INSTRUCTOR'S SOLUTIONS MANUAL

FOUNDATIONS OF GEOMETRY

SECOND EDITION

Gerard A. Venema

Calvin College

PEARSON

Boston Columbus Indianapolis New York San Francisco Upper Saddle River
Amsterdam Cape Town Dubai London Madrid Milan Munich Paris Montreal Toronto
Delhi Mexico City Sao Paulo Sydney Hong Kong Seoul Singapore Taipei Tokyo

Reproduced by Pearson from electronic files supplied by the author.

Publishing as Pearson, 75 Arlington Street, Boston, MA 02116.

ISBN 0136020593

1 2 3 4 5 6 XX 15 14 13 12 11

www.pearsonhighered.com

PEARSON

Contents

Solutions to Exercises in Chapter 1

1.6.1 Check that the formula $A = \frac{1}{4}(a + c)(b + d)$ works for rectangles but not for parallelograms.

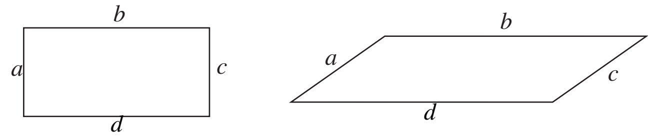

FIGURE S1.1: Exercise 1.6.1. A rectangle and a parallelogram

For rectangles and parallelograms, $a = c$ and $b = d$ and $Area = base{*}height$.
For a rectangle, the base and the height will be equal to the lengths of two adjacent sides. Therefore $A = a * d = \frac{1}{2}(a + a) * \frac{1}{2}(b + b) = \frac{1}{4}(a + c)(b + d)$
In the case of a parallelogram, the height is smaller than the length of the side so the formula does not give the correct answer.

1.6.2 The area of a circle is given by the formula $A = \pi(\frac{d}{2})^2$. According the Egyptians, A is also equal to the area of a square with sides equal to $\frac{8}{9}d$; thus $A = (\frac{8}{9})^2 d^2$. Equating and solving for π gives

$$\pi = \frac{(\frac{8}{9})^2 d^2}{\frac{1}{4}d^2} = \frac{\frac{64}{81}}{\frac{1}{4}} = \frac{256}{81} \approx 3.160494.$$

1.6.3 The sum of the measures of the two acute angles in $\triangle ABC$ is 90°, so the first shaded region is a square. We must show that the area of the shaded region in the first square (c^2) is equal to the area of the shaded region in the second square $(a^2 + b^2)$.
The two large squares have the same area because they both have side length $a + b$. Also each of these squares contains four copies of triangle $\triangle ABC$ (in white). Therefore, by subtraction, the shadesd regions must have equal area and so $a^2 + b^2 = c^2$.

1.6.4 **(a)** Suppose $a = u^2 - v^2$, $b = 2uv$ and $c = u^2 + v^2$. We must show that $a^2 + b^2 = c^2$. First, $a^2 + b^2 = (u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 = u^4 + 2u^2v^2 + v^4$ and, second, $c^2 = (u^2 + v^2)^2 = u^4 + 2u^2v^2 + v^4 = u^4 + 2u^2v^2 + v^4$. So $a^2 + b^2 = c^2$.

(b) Let u and v be odd. We will show that a, b and c are all even. Since u and v are both odd, we know that u^2 and v^2 are also odd. Therefore $a = u^2 - v^2$ is even (the difference between two odd numbers is even). It is obvious that $b = 2uv$ is even, and $c = u^2 + v^2$ is also even since it is the sum of two odd numbers.

(c) Suppose one of u and v is even and the other is odd. We will show that a, b, and c do not have any common prime factors. Now a and c are both odd, so 2 is not a factor of a or c. Suppose $x \neq 2$ is a prime factor of b. Then either x divides u or x divides v, but not both because u and v are relatively prime. If x divides u, then it also divides u^2 but not v^2. Thus x is not a factor of a or c.

If x divides v, then it divides v^2 but not u^2. Again x is not a factor of a or c. Therefore (a, b, c) is a primitive Pythagorean triple.

