INSTRUCTOR'S SOLUTIONS MANUAL

DIFFERENTIAL EQUATIONS & LINEAR ALGEBRA

FOURTH EDITION

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Pearson

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CONTENTS

PREFACE

This is a solutions manual to accompany the textbook **DIFFERENTIAL EQUATIONS & LINEAR ALGEBRA** (4th edition, 2018) by C. Henry Edwards, David E. Penney, and David T. Calvis. We include solutions to most of the problems in the text. The corresponding **Student's Solutions Manual** contains solutions to most of the odd-numbered solutions in the text.

Our goal is to support teaching of the subject of differential equations with linear algebra in every way that we can. We therefore invite comments and suggested improvements for future printings of this manual, as well as advice regarding features that might be added to increase its usefulness in subsequent editions. Additional supplementary material can be found at the Expanded Applications website listed below.

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CHAPTER 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

SECTION 1.1

DIFFERENTIAL EQUATIONS AND MATHEMATICAL MODELS

The main purpose of Section 1.1 is simply to introduce the basic notation and terminology of differential equations, and to show the student what is meant by a solution of a differential equation. Also, the use of differential equations in the mathematical modeling of real-world phenomena is outlined.

Problems 1-12 are routine verifications by direct substitution of the suggested solutions into the given differential equations. We include here just some typical examples of such verifications.

3. If $y_1 = \cos 2x$ and $y_2 = \sin 2x$, then $y_1' = -2\sin 2x$ $y_2' = 2\cos 2x$, so $y_1'' = -4\cos 2x = -4y_1$ and $y_2'' = -4\sin 2x = -4y_2$. Thus $y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$.

4. If $y_1 = e^{3x}$ and $y_2 = e^{-3x}$, then $y_1 = 3e^{3x}$ and $y_2 = -3e^{-3x}$, so $y_1'' = 9e^{3x} = 9y_1$ and $y_2'' = 9e^{-3x} = 9y_2$.

5. If $y = e^x - e^{-x}$, then $y' = e^x + e^{-x}$, so $y' - y = \left(e^x + e^{-x}\right) - \left(e^x - e^{-x}\right) = 2e^{-x}$. Thus $y' = y + 2e^{-x}$.

6. If $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, then $y_1' = -2e^{-2x}$, $y_1'' = 4e^{-2x}$, $y_2' = e^{-2x} - 2xe^{-2x}$, and $y_2'' = -4e^{-2x} + 4xe^{-2x}$. Hence

$$y_1'' + 4y_1' + 4y_1 = \left(4e^{-2x}\right) + 4\left(-2e^{-2x}\right) + 4\left(e^{-2x}\right) = 0$$

and

$$y_2'' + 4y_2' + 4y_2 = \left(-4e^{-2x} + 4xe^{-2x}\right) + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4\left(xe^{-2x}\right) = 0.$$

8. If $y_1 = \cos x - \cos 2x$ and $y_2 = \sin x - \cos 2x$, then $y_1' = -\sin x + 2\sin 2x$, $y_1'' = -\cos x + 4\cos 2x$, $y_2' = \cos x + 2\sin 2x$, and $y_2'' = -\sin x + 4\cos 2x$. Hence

$$y_1'' + y_1 = \left(-\cos x + 4\cos 2x\right) + \left(\cos x - \cos 2x\right) = 3\cos 2x$$

and

$$y_2'' + y_2 = \left(-\sin x + 4\cos 2x\right) + \left(\sin x - \cos 2x\right) = 3\cos 2x.$$

11. If $y = y_1 = x^{-2}$, then $y' = -2x^{-3}$ and $y'' = 6x^{-4}$, so

$$x^2 y'' + 5x y' + 4y = x^2\left(6x^{-4}\right) + 5x\left(-2x^{-3}\right) + 4\left(x^{-2}\right) = 0.$$

