(b) $5 x_{1}-2 x_{2}+x_{3}=3$
$2 x_{1}+3 x_{2}-4 x_{3}=0$
(c) $2 x_{1}+x_{2}+4 x_{3}=-1$ $4 x_{1}-2 x_{2}+3 x_{3}=4$ $5 x_{1}+2 x_{2}+6 x_{2}=-1$
(d) $4 x_{1}-3 x_{2}+x_{3}+2 x_{4}=4$ $3 x_{1}+x_{2}-5 x_{3}+6 x_{4}=5$
$x_{1}+x_{2}+2 x_{3}+4 x_{4}=8$
$5 x_{1}+x_{2}+3 x_{3}-2 x_{4}=7$
4. Given the system

$$
\begin{aligned}
& -m_{1} x_{1}+x_{2}=b_{1} \\
& -m_{2} x_{1}+x_{2}=b_{2}
\end{aligned}
$$

one can eliminate the variable $x_{2}$ by subtracting the first row from the second. One then obtains the equivalent system

$$
\begin{aligned}
-m_{1} x_{1}+x_{2} & =b_{1} \\
\left(m_{1}-m_{2}\right) x_{1} & =b_{2}-b_{1}
\end{aligned}
$$

(a) If $m_{1} \neq m_{2}$, then one can solve the second equation for $x_{1}$

$$
x_{1}=\frac{b_{2}-b_{1}}{m_{1}-m_{2}}
$$

One can then plug this value of $x_{1}$ into the first equation and solve for $x_{2}$. Thus, if $m_{1} \neq m_{2}$, there will be a unique ordered pair $\left(x_{1}, x_{2}\right)$ that satisfies the two equations.
(b) If $m_{1}=m_{2}$, then the $x_{1}$ term drops out in the second equation

$$
0=b_{2}-b_{1}
$$

This is possible if and only if $b_{1}=b_{2}$.
(c) If $m_{1} \neq m_{2}$, then the two equations represent lines in the plane with different slopes. Two nonparallel lines intersect in a point. That point will be the unique solution to the system. If $m_{1}=m_{2}$ and $b_{1}=b_{2}$, then both equations represent the same line and consequently every point on that line will satisfy both equations. If $m_{1}=m_{2}$ and $b_{1} \neq b_{2}$, then the equations represent parallel lines. Since parallel lines do not intersect, there is no point on both lines and hence no solution to the system.
10. The system must be consistent since $(0,0)$ is a solution.
11. A linear equation in 3 unknowns represents a plane in three space. The solution set to a $3 \times 3$ linear system would be the set of all points that lie on all three planes. If the planes are parallel or one plane is parallel to the line of intersection of the other two, then the solution set will be empty. The three equations could represent the same plane or the three planes could all intersect in a line. In either case the solution set will contain infinitely many points. If the three planes intersect in a point, then the solution set will contain only that point.

## 2 ROW ECHELON FORM

2. (b) The system is consistent with a unique solution $(4,-1)$.
3. (b) $x_{1}$ and $x_{3}$ are lead variables and $x_{2}$ is a free variable.
(d) $x_{1}$ and $x_{3}$ are lead variables and $x_{2}$ and $x_{4}$ are free variables.
(f) $x_{2}$ and $x_{3}$ are lead variables and $x_{1}$ is a free variable.
4. (l) The solution is $(0,-1.5,-3.5)$.
5. (c) The solution set consists of all ordered triples of the form $(0,-\alpha, \alpha)$.
6. A homogeneous linear equation in 3 unknowns corresponds to a plane that passes through the origin in 3 -space. Two such equations would correspond to two planes through the origin. If one equation is a multiple of the other, then both represent the same plane through the origin and every point on that plane will be a solution to the system. If one equation is not a multiple of the other, then we have two distinct planes that intersect in a line through the
origin. Every point on the line of intersection will be a solution to the linear system. So in either case the system must have infinitely many solutions.