1.6.5 Let $h + x$ be the height of the entire (untruncated) pyramid. We know that

$$\frac{h + x}{x} = \frac{a}{b}$$

(by the Similar Triangles Theorem), so $x = h\dfrac{b}{a - b}$ (algebra). The volume of the truncated pyramid is the volume of the whole pyramid minus the volume of the top pyramid. Therefore

$$\begin{aligned}
V &= \frac{1}{3}(h + x)a^2 - \frac{1}{3}xb^2 \\
&= \frac{1}{3}(h + h\frac{b}{a - b})a^2 - \frac{1}{3}h(\frac{b^3}{a - b}) \\
&= \frac{h}{3}(a^2 + \frac{a^2 b}{a - b}) - \frac{h}{3}(\frac{b^3}{a - b}) \\
&= \frac{h}{3}(a^2 + \frac{a^2 b - b^3}{a - b}) \\
&= \frac{h}{3}(a^2 + \frac{(a - b)(ab + b^2)}{(a - b)}) \\
&= \frac{h}{3}(a^2 + ab + b^2).
\end{aligned}$$

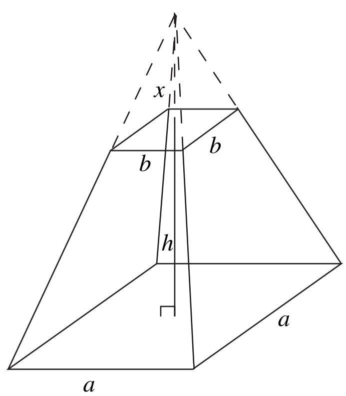

FIGURE S1.2: Exercise 1.6.5. A truncated pyramid.

1.6.6 Constructions using a compass and a straightedge. There are numerous ways in which to accomplish each of these constructions; just one is indicated in each case.

(a) The perpendicular bisector of a line segment $\overline{AB}$.
Using the compass, construct two circles, the first about A through B, the second about B through A. Then use the straightedge to construct a line through the two points created by the intersection of the two circles.

(b) A line through a point P perpendicular to a line ℓ.
Use the compass to construct a circle about P, making sure the circle is big enough so that it intersects ℓ at two points, A and B. Then construct the perpendicular bisector of segment $\overline{AB}$ as in part (a).

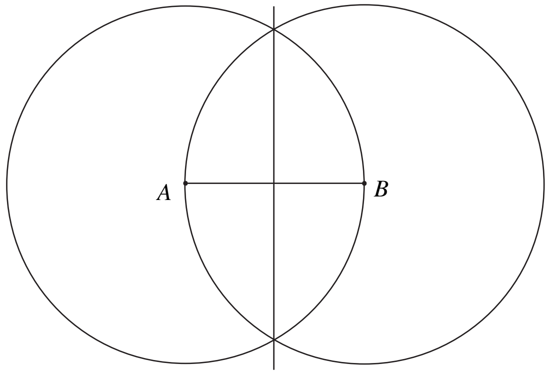

FIGURE S1.3: Exercise 1.6(a) Construction of a perpendicular bisector

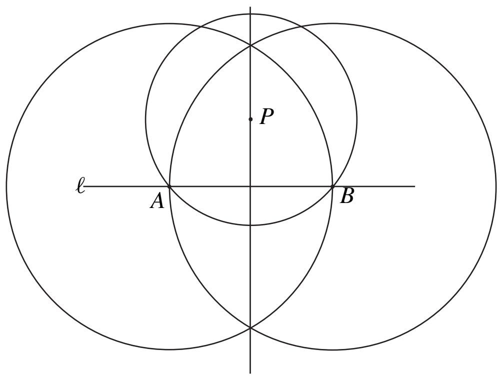

FIGURE S1.4: Exercise 1.6.6(b) Construction of a line through P, perpendicular to ℓ

(c) The angle bisector of $\angle BAC$.
Using the compass, construct a circle about A that intersects $\overline{AB}$ and $\overline{AC}$. Call those points of intersection D and E respectively. Then construct the perpendicular bisector of $\overline{DE}$. This line is the angle bisector.