If $y = y_2 = x^{-2}\ln x$, then $y' = x^{-3} - 2x^{-3}\ln x$ and $y'' = -5x^{-4} + 6x^{-4}\ln x$, so

$$\begin{aligned} x^2 y'' + 5x y' + 4y &= x^2\left(-5x^{-4} + 6x^{-4}\ln x\right) + 5x\left(x^{-3} - 2x^{-3}\ln x\right) + 4\left(x^{-2}\ln x\right) \\ &= \left(-5x^{-2} + 5x^{-2}\right) + \left(6x^{-2} - 10x^{-2} + 4x^{-2}\right)\ln x = 0. \end{aligned}$$

13. Substitution of $y = e^{rx}$ into $3y' = 2y$ gives the equation $3re^{rx} = 2e^{rx}$, which simplifies to $3r = 2$. Thus $r = 2/3$.

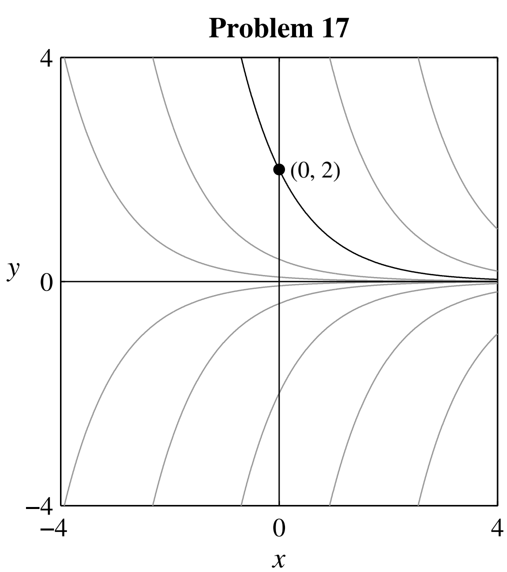
14. Substitution of $y = e^{rx}$ into $4y'' = y$ gives the equation $4r^2 e^{rx} = e^{rx}$, which simplifies to $4r^2 = 1$. Thus $r = \pm 1/2$.

15. Substitution of $y = e^{rx}$ into $y'' + y' - 2y = 0$ gives the equation $r^2 e^{rx} + re^{rx} - 2e^{rx} = 0$, which simplifies to $r^2 + r - 2 = (r+2)(r-1) = 0$. Thus $r = -2$ or $r = 1$.

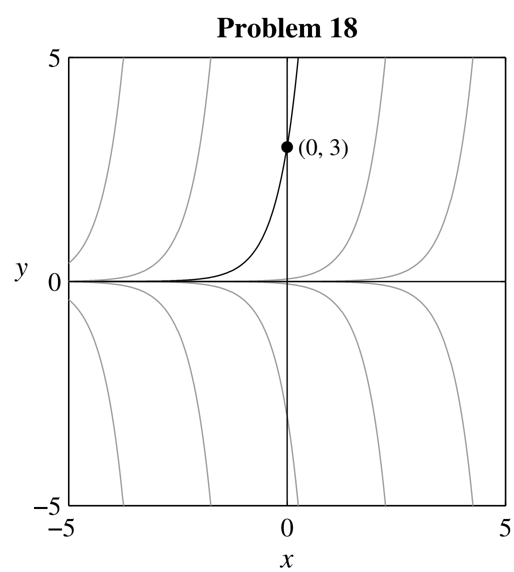
16. Substitution of $y = e^{rx}$ into $3y'' + 3y' - 4y = 0$ gives the equation $3r^2 e^{rx} + 3re^{rx} - 4e^{rx} = 0$, which simplifies to $3r^2 + 3r - 4 = 0$. The quadratic formula then gives the solutions $r = \left(-3 \pm \sqrt{57}\right)/6$.

The verifications of the suggested solutions in Problems 17-26 are similar to those in Problems 1-12. We illustrate the determination of the value of C only in some typical cases. However, we illustrate typical solution curves for each of these problems.

