INSTRUCTOR'S SOLUTIONS MANUAL

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to accompany

A PROBLEM SOLVING APPROACH TO MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS

THIRTEENTH EDITION

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Pearson

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ISBN-13: 978-0-13-499560-1
ISBN-10: 0-13-499560-0

Contents

CHAPTER 1

AN INTRODUCTION TO PROBLEM SOLVING

Assessment 1-1A: Mathematics and Problem Solving

1. (a) List the numbers:

$$\begin{array}{rcrcccrcr} 1 & + & 2 & + & \cdots & + & 98 & + & 99 \\ 99 & + & 98 & + & \cdots & + & 2 & + & 1 \\ \hline 100 & + & 100 & + & \cdots & + & 100 & + & 100 \end{array}$$

There are 99 sums of 100. Thus the total can be found by computing $\frac{99 \cdot 100}{2} = \mathbf{4950}$. (Another way of looking at this problem is to realize there are $\frac{99}{2} = 49.5$ pairs of sums, each of 100; thus $49.5 \cdot 100 = 4950$.)

(b) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{1001-1}{2} + 1 = 501$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 1 & + & 3 & + & \cdots & + & 999 & + & 1001 \\ 1001 & + & 999 & + & \cdots & + & 3 & + & 1 \\ \hline 1002 & + & 1002 & + & \cdots & + & 1002 & + & 1002 \end{array}$$

There are 501 sums of 1002. Thus the total can be found by computing $\frac{501 \cdot 1002}{2} = \mathbf{251{,}001}$.

(c) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{300-3}{3} + 1 = 100$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 3 & + & 6 & + & \cdots & + & 297 & + & 300 \\ 300 & + & 297 & + & \cdots & + & 6 & + & 3 \\ \hline 303 & + & 303 & + & \cdots & + & 303 & + & 303 \end{array}$$

There are 100 sums of 303. Thus the total can be found by computing $\frac{100 \cdot 303}{2} = \mathbf{15{,}150}$.

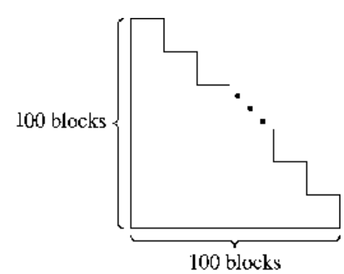
(d) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{400-4}{4} + 1 = 100$ terms.

List the numbers:

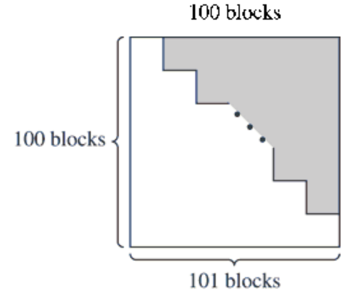
$$\begin{array}{rcrcccrcr} 4 & + & 8 & + & \cdots & + & 396 & + & 400 \\ 400 & + & 396 & + & \cdots & + & 8 & + & 4 \\ \hline 404 & + & 404 & + & \cdots & + & 404 & + & 404 \end{array}$$

There are 100 sums of 404. Thus the total can be found by computing $\frac{100 \cdot 404}{2} = \mathbf{20{,}200}$.

2. (a)



(b)



When the stack in (a) and a stack of the same size is placed differently next to the original stack in (a), a rectangle containing 100 (101) blocks is created. Since each block is represented twice, the desired sum is $100(101)/2 = \mathbf{5050}$.

While the above represents a specific example, the same thinking can be used for any natural number n to arrive at a formula $n(n+1)/2$.

3. There are $\frac{147-36}{1} + 1 = 112$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 36 & + & 37 & + & \cdots & + & 146 & + & 147 \\ 147 & + & 146 & + & \cdots & + & 37 & + & 36 \\ \hline 183 & + & 183 & + & \cdots & + & 183 & + & 183 \end{array}$$

There are 112 sums of 183. Thus the total can be found by computing $\frac{112 \cdot 183}{2} = \mathbf{10,248}$.

4. **(a)** Make a table as follows; there are 9 rows so there are **9 different ways**.

6-cookie packages	2-cookie packages	single-cookie packages
1	2	0
1	1	2
1	0	4
0	5	0
0	4	2
0	3	4
0	2	6
0	1	8
0	0	10

(b) Make a table as follows; there are 12 rows so there are **12 different ways**.

6-cookie packages	2-cookie packages	single-cookie packages
2	0	0
1	3	0
1	2	2
1	1	4
1	0	6
0	6	0
0	5	2
0	4	4
0	3	6
0	2	8
0	1	10
0	0	12

5. If each layer of boxes has 7 more than the previous layer we can add powers of 7:

$7^0 = 1$ (red box)

$7^1 = 7$ (blue boxes)

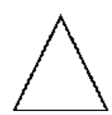
$7^2 = 49$ (black boxes)

$7^3 = 343$ (yellow boxes)

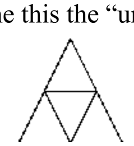
$7^4 = 2401$ (gold boxes)

$1 + 7 + 49 + 343 + 2401 =$ **2801 boxes** altogether.

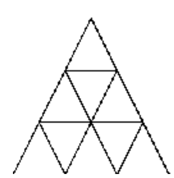
6. Using strategies from Poyla's problem solving list identify subgoals (solve simpler problems) and make diagrams to solve the original problem.



1 triangle; name this the "unit" triangle.

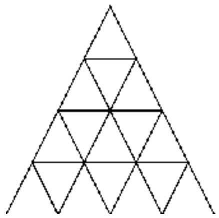


This triangle is made of 4 unit triangles. Counting the large triangle there are 5 triangles



Unit triangles	4 unit triangles	9 unit triangles
9	3	1

13 total triangles



Unit triangles	4 unit triangles	9 unit triangles	16 unit triangles
16	7	3	1

There are 27 triangles in the original figure.

7. Observe that $E = (1+1)+(3+1)+\cdots+(97+1) = O+49$. Thus, E is 49 more than O.

Alternative strategy:

$$O + E = 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 97 + 98$$
$$= \frac{98(99)}{2} = 49(99)$$
$$E = 2(1 + 2 + 3 + 4 + \cdots + 49) = 2\left(\frac{49(50)}{2}\right) = 49(50)$$
$$O = O + E - E = 49(99) - 49(50) = 49(49).$$

So O is 49 less than E.

8. Bubba is last; Cory must be between Alababa and Dandy; Dandy is faster than Cory. Listing from fastest to slowest, the finishing order is then **Dandy, Cory, Alababa, and Bubba**.

9. Make a table.

\$20 bills	\$10 bills	\$5 bills
2	1	0
2	0	2
1	3	0
1	2	2
1	1	4
1	0	6
0	5	0
0	4	2
0	3	4
0	2	6
0	1	8
0	0	10

There are twelve rows so there are **twelve different ways**.

10. The diagonal from the left, top corner to the right, bottom corner sums to $17 + 22 + 27 = 66$.

The first row sums to $17 + a + 7 = 24 + a$. So $a = 66 - 24 = \mathbf{42}$. The last column sums to $7 + b + 27 = 34 + b$. So $b = 66 - 34 = \mathbf{32}$. The first column sums to $17 + 12 + c = 29 + c$. So $c = 66 - 29 = \mathbf{37}$. The second column sums to $42 + 22 + d = 64 + d$. So $d = 66 - 64 = \mathbf{2}$.

11. Debbie and Amy began reading on the same day, since 72 pages for Debbie ÷ 9 pages per day = 8 days. Thus Amy is on 6 pages per day $\times$ 8 days $=$ **page 48**.

12. The last three digits must sum to 20, so the second to last digit must be $20 - (7 + 4) = 9$. Since the sum of the 11^{th}, 12^{th}, and 13^{th} digits is also 20, the 11^{th} digit is $20 - (7 + 9) = 4$.

A		7								4	7	9	4

We can continue in this fashion until we find that A is 9, or we can observe the repeating pattern from back to front, 4, 9, 7, 4, 9, 7, … and discover that **A is 9**.

13. Choose the box labeled Oranges and Apples (Box B). Retrieve a fruit from Box B. Since Box B is mislabeled, Box B should be labeled as having the fruit you retrieved. For example, if you retrieved an apple, then Box B should be labeled Apples. Since Box A is mislabeled, the Oranges and Apples label should be placed on Box A. These leave only one possibility for Box C; it should be labeled Oranges. If an orange was retrieved from Box B, then Box C would be labeled Oranges and Apples and Box A should be labeled Apples.

14. The electrician made \$1315 for 4 days at \$50 per hour. She spent \$15 per day on gasoline so 4 • \$15 = \$60 on gasoline. The total is then \$1315 + \$60 = \$ 1375. At \$50 per hour, she worked $\frac{1375}{50}$ = **27.5 hours**.

15. Working backward: Top – 6 rungs – 7 rungs + 5 rungs – 3 rungs = top – 11 rungs, which is located at the middle. From the middle rung travel up 11 to the top or down 11 to the bottom. Along with the starting rung, then, there are 11 + 11 + 1 = **23 rungs.**

16. There are several different ways to solve this. One is to use a variable. So, let a be equal to the number of apple pies that are baked. This means the number of cherry pies that are baked is represented by $4 - a$. So, using 9 slices for an apple pie and 7 slices for a cherry pie, we get:

$9a + 7(4 - a) = 34$

$9a + 28 - 7a = 34$

$2a = 6$

$a = 3$

So the number of **apple pies is 3**. Therefore, the number of **cherry pies is 1**.

Since the number of pies is small, another solution strategy is to make a table of all possible cases:

Apple Pies	Cherry Pies	Apple slices	Cherry slices	Total slices
0	4	0	28	28
1	3	9	21	30
2	2	18	14	32
3	**1**	**27**	**7**	**34**
4	0	36	0	36

From the table, we can see there are 34 slices when there are three apple pies and one cherry pie.

17. **Al made \$50**. Examine it step by step. First, Al spent \$100 on the CD player. At this point, he is down \$100. Once he sold it for \$125, he is now \$25 ahead. When he later bought it back for \$150, he was down \$125; but after selling it again for \$175, he is now ahead \$50.

18. Since the bat is \$49 more than the ball, and the total spent is \$50, we can use the guess and check method to solve. The table below represents possible guesses:

Cost of bat	Cost of ball	sum
\$49.00	\$0.00	\$49.00
\$49.25	\$0.25	\$49.50
\$49.50	**\$0.50**	**\$50.00**

Another method would be to use a variable. Let the cost of the ball (in dollars) be the variable b. The cost of a bat in dollars, therefore, would be $b + 49$. Together the two costs must add up to 50, so:

$b + (b + 49) = 50$

$2b + 49 = 50$

$2b = 1$

$b = ½$. In terms of money, $b = \$0.50$, **50 cents**. Since the ball costs 50 cents, **the bat must cost \$49.50**, and the sum of the price of bat and ball does equal \$50.

Assessment 1-1B

1. **(a)** List the numbers:

$$\begin{array}{rcrcccrcr} 1 & + & 2 & + & \cdots & + & 48 & + & 49 \\ 49 & + & 48 & + & \cdots & + & 2 & + & 1 \\ \hline 50 & + & 50 & + & \cdots & + & 50 & + & 50 \end{array}$$

There are 49 sums of 50. Thus the total can be found by computing $\frac{49 \cdot 50}{2} = \mathbf{1225}$. (Another way of looking at this problem is to realize there are $\frac{49}{2} = 24.5$ pairs of sums, each of 50; thus $24.5 \cdot 50 = 1225$.)