1.6.7 **(a)** No. Euclid's postulates say nothing about the number of points on a line.
(b) No.
(c) No. The postulates only assert that there is a line; they do not say there is only one.

1.6.8 The proof of Proposition 29.

1.6.9 Let $\square ABCD$ be a rhombus (all four sides are equal), and let E be the point of intersection between $\overline{AC}$ and $\overline{DB}$.[1] We must show that $\triangle AEB \cong \triangle CEB \cong \triangle CED \cong \triangle AED$. Now $\angle BAC \cong \angle ACB$ and $\angle CAD \cong \angle ACD$ by Proposition 5. By addition we can see that $\angle BAD \cong \angle BCD$ and similarly, $\angle ADC \cong \angle ABC$. Now we know that $\triangle ABC \cong \triangle ADC$ by Proposition 4. Similarly, $\triangle DBA \cong \triangle DBC$. This implies that $\angle BAC \cong \angle DAC \cong \angle BCA \cong \angle DCA$ and $\angle BDA \cong \angle DBA \cong \angle BDC \cong \angle DBC$.

[1] In this solution and the next, the existence of the point E is taken for granted. Its existence is obvious from the diagram. Proving that E exists is one of the gaps that must be filled in these proofs. This point will be addressed in Chapter 6.

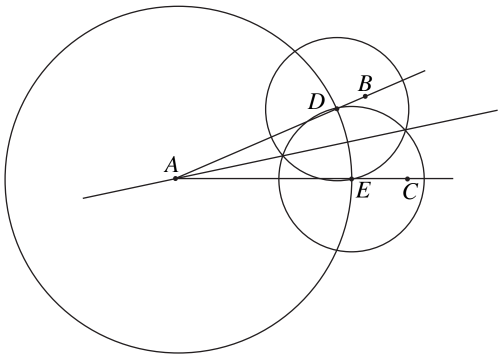

FIGURE S1.5: Exercise 1.6.6(c) Construction of an angle bisector

Thus $\triangle AEB \cong \triangle AED \cong \triangle CEB \cong \triangle CED$, again by Proposition 4.[2]

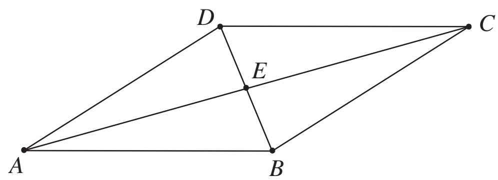

FIGURE S1.6: Exercise 1.6.9 Rhombus $\square ABCD$

1.6.10 Let $\square ABCD$ be a rectangle, and let E be the point of intersection of $\overline{AC}$ and $\overline{BD}$. We must prove that $\overline{AC} \cong \overline{BD}$ and that $\overline{AC}$ and $\overline{BD}$ bisect each other (i.e., $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$). By Proposition 28, $\overleftrightarrow{DA} \parallel \overleftrightarrow{CB}$ and $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$. Therefore, by Proposition 29, $\angle CAB \cong \angle ACD$ and $\angle DAC \cong \angle ACB$. Hence $\triangle ABC \cong \triangle CDA$ and $\triangle ADB \cong \triangle CBD$ by Proposition 26 (ASA). Since those triangles are congruent we know that opposite sides of the rectangle are congruent and $\triangle ABD \cong \triangle BAC$ (by Proposition 4), and therefore $\overline{BD} \cong \overline{AC}$.
Now we must prove that the segments bisect each other. By Proposition 29, $\angle CAB \cong \angle ACD$ and $\angle DBA \cong \angle BDC$. Hence $\triangle ABE \cong \triangle CDE$ (by Proposition 26) which implies that $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$. Therefore the diagonals are equal and bisect each other.