In the case of a nonhomogeneous $2 \times 3$ linear system, the equations correspond to planes that do not both pass through the origin. If one equation is a multiple of the other, then both represent the same plane and there are infinitely many solutions. If the equations represent planes that are parallel, then they do not intersect and hence the system will not have any solutions. If the equations represent distinct planes that are not parallel, then they must intersect in a line and hence there will be infinitely many solutions. So the only possibilities for a nonhomogeneous $2 \times 3$ linear system are 0 or infinitely many solutions.
9. (a) Since the system is homogeneous it must be consistent.
13. A homogeneous system is always consistent since it has the trivial solution $(0, \ldots, 0)$. If the reduced row echelon form of the coefficient matrix involves free variables, then there will be infinitely many solutions. If there are no free variables, then the trivial solution will be the only solution.
14. A nonhomogeneous system could be inconsistent in which case there would be no solutions. If the system is consistent and underdetermined, then there will be free variables and this would imply that we will have infinitely many solutions.
16. At each intersection, the number of vehicles entering must equal the number of vehicles leaving in order for the traffic to flow. This condition leads to the following system of equations

$$
\begin{aligned}
& x_{1}+a_{1}=x_{2}+b_{1} \\
& x_{2}+a_{2}=x_{3}+b_{2} \\
& x_{3}+a_{3}=x_{4}+b_{3} \\
& x_{4}+a_{4}=x_{1}+b_{4}
\end{aligned}
$$

If we add all four equations, we get

$$
x_{1}+x_{2}+x_{3}+x_{4}+a_{1}+a_{2}+a_{3}+a_{4}=x_{1}+x_{2}+x_{3}+x_{4}+b_{1}+b_{2}+b_{3}+b_{4}
$$

and hence

$$
a_{1}+a_{2}+a_{3}+a_{4}=b_{1}+b_{2}+b_{3}+b_{4}
$$

17. If $\left(c_{1}, c_{2}\right)$ is a solution, then

$$
\begin{aligned}
& a_{11} c_{1}+a_{12} c_{2}=0 \\
& a_{21} c_{1}+a_{22} c_{2}=0
\end{aligned}
$$

Multiplying both equations through by $\alpha$, one obtains

$$
\begin{aligned}
& a_{11}\left(\alpha c_{1}\right)+a_{12}\left(\alpha c_{2}\right)=\alpha \cdot 0=0 \\
& a_{21}\left(\alpha c_{1}\right)+a_{22}\left(\alpha c_{2}\right)=\alpha \cdot 0=0
\end{aligned}
$$

Thus $\left(\alpha c_{1}, \alpha c_{2}\right)$ is also a solution.
18. (a) If $x_{4}=0$, then $x_{1}, x_{2}$, and $x_{3}$ will all be 0 . Thus if no glucose is produced, then there is no reaction. $(0,0,0,0)$ is the trivial solution in the sense that if there are no molecules of carbon dioxide and water, then there will be no reaction.
(b) If we choose another value of $x_{4}$, say $x_{4}=2$, then we end up with solution $x_{1}=12$, $x_{2}=12, x_{3}=12, x_{4}=2$. Note the ratios are still 6:6:6:1.

## 3 MATRIX ARITHMETIC

1. (e) $\left(\begin{array}{rrr}8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6\end{array}\right)$
(g) $\left(\begin{array}{rrr}5 & -10 & 15 \\ 5 & -1 & 4 \\ 8 & -9 & 6\end{array}\right)$
2. (d) $\left(\begin{array}{rrr}36 & 10 & 56 \\ 10 & 3 & 16\end{array}\right)$
3. (a) $5 A=\left(\begin{array}{rr}15 & 20 \\ 5 & 5 \\ 10 & 35\end{array}\right)$

$$
2 A+3 A=\left(\begin{array}{rr}
6 & 8 \\
2 & 2 \\
4 & 14
\end{array}\right)+\left(\begin{array}{rr}
9 & 12 \\
3 & 3 \\
6 & 21
\end{array}\right)=\left(\begin{array}{rr}
15 & 20 \\
5 & 5 \\
10 & 35
\end{array}\right)
$$

(b) $6 A=\left(\begin{array}{rr}18 & 24 \\ 6 & 6 \\ 12 & 42\end{array}\right)$

$$
3(2 A)=3\left(\begin{array}{rr}
6 & 8 \\
2 & 2 \\
4 & 14
\end{array}\right)=\left(\begin{array}{rr}
18 & 24 \\
6 & 6 \\
12 & 42
\end{array}\right)
$$