17. $C = 2$

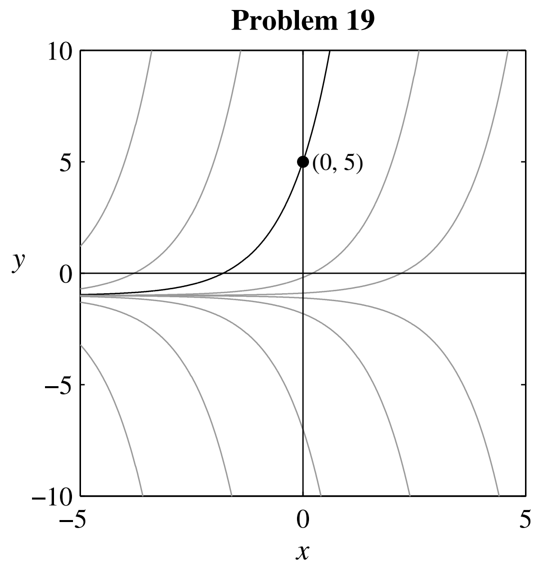


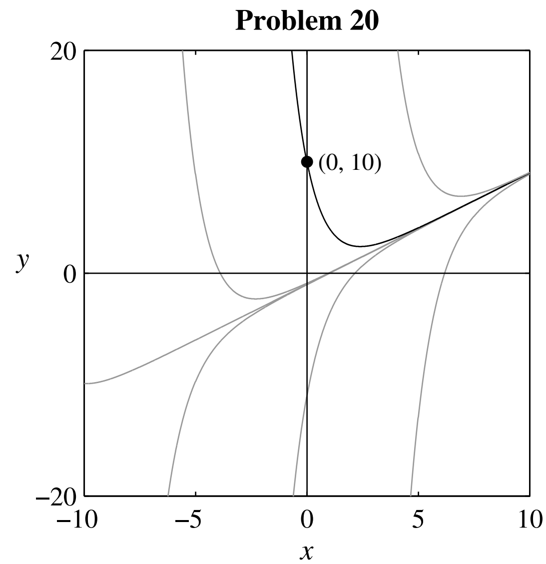
18. $C = 3$



19. If $y(x) = Ce^x - 1$, then $y(0) = 5$ gives $C - 1 = 5$, so $C = 6$.

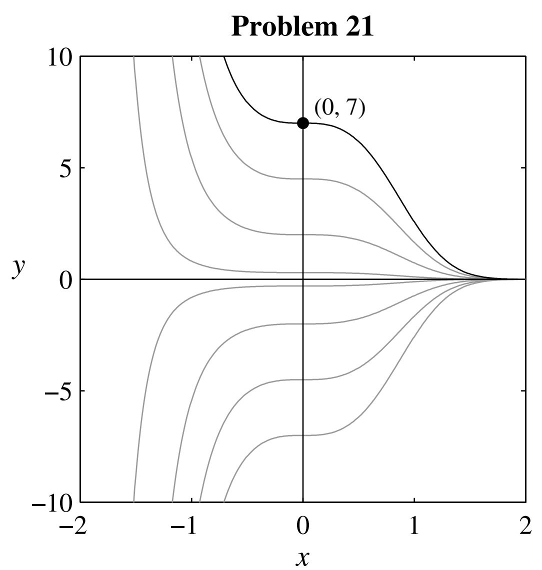
20. If $y(x) = Ce^{-x} + x - 1$, then $y(0) = 10$ gives $C - 1 = 10$, or $C = 11$.

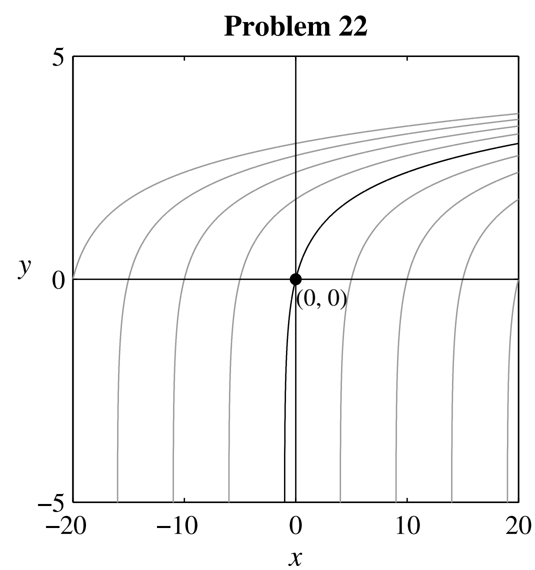




21. $C = 7$.