(b) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{2009 - 1}{2} + 1 = 1005$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 1 & + & 3 & + & \cdots & + & 2007 & + & 2009 \\ 2009 & + & 2007 & + & \cdots & + & 3 & + & 1 \\ \hline 2010 & + & 2010 & + & \cdots & + & 2010 & + & 2010 \end{array}$$

There are 1005 sums of 2010. Thus the total can be found by computing $\frac{1005 \cdot 2010}{2} = \mathbf{1,010,025}$.

(c) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{600 - 6}{6} + 1 = 100$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 6 & + & 12 & + & \cdots & + & 594 & + & 600 \\ 600 & + & 594 & + & \cdots & + & 12 & + & 6 \\ \hline 606 & + & 606 & + & \cdots & + & 606 & + & 606 \end{array}$$

There are 100 sums of 606. Thus the total can be found by computing $\frac{606 \cdot 100}{2} = \mathbf{30,300}$.

(d) The number of terms in any sequence of numbers may be found by subtracting the first term from the last (or the last from the first if the first is greater than the last), dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{1000-5}{5}+1=200$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 1000 & + & 995 & + & \cdots & + & 10 & + & 5 \\ 5 & + & 10 & + & \cdots & + & 995 & + & 1000 \\ \hline 1005 & + & 1005 & + & \cdots & + & 1005 & + & 1005 \end{array}$$

There are 200 sums of 1005. Thus the total can be found by computing $\frac{1005 \cdot 200}{2} = \mathbf{100{,}500}$.

2. **(a)** The diagram illustrates how the numbers can be paired to form 50 sums of 101. The sum of the first 100 natural numbers is $50(101) = \mathbf{5050}$.

(b) A diagram similar to the one in 2a would illustrate how the numbers can be paired to form 100 sums of 202. Because there are an odd number of terms, the middle term, 101, is left unpaired. So, the sum of the first 201 natural numbers is 100 • 202 + 101 = **20,301**.

3. There are $\frac{203-58}{1}+1 = 146$ terms. terms

List the numbers:

$$\begin{array}{rcrcccrcr} 58 & + & 59 & + & \cdots & + & 202 & + & 203 \\ 203 & + & 202 & + & \cdots & + & 59 & + & 58 \\ \hline 261 & + & 261 & + & \cdots & + & 261 & + & 261 \end{array}$$

There are 146 sums of 261. Thus the total can be found by computing $\frac{146 \cdot 261}{2} = \mathbf{19{,}053}$.

4. There are many answers to this problem. A systematic list is a good approach. Using only two numbers and addition, 5 rows give 5 different ways.

One number	Eleven minus the number
1	10
2	9
3	8
4	7
5	6

We can view 6 + 5 as a different way than 5 + 6 and continue in this manner to find 10 different ways. Or, we can find 7 more ways using three numbers as follows.

1	1	9
2	1	8
3	1	7
4	1	6
5	1	5
1	2	8

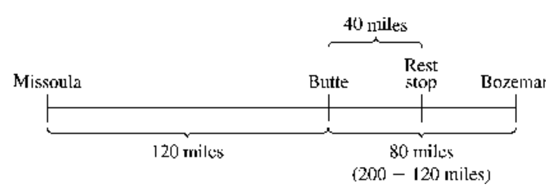
5. Since there are two different color socks in the drawer, drawing only two socks does not guarantee finding a matching pair; you could get one sock of each color. However, once you draw a third sock, you are guaranteed to have a matching pair, since the third sock must match one or the other of the previous two socks.

6. There are 13 squares of one unit each; 4 squares of four units each; and one square 9 units; for a total of 18 squares.

7.
$$\begin{aligned} P &= 1+3+5+7+\ldots+99 \\ Q &= \quad\quad 5+7+\ldots+99+101 \\ Q-P &= (5+\ldots+99+101) - \\ &\quad\ (1+3+5+\ldots+99) \\ &= (101)-(1+3) \\ &= 97 \end{aligned}$$

Q is larger than P by 97.

8. A diagram will help.



The next step is 120 miles + 40 miles = **160 miles** from Missoula.

9. **(a)** Marc must have five pennies to make an even \$1.00. The minimum number of coins would have as many quarters as possible, or three quarters. The remaining $20¢$ must consist of at least one dime and one nickel; the only possibility is one dime and two nickels. The minimum is 5 pennies, 2 nickels, 1 dime, and 3 quarters, or **11 coins**.

(b) The maximum number of coins is achieved by having as many pennies as possible. It is a requirement to have one quarter, one dime, and one nickel $= 40¢$, so there may then be 60 pennies for a total of **63 coins**.

10. Adding all the numbers gives 99. This means that each row, diagonal, and column must add to $99 \div 3 = 33$. Write 33 as a sum of the numbers in all possible ways:

$19 + 11 + 3$

$19 + 9 + 5$

$17 + 13 + 3$

$17 + 11 + 5$

$17 + 9 + 7$

$15 + 13 + 5$

$15 + 11 + 7$

$13 + 11 + 9$

Summarizing the pattern:

Number	Nr. sums with number
3	2
5	3
7	2
9	3
11	4
13	3
15	2
17	3
19	2

Thus 11 must be in the center of the square and 5, 9, 13, and 17 must be in the corners. One solution would be:

17	7	9
3	11	19
13	15	5

11. Answers may vary; two solutions might be to:

(a) Put four marbles on each tray of the balance scale. Take the heavier four and weigh two on each tray. Take the heavier two and weigh one on each tray; the heavier marble will be evident on this third weighing.

(b) This alternative shows the heavier marble can be found more efficiently, two steps rather than three. Put three marbles in each tray of the balance scale.

(***i***) If the two trays are the same weight, the heavier marble is one of the remaining two. Weigh them to find the heavier.

(***ii***) If one side is heavier, take two of the three marbles and weigh them. If they are the same weight, the remaining marble is the heavier. If not, the heavier will be evident on this second weighing.

12. **(a)** There are:

1 partridge $\times$ 12 days $=$ 12 gifts;
2 doves $\times$ 11 days $=$ 22 gifts;
3 hens $\times$ 10 days $=$ 30 gifts;
4 birds $\times$ 9 days $=$ 36 gifts;
5 rings $\times$ 8 days $=$ 40 gifts;
6 geese $\times$ 7 days $=$ 42 gifts;
7 swans $\times$ 6 days $=$ 42 gifts;
8 maids $\times$ 5 days $=$ 40 gifts;
9 ladies $\times$ 4 days $=$ 36 gifts;
10 lords $\times$ 3 days $=$ 30 gifts;
11 pipers $\times$ 2 days $=$ 22 gifts; and
12 drummers $\times$ 1 day $=$ 12 gifts.

So the **gifts given the most by your true love was 42 geese and 42 swans**.

(b) $12 + 22 + \cdots + 22 + 12 =$ **364 gifts total**.

13. **(a)** There must be 1 or 3 quarters for an amount ending in 5. Then dimes can add to \$1.15 plus 4 pennies to realize \$1.19. Thus:

Quarters	Dimes	Pennies	Total
3	**4**	**4**	**\$1.19**
1	**9**	**4**	**\$1.19**

and in neither case can change for \$1.00 be made.

(b) Two or zero quarters would allow an amount ending in 0. Then more combinations of dimes or pennies could add to \$1.00.

14. If the price of 15 sandwiches equals the price of 20 salads, each sandwich will buy $\frac{20}{15} = \frac{4}{3}$ salads. Thus 3 sandwiches $= 3\left(\frac{4}{3}\right) =$ **4 salads**.

15. Use a variable and a table

12 AM	5 AM	9 AM	12 PM
T	T − 15	2(T − 15)	2(T − 15) + 10

$$\begin{aligned} \text{So,}\quad 2(T - 15) + 10 &= 32 \\ 2T - 30 + 10 &= 32 \\ 2T - 20 &= 32 \\ 2T &= 52 \\ T &= \textbf{26 degrees}. \end{aligned}$$

16. One way to solve this problem is to write an equation using a variable. For example, let p equal the number of puzzles Seth bought. So the number of trucks Seth bought would be $5 - p$. Since each puzzle cost \$9 and each truck cost \$5, we get

$9p + 5(5 - p) = 33$

$9p + 25 - 5p = 33$

$4p = 8$

$p = 2$

So Seth bought **2 puzzles**. Since he bought 5 gifts all together, he also bought **3 trucks.**

Another way to solve it is to guess and check, making a table to keep track of results

Puzzles	Trucks	Cost of puzzles	Cost of Trucks	Total cost
0	5	\$0	\$25	\$25
1	4	\$9	\$20	\$29
2	**3**	**\$18**	**\$15**	**\$33**
3	2	\$27	\$10	\$37

The table shows that by buying **2 puzzles and 3 trucks**, Seth will spend \$33.

17. The first line tells us that hamburgers equal \$10. Using that information, we can use the second line to figure out 4 hot dogs cost \$8, so each hot dog costs \$2. Going to the third line, since two hot dogs cost \$4, each drink must cost \$1. So on the fourth line, one drink and one hamburger cost \$11. **The value of the question mark is \$11**.

18. This is difficult to visualize, so the strategy of examining a simpler case and looking for a pattern might be the best method to solve this problem. It would also help if you are able to make a model of the shapes and act out the situation.

Imagine a $3 \times 3 \times 3$ large cube. It would be made up of a total of 27 small cubes. Taking off one layer all around would involve taking off the top layer, the bottom layer, the front, the back, the left side, and the right side. Doing this one step at a time: taking off the top layer takes away 9 cubes. The same if you take off the bottom layer, so you've removed 18 cubes so far. Taking off the front and the back would remove 3 more cubes each (the other 6 cubes on those faces were removed when the top and bottom were removed). Finally, taking off the left and right sides removes one more cube each. In total, 26 cubes are removed, leaving only 1 cube.

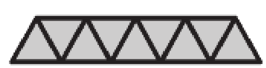
Now, imagine a $4 \times 4 \times 4$ cube. There would be a total of 64 small cubes making it. Removing the top and bottom layers takes a total of 32 cubes away; removing the front and the back would take away 16 more cubes; finally, removing the left and right sides takes away 8 more cubes, leaving only 8 cubes.

So, when starting with a $3 \times 3 \times 3$ large cube, we are left with 1 small cube $(1 \times 1 \times 1 = 1)$; when starting with a $4 \times 4 \times 4$ large cube, we are left with 8 small cubes $(2 \times 2 \times 2 = 8)$. Therefore, the pattern seems to be, when given large cube of dimension $n \times n \times n$, to find the number of small cubes that make it after removing one layer of small cubes all around the larger cube, you would take $(n - 2) \times (n - 2) \times (n - 2)$. So for a $10 \times 10 \times 10$ large cube, the number of small cubes left would be $8 \times 8 \times 8 =$ **512 small cubes.**

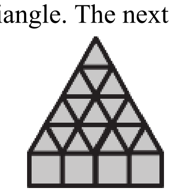
Assessment 1-2A: Explorations with Patterns

1. (a) Each figure in the sequence adds one box each to the top and bottom rows. The next would be:

(b) Each figure in the sequence adds one upright and one inverted triangle. The next would be:



(c) Each figure in the sequence adds one box to the base and one row to the overall triangle. The next would be:



2. **(a)** Terms that continue a pattern are **17, 21, 25**, , … . This is an **arithmetic** sequence because each successive term is obtained from the previous term by addition of 4.