(c) $A^{T}=\left(\begin{array}{lll}3 & 1 & 2 \\ 4 & 1 & 7\end{array}\right)$

$$
\left(A^{T}\right)^{T}=\left(\begin{array}{lll}
3 & 1 & 2 \\
4 & 1 & 7
\end{array}\right)^{T}=\left(\begin{array}{ll}
3 & 4 \\
1 & 1 \\
2 & 7
\end{array}\right)=A
$$

6. (a) $A+B=\left(\begin{array}{lll}5 & 4 & 6 \\ 0 & 5 & 1\end{array}\right)=B+A$
(b) $3(A+B)=3\left(\begin{array}{lll}5 & 4 & 6 \\ 0 & 5 & 1\end{array}\right)=\left(\begin{array}{rrr}15 & 12 & 18 \\ 0 & 15 & 3\end{array}\right)$
$3 A+3 B=\left(\begin{array}{rrr}12 & 3 & 18 \\ 6 & 9 & 15\end{array}\right)+\left(\begin{array}{rrr}3 & 9 & 0 \\ -6 & 6 & -12\end{array}\right)$

$$
=\left(\begin{array}{rrr}
15 & 12 & 18 \\
0 & 15 & 3
\end{array}\right)
$$

(c) $(A+B)^{T}=\left(\begin{array}{lll}5 & 4 & 6 \\ 0 & 5 & 1\end{array}\right)^{T}=\left(\begin{array}{ll}5 & 0 \\ 4 & 5 \\ 6 & 1\end{array}\right)$

$$
A^{T}+B^{T}=\left(\begin{array}{ll}
4 & 2 \\
1 & 3 \\
6 & 5
\end{array}\right)+\left(\begin{array}{rr}
1 & -2 \\
3 & 2 \\
0 & -4
\end{array}\right)=\left(\begin{array}{ll}
5 & 0 \\
4 & 5 \\
6 & 1
\end{array}\right)
$$

7. (a) $3(A B)=3\left(\begin{array}{rr}5 & 14 \\ 15 & 42 \\ 0 & 16\end{array}\right)=\left(\begin{array}{rr}15 & 42 \\ 45 & 126 \\ 0 & 48\end{array}\right)$

$$
(3 A) B=\left(\begin{array}{rr}
6 & 3 \\
18 & 9 \\
-6 & 12
\end{array}\right)\left(\begin{array}{ll}
2 & 4 \\
1 & 6
\end{array}\right)=\left(\begin{array}{rr}
15 & 42 \\
45 & 126 \\
0 & 48
\end{array}\right)
$$

$$
A(3 B)=\left(\begin{array}{rr}
2 & 1 \\
6 & 3 \\
-2 & 4
\end{array}\right)\left(\begin{array}{ll}
6 & 12 \\
3 & 18
\end{array}\right)=\left(\begin{array}{rr}
15 & 42 \\
45 & 126 \\
0 & 48
\end{array}\right)
$$

(b) $(A B)^{T}=\left(\begin{array}{rr}5 & 14 \\ 15 & 42 \\ 0 & 16\end{array}\right)^{T}=\left(\begin{array}{rrr}5 & 15 & 0 \\ 14 & 42 & 16\end{array}\right)$ $B^{T} A^{T}=\left(\begin{array}{ll}2 & 1 \\ 4 & 6\end{array}\right)\left(\begin{array}{rrr}2 & 6 & -2 \\ 1 & 3 & 4\end{array}\right)=\left(\begin{array}{rrr}5 & 15 & 0 \\ 14 & 42 & 16\end{array}\right)$
8. (a) $(A+B)+C=\left(\begin{array}{ll}0 & 5 \\ 1 & 7\end{array}\right)+\left(\begin{array}{ll}3 & 1 \\ 2 & 1\end{array}\right)=\left(\begin{array}{ll}3 & 6 \\ 3 & 8\end{array}\right)$

$$
A+(B+C)=\left(\begin{array}{ll}
2 & 4 \\
1 & 3
\end{array}\right)+\left(\begin{array}{ll}
1 & 2 \\
2 & 5
\end{array}\right)=\left(\begin{array}{ll}
3 & 6 \\
3 & 8
\end{array}\right)
$$