22. If $y(x) = \ln(x + C)$, then $y(0) = 0$ gives $\ln C = 0$, so $C = 1$.

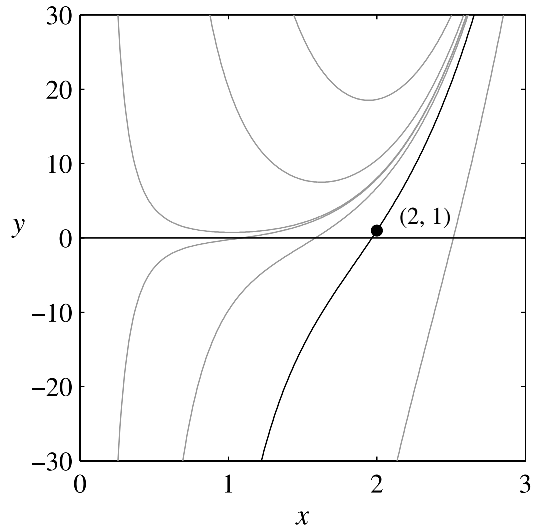




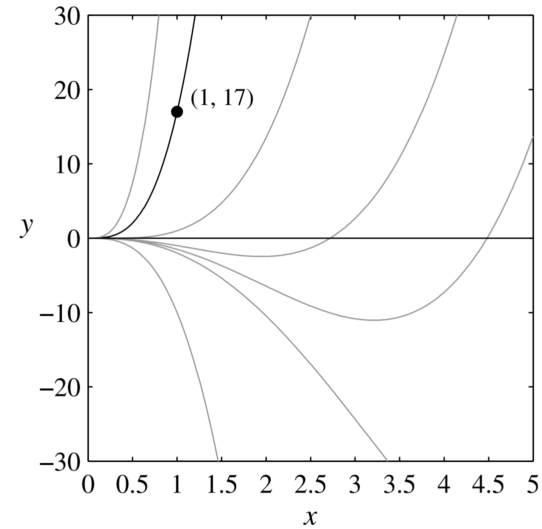
23. If $y(x) = \frac{1}{4}x^5 + Cx^{-2}$, then $y(2) = 1$ gives $\frac{1}{4} \cdot 32 + C \cdot \frac{1}{8} = 1$, or $C = -56$.

24. $C = 17$.