(b) Terms that continue a pattern are **220, 270, 320**, … . This is **arithmetic** because each successive term is obtained from the previous term by addition of 50.

(c) Terms that continue a pattern are **27, 81, 243**, … . This is **geometric** because each successive term is obtained from the previous term by multiplying by 3.

(d) Terms that continue a pattern are $\mathbf{10^9, 10^{11}, 10^{13}}$, … . This is **geometric** because each successive term is obtained from the previous term by multiplying by 10^2.

(e) Terms that continue a pattern are $\mathbf{193+10\times 2^{30}, 193+11\times 2^{30}, 193+12\times 2^{30}}$, … . This is **arithmetic** because each successive term is obtained from the previous term by addition of 2^{30}.

3. In these problems, let a_n represent the nth term in a sequence, a_1 represent the first term, d represent the common difference between terms in an arithmetic sequence, and r represent the common ratio between terms in a geometric sequence. In an arithmetic sequence, $a_n = a_1 + (n-1)d$; in a geometric sequence $a_n = a_1 r^{n-1}$. Thus:

(a) Arithmetic sequence: $a_1 = 1$ and $d = 4$:

(***i***) $a_{100} = 1 + (100-1)\cdot 4$
$= 1 + 99\cdot 4 = \mathbf{397}$.

(***ii***) $a_n = 1 + (n-1)\cdot 4$
$= 1 + 4n - 4 = \mathbf{4n - 3}$.

(b) Arithmetic sequence:
$a_1 = 70$ and $d = 50$:

(***i***) $a_{100} = 70 + (100-1)\cdot 50$
$= 70 + 99\cdot 50 = \mathbf{5020}$.

(***ii***) $a_n = 70 + (n-1)\cdot 50$
$= 70 + 50n - 50$ or $\mathbf{50n + 20}$.

(c) Geometric sequence: $a_1 = 1$ and $r = 3$:

(***i***) $a_{100} = 1\cdot 3^{100-1} = \mathbf{3^{99}}$.

(***ii***) $a_n = 1\cdot 3^{n-1} = \mathbf{3^{n-1}}$.

(d) Geometric sequence:
$a_1 = 10$ and $r = 10^2$:

(i) $a_{100} = 10\cdot (10^2)^{(100-1)} = 10\cdot(10^2)^{99}$
$= 10\cdot 10^{198} = \mathbf{10^{199}}$.

(ii) $a_n = 10\cdot(10^2)^{(n-1)}$
$= 10\cdot 10^{(2n-2)} = \mathbf{10^{2n-1}}$.

(e) Arithmetic sequence:
$a_1 = 193 + 7\cdot 2^{30}$ and $d = 2^{30}$:

(i)
$a_{100} = 193 + 7\cdot 2^{30} + (100-1)\cdot 2^{30}$
$= 193 + 7\cdot 2^{30} + 99\cdot 2^{30}$
$= \mathbf{193 + 106 \times 2^{30}}$.

(***ii***) $a_n = 193 + 7\cdot 2^{30} + (n-1)\cdot 2^{30}$
$= \mathbf{193 + (n+6)\times 2^{30}}$.

4. **2, 7, 12**, … . Each term is the 5th number on a clock face (clockwise) from, the preceding term.

5. **(a)** Make a table.

Number of term	Term
1	$1\cdot 1\cdot 1 = 1$
2	$2\cdot 2\cdot 2 = 8$
3	$3\cdot 3\cdot 3 = 27$
4	$4\cdot 4\cdot 4 = 64$
5	$5\cdot 5\cdot 5 = 125$
6	$6\cdot 6\cdot 6 = 216$
7	$7\cdot 7\cdot 7 = 343$
8	$8\cdot 8\cdot 8 = 512$
9	$9\cdot 9\cdot 9 = 729$
10	$10\cdot 10\cdot 10 = 1000$
11	$11\cdot 11\cdot 11 = 1331$

The 11^{th} term **1331** is the least 4-digit number greater than 1000.

(b) The 9^{th} term **729** is the greatest 3-digit number in this pattern.

(c) $10^4 = 10{,}000$; The greatest number less than 10^4 is $21\cdot 21\cdot 21 = \mathbf{9261}$.

(d) The cell A14 corresponds to the 14th term, which is $14\cdot 14\cdot 14 = \mathbf{2744}$.

6. **(a)** The number of matchstick squares in each windmill form an arithmetic sequence with $a_1 = 5$ and $d = 4$. The number of matchstick squares required to build the

10th windmill is thus $5 + (10 - 1) \cdot 4 = 5 + 9 \cdot 4 = \mathbf{41\ squares}$.

(b) The nth windmill would require $5 + (n - 1) \cdot 4 = 5 + 4n - 4 = \mathbf{4n + 1}$ **squares**.

(c) There are 16 matchsticks in the original windmill. Each additional windmill adds 12 matchsticks.

This is an arithmetic sequence with $a_1 = 16$ and $d = 12$, so $a_n = 16 + (n - 1) \cdot 12 = \mathbf{12n + 4}$ **matchsticks**.

7. (a) Each cube adds four squares to the preceding figure; or 6, 10, 14, … . This is an arithmetic sequence with $a_1 = 6$ and $d = 4$. Thus $a_{15} = 6 + (15 - 1) \cdot 4 =$ **62 squares** to be painted in the 10th figure.

(b) This is an arithmetic sequence with $a_1 = 6$ and $d = 6$. The nth term is thus: $a_n = 6 + (n - 1) \cdot 4 = \mathbf{4n + 2}$.

8. Since the first year begins with 700 students, after the first year there would be 760, after the second there would be 820, … , and after the twelfth year the number of students would be the 13th term in the sequence.

This then is an arithmetic sequence with $a_1 = 700$ and $d = 60$, so the 13th term (current enrollment + twelve more years) is: $700 + (13 - 1) \cdot 60 =$ **1420 students.**

9. Using the general expression for the nth term of an arithmetic sequence with $a_1 = 24{,}000$ and $a_9 = 31{,}680$ yields:

$$31680 = 24000 + (9 - 1)d$$
$$31680 = 24000 + 8d \Rightarrow d = 960,$$

the amount by which Juan's income increased each year.

To find the year in which his income was $45120:

$$45120 = 24000 + (n - 1) \cdot 960$$
$$45120 = 23040 + 960n$$
$$\Rightarrow n = 23.$$

Juan's income was $45,120 in his **23rd year**.

10. The number that fits into the last triangle is **8**. The numbers inside the triangle are found by multiplying the number at the top of the triangle by the number at the bottom left of the triangle; then subtracting from that the number at the bottom right of the triangle. So, $2 \times 5 - 2 = \mathbf{8}$.

11. (a) To build an up-down-up staircase with 3 steps up and 3 steps down, each current step will have a block placed on it; plus a block will be added to each end. In total, there will be five more blocks, for a total of 9 blocks used. To build an up-down-up staircase with 4 steps up and 4 steps down, each step from the previous iteration will have a block placed on it (5 blocks) plus a block at each end (2 blocks), adding seven total blocks, bringing the total to 9 + 7 = 16 blocks. The table below illustrates the pattern:

Steps up/down	Blocks added	Total blocks
1	0	1
2	3	4
3	5	9
4	7	16

Therefore, to build an up-down-up staircase with 5 steps up and 5 steps down, 9 blocks need to be added to the previous 16 blocks to arrive at a total of **25 blocks**.

(b) Based on the table and answer above, the total blocks are always the square of the number of steps up and down. So, if the number of steps up and down are n, then the total number of blocks will be $\boldsymbol{n^2}$.

12. (a) Using the general expression for the nth term of an arithmetic sequence with $a_1 = 51, a_n = 251$, and $d = 1$ yields:

$$251 = 51 + (n - 1) \cdot 1 \Rightarrow$$
$$251 = 50 + n \Rightarrow n = 201.$$

There are **201 terms** in the sequence.

(b) Using the general expression for the ***n***th term of a geometric sequence with $a_1 = 1$, $a_n = 2^{60}$, and $r = 2$ yields:

$$2^{60} = 1(2)^{n-1} = 2^{n-1}$$
$$\Rightarrow n - 1 = 60 \Rightarrow n = 61.$$

There are **61 terms** in the sequence.

(c) Using the general expression for the ***n***th term of an arithmetic sequence with $a_1 = 10$, $a_n = 2000$, and $d = 10$ yields:

$$2000 = 10 + (n - 1) \cdot 10 \Rightarrow$$
$$2000 = 10n \Rightarrow n = 200.$$

There are **200 terms** in the sequence.

(d) Using the general expression for the n^{th} term of a geometric sequence with $a_1 = 1$, $a_n = 1024$, and $r = 2$ yields:

$1024 = 1(2)^{n-1} \Rightarrow$

$2^{10} = 2^{n-1} \Rightarrow n - 1 = 10$

$\Rightarrow n = 11.$

There are **11 terms** in the sequence.

13. (a) First term: $(1)^2 + 2 = \mathbf{3}$;

Second term: $(2)^2 + 2 = \mathbf{6}$;

Third term: $(3)^2 + 2 = \mathbf{11}$;

Fourth term: $(4)^2 + 2 = \mathbf{18}$; and

Fifth term: $(5)^2 + 2 = \mathbf{27}$.

(b) First term: $5(1) + 1 = \mathbf{6}$;

Second term: $5(2) + 1 = \mathbf{11}$;

Third term: $5(3) + 1 = \mathbf{16}$;

Fourth term: $5(4) + 1 = \mathbf{21}$; and

Fifth term: $5(5) + 1 = \mathbf{26}$.

(c) First term: $10^{(1)} - 1 = \mathbf{9}$;

Second term: $10^{(2)} - 1 = \mathbf{99}$;

Third term: $10^{(3)} - 1 = \mathbf{999}$;

Fourth term: $10^{(4)} - 1 = \mathbf{9999}$; and

Fifth term: $10^{(5)} - 1 = \mathbf{99999}$.

(d) First term: $3(1) - 2 = \mathbf{1;}$

Second term: $3(2) - 2 = \mathbf{4;}$

Third term: $3(3) - 2 = \mathbf{7;}$

Fourth term: $3(4) - 2 = \mathbf{10;}$ and

Fifth term: $3(5) - 2 = \mathbf{13.}$

14. Answers may vary; examples are:

(a) If $n = 5$, then $\frac{5+5}{5} = 2 \neq 5 + 1 = 6$.

(b) If $n = 2$, then $(2 + 4)^2 = 6^2 = 36$ does not equal $2^2 + 4^2 = 20$.

15. (a) There are 1, 5, 11, 19, 29 tiles in the five figures. Each figure adds 2n tiles to the preceding figure, thus a_6 the 6th term has $29 + 12 = $ **41 tiles**.

(b) $n^2 = 1, 4, 9, 16, 25, \ldots$. Adding $(n - 1)$ to n^2 yields $1, 5, 11, 19, 29, \ldots$, which is the proper sequence. Thus the n*th* term has $\mathbf{n^2 + n - 1}$.