(b) $(A B) C=\left(\begin{array}{ll}-4 & 18 \\ -2 & 13\end{array}\right)\left(\begin{array}{ll}3 & 1 \\ 2 & 1\end{array}\right)=\left(\begin{array}{ll}24 & 14 \\ 20 & 11\end{array}\right)$

$$
A(B C)=\left(\begin{array}{ll}
2 & 4 \\
1 & 3
\end{array}\right)\left(\begin{array}{rr}
-4 & -1 \\
8 & 4
\end{array}\right)=\left(\begin{array}{ll}
24 & 14 \\
20 & 11
\end{array}\right)
$$

(c) $A(B+C)=\left(\begin{array}{ll}2 & 4 \\ 1 & 3\end{array}\right)\left(\begin{array}{ll}1 & 2 \\ 2 & 5\end{array}\right)=\left(\begin{array}{rr}10 & 24 \\ 7 & 17\end{array}\right)$

$$
A B+A C=\left(\begin{array}{ll}
-4 & 18 \\
-2 & 13
\end{array}\right)+\left(\begin{array}{rr}
14 & 6 \\
9 & 4
\end{array}\right)=\left(\begin{array}{rr}
10 & 24 \\
7 & 17
\end{array}\right)
$$

(d) $(A+B) C=\left(\begin{array}{ll}0 & 5 \\ 1 & 7\end{array}\right)\left(\begin{array}{ll}3 & 1 \\ 2 & 1\end{array}\right)=\left(\begin{array}{ll}10 & 5 \\ 17 & 8\end{array}\right)$
$A C+B C=\left(\begin{array}{rr}14 & 6 \\ 9 & 4\end{array}\right)+\left(\begin{array}{rr}-4 & -1 \\ 8 & 4\end{array}\right)=\left(\begin{array}{ll}10 & 5 \\ 17 & 8\end{array}\right)$
9. (b) $\mathbf{x}=(2,1)^{T}$ is a solution since $\mathbf{b}=2 \mathbf{a}_{1}+\mathbf{a}_{2}$. There are no other solutions since the echelon form of $A$ is strictly triangular.
(c) The solution to $A \mathbf{x}=\mathbf{c}$ is $\mathbf{x}=\left(-\frac{5}{2},-\frac{1}{4}\right)^{T}$. Therefore $\mathbf{c}=-\frac{5}{2} \mathbf{a}_{1}-\frac{1}{4} \mathbf{a}_{2}$.
11. The given information implies that

$$
\mathbf{x}_{1}=\left(\begin{array}{l}
1 \\
1 \\
0
\end{array}\right) \quad \text { and } \quad \mathbf{x}_{2}=\left(\begin{array}{l}
0 \\
1 \\
1
\end{array}\right)
$$

are both solutions to the system. So the system is consistent and since there is more than one solution, the row echelon form of $A$ must involve a free variable. A consistent system with a free variable has infinitely many solutions.
12. The system is consistent since $\mathbf{x}=(1,1,1,1)^{T}$ is a solution. The system can have at most 3 lead variables since $A$ only has 3 rows. Therefore, there must be at least one free variable. A consistent system with a free variable has infinitely many solutions.
13. (a) It follows from the reduced row echelon form that the free variables are $x_{2}, x_{4}, x_{5}$. If we set $x_{2}=a, x_{4}=b, x_{5}=c$, then

$$
\begin{aligned}
& x_{1}=-2-2 a-3 b-c \\
& x_{3}=5-2 b-4 c
\end{aligned}
$$

and hence the solution consists of all vectors of the form

$$
\mathbf{x}=(-2-2 a-3 b-c, a, 5-2 b-4 c, b, c)^{T}
$$

(b) If we set the free variables equal to 0 , then $\mathbf{x}_{0}=(-2,0,5,0,0)^{T}$ is a solution to $A \mathbf{x}=\mathbf{b}$ and hence

$$
\mathbf{b}=A \mathbf{x}_{0}=-2 \mathbf{a}_{1}+5 \mathbf{a}_{3}=(8,-7,-1,7)^{T}
$$