Problem 23

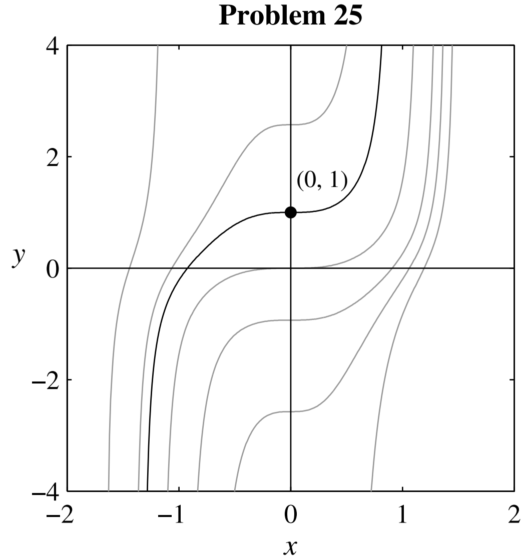


Problem 24

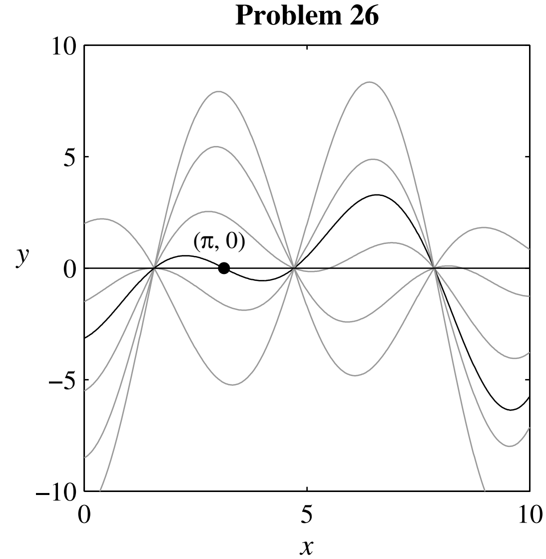


25. If $y = \tan\left(x^3 + C\right)$, then $y(0) = 1$ gives the equation $\tan C = 1$. Hence one value of C is $C = \pi/4$, as is this value plus any integral multiple of π.

Problem 25



Problem 26



26. Substitution of $x = \pi$ and $y = 0$ into $y = (x + C)\cos x$ yields $0 = (\pi + C)(-1)$, so $C = -\pi$.

27. $y' = x + y$

28. The slope of the line through (x, y) and $(x/2, 0)$ is $y' = \dfrac{y - 0}{x - x/2} = 2y/x$, so the differential equation is $xy' = 2y$.

29. If $m = y'$ is the slope of the tangent line and m' is the slope of the normal line at (x, y), then the relation $m\,m' = -1$ yields $m' = -1/y' = (y-1)/(x-0)$. Solving for y' then gives the differential equation $(1-y)\,y' = x$.

30. Here $m = y'$ and $m' = D_x(x^2 + k) = 2x$, so the orthogonality relation $m\,m' = -1$ gives the differential equation $2xy' = -1$.

31. The slope of the line through (x, y) and $(-y, x)$ is $y' = (x-y)/(-y-x)$, so the differential equation is $(x+y)y' = y - x$.

In Problems 32-36 we get the desired differential equation when we replace the "time rate of change" of the dependent variable with its derivative with respect to time *t*, the word "is" with the = sign, the phrase "proportional to" with *k*, and finally translate the remainder of the given sentence into symbols.

32. $dP/dt = k\sqrt{P}$

33. $dv/dt = kv^2$

34. $dv/dt = k(250 - v)$

35. $dN/dt = k(P - N)$

36. $dN/dt = kN(P - N)$

37. The second derivative of any linear function is zero, so we spot the two solutions $y(x) \equiv 1$ and $y(x) = x$ of the differential equation $y'' = 0$.

38. A function whose derivative equals itself, and is hence a solution of the differential equation $y' = y$, is $y(x) = e^x$.

39. We reason that if $y = kx^2$, then each term in the differential equation is a multiple of x^2. The choice $k = 1$ balances the equation and provides the solution $y(x) = x^2$.

40. If *y* is a constant, then $y' \equiv 0$, so the differential equation reduces to $y^2 = 1$. This gives the two constant-valued solutions $y(x) \equiv 1$ and $y(x) \equiv -1$.

41. We reason that if $y = ke^x$, then each term in the differential equation is a multiple of e^x. The choice $k = \frac{1}{2}$ balances the equation and provides the solution $y(x) = \frac{1}{2}e^x$.

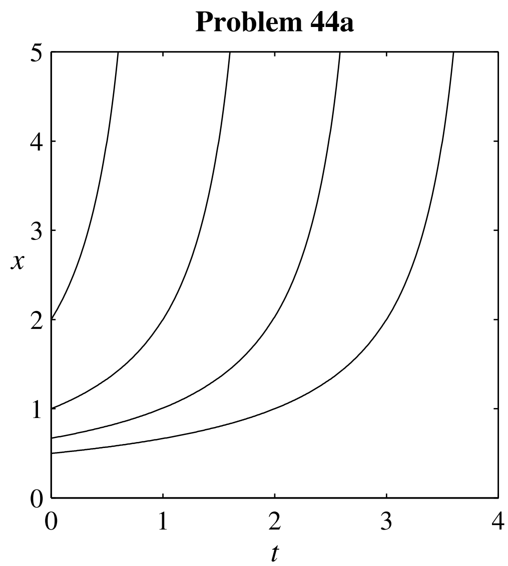
42. Two functions, each equaling the negative of its own second derivative, are the two solutions $y(x) = \cos x$ and $y(x) = \sin x$ of the differential equation $y'' = -y$.