(c) If $n^2 + (n - 1) = 1259$;

Then $n^2 + n - 1260 = 0$. This implies $(n - 35)(n + 36) = 0$, so $n = 35$.

There are 1259 tiles in the **35th figure**.

16. The nth term of the arithmetic sequence is $200 + n(200)$. The sequence can also be generated by adding 200 to the previous term. The n^{th} term of the geometric sequence is 2^n. The sequence can also be generated by multiplying the previous term by 2. Make a table.

Number of the term	Arithmetic term	Geometric term
7	1600	128
8	1800	256
9	2000	512
10	2200	1024
11	2400	2048
12	2600	4096

With the **12th term**, the geometric sequence is greater.

17. (a) Start with one piece of paper. Cutting it into five pieces gives us 5. Taking each of the pieces and cutting it into five pieces again gives $5 \cdot 5 = 25$ pieces. Continuing this process gives a geometric sequence: 1, 5, 25, 125, … . After the 5^{th} cut there are $\mathbf{5^5 = 3125}$ pieces of paper.

(b) The number of pieces after the nth cut would be $\mathbf{5^n}$.

18. (a) For an arithmetic sequence there is a common difference between the terms. Between 39 and 69 there are three differences so we can find the common difference by subtracting 39 from 69 and dividing the answer by three: $69 - 39 = 30$ and $30 \div 3 = 10$. The common difference is 10 and we can find the missing terms: 39 – 10 = **29** and 39 + 10 = **49** and 49 + 10 = **59**.

(b) For an arithmetic sequence there is a common difference between the terms. Between 200 and 800 there are three differences so we can find the common difference by subtracting 200 from 800 and dividing the answer by three: $800 - 200 = 600$ and $600 \div 3 = 200$. The common difference is 200 and we can find the missing terms: 200 – 200 = **0** and 200 + 200 = **400** and 400 + 200 = **600**.

(c) For a geometric sequence there is a common ration between the terms. Between 5^4 and 5^{10} there are three common ratios used so we can find the common ratio by dividing 5^{10} by 5^4 and then taking the cube root:

$5^{10} \div 5^4 = 5^6$ and $(5^6)^{\left(\frac{1}{3}\right)} = 5^2$. The common ratio is 52 and we can find the missing terms:

$5^4 \div 5^2 = \mathbf{5^2}, 5^4 \cdot 5^2 = \mathbf{5^6}, 5^6 \cdot 5^2 = \mathbf{5^8}$.

19. **(a)** Let's call the missing terms a, b, c, d, e and f, then the sequence becomes:

$a, b, 1, 1, c, d, e, f$

$b + 1 = 1 \to b = 0$

$a + b = 1 \to a + 0 = 1 \to a = 1$

$1 + 1 = c \to c = 2$

$1 + c = d \to 1 + 2 = d \to d = 3$

$c + d = e \to 2 + 3 = e \to e = 5$

$d + e = f \to 3 + 5 = f \to= 8.$

The missing terms are **1, 0, 2, 3, 5**, and **8**.

(b) Let's call the missing terms a, b, c, and d, then the sequence becomes:

$a, b, c, 10, 13, d, 36, 59$

$c + 10 = 13 \to c = 3$

$b + c = 10 \to b + 3 = 10 \to b = 7$

$a + b = c \to a + 7 = 3 \to= {}^{-}4$

$10 + 13 = d \to d = 23$

The missing terms are **-4, 7, 3**, and **23**.

(c) If a Fibonacci-type sequence is a sequence in which the first two terms are arbitrary and in which every term starting from the third is the sum of the previous two terms, then we can add 0 and 2 to get the third term and continue the pattern:

$0 + 2 = 2$

$2 + 2 = 4$

$2 + 4 = 6$

$4 + 6 = 10$

$6 + 10 = 16$

$10 + 16 = 26$

The missing terms are **2, 4, 6, 10, 16**, and **26**.

20. **(a)**

Year 1 $80 + .05(80) = 84$

Year 2 $84 + .05(84) = 88.2$

Year 3 $88.2 + .05(88.2) = 92.61$

Year 4 $92.61 + .05(92.61) = 97.2405$

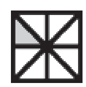
Year 5 $97.2405 + .05(97.2405) = 102.102525$

$\approx$ **\$102.10**.

(b) This is a geometric sequence with $a_1 = 80$ and $\boldsymbol{r} = 1.05$, so the price after $\boldsymbol{n}$ years is $\mathbf{80 \cdot 1.05^{n}}$.

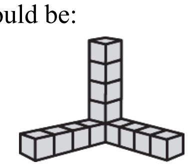
Assessment 1-2B

1. **(a)** In a clockwise direction, the shaded area moves to a new position separated from the original by one open space, then two open spaces, then by three, etc. The separation in each successive step increases by one unit; next would be:



(b) Each figure in the sequence adds one row of boxes to the base. Next would be:

(c) Each figure in the sequence adds one box to the top and each leg of the figure. Next would be:



2. **(a)** Terms that continue a pattern are **18, 22, 26**, … . This is an **arithmetic** sequence because each successive term is obtained from the previous term by addition of 4.

(b) Terms that continue a pattern are **39, 52, 65**, … . This is an **arithmetic** sequence because each successive term is obtained from the previous term by addition of 13.

(c) Terms that continue a pattern are $\mathbf{4^4}$, $\mathbf{4^5}$, $\mathbf{4^6}$, … . This is a **geometric** sequence because each successive term is obtained from the previous term by multiplying by 4.

(d) Terms that continue a pattern are $\mathbf{2^{14}}$, $\mathbf{2^{18}}$, $\mathbf{2^{22}}$, … . This is a **geometric** sequence because each successive term is obtained from the previous term by multiplying by 24.

(e) Terms that continue a pattern are $\mathbf{100+10\times 2^{50}, 100+12\times 2^{50}, 100+14\times 2^{50}}$,.... This is an **arithmetic** sequence because each successive term is obtained from the previous term by adding by $2 \cdot 2^{50}$.

3. In these problems, a_n represents the nth term in a sequence, a_1 represents the first term, d represent the common difference between terms in an arithmetic sequence, and r represents the common ratio between terms in a geometric sequence.

In an arithmetic sequence, $a_n = a_1 + (n-1)d$; in a geometric sequence, $a_n = a_1 r^{n-1}$. Thus:

(a) Arithmetic sequence: $a_1 = 2$ and $d = 4$.

(*i*) $a_{100} = 2 + (100-1)\cdot 4 = \mathbf{398}$.

(*ii*) $a_n = 2 + (n-1)\cdot 4 = 2 + 4n - 4 = \mathbf{4n - 2}$.

(b) Arithmetic sequence: $a_1 = 0$ and $d = 13$.

(*i*) $a_{100} = 0 + (100-1)\cdot 13 = \mathbf{1287}$.

(*ii*) $a_n = 0 + (n-1)\cdot 13 = \mathbf{13n - 13}$.

(c) Geometric sequence: $a_1 = 4$ and $r = 4$.

(*i*) $a_{100} = 4 \cdot 4^{99} = \mathbf{4^{100}}$.

(*ii*) $a_n = 4 \cdot 4^{n-1} = \mathbf{4^n}$.

(d) Geometric sequence: $a_1 = 2^2$ and $r = 2^4$.

(*i*)

$$a_{100} = 2^2 \cdot (2^4)^{99} = 2^2 \cdot 2^{396} = \mathbf{2^{398}}.$$

(*ii*) $a_n = 2^2 \cdot (2^4)^{(n-1)} = 2^2 \cdot 2^{4n-4} = \mathbf{2^{4n-2}}$.

(e) Arithmetic sequence:

$a_1 = 100 + 4 \cdot 2^{50}$ and $d = 2 \cdot 2^{50} = 2^{51}$:

(*i*)

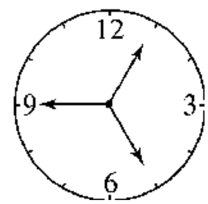
$$\begin{aligned} a_{100} &= 100 + 4 \cdot 2^{50} + (100-1)\cdot 2^{51} \\ &= 100 + 2 \cdot 2^{51} + 99 \cdot 2^{51} \\ &= 100 + 101 \cdot 2^{51} \\ &= 100 + 101 \cdot 2 \cdot 2^{50} \\ &= \mathbf{100 + 202 \times 2^{50}}. \end{aligned}$$

(*ii*)

$$\begin{aligned} a_n &= 100 + 4 \cdot 2^{50} + (n-1)\cdot 2^{51} \\ &= 100 + 2 \cdot 2^{51} + (n-1)\cdot 2^{51} \\ &= 100 + (n+1)\cdot 2^{51} \\ &= 100 + (n+1)\cdot 2 \cdot 2^{50} \\ &= \mathbf{100 + 2(n+1)2^{50}}. \end{aligned}$$

4. The hands must move 8 hours to move from 1 to 9 on the clock face. To move from 9 to 5, the hand must move 8 hours also. To move from 5 to 1, the hand must move another 8 hours. If we add 8 hours to 1 o'clock, we will land on the 9.

This pattern will continue, so the next three terms are **9, 5, 1**.



5. (a) Answers may vary: two possible answers are:

(*i*) **The sum of the first *n* odd numbers is $\mathbf{n^2}$;** e.g., $1 + 3 + 5 + 7 = 4^2$.

(*ii*) **Square the average of the first and last terms;** e.g.,

$$1 + 3 + 5 + 7 = \left(\frac{1+7}{2}\right)^2 = 4^2.$$

(b) There are $\frac{35-1}{2} + 1 = 18$ terms in this sequence.

(*i*) $1 + 3 + 5 + 7 + \cdots + 35 = 18^2 = \mathbf{324}$.

(*ii*) $\left(\frac{1+35}{2}\right)^2 = 18^2 = \mathbf{324}$.

6. (a) Note that 5 toothpicks are added to form each succeeding hexagon. This is an arithmetic sequence $a_1 = 6$ and $d = 5$,

so $a_{10} = 6 + (10 - 1) \cdot 5 =$
$6 + 9 \cdot 5 =$ **51 toothpicks**.

(b) n hexagons would require $6 + (n - 1) \cdot 5 = 6 + 5n - 5 = \mathbf{5n + 1}$ **toothpicks**.

7. (a) Looking at the second figure, there are 3 + 1 = 4 triangles. In the third figure, there are $5 + 3 + 1 = 9$ triangles. The fourth figure would then have $7 + 5 + 3 + 1 = 16$ triangles. An alternative to simply adding $7, 5, 3,$ and 1 together is to note that $7 + 1 = 8$ and $5 + 3 = 8.$ There are $\frac{4}{2} = 2$ of these sums, and $2 \cdot 8 = 16.$ Then the 100th figure would have $100 + 99 = 199$ triangles in the base, $99 + 98 = 197$ triangles in the second row, and so on until the 100th row where there would be 1 triangle. $199 + 1 = 200;$ $197 + 3 = 200;$ etc. and so the sum of each pair is 200 and there are $\frac{100}{2} = 50$ of these pairs. $50 \cdot 200 = 10{,}000,$ or **10,000 triangles** in the 100th figure.