14. If $w_{3}$ is the weight given to professional activities, then the weights for research and teaching should be $w_{1}=3 w_{3}$ and $w_{2}=2 w_{3}$. Note that

$$
1.5 w_{2}=3 w_{3}=w_{1}
$$

so the weight given to research is 1.5 times the weight given to teaching. Since the weights must all add up to 1 , we have

$$
1=w_{1}+w_{2}+w_{3}=3 w_{3}+2 w_{3}+w_{3}=6 w_{3}
$$

and hence it follows that $w_{3}=\frac{1}{6}, w_{2}=\frac{1}{3}, w_{1}=\frac{1}{2}$. If $C$ is the matrix in the example problem from the Analytic Hierarchy Process Application, then the rating vector $\mathbf{r}$ is computed by multiplying $C$ times the weight vector w.

$$
\mathbf{r}=C \mathbf{w}=\left(\begin{array}{ccc}
\frac{1}{2} & \frac{1}{5} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{4} & \frac{3}{10} & \frac{1}{4}
\end{array}\right)\left(\begin{array}{c}
\frac{1}{2} \\
\frac{1}{3} \\
\frac{1}{6}
\end{array}\right)=\left(\begin{array}{c}
\frac{43}{120} \\
\frac{45}{120} \\
\frac{32}{120}
\end{array}\right)
$$

15. $A^{T}$ is an $n \times m$ matrix. Since $A^{T}$ has $m$ columns and $A$ has $m$ rows, the multiplication $A^{T} A$ is possible. The multiplication $A A^{T}$ is possible since $A$ has $n$ columns and $A^{T}$ has $n$ rows.
16. If $A$ is skew-symmetric, then $A^{T}=-A$. Since the $(j, j)$ entry of $A^{T}$ is $a_{j j}$ and the $(j, j)$ entry of $-A$ is $-a_{j j}$, it follows that $a_{j j}=-a_{j j}$ for each $j$ and hence the diagonal entries of $A$ must all be 0 .
17. The search vector is $\mathbf{x}=(1,0,1,0,1,0)^{T}$. The search result is given by the vector

$$
\mathbf{y}=A^{T} \mathbf{x}=(1,2,2,1,1,2,1)^{T}
$$

The $i$ th entry of $\mathbf{y}$ is equal to the number of search words in the title of the $i$ th book.
18. If $\alpha=a_{21} / a_{11}$, then

$$
\left(\begin{array}{ll}
1 & 0 \\
\alpha & 1
\end{array}\right)\left(\begin{array}{cc}
a_{11} & a_{12} \\
0 & b
\end{array}\right)=\left(\begin{array}{cc}
a_{11} & a_{12} \\
\alpha a_{11} & \alpha a_{12}+b
\end{array}\right)=\left(\begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & \alpha a_{12}+b
\end{array}\right)
$$

The product will equal $A$ provided

$$
\alpha a_{12}+b=a_{22}
$$

Thus we must choose

$$
b=a_{22}-\alpha a_{12}=a_{22}-\frac{a_{21} a_{12}}{a_{11}}
$$

## MATRIX ALGEBRA

1. (a) $(A+B)^{2}=(A+B)(A+B)=(A+B) A+(A+B) B=A^{2}+B A+A B+B^{2}$

For real numbers, $a b+b a=2 a b$; however, with matrices $A B+B A$ is generally not equal to $2 A B$.
(b)

$$
\begin{aligned}
(A+B)(A-B) & =(A+B)(A-B) \\
& =(A+B) A-(A+B) B \\
& =A^{2}+B A-A B-B^{2}
\end{aligned}
$$

For real numbers, $a b-b a=0$; however, with matrices $A B-B A$ is generally not equal to $O$.
2. If we replace $a$ by $A$ and $b$ by the identity matrix, $I$, then both rules will work, since

$$
(A+I)^{2}=A^{2}+I A+A I+B^{2}=A^{2}+A I+A I+B^{2}=A^{2}+2 A I+B^{2}
$$

and

$$
(A+I)(A-I)=A^{2}+I A-A I-I^{2}=A^{2}+A-A-I^{2}=A^{2}-I^{2}
$$

3. There are many possible choices for $A$ and $B$. For example, one could choose

$$
A=\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \quad \text { and } \quad B=\left(\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right)
$$