1.6.11 The argument works for the first case. This is the case in which the triangle actually is isosceles. The second case never occurs (D is never inside the triangle). The flaw lies in the third case (D is outside the triangle). If the triangle is not isosceles then either E will be outside the triangle and F will be on the edge $\overline{AC}$, or E will be on the edge $\overline{AB}$ and F will be outside. They cannot both be outside as shown in the diagram. This can be checked by drawing a careful diagram by hand or by drawing the diagram using GeoGebra (or similar software).

[2]It should be noted that the fact about rhombi can be proved using just propositions that come early in Book I and do not depend on the Fifth Postulate, whereas the proof in the next exercise requires propositions about parallelism that Euclid proves much later in Book I using his Fifth Postulate.

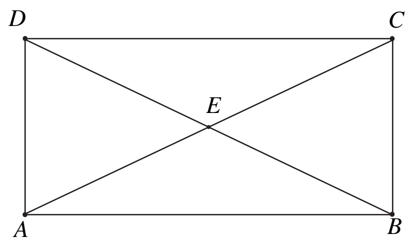

FIGURE S1.7: Exercise 1.6.10 Rectangle $\square ABCD$

Solutions to Exercises in Chapter 2

2.4.1 This is not a model for Incidence Geometry since it does not satisfy Incidence Axiom 3. This example is isomorphic to the 3-point line.

2.4.2 One-point Geometry satisfies Axioms 1 and 2 but not Axiom 3. Every pair of distinct points defines a unique line (vacuously—there is no pair of distinct points). There do not exist three distinct points, so there cannot be three noncollinear points. One-point Geometry satisfies all three parallel postulates (vacuously—there is no line).

2.4.3 It helps to draw a schematic diagram of the relationships.

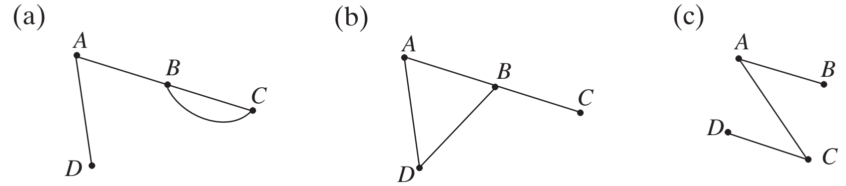

FIGURE S2.1: A schematic representation of the committee structures

- **(a)** Not a model. There is no line containing B and D. There are two lines containing B and C.
- **(b)** Not a model. There is no line containing C and D.
- **(c)** Not a model. There is no line containing A and D.

2.4.4
- **(a)** The Three-point plane is a model for Three-point geometry.
- **(b)** Every model for Three-point geometry has 3 lines. If there are 3 points, then there are also 3 pairs of points
- **(c)** Suppose there are two models for Three-point geometry, model A and model B. Choose any 1-1 correspondence of the points in model A to the points in model B. Any line in A is determined by two points. These two points correspond to two points in B. Those two points determine a line in B. The isomorphism should map the given line in A to this line in B. Then the function will preserve betweennes. Therefore models A and B are isomorphic to one another.

2.4.5 Axiom 1 does not hold, but Axioms 2 and 3 do. The Euclidean Parallel Postulate holds. The other parallel postulates are false in this interpretation.

2.4.6 See Figure S2.2.

2.4.7 Fano's Geometry satisfies the Elliptic Parallel Postulate because every line shares at least one point with every other line; there are no parallel lines. It does not satisfy either of the other parallel postulates.

2.4.8 The three-point line satisfies all three parallel postulates (vacuously).

2.4.9 If there are so few points and lines that there is no line with an external point, then all three parallel postulates are satisfied (vacuously). If there is a line with an external point, then there will either be a parallel line through the external point or there will not be. Hence at most one of the parallel postulates can be satisfied in

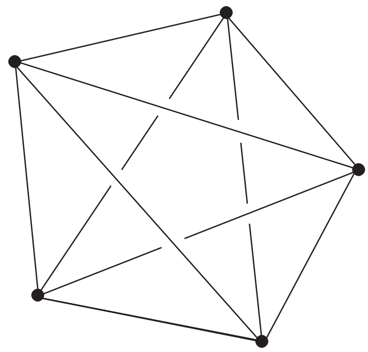

FIGURE S2.2: Five-point Geometry

that case. Since every incidence geometry contains three noncollinear points, there must be a line with an external point. Hence an incidence geometry can satisfy at most one of the parallel postulates.