(b) The number of triangles in the nth figure is $\frac{n}{2}$(number of triangles in base + 1). The number of triangles in the base is $n + (n - 1),$ or $2n - 1.$ $(2n - 1) + 1 = 2n.$ Then $\frac{n}{2}(2n) = n^2$, or $\boldsymbol{n^2}$ **triangles** in the nth figure.

8. This is a geometric sequence with $a_1 = \dfrac{15{,}360}{2}$ and $r = \frac{1}{2}$. The nth term of a geometric sequence is $a_n = a_1 r^{n-1};$ thus the 10th term would be $15360\left(\frac{1}{2}\right)^{10} =$ **15 liters**.

Note the progression of terms in the following table:

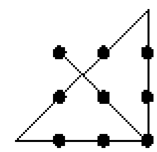
After Day	Amount of Water Remaining
1	$15{,}360 \cdot \frac{1}{2} = 7680$ liters
2	$7680 \cdot \frac{1}{2} = 3840$ liters
$\vdots$	$\vdots$
9	$60 \cdot \frac{1}{2} = 30$ liters
10	$30 \cdot \frac{1}{2} = 15$ liters

9. This is an arithmetic sequence with $a_1 = 8\frac{1}{6}$ (i.e., 8 a.m. plus 10 minutes, or $\frac{10}{60}$ of an hour) and $d = \frac{5}{6}$ (or $\frac{50}{60}$ of an hour). Thus

$a_8 = 8\frac{1}{6} + (8 - 1) \cdot \frac{5}{6} = 14,$ or **2:00 p.m.**

(14 is 2:00 p.m. on a 24-hour clock.)

10. Answers will be a rotation of the following figure:



11. (a) In the first drawing, there are 6 toothpicks; in the second drawing, there are 10, and the third drawing has 14 toothpicks. This it is an arithmetic sequence with $a_1 = 6,$ $d = 4.$ So, for the tenth figure, we would have

$a_{10} = 6 + (10 - 1) \cdot 4$
$a_{10} = 6 + 9 \cdot 4$
$a_{10} =$ **42.**

(b) The n^{th} term for this arithmetic sequence is

$a_n = 6 + (n - 1) \cdot 4$
$a_n = 6 + 4n - 4$
$a_n = \mathbf{4n + 2.}$

(c) For a total of 102 toothpicks, to find the figure, we have

$102 = 4n + 2$
$100 = 4n$
$\mathbf{25} = n.$

12. (a) The n^{th} term for this geometric sequence is $3^{n-1}.$ Thus $3^{99} = 3^{n-1}.$
So $99 = n - 1,$ and $n = 100.$
There are **100 terms** in the sequence.

(b) The n^{th} term for this arithmetic sequence is $9 + (n - 1) \cdot 4.$ Thus $353 = 9 + (n - 1) \cdot 4.$ Solving for $n,$ $n = 87.$ There are **87 terms** in the sequence.

(c) The n^{th} term for this arithmetic sequence is $38 + (n - 1) \cdot 1.$ Thus $238 = 38 + (n - 1) \cdot 1.$ Solving for $n,$ $n = 201.$ There are **201 terms** in the sequence.

13. **(a)** First term: $5(1) - 1 = \mathbf{4}$

Second term: $5(2) - 1 = \mathbf{9}$

Third term: $5(3) - 1 = \mathbf{14}$

Fourth term: $5(4) - 1 = \mathbf{19}$

Fifth term: $5(5) - 1 = \mathbf{24}$

(b) First term: $6(1) - 2 = \mathbf{4}$

Second term: $6(2) - 2 = \mathbf{10}$

Third term: $6(3) - 2 = \mathbf{16}$

Fourth term: $6(4) - 2 = \mathbf{22}$

Fifth term: $6(5) - 2 = \mathbf{28}$

(c) First term: $5 \cdot 1 + 1 = \mathbf{6}$

Second term: $5 \cdot 2 + 1 = \mathbf{11}$

Third term: $5 \cdot 3 + 1 = \mathbf{16}$

Fourth term: $5 \cdot 4 + 1 = \mathbf{21}$

Fifth term: $5 \cdot 5 + 1 = \mathbf{26}$

(d) First term: $1^2 - 1 = \mathbf{0}$

Second term: $2^2 - 1 = \mathbf{3}$

Third term: $3^2 - 1 = \mathbf{8}$

Fourth term: $4^2 - 1 = \mathbf{15}$

Fifth term: $5^2 - 1 = \mathbf{24}$

14. Answers may vary; examples are:

(a) If $n = 6$, then $\frac{3+6}{3} = 3 \neq 6$.

(b) If $n = 4$, then $(4 - 2)^2 = 4 \neq 4^2 - 2^2 = 12$.

15. **(a)** The first figure has 2 tiles, the second has 5 tiles, the third has 8 tiles, … . This is an arithmetic sequence where the n^{th} term is $2 + (n - 1) \cdot 3$.

Thus the 7^{th} term has $2 + (7 - 1) \cdot 3 = \mathbf{20}$ **tiles.**

(b) The nth term is $2 + (n - 1) \cdot 3 = 2 + 3n - 3 = \mathbf{3n - 1}$.

(c) The question can be written as: Is there an n such that $3n - 1 = 449$. Since $3n - 1 = 449 \Rightarrow 3n = 450 \Rightarrow n = 150$, the answer is **yes, the 150th figure**.

16. The n^{th} term of the arithmetic sequence is $-100 + n(300)$. The sequence can also be generated by adding 300 to the previous term. The n^{th} term of the geometric sequence is 3^{n-1}. The sequence can also be generated by multiplying the previous term by 3. Make a table

Number of the term	Arithmetic term	Geometric term
7	2000	729
8	2300	2187
9	2600	6561

With the **9th term**, the geometric sequence is greater.

17. Use a table of Fibonacci numbers to find the pattern, F_n is the nth Fibonacci number:

Generation	Male	Female	Number in Generation	Total
1	1	0	1	1
2	0	1	1	2
3	1	1	2	4
4	1	2	3	7
5	2	3	5	12
6	3	5	8	20
⋮	⋮	⋮	⋮	⋮
n	F_{n-2}	F_{n-1}	F_n	$F_{n+2}-1$

The sum of the first n Fibonacci numbers is $F_{n+2} - 1$. $F_{12} = 144$, so there are **143 bees** in all 10 generations.

18. **(a)** For an arithmetic sequence there is a common difference between the terms. Between 49 and 64 there are three differences so we can find the common difference by subtracting 49 from 64 and dividing the answer by three: $64 - 49 = 15 \text{ and } 15 \div 3 = 5$. The common difference is 5 and we can find the missing terms: 49 – 5 = **44** and 49 + 5 = **54** and 54 + 5 = **59**.

(b) For a geometric sequence there is a common ratio between the terms. Between 1 and 625 there are four common ratios used so we can find the common ratio by dividing 625 by 1 and then taking the fourth root: $625 \div 1 = 625 \text{ and } 625^{\left(\frac{1}{4}\right)} = 5$. The common ratio is 5 and we can find the missing terms: $1 \cdot 5 = \mathbf{5}, 5 \cdot 5 = \mathbf{25}, 25 \cdot 5 = \mathbf{125}$.

(c) For a geometric sequence there is a common ratio between the terms. Between 310 and 319 there are three common ratios used so we can find the common ratio by dividing 319 by 310 and then taking the cube root: $3^{19} \div 3^{10} = 3^9$ and $(3^9)^{\left(\frac{1}{3}\right)} = 3^3$. The common ratio is 33 and we can find the missing terms: $3^{10} \div 3^3 = \mathbf{3^7}, 3^{10} \cdot 3^3 = \mathbf{3^{13}}, 3^{13} \cdot 3^3 = \mathbf{3^{16}}$.

(d) For an arithmetic sequence there is a common difference between the terms. Between ***a*** and 5***a*** there are four differences so we can find the common difference by subtracting ***a*** from 5***a*** and dividing the answer by four: $5a - a = 4a$ and $4a \div 4 = a$. The common difference is ***a*** and we can find the missing terms: ***a*** + ***a*** = **2*a***, 2***a*** + ***a*** = **3*a***, 3***a*** + ***a*** = **4*a***.

19. (a) Let's call the missing terms *x* and *y,* then the sequence becomes 1, *x*, *y*, 7, 11 and if it is a Fibonacci-type sequence then:

$1 + x = y$

$x + y = 7$

$y + 7 = 11 \to y = 11 - 7 = 4$

and $x + y = 7 \to x + 4 = 7 \to x = 3$.

The missing terms are 3 and 4.

(b) Let's call the missing terms *x*, *y*. and *z*, then the sequence becomes *x*, 2, *y*, 4, *z* and if it is a Fibonacci-type sequence then:

$x + 2 = y$

$2 + y = 4 \to y = 2$

$y + 4 = z \to 2 + 4 = z \to z = 6$

and $x + 2 = 2 \to x = 0$.

The missing terms are 0, 2, and 6.

(c) Let's call the missing terms *x*, *y*. and *z*, then the sequence becomes *x*, *y*, 3, 4, *z* and if it is a Fibonacci-type sequence then:

$x + y = 3$

$y + 3 = 4 \to y = 4 - 3 = 1$

$3 + 4 = 7$

and $x + y = 3 \to x + 1 = 3 \to x = 2$.

The missing terms are 2, 1, and 7.

Mathematical Connections 1-2: Review Problems

15. Order the teams from 1 to 10, and consider a simpler problem of counting how many games are played if each team plays each other once. The first team plays nine teams. The second team also plays nine teams, but one of these games has already been counted. The third team also plays 9 teams, but two of these games were counted in the previous two summands. Continuing in this manner, the total is 10 + 9 + 8 + … + 3 + 2 + 1 = 9(10) / 2 = 45 games. Double this amount to obtain **90 games must be played** for each team to play each other twice.

16. 7 ways. Make a table:

Quarters	Dimes	Nickels
1	1	1
1	0	3
0	4	0
0	3	2
0	2	4
0	1	6
0	0	8

17. If the problem is interpreted to stated that at least one 12-person tent is used, **then there are 10 ways**. This can be seen by the table below, which illustrates the ways 2-,3-,5-, and 6-person tents can be combined accommodate 14 people.

6-Person	**5-Person**	**3-Person**	**2-Person**
2	0	0	1
1	1	1	0
1	0	2	1
1	0	0	4
0	2	0	2
0	1	3	0
0	1	1	3
0	0	4	1
0	0	2	4
0	0	0	7

Chapter 1 Review

1. Make a plan. Every 7 days (every week) the day will change from Sunday to Sunday. 365 days per year $\div 7$ days per week ≈ 52 weeks per year $+\frac{1}{7}$ weeks per year. Thus the day of the week will change from Sunday to Sunday 52 times and then change from Sunday to Monday. **July 4 will be a Monday**.

2. $\$5.90 \div 2 = \$2.95$ more on one of the items. That is $\$20 + \$2.95 =$ **$22.95 for the more expensive item** and $\$20 - \$2.95 = \$17.05$ for the less expensive item. Check that both items add up to $40: $\$22.95 + \$17.05 = \$40$.