More generally if

$$
A=\left(\begin{array}{cc}
a & b \\
c a & c b
\end{array}\right) \quad B=\left(\begin{array}{rr}
d b & e b \\
-d a & -e a
\end{array}\right)
$$

then $A B=O$ for any choice of the scalars $a, b, c, d, e$.
4. To construct nonzero matrices $A, B, C$ with the desired properties, first find nonzero matrices $C$ and $D$ such that $D C=O$ (see Exercise 3). Next, for any nonzero matrix $A$, set $B=A+D$. It follows that

$$
B C=(A+D) C=A C+D C=A C+O=A C
$$

5. A $2 \times 2$ symmetric matrix is one of the form

$$
A=\left(\begin{array}{ll}
a & b \\
b & c
\end{array}\right)
$$

Thus

$$
A^{2}=\left(\begin{array}{ll}
a^{2}+b^{2} & a b+b c \\
a b+b c & b^{2}+c^{2}
\end{array}\right)
$$

If $A^{2}=O$, then its diagonal entries must be 0 .

$$
a^{2}+b^{2}=0 \quad \text { and } \quad b^{2}+c^{2}=0
$$

Thus $a=b=c=0$ and hence $A=O$.
6. Let

$$
D=(A B) C=\left(\begin{array}{ll}
a_{11} b_{11}+a_{12} b_{21} & a_{11} b_{12}+a_{12} b_{22} \\
a_{21} b_{11}+a_{22} b_{21} & a_{21} b_{12}+a_{22} b_{22}
\end{array}\right)\left(\begin{array}{ll}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{array}\right)
$$

It follows that

$$
\begin{aligned}
d_{11} & =\left(a_{11} b_{11}+a_{12} b_{21}\right) c_{11}+\left(a_{11} b_{12}+a_{12} b_{22}\right) c_{21} \\
& =a_{11} b_{11} c_{11}+a_{12} b_{21} c_{11}+a_{11} b_{12} c_{21}+a_{12} b_{22} c_{21} \\
d_{12} & =\left(a_{11} b_{11}+a_{12} b_{21}\right) c_{12}+\left(a_{11} b_{12}+a_{12} b_{22}\right) c_{22} \\
& =a_{11} b_{11} c_{12}+a_{12} b_{21} c_{12}+a_{11} b_{12} c_{22}+a_{12} b_{22} c_{22} \\
d_{21} & =\left(a_{21} b_{11}+a_{22} b_{21}\right) c_{11}+\left(a_{21} b_{12}+a_{22} b_{22}\right) c_{21} \\
& =a_{21} b_{11} c_{11}+a_{22} b_{21} c_{11}+a_{21} b_{12} c_{21}+a_{22} b_{22} c_{21} \\
d_{22} & =\left(a_{21} b_{11}+a_{22} b_{21}\right) c_{12}+\left(a_{21} b_{12}+a_{22} b_{22}\right) c_{22} \\
& =a_{21} b_{11} c_{12}+a_{22} b_{21} c_{12}+a_{21} b_{12} c_{22}+a_{22} b_{22} c_{22}
\end{aligned}
$$

If we set

$$
E=A(B C)=\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)\left(\begin{array}{ll}
b_{11} c_{11}+b_{12} c_{21} & b_{11} c_{12}+b_{12} c_{22} \\
b_{21} c_{11}+b_{22} c_{21} & b_{21} c_{12}+b_{22} c_{22}
\end{array}\right)
$$