2.10 Start with a line with three points on it. There must exist another point not on that line (Incidence Axiom 3). That point, together with the points on the original line, determines three more lines (Incidence Axiom 1). But each of those lines must have a third point on it. So there must be at least three more points, for a total of at least seven points. Since Fano's Geometry has exactly seven points, seven is the minimum.

2.4.11 See Figures S2.3 and S2.4.

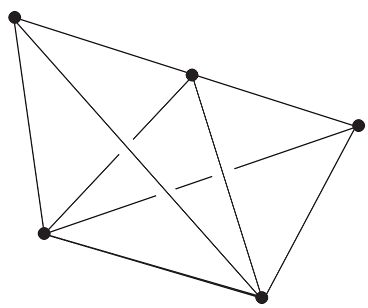

FIGURE S2.3: An unbalanced geometry

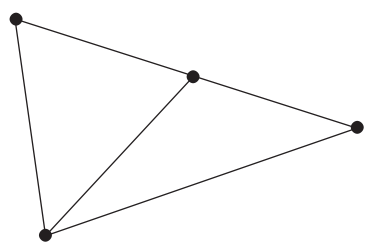

FIGURE S2.4: A simpler example

2.4.12 **(a)** The three-point line (Example 2.2.3).

(b) The square (Exercise 2.4.5) or the sphere (Example 2.2.9).

(c) One-point geometry (Exercise 2.4.2).

2.4.13 **(a)** The dual of the Three-point plane is another Three-point plane. It is a model for incidence geometry.

(b) The dual of the Three-point line is a point which is incident with 3 lines. This is not a model for incidence geometry.

(c) The dual of Four-point Geometry has 6 points and 4 lines. Each point is incident with exactly 2 lines, and each line is incident with 3 points. It is not a model for incidence geometry because it does not satisfy Incidence Axiom 1.

(d) The dual of Fano's Geometry is isomorphic to Fano's Geometry, so it is a model for incidence geometry.

2.5.1 **(a)** H : it rains
C : I get wet

(b) H : the sun shines
C : we go hiking and biking

(c) H : $x > 0$
C : $\exists$ a y such that $y^2 = 0$

(d) H : $2x + 1 = 5$
C : $x = 2$ or $x = 3$

2.5.2 **(a)** Converse : If I get wet, then it rained.
Contrapositive : If I do not get wet, then it did not rain.

(b) Converse : If we go hiking and biking, then the sun shines.
Contrapositive : If we do not go hiking and biking, then the sun does not shine.

(c) Converse : If $\exists$ a y such that $y^2 = 0$, then $x > 0$.
Contrapositive : If $\forall\, y$, $y^2 \neq 0$, then $x \leq 0$.

(d) Converse : If $x = 2$ or $x = 3$, then $2x + 1 = 5$.
Contrapositive : If $x \neq 2$ and $x \neq 3$, then $2x + 1 \neq 5$

2.5.3 **(a)** It rains and I do not get wet.

(b) The sun shines but we do not go hiking or biking.

(c) $x > 0$ and $\forall y$, $y^2 \neq 0$.

(d) $2x + 1 = 5$ but x is not equal to 2 or 3.

2.5.4 **(a)** If the grade is an A, then the score is at least 90%.

(b) If the score is at least 50%, then the grade is a passing grade.

(c) If you fail, then you scored less than 50%.

(d) If you try hard, then you succeed.

2.5.5 **(a)** Converse: If the score is at least 90%, then the grade is an A.
Contrapositive: If the score is less than 90%, then the grade is not an A.

(b) Converse: If the grade is a passing grade, then the score is at least 50%.
Contrapositive: If the grades is not a passing grade, then the score is less than 50%.

(c) Converse: If you score less than 50%, then you fail.
Contrapositive: If you score at least 50%, then you pass.