3. **(a)** **15, 21, 28. Neither**. The successive differences of terms increases by one; e.g., 10 + 5, 15 + 6, … .
   (b) **32, 27, 22. Arithmetic** Subtract 5 from each term to obtain the subsequent term.
   (c) **400, 200, 100. Geometric** Each term is half the previous term.
   (d) **21, 34, 55. Neither** Each term is the sum of the previous two terms—this is the Fibonacci sequence.
   (e) **17, 20, 23. Arithmetic** Add 3 to each term to obtain the subsequent term.
   (f) **256, 1024, 4096. Geometric** Multiply each term by 4 to obtain the subsequent term.
   (g) **16, 20, 24. Arithmetic** Add 4 to each term to obtain the subsequent term.
   (h) **125, 216, 343. Neither** Each term is the 3rd power of the counting numbers = 13, 23, 33, … .

4. **(a)** The successive differences are 3. Each term is 3 more than the previous term. This suggests that it is an arithmetic sequence of the form $3n + ?$. Since the first term is 5, $3(1) + ? = 5$. The n^{th} term would be $\mathbf{3n + 2}$.
   (b) Each term given is 3 times the previous term. This suggests that the sequence is geometric. $\boldsymbol{n^{th}}$ term will be $\mathbf{3^n}$.
   (c) The only number that changes in successive terms is the exponent. For the first term, the exponent is 2; for the second term, the exponent is 3; for the third term, the exponent is 4; and so on. So, for the n^{th} term, the exponent will be $n + 1$. Therefore, the n^{th} term will be $\mathbf{2^{n+1} - 1}$.

5. **(a)** $3(1) - 2 = \mathbf{1}$;
   $3(2) - 2 = \mathbf{4}$;
   $3(3) - 2 = \mathbf{7}$;
   $3(4) - 2 = \mathbf{10}$; and
   $3(5) - 2 = \mathbf{13}$.
   (b) $1^2 + 1 = \mathbf{2}$;
   $2^2 + 2 = \mathbf{6}$;
   $3^2 + 3 = \mathbf{12}$;
   $4^2 + 4 = \mathbf{20}$; and
   $5^2 + 5 = \mathbf{30}$.
   (c) $4(1) - 1 = \mathbf{3}$;
   $4(2) - 1 = \mathbf{7}$;
   $4(3) - 1 = \mathbf{11}$;
   $4(4) - 1 = \mathbf{15}$; and
   $4(5) - 1 = \mathbf{19}$.

6. **(a)** $a_1 = 2, d = 2, a_n = 200$.
   So $200 = 2 + (n - 1) \cdot 2 \Rightarrow n = 100$.
   Sum is $\frac{100(2+200)}{2} = \mathbf{10{,}100}$.
   (b) $a_1 = 51, \ d = 1, a_n = 151$.
   So $151 = 51 + (n - 1) \cdot 1 \Rightarrow n = 101$.
   Sum is $\frac{101 \cdot (51+151)}{2} = \mathbf{10{,}201}$.

7. **(a)** Answers will vary; for example 5 and 3 are odd numbers, but $5 + 3 = 8$, which is not odd.
   (b) 15 is odd; and it does not end in a 1 or a 3.
   (c) The sum of any two even numbers is always even. An even number is one divisible by 2, so any even number can be represented by $2 + 2 + 2 + \cdots$.
   Regardless of how many twos are added, the result is always a multiple of 2, or an even number.

8. All rows, columns, and diagonals must add to 34; i.e., the sum of the digits in row 1. Complete rows or columns with one number missing, then two, etc. to work through the square:

16	3	2	13
5	10	**11**	**8**
9	**6**	7	12
4	**15**	14	**1**

9. The ten middle tables will hold two each and the two end tables will hold three each, totaling **26 people.**

10. **(a)** $\square + 2^{60} = 2^{61}$

$\square = 2^{61} - 2^{60}$

$\square = 2(2^{60}) - 2^{60}$

$\square = 2^{60}(2-1)$

$\square = \mathbf{2^{60}}$

(b) $\square^2 = 625$

$\sqrt{\square^2} = \sqrt{625}$

$\square = \mathbf{25}$

11. $100 \div 5 = 20$ plus $1 =$ **21 posts**. 1 must be added because both end posts must be counted.

12. 1 mile $=$ 5280 feet.

$5280 \div 6$ feet $=$ 880 turns per mile.

880×50000 miles $=$ **44,000,000 turns**.

13. There are 9 students between 7 and 17 (8 through 16). There must be 9 between them in both directions, since they are direct opposites. $9 + 9 + 2 =$ **20 students**.

14. Let l be a large box, m be a medium box, and s be a small box:

$3l + (3l \times 2m \text{ each}) + [(3 \times 2)m \times 5s \text{ each}]$

$3l + 6m + 30s =$ **39 total boxes**.

15. Extend the pattern of doubling the number of ants each day. This is a geometric sequence with $a_1 = 1500$, $a_n = 100{,}000$, and $r = 2$.

$100{,}000 = 1500 \cdot 2^{n-1} \Rightarrow$

$66\frac{2}{3} = 2^{n-1}$.

Since $2^{7-1} < 66\frac{2}{3}$ and $2^{8-1} > 66\frac{2}{3}$, the ant farm will fill sometime **between the 7th and 8th day**.

16. The best strategy would be one of guessing and checking:

(***i***) Ten 3's $+$ two 5's $= 40\ldots$ close but too low.

(***ii***) Nine 3's $+$ three 5's $= 42\ldots$ still too low.

(***iii***) Eight 3's $+$ Four 5's $= 44$.

They must have answered **four 5-point questions**.

17. Let $\ell =$ length of the longest piece,

$m =$ length of the middle-sized piece, and

$s =$ lenth of the shortest piece.

Then $\ell = 3m$ and $s = m - 10$.

So $\ell + m + s = 90 \Rightarrow$

$3m + m + (m - 10) = 90 \Rightarrow$

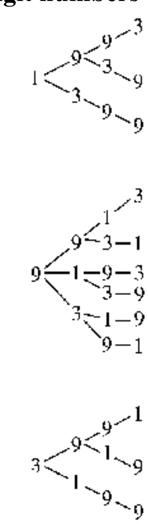
$5m = 100$.

Thus $m =$ **20 cm**;

$\ell = 3m =$ **60 cm**; and

$s = m - 10 =$ **10 cm**.

18. Make a diagram that demonstrates all the ways four-digit numbers can be formed from left (thousands place) to right (ones place). **12 four-digit numbers** can be formed.



19. Answer may vary. Fill the 4-cup container with water and pour the water into the 7-cup container. Fill the 4-cup container again and pour water into the 7-cup container until it is full. Four minus three (1) cups of water will remain in the 4-cup container. Empty the 7-cup container and pour the contents of the 4-cup container into the 7- cup container. The 7-cup container now holds 1 cup of water. Refill the 4-cup container and pour it into the 7-cup container. The 7-cup container now contains exactly 5 cups of water.

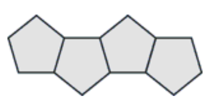
20. A possible pattern is to increase each rectangle by one row of dots and one column of dots to obtain the next term in the sequence. Make a table.

Number of the term	Row of dots	Column of dots	Term (row × column)
1	1	2	2
2	2	3	6
3	3	4	12
4	4	5	20
5	5	6	30
6	6	7	42
7	7	8	56
⋮			
100	100	101	10100
⋮			
n	n	$n+1$	$n(n+1)$

We also observe that the number of the term corresponds to the number of rows in the arrays and that the number of columns in the array is the number of the term plus one. Thus, the next three terms are **30, 42, and 56**. **The 100th term is 10,100 and the n^{th} term is $n(n+1)$**.

21. A possible pattern is that each successive figure is constructed by adjoining another pentagon to the previous figure

(a)



(b) Observe that the perimeter of the first figure is 5 and that when a new pentagon is adjoined 4 new sides are added and one side (where the new pentagon is adjoined) is lost. Make a table.

Number of terms	1-unit sides (perimeter)
1	**5**
2	$5-1+4=$ **8**
3	$8-1+4=$ **11**
4	$11-1+4=$ **14**

(c) and **(d)** Looking at the terms in the sequence and noting that the difference of terms is three, we suspect that the sequence is arithmetic and conjecture that the n^{th} term is $3n+2$. However, we need to be sure. Looking at the 4th term (part a) we observe that the pentagons on the end Contribute 4 sides to the perimeter and the "middle" pentagons contribute 3 sides. Thus, in the n^{th} figure there will be 2 "end" pentagons that contribute 4 1-unit sides and $n-2$ "middle" pentagons that contribute $3(n-2)$ 1 unit sides. The total will be $3(n-2)+2(4)=$ **3n + 2 units**. Thus the 100th term is $3(100)+2=$ **302**.

22. **(a)** The circled terms will constitute an arithmetic sequence because the common difference will be twice the difference in the original series.

(b) The new sequence will be a geometric sequence because the ratio will be the square of the ratio of the original series.

23. When $n=1$, then $n^2-n=1^2-1=0$, so the first term, $a_1=0$. When $n=2$, then $n^2-n=2^2-2$ or 2. Thus $a_1+a_2=2$; hence $a_2=2$. For $n=3$, $n^2-n=3^2-3=6$. Substituting for a_1 and a_2, we get $a_3=6-2=4$. For $n=4$, $4^2-4=12$. Substituting for a_1, a_2, and a_3, we get $0+2+4+a_4=12$. Hence $\boldsymbol{a_4=6}$.

24. **(a)** Let's call the missing terms a, and b then the sequence becomes:

$13, a, b, 27$

$13+a=b \rightarrow a=b-13$

$a+b=27 \rightarrow b-13+b=27$

$\rightarrow 2b=40$

$\rightarrow b=20$

$a=b-13 \rightarrow a=7$

So **7 and 20** are the missing terms.

(b) Let's call the missing terms a, and b then the sequence becomes:

$137, a, b, 163$

$137+a=b \rightarrow a=b-137$

$a+b=163 \rightarrow b-137+b=163$

$\rightarrow 2b=300$

$\rightarrow b=150$

$a=b-137 \rightarrow a=13$

So **13 and 150** are the missing terms.

(c) Let's call the missing terms x, and y, then the sequence becomes:

$$b, x, y, a$$

$$b + x = y \to x = y - b$$

$$x + y = a \to y - b + y = a$$

$$\to 2y = a + b$$

$$\to y = \frac{a+b}{2}$$

$$x = \frac{a+b}{2} - b \to x = \frac{a+b}{2} - \frac{2b}{2}$$

$$\to x = \frac{a-b}{2}$$

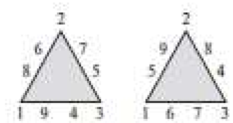
So the missing terms are $\boldsymbol{\frac{a-b}{2}}$ **and** $\boldsymbol{\frac{a+b}{2}}$.

25. The first line tells us that since three cylinders =15, each cylinder must be worth 5. Using that information, we can use the second line to figure out that a circle must be worth 4. That information can be used on the third line to determine that the cup-like shape must be equal to 1. So, putting it all together in the fourth line, we have 4 + 5 + 1 = **10**.