then it follows that

$$
\begin{aligned}
e_{11} & =a_{11}\left(b_{11} c_{11}+b_{12} c_{21}\right)+a_{12}\left(b_{21} c_{11}+b_{22} c_{21}\right) \\
& =a_{11} b_{11} c_{11}+a_{11} b_{12} c_{21}+a_{12} b_{21} c_{11}+a_{12} b_{22} c_{21} \\
e_{12} & =a_{11}\left(b_{11} c_{12}+b_{12} c_{22}\right)+a_{12}\left(b_{21} c_{12}+b_{22} c_{22}\right) \\
& =a_{11} b_{11} c_{12}+a_{11} b_{12} c_{22}+a_{12} b_{21} c_{12}+a_{12} b_{22} c_{22} \\
e_{21} & =a_{21}\left(b_{11} c_{11}+b_{12} c_{21}\right)+a_{22}\left(b_{21} c_{11}+b_{22} c_{21}\right) \\
& =a_{21} b_{11} c_{11}+a_{21} b_{12} c_{21}+a_{22} b_{21} c_{11}+a_{22} b_{22} c_{21} \\
e_{22} & =a_{21}\left(b_{11} c_{12}+b_{12} c_{22}\right)+a_{22}\left(b_{21} c_{12}+b_{22} c_{22}\right) \\
& =a_{21} b_{11} c_{12}+a_{21} b_{12} c_{22}+a_{22} b_{21} c_{12}+a_{22} b_{22} c_{22}
\end{aligned}
$$

Thus

$$
d_{11}=e_{11} \quad d_{12}=e_{12} \quad d_{21}=e_{21} \quad d_{22}=e_{22}
$$

and hence

$$
(A B) C=D=E=A(B C)
$$

9. 

$$
A^{2}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) \quad A^{3}=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

and $A^{4}=O$. If $n>4$, then

$$
A^{n}=A^{n-4} A^{4}=A^{n-4} O=O
$$

10. (a) The matrix $C$ is symmetric since

$$
C^{T}=(A+B)^{T}=A^{T}+B^{T}=A+B=C
$$

(b) The matrix $D$ is symmetric since

$$
D^{T}=(A A)^{T}=A^{T} A^{T}=A^{2}=D
$$

(c) The matrix $E=A B$ is not symmetric since

$$
E^{T}=(A B)^{T}=B^{T} A^{T}=B A
$$

and in general, $A B \neq B A$.
(d) The matrix $F$ is symmetric since

$$
F^{T}=(A B A)^{T}=A^{T} B^{T} A^{T}=A B A=F
$$

(e) The matrix $G$ is symmetric since

$$
G^{T}=(A B+B A)^{T}=(A B)^{T}+(B A)^{T}=B^{T} A^{T}+A^{T} B^{T}=B A+A B=G
$$

(f) The matrix $H$ is not symmetric since

$$
H^{T}=(A B-B A)^{T}=(A B)^{T}-(B A)^{T}=B^{T} A^{T}-A^{T} B^{T}=B A-A B=-H
$$

11. (a) The matrix $A$ is symmetric since

$$
A^{T}=\left(C+C^{T}\right)^{T}=C^{T}+\left(C^{T}\right)^{T}=C^{T}+C=A
$$

(b) The matrix $B$ is not symmetric since

$$
B^{T}=\left(C-C^{T}\right)^{T}=C^{T}-\left(C^{T}\right)^{T}=C^{T}-C=-B
$$

(c) The matrix $D$ is symmetric since

$$
A^{T}=\left(C^{T} C\right)^{T}=C^{T}\left(C^{T}\right)^{T}=C^{T} C=D
$$

(d) The matrix $E$ is symmetric since

$$
\begin{aligned}
E^{T} & =\left(C^{T} C-C C^{T}\right)^{T}=\left(C^{T} C\right)^{T}-\left(C C^{T}\right)^{T} \\
& =C^{T}\left(C^{T}\right)^{T}-\left(C^{T}\right)^{T} C^{T}=C^{T} C-C C^{T}=E
\end{aligned}
$$

(e) The matrix $F$ is symmetric since

$$
F^{T}=\left((I+C)\left(I+C^{T}\right)\right)^{T}=\left(I+C^{T}\right)^{T}(I+C)^{T}=(I+C)\left(I+C^{T}\right)=F
$$

(e) The matrix $G$ is not symmetric.

$$
\begin{aligned}
F & =(I+C)\left(I-C^{T}\right)=I+C-C^{T}-C C^{T} \\
F^{T} & =\left((I+C)\left(I-C^{T}\right)\right)^{T}=\left(I-C^{T}\right)^{T}(I+C)^{T} \\
& =(I-C)\left(I+C^{T}\right)=I-C+C^{T}-C C^{T}
\end{aligned}
$$