(d) Converse: If you succeed, then you tried hard.
Contrapositive: If you do not succeed, then you did not try hard.

2.5.6 **(a)** The grade is an A but the score is less than 90%.

(b) The grade is at least 50% but the grade is not a passing grade.
(c) You fail and your score is at least 50%.
(d) You try hard but do not succeed.

2.5.8 **(a)** H : I pass geometry
C : I can take topology
(b) H : it rains
C : I get wet
(c) H : the number x is divisible by 4
C : x is even

2.5.9 **(a)** $\forall$ triangles T, the angle sum of T is $180°$.
(b) $\exists$ triangle T such that the angle sum of T is less than $180°$.
(c) $\exists$ triangle T such that the angle sum of T is not equal to $180°$.
(d) $\forall$ great circles α and β, $\alpha \cap \beta \neq \emptyset$.
(e) $\forall$ point P and $\forall$ line ℓ $\exists$ line m such that P lies on ℓ and $m \perp \ell$.

2.5.10 **(a)** $\forall$ model for incidence geometry, the Euclidean Parallel Postulate does not hold in that model.
(b) $\exists$ a model for incidence geometry in which there are not exactly 7 points (the number of points is either $\leqslant 6$ or $\geqslant 8$).
(c) $\exists$ a triangle whose angle sum is not $180°$.
(d) $\exists$ a triangle whose angle sum is at least $180°$.
(e) It is not hot or it is not humid outside.
(f) My favorite color is not red and it is not green.
(g) The sun shines and (but?) we do not go hiking. (See explanation in last full paragraph on page 36.)
(h) $\exists$ a geometry student who does not know how to write proofs.

2.5.11 **(a)** Negation of Euclidean Parallel Postulate. There exist a line ℓ and a point P not on ℓ such that either there is no line m such that P lies on m and m is parallel to ℓ or there are (at least) two lines m and n such that P lies on both m and n, $m \parallel \ell$, and $n \parallel \ell$.
(b) Negation of Elliptic Parallel Postulate. There exist a line ℓ and a point P that does not lie on ℓ such that there is at least one line m such that P lies on m and $m \parallel \ell$.
(c) Negation of Hyperbolic Parallel Postulate. There exist a line ℓ and a point P that does not lie on ℓ such that either there is no line m such that P lies on m and $m \parallel \ell$ or there is exactly one line m with these properties.

Note. You could emphasize the separate existence of ℓ and P by starting each of the statements above with, "There exist a line ℓ and there exists a point P not on ℓ such that ...."

2.5.12 *not* (S *and* T) ≡ (*not* S) *or* (*not* T).

S	T	S and T	*not* (S and T)	*not* S	*not* T	(*not* C) or (*not* H)
True	True	True	False	False	False	False
True	False	False	True	False	True	True
False	True	False	True	True	False	True
False	False	False	True	True	True	True

not $(S$ *or* $T) \equiv$ (*not* S) *and* (*not* T).

S	T	S or T	*not* (S or T)	*not* S	*not* T	(*not* C) and (*not* H)
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

2.5.13 $H \Rightarrow C \equiv$ (*not* H) *or* C.

H	C	$H \Rightarrow C$	*not* H	(*not* H) or C
True	True	True	False	True
True	False	False	False	False
False	True	True	True	True
False	False	True	True	True

By De Morgan, *not* (H *and* (*not* C)) is logically equivalent to (*not* H) *or* C.

2.5.14 *not* $(H \Rightarrow C) \equiv$ (H *and* (*not* C)).

H	C	$H \Rightarrow C$	*not*($H \Rightarrow C$)	*not* C	H *and* (*not* C)
True	True	True	False	False	False
True	False	False	True	True	True
False	True	True	False	False	False
False	False	True	False	True	False

2.5.15 If $\triangle ABC$ is right triangle with right angle at vertex C, and a, b, and c are the lengths of the sides opposite vertices A, B, and C respectively, then $a^2 + b^2 = c^2$.

2.5.16 **(a)** Euclidean. If ℓ is a line and P is a point that does not lie on ℓ, then there exists exactly one line m such that P lies on m and $m \parallel \ell$.