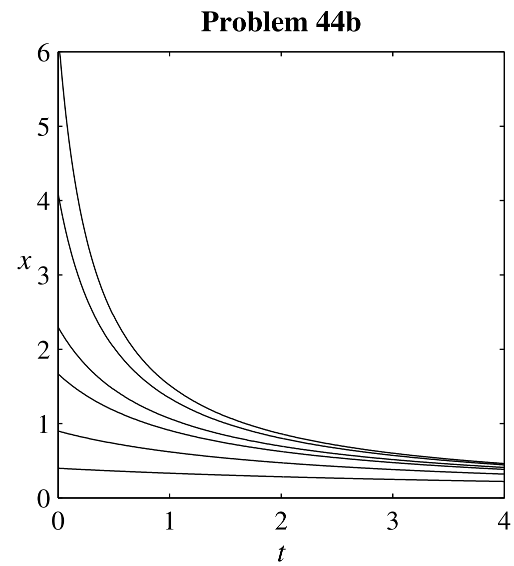
43. **(a)** We need only substitute $x(t) = 1/(C - kt)$ in both sides of the differential equation $x' = kx^2$ for a routine verification.

(b) The zero-valued function $x(t) \equiv 0$ obviously satisfies the initial value problem $x' = kx^2$, $x(0) = 0$.

44. **(a)** The figure shows typical graphs of solutions of the differential equation $x' = \frac{1}{2}x^2$.

(b) The figure shows typical graphs of solutions of the differential equation $x' = -\frac{1}{2}x^2$. We see that—whereas the graphs with $k = \frac{1}{2}$ appear to "diverge to infinity"—each solution with $k = -\frac{1}{2}$ appears to approach 0 as $t \to \infty$. Indeed, we see from the Problem 43(a) solution $x(t) = 1/(C - \frac{1}{2}t)$ that $x(t) \to \infty$ as $t \to 2C$. However, with $k = -\frac{1}{2}$ it is clear from the resulting solution $x(t) = 1/(C + \frac{1}{2}t)$ that $x(t)$ remains bounded on any bounded interval, but $x(t) \to 0$ as $t \to +\infty$.





45. Substitution of $P' = 1$ and $P = 10$ into the differential equation $P' = kP^2$ gives $k = \frac{1}{100}$, so Problem 43(a) yields a solution of the form $P(t) = 1/(C - \frac{1}{100}t)$. The initial condition $P(0) = 2$ now yields $C = \frac{1}{2}$, so we get the solution

$$P(t) = \frac{1}{\frac{1}{2} - \frac{t}{100}} = \frac{100}{50 - t}.$$

We now find readily that $P = 100$ when $t = 49$ and that $P = 1000$ when $t = 49.9$. It appears that P grows without bound (and thus "explodes") as t approaches 50.

46. Substitution of $v' = -1$ and $v = 5$ into the differential equation $v' = kv^2$ gives $k = -\frac{1}{25}$, so Problem 43(a) yields a solution of the form $v(t) = 1/(C + t/25)$. The initial condition $v(0) = 10$ now yields $C = \frac{1}{10}$, so we get the solution

$$v(t) = \frac{1}{\frac{1}{10} + \frac{t}{25}} = \frac{50}{5 + 2t}.$$

We now find readily that $v = 1$ when $t = 22.5$ and that $v = 0.1$ when $t = 247.5$. It appears that v approaches 0 as t increases without bound. Thus the boat gradually slows, but never comes to a "full stop" in a finite period of time.

47. **(a)** $y(10) = 10$ yields $10 = 1/(C - 10)$, so $C = 101/10$.

(b) There is no such value of C, but the constant function $y(x) \equiv 0$ satisfies the conditions $y' = y^2$ and $y(0) = 0$.

(c) It is obvious visually (in Fig. 1.1.8 of the text) that one and only one solution curve passes through each point (a, b) of the xy-plane, so it follows that there exists a unique solution to the initial value