26. The pattern involves taking two squares that are diagonal, summing their numbers, and putting that sum in the square that is below one of the diagonal squares and to the left of the other diagonal square. For example, the top square is 4; the square diagonal to it (to the right) is 2; their sum is 6, which is the number that appears in the square below the 4 and to the left of the 2. You can also see it where 9 + 10 = 19, and 1 + 6 =7. So using this pattern, the question mark must be **equal to 3.**

27. Since 1, 2, and 3 are already on the triangle, the numbers 4-9 will be used to fill in the six question marks. On the left side, 1+2 = 3, so the two question marks on that side must sum to 14. Similarly on the bottom, the two question marks must sum to 13, and on the right side, the two question marks must sum to 12. So on the right side with the available numbers, only two possibilities, 5+7, or 4+8, sum to 12.

If the two numbers on the right side are 5 and 7, the bottom two numbers (which must sum to 13) are 4 and 9, and the two numbers on the left side must be 6 and 8. If the two numbers on the right side are 4 and 8, the two bottom numbers must be 6 and 7, leaving 5 and 9 to be the numbers on the right. Below are the two solutions presented on the triangle:



CHAPTER 2

INTRODUCTION TO LOGIC AND SETS

Assessment 2-1A: Reasoning and Logic: An Introduction

1. **(a)** **False statement**. A statement is a sentence that is either true or false, but not both.
   (b) **False statement.** Los Angeles is a city, not a state.
   (c) **Not a statement.** Questions are not statements.
   (d) **True statement.**
2. **(a)** **There exists at least one** natural number n such that $n + 8 = 11$.
   (b) **There exists at least one** natural number n such that $n^2 = 4$.
   (c) For **all** natural numbers n, $n + 3 = 3 + n$.
   (d) For **all** natural numbers n, $5n + 4n = 9n$.
3. **(a)** For **all** natural numbers n, $n + 8 = 11$.
   (b) For **all** natural numbers n, $n^2 = 4$.
   (c) There is **no** natural number x such that $x + 3 = 3 + x$.
   (d) There is **no** natural number x such that $5x + 4x = 9x$.
4. **(a)** The book **does not have** 500 pages.
   (b) $3 \cdot 5 \neq 15$.
   (c) **Some** dogs **do not have** four legs.
   (d) **No** rectangles are squares.
   (e) **All** rectangles are squares.
   (f) **Some** dogs have fleas.
5. **(a)** If $n = 4$, or $n = 5$, then $n < 6$ *and* $n > 3$, so the statement is **true**, since it can be shown to work for some natural numbers n.
   (b) All natural numbers are greater than zero; so, since the condition is that $n > 0$ *or* $n < 5$, the statement is **true**.
6. **(a)**

p	$\sim p$	$\sim(\sim p)$
T	**F**	**T**
F	**T**	**F**

   (b)

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	**F**	**T**	**F**
F	**T**	**T**	**F**

   (c) **Yes**. The truth table entries are the same.
   (d) **No**. The truth table entries are not the same.
7. **(a)**

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	
T	**T**	**T**	**F**	**T**	
T	**F**	**F**	**F**	**F**	
F	**T**	**T**	**T**	**T**	
F	**F**	**T**	**T**	**T**	

   (b) $p \to q \equiv \sim p \vee q$. Answers will vary. Here are two possible examples:1. Let p be "the Bobcats win" and q be "the Bobcats make the playoffs". Then Column 3 would read "If the Bobcats win, then the Bobcats make the playoffs". Column 5 would read "The Bobcats lose or the Bobcats make the playoffs."2. Let p be "it is summer vacation" and q be "I am at home". Then Column 3 would read "If it is summer vacation, then I am at home". Column 5 would read "It is not summer vacation or I am at home."
   (c) In this problem, p is the statement "2 + 3 = 5" and q is the statement "4 + 6 = 10". That would make the statement in this problem be in the form of $\sim p \vee q$. So, it will be logically equivalent to a statement in the form of $p \to q$ or "if 2 + 3 = 5, then 4 + 6 = 10."
8. **(a)** $\mathbf{q \wedge r}$. Both q and r are true.
   (b) $\mathbf{r \vee \sim q}$. r is true or q is not true.
   (c) $\mathbf{\sim(q \wedge r)}$. q and r are not both true.
   (d) $\mathbf{\sim q}$. q is not true.
9. **(a)** **False**. The statement is a conjunction. In order for a conjunction to be true, both p and q must be true; otherwise, the conjunction is false. The two parts of the conjunction could be stated as such: p is the statement $2 + 3 = 5$ and q is the

statement $4 + 7 = 10$. In this situation, p is true, but q is false.

(b) **False**. The United States Supreme Court has nine justices when every seat is filled.

(c) **True**. The only triangles that have three sides of the same length are equilateral triangles. In every case, an equilateral triangle will have two sides the same length as well.

(d) **False**. Isosceles triangles have two sides equal in length, but the third side does not have to be equal to the other two.

10. **(a)** By DeMorgan's Laws, the negation of $p \wedge q$ is $\sim p \vee \sim q$. Therefore, the answer is $\mathbf{2 + 3 \neq 5}$ **or** $\mathbf{4 + 7 \neq 10}$.

(b) **With every seat filled, the Supreme Court of the United States does not have 12 justices**.

(c) By problem #7 and DeMorgan's Laws, the negation of $p \rightarrow q$ is $p \wedge \sim q$. So, the statement would read: **the triangle has three sides of the same length and the triangle does not have two sides of the same length**.

(d) **The triangle has two sides of the same length and the triangle does not have three sides of the same length**.

In both (c) and (d) above, the negation of a conditional statement $p \rightarrow q$ is $p \wedge \sim q$.

11. **(a)**

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \vee q)$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	T

Since the truth values for $\sim p \vee \sim q$ are not the same as for $\sim(p \vee q)$, the statements are **not logically equivalent**.

(b)

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Since the truth values for $\sim(p \wedge q)$ are not the same as for $\sim p \wedge \sim q$, the statements are **not logically equivalent**.

12. Alex is a male teacher who does not teach math and is 30 years old or younger.

13. If p = "it is raining" and q = "the grass is wet":

(a) $\boldsymbol{p \rightarrow q}$. **(b)** $\boldsymbol{\sim p \rightarrow q}$.

(c) $\boldsymbol{p \rightarrow \sim q}$.

(d) $\boldsymbol{p \rightarrow q}$. The hypothesis is "it is raining;" the conclusion is "the grass is wet."

(e) $\boldsymbol{\sim q \rightarrow \sim p}$. **(f)** $\boldsymbol{q \leftrightarrow p}$.

14. **(a)** **Converse**: If a triangle has no two sides of the same length, then the triangle is scalene. **Inverse**: If a triangle is not scalene, then the triangle has (at least) two sides of the same length. **Contrapositive**: If a triangle does not have two sides of the same length, then the triangle is not scalene.

(b) **Converse:** If an angle is a right angle, then it is not acute. **Inverse**: If an angle is acute, then it is not a right angle. **Contrapositive:** If an angle is not a right angle, then it is acute. Note that the original statement and the contrapositive are not true, while the converse and inverse are true.

(c) **Converse**: If Maria is not a citizen of Cuba, then she is a U.S. citizen. **Inverse**: If Maria is not a U.S. citizen, then she is a citizen of Cuba. **Contrapositive**: If Maria is a citizen of Cuba, then she is not a U.S. citizen. Note that the original statement and the contrapositive are true, while the converse and inverse are not true

(d) **Converse**: If a number is not a natural number, then it is a whole number. **Inverse**: If a number is not a whole number, then it is a natural number. **Contrapositive**: If a number is a natural number, then it is not a whole number.

15. The statements **are negations** of each other.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$p \rightarrow q$	$\sim(p \rightarrow q)$
T	T	F	F	T	T	F
T	F	T	T	F	F	T
F	T	F	F	T	T	F
F	F	T	F	T	T	F

16. The **contrapositive** is logically equivalent: "**If a number is not a multiple of 4 then it is not a multiple of 8**."

17. **(a)** **Valid.** This is valid by the transitivity property. "All squares are quadrilaterals" is $p \rightarrow q$; "all quadrilaterals are polygons" is $q \rightarrow r$; and "all squares are polygons" is $p \rightarrow r$.

(b) **Invalid.** We do not know what will happen to students who are not freshman. There is no statement "sophomores, juniors, and seniors do not take mathematics."

18. **(a)** Since all students in Integrated Mathematics I make A's, and some of those students are in Beta Club, then **some Beta Club students make A's**.

(b) Let $p =$ I study for the final, $q =$ I pass the final, $r =$ I pass the course, $s =$ I look for a teaching job. Then $p \rightarrow q$, if I study for the final, then I will pass the final. $q \rightarrow r$, if I pass the final, then I will pass the course. $r \rightarrow s$, if I pass the course, I will look for a teaching job. So $p \rightarrow s$, **if I study for the final, then I will look for a teaching job**.

(c) The first statement could be rephrased as "If a triangle is equilateral, then it is isosceles." Let $p =$ equilateral triangle and $q =$ isosceles triangle. So the first statement is $p \rightarrow q$,. The second statement is simply p; then the conclusion should be q or **there exist triangles that are isosceles**.

19. **(a)** If a figure is a square, then it is a rectangle.

(b) If a number is an integer, then it is a rational number.

(c) If a polygon has exactly three sides, then it is a triangle.

20. **(a)** If $\sim p \vee \sim q \equiv \sim(p \wedge q)$, then

$$\mathbf{3 \cdot 2 \neq 6} \; \textbf{\textit{or}} \; \mathbf{1 + 1 = 3}.$$

(b) If $\sim p \wedge \sim q \equiv \sim(p \vee q)$, then **you cannot pay me now and you cannot pay me later**.

Assessment 2-1B

1. **(a)** **Not a statement**. A statement is a sentence that is either true or false.

(b) **Not a statement**. A statement must be either true of false; this could be either since "he" and "town" are not specified.

(c) **True statement**.

(d) **False statement**. $2 + 3 \neq 8$.

(e) **Not a statement**. A statement is a sentence that is either true or false.

2. **(a)** **For all** natural numbers n, $n + 0 = n$.

(b) **There exists no** natural number n such that $n + 1 = n + 2$.

(c) **There exists at least one** natural number n such that $3(n + 2) = 12$.

(d) **There exists at least one** natural number n such that $n^3 = 8$.

3. **(a)** **There is no** natural number n such that $n + 0 = n$.

(b) **There exists at least one** natural number n such that $n + 1 = n + 2$.

(c) **For all** natural numbers n, $3 \cdot (n + 2) \neq 12$.

(d) **For all** natural numbers n, $n^3 \neq 8$.

4. **(a)** **Six is greater than or equal to 8**. Another possible answer is **6 is not less than 8.**

(b) **All cats have nine lives**. Another way to express this would be to say that **no cats do not have nine lives**.

(c) **There exists a square that is not a rectangle**. Another way to express this would be to say **some squares are not rectangles**.

(d) **All numbers are positive**.

(e) **No people have blond hair**.

5. **(a)** If $n = 10$, then $n > 5$ and $n > 2$, so the statement is **true**.

(b) n could equal 5, so the statement is **false**.

6. **(a)** $p \vee q$ is false only if both p and q are false, so if p is true the statement is **true** regardless of the truth value of q.

(b) An implication is false only when p is true and q is false, so if p is false then the statement is **true** regardless of the truth value of q.

7.

p	q	$\sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$	
T	**T**	**F**	**T**	**F**	
T	**F**	**T**	**T**	**F**	
F	**T**	**F**	**F**	**T**	
F	**F**	**T**	**T**	**F**	

8. **(a)** $q \wedge r$.