(b) Elliptic. If ℓ is a line and P is a point that does not lie on ℓ, then there does not exist a line m such that P lies on m and $m \parallel \ell$.

(c) Hyperbolic. If ℓ is a line and P is a point that does not lie on ℓ, then there exist at least two distinct lines m and n such that P lies on both m and n and ℓ is parallel to both m and n.

2.6.1 Converse to Theorem 2.6.2.

Proof. Let ℓ and m be two lines (notation). Assume there exists exactly one point that lies on both ℓ and m (hypothesis). We must show that $\ell \neq m$ and $\ell \nparallel m$.

Suppose $\ell = m$ (RAA hypothesis). There exists two distinct points Q and R that lie on ℓ (Incidence Axiom 2). Since $m = \ell$, Q and R lie on both ℓ and m. This contradicts the hypothesis that ℓ and m intersect in exactly one point, so we can reject the RAA hypothesis and conclude that $\ell \neq m$.

Since ℓ and m have a point in common, $\ell \nparallel m$ (definition of parallel). □

Note. In the next few proofs it is convenient to introduce the notation $\overleftrightarrow{AB}$ for the line determined by A and B (Incidence Axiom 1). This notation is not defined in the textbook until Chapter 3, but it fits with Incidence Axiom 1 and allows the proofs below to be written more succinctly.

2.6.2 Theorem 2.6.3.

Proof. Let ℓ be a line (hypothesis). We must prove that there exists a point P such that P does not lie on ℓ. There exist three noncollinear points A, B, and C (Axiom 3). The three points A, B and C cannot all lie on ℓ because if they did then they would be collinear (definition of collinear). Hence at least one of them does not lie on ℓ and the proof is complete. □

Note. Many (or most) students will get the last proof wrong. The reason is that they want to assume some relationship between the line given in the hypothesis of the theorem and the points given by Axiom 3. In particular, many students will start with the first three sentences of the proof above, but will then either assert or assume that $\ell = \overleftrightarrow{AB}$.

This is an important opportunity to point out that the axioms mean exactly what they say and we must be careful not to impose our own assumptions on them. The fact that we are imposing an unstated assumption on the axioms is often covered up by the fact that the same letters are used for the three points given by Axiom 3 and the two points given by Axiom 2. So this exercise is also an opportunity to discuss notation and how we assign names to mathematical objects. (The fact that we happen to use the same letter to name two different points does not make them the same point.)

Similar comments apply to the next three proofs.

2.6.3 Theorem 2.6.4.

Proof. Let P be a point (hypothesis). We must show that there are two distinct lines ℓ and m such that P lies on both ℓ and m. There exist three noncollinear points A, B, and C (Axiom 3). There are two cases to consider: either P is equal to one of the three points A, B, and C or it is not.

Suppose, first, that $P = A$. Define $\ell = \overleftrightarrow{AB}$ and $m = \overleftrightarrow{AC}$ (Axiom 1). Obviously $P = A$ lies on both these lines. It cannot be that $\ell = m$ because in that case A, B, and C would be collinear. So the proof of this case is complete. The proofs of the cases in which $P = B$ and $P = C$ are similar.

Now suppose that P is distinct from all three of the points A, B, and C. In that case we can define three lines $\ell = \overleftrightarrow{PA}$, $m = \overleftrightarrow{PB}$, and $n = \overleftrightarrow{PC}$ (Axiom 1). These three lines cannot all be the same because if they were then A, B, and C would be collinear. Therefore at least two of them are distinct and the proof is complete. □

Note. Many students will assert that the three lines in the last paragraph are distinct. But that is not necessarily the case, as Fig. S2.5 shows.

Outline of an alternative proof: Find one line ℓ such that P lies on ℓ. Then use the previous theorem to find a point R that does not lie on ℓ. The line through P and R is the second line. This proof is simpler and better than the one given above, but most students do not think of it. You might want to lead them in that direction by suggesting that they prove the existence of one line first rather than proving the existence of both at the same time.