$F$ and $F^{T}$ are not the same. The two middle terms $C-C^{T}$ and $-C+C^{T}$ do not agree.
12. If $d=a_{11} a_{22}-a_{21} a_{12} \neq 0$, then

$$
\begin{aligned}
& \frac{1}{d}\left(\begin{array}{rr}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{array}\right)\left(\begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)=\left(\begin{array}{cc}
\frac{a_{11} a_{22}-a_{12} a_{21}}{d} & 0 \\
0 & \frac{a_{11} a_{22}-a_{12} a_{21}}{d}
\end{array}\right)=I \\
& \left(\begin{array}{cr}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)\left[\frac{1}{d}\left(\begin{array}{rr}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{array}\right)\right]=\left(\begin{array}{cc}
\frac{a_{11} a_{22}-a_{12} a_{21}}{d} & 0 \\
0 & \frac{a_{11} a_{22}-a_{12} a_{21}}{d}
\end{array}\right)=I
\end{aligned}
$$

Therefore

$$
\frac{1}{d}\left(\begin{array}{rr}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{array}\right)=A^{-1}
$$

13. (b) $\left(\begin{array}{rr}-3 & 5 \\ 2 & -3\end{array}\right)$
14. If $A$ were nonsingular and $A B=A$, then it would follow that $A^{-1} A B=A^{-1} A$ and hence that $B=I$. So if $B \neq I$, then $A$ must be singular.
15. Since

$$
A^{-1} A=A A^{-1}=I
$$

it follows from the definition that $A^{-1}$ is nonsingular and its inverse is $A$.
16. Since

$$
\begin{aligned}
A^{T}\left(A^{-1}\right)^{T} & =\left(A^{-1} A\right)^{T}=I \\
\left(A^{-1}\right)^{T} A^{T} & =\left(A A^{-1}\right)^{T}=I
\end{aligned}
$$

it follows that

$$
\left(A^{-1}\right)^{T}=\left(A^{T}\right)^{-1}
$$

17. If $A \mathbf{x}=A \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then $A$ must be singular, for if $A$ were nonsingular, then we could multiply by $A^{-1}$ and get

$$
\begin{aligned}
A^{-1} A \mathbf{x} & =A^{-1} A \mathbf{y} \\
\mathbf{x} & =\mathbf{y}
\end{aligned}
$$

18. For $m=1$,

$$
\left(A^{1}\right)^{-1}=A^{-1}=\left(A^{-1}\right)^{1}
$$

Assume the result holds in the case $m=k$, that is,

$$
\left(A^{k}\right)^{-1}=\left(A^{-1}\right)^{k}
$$

It follows that

$$
\left(A^{-1}\right)^{k+1} A^{k+1}=A^{-1}\left(A^{-1}\right)^{k} A^{k} A=A^{-1} A=I
$$

and

$$
A^{k+1}\left(A^{-1}\right)^{k+1}=A A^{k}\left(A^{-1}\right)^{k} A^{-1}=A A^{-1}=I
$$

Therefore

$$
\left(A^{-1}\right)^{k+1}=\left(A^{k+1}\right)^{-1}
$$

and the result follows by mathematical induction.
19. If $A^{2}=O$, then

$$
(I+A)(I-A)=I+A-A+A^{2}=I
$$

and

$$
(I-A)(I+A)=I-A+A+A^{2}=I
$$

Therefore $I-A$ is nonsingular and $(I-A)^{-1}=I+A$.
20. If $A^{k+1}=O$, then

$$
\begin{aligned}
\left(I+A+\cdots+A^{k}\right)(I-A) & =\left(I+A+\cdots+A^{k}\right)-\left(A+A^{2}+\cdots+A^{k+1}\right) \\
& =I-A^{k+1}=I
\end{aligned}
$$

and

$$
\begin{aligned}
(I-A)\left(I+A+\cdots+A^{k}\right) & =\left(I+A+\cdots+A^{k}\right)-\left(A+A^{2}+\cdots+A^{k+1}\right) \\
& =I-A^{k+1}=I
\end{aligned}
$$