(b) $q \wedge \sim r$.

(c) $\sim r \vee \sim q$.

(d) $\sim(q \wedge r)$.

9. **(a)** **True**. This statement is a disjunction. The two parts could be stated as such: p is the statement "4 + 6 = 10", while q is the statement "2 + 3 = 5". In this situation p is true, and q is true. In order for a disjunction to be true, either p or q (or both) have to be true; the only way a disjunction can be false is if both p and q are false. So therefore, this statement is true.

(b) **Answers may vary.** If a team has more than 11 players on the field, it is a penalty and the play will not count; so, an answer of **False** could be given. However, just because it is a penalty doesn't mean a team can't have 12 or more players on the field; so, you could say the answer is **True**. You could also say **True** if you consider the time between plays when players are substituting in for other players; oftentimes, there are more than 11 players on the field during substitutions.

(c) **True**.

(d) **False**. To see, sketch a drawing where three sides are the same length, but with the two angles where the sides intersect being different measures (in fact, make one a right angle, the other an obtuse angle). You should easily make a quadrilateral with a side length different from the other three.

(e) **True**. If a rectangle has four sides of the same length, then by default it must have three sides the same length. Of course, a rectangle with four equal sides is a square!

10. **(a)** By DeMorgan's Laws, the negation of the disjunction $p \vee q$ is $\sim p \wedge \sim q$. **So, the statement would be** $4 + 6 \neq 10$ **and** $2 + 3 \neq 5$

(b) **A National Football League team cannot have more than 11 players on the field while a game is in progress**.

(c) **The first president of the United States was not George Washington**.

(d) **A quadrilateral has three sides of the same length and the quadrilateral does not have four sides of the same length**.

(e) **A rectangle has four sides of the same length and that rectangle does not have three sides of the same length**.

In both (d) and (e), the negation of the conditional statement $p \to q$ is $p \wedge \sim q$.

11. **(a)**

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Since the truth values for $\sim(p \vee q)$ are the same as for $\sim p \wedge \sim q$, the statements **are logically equivalent**.

(b)

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Since the truth values for $\sim(p \wedge q)$ are the same as for $\sim p \vee \sim q$, the statements are **logically equivalent**.

12. **Megan is a female brunette who owns a car. She is not single**.

13. **(a)** $p \to q$.

(b) $\sim p \to q$.

(c) $p \to \sim q$.

(d) q if p, or $p \to q$.

(e) $\sim p \to \sim q$.

(f) $\sim q \to \sim p$.

14. **(a)** Converse: **If $x^2 = 9$, then $x = 3$**.

Inverse: **If $x \neq 3$, then $x^2 \neq 9$**.

Contrapositive: **If $x^2 \neq 9$, then $x \neq 3$**.

(b) Converse**: If classes are canceled, then it snowed**.

Inverse: **If it does not snow, then classes are not canceled**.

Contrapositive: **If classes are not canceled, then it did not snow**.

15. **No**. This is the inverse; i.e., if it does not rain then Iris can either go to the movies or not without making her statement false.

16. **(a)** **Valid**. Use modus ponens: Hypatia was a woman $\rightarrow$ all women are mortal $\rightarrow$ Hypatia was mortal.

(b) **Valid**. Since *Dirty Harry* was not written by J.K. Rowling, and she wrote all the *Harry Potter* books, then *Dirty Harry* cannot be a *Harry Potter* book.

(c) **Not valid**.

17. **(a)** Since all students in Integrated Mathematics I are in Kappa Mu Epsilon, and Helen is in Integrated Mathematics I, then the conclusion is that **Helen is in Kappa Mu Epsilon**.

(b) Let p = all engineers need mathematics and q = Ron needs mathematics.

Then $p \rightarrow q$, or if all engineers need mathematics then Ron needs mathematics.

p is true, but q is false, Ron does not need mathematics.

So **Ron is not an engineer**.

(c) Since all bicycles have tires and all tires use rubber, then the conclusion is **all bicycles use rubber.**

18. **(a)** **If** a number is a natural number, **then** it is a real number.

(b) **If** a figure is a circle, **then** it is a closed figure.

19. DeMorgan's Laws are that:

$\sim(p \wedge q)$ is the logical equivalent of $\sim p \vee \sim q$.

$\sim(p \vee q)$ is the logical equivalent of $\sim p \wedge \sim q$.

Thus:

(a) The negation is $\mathbf{3 + 5 = 9}$ **or** $\mathbf{3 \cdot 5 \neq 15}$.

(b) The negation is **I am not going and she is not going**.

Assessment 2-2A: Describing Sets

1. **(a)** Either a list or set-builder notation may be used: $\{\mathbf{a, s, e, m, n, t}\}$ or $\{\, x \mid x$ **is a letter in the word *assessment***$\}$.

(b) $\{\mathbf{21, 22, 23, 24, \ldots}\}$ or $\{\, x \mid x$ **is a natural number and** $\mathbf{x > 20}\}$ or $\{x \mid x \in N$ **and** $\mathbf{x > 20}\}$.

2. **(a)** $\boldsymbol{P = \{p, q, r, s\}}$.

(b) $\{\mathbf{1,2}\} \subset \{\mathbf{1,2,3}\}$. The symbol $\subset$ refers to a proper subset.

(c) $\{\mathbf{0,1}\} \not\subseteq \{\mathbf{1,2,3}\}$. The symbol $\subseteq$ refers to a subset.

3. **(a)** **Yes**. $\{1,2,3,4,5\} \sim \{m,n,o,p,q\}$ because both sets have the same number of elements and thus exhibit a one-to-one correspondence.

(b) **Yes**. $\{a,b,c,d,e,f,\ldots,m\} \sim \{1,2,3,\ldots,13\}$ because both sets have the same number of elements.

(c) **No**. $\{x \mid x$ is a letter in the word *mathematics*$\} \nsim \{1,2,3,4,\ldots,11\}$; there are only eight unduplicated letters in the word *mathematics*.

4. **(a)** The first element of the first set can be paired with any of the six in the second set, leaving five possible pairings for the second element, four for the third, three for the fourth, two for the fifth, and one for the sixth. Thus there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{720}$ one-to-one correspondences.

(b) There are $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = \mathbf{n!}$ possible one-to-one correspondences. The first element of the first set can be paired with any of the n elements of the second set; for each of those n ways to make the first pairing, there are $n-1$ ways the second element of the first set can be paired with any element of the second set; which means there are $n-2$ ways the third element of the first set can be paired with any element of the third set; and so on. The Fundamental Counting Principle states that the choices can be multiplied to find the total number of correspondences.

5. **(a)** If x must correspond to 5, then y may correspond to any of the four remaining elements of $\{1,2,3,4,5\}$, z may correspond to any of the three remaining, etc. Then $1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{24}$ **one-to-one correspondences**.

(b) There would be $1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = $ **6 one-to-one correspondences**.

(c) The set $\{x, y, z\}$ could correspond to the set $\{1,3,5\}$ in $3 \cdot 2 \cdot 1 = 6$ ways. The set $\{u, v\}$ could correspond with the set $\{2,4\}$ in $2 \cdot 1 = 2$ ways. There would then be $6 \cdot 2 = $ **12 one-to-one correspondences**.

6. There are three pairs of equal sets:

(i) $\boldsymbol{A = C}$. The order of the elements does not matter.

(ii) $\boldsymbol{E = H}$. They are both the null set.

(iii) $\boldsymbol{I = L}$. Both represent the numbers $1,3,5,7,\ldots$.

7. (a) Assume an arithmetic sequence with $a_1 = 201, a_n = 1100,$ and $d = 1.$ Thus $1100 = 201 + (n-1) \cdot 1;$ solving, $n = 900.$ The cardinal number of the set is therefore **900**.

(b) Assume an arithmetic sequence with $a_1 = 1, a_n = 101,$ and $d = 2.$ Thus $101 = 1 + (n-1) \cdot 2;$ solving, $n = 51.$ The cardinal number of the set is therefore **51**.

(c) Assume a geometric sequence with $a_1 = 1, a_n = 1024,$ and $r = 2.$ Thus $1024 = 1 \cdot 2^{n-1} \Rightarrow 2^{10} = 2^{n-1} \Rightarrow n-1 = 10 \Rightarrow n = 11.$ The cardinal number of the set is therefore **11**.

(d) If $k = 1,2,3,\ldots,100,$ the cardinal number of the set $\{x | x = k^3, k = 1,2,3,\ldots,100\} = \mathbf{100},$ since there are 100 elements in the set.

8. $\overline{A}$ represents all elements in U that are not in A, or **the set of all college students with at least one grade that is not an *A***.

9. (a) A proper subset must have at least one less element than the set, so the maximum $\boldsymbol{n(B) = 7}$.

(b) Since $B \subset C,$ and $n(B) = 8$ then C could have **any number of elements in it, so long as it was greater than eight**.

10. (a) If $C \subseteq D$ and $D \subseteq C$, then the sets arc equal; so $\boldsymbol{n(D) = 5}$.

(b) Answers vary. For example, **the sets are equal** and/or **the sets are equivalent**.

11. (a) A has 5 elements, thus $2^5 = $ **32 subsets**.

(b) Since A is a subset of A and A is the only subset of A that is not proper, A has $2^5 - 1 = $ **31 proper subsets**.

(c) Let $B = \{b,c,d\}$. Since $B \subset A,$ the subsets of B are all subsets of A that do not contain a and $e.$ There are $2^3 = 8$ of these subsets. If we join (union) a and e to each of these subsets there are still **8 subsets**.

Alternative. Start with $\{a,e\}$. For each element $b, c,$ and d there are two options: include the element or do not include the element. So there are $2 \cdot 2 \cdot 2 = 8$ ways to create subsets of A that include a and $e.$

12. If there are n elements in a set, 2^n subsets can be formed. This includes the set itself. So if there are 127 proper subsets, then there are 128 subsets. Since $2^7 = 128,$ the set has **7 elements**.

13. In roster format, $A = \{3,6,9,12,\ldots\}, B = \{6,12,18,24,\ldots\},$ and $C = \{12,24,36,\ldots\}$. Thus, $\boldsymbol{B \subset A, C \subset A,}$ **and** $\boldsymbol{C \subset B}$.

Alternatively: $12n = 6(2n) = 3(4n).$ Since $2n$ and $4n$ are natural number $\boldsymbol{C \subset A, C \subset B},$ and $\boldsymbol{B \subset A}$.

14. (a) $\notin$. There are no elements in the empty set.

(b) $\in$. $1024 = 2^{10}$ and $10 \in N.$

(c) $\in$. $3(1001) - 1 = 3002$ and $1001 \in N.$

(d) $\notin$. For example, $x = 3$ is not an element because for $3 = 2^n$, $n \notin N.$

15. (a) $\not\subseteq$. 0 is not a set so cannot be a subset of the empty set, which has only one subset, $\varnothing$.

(b) $\not\subseteq$. 1024 is an element, not a subset.

(c) $\not\subseteq$. 3002 is an element, not a subset.

(d) $\not\subseteq$. x is an element, not a subset.

16. (a) **Yes**. Any set is a subset of itself, so if $A = B$ then $A \subseteq B.$

(b) **No**. A could equal $B;$ then A would be a subset but not a proper subset of $B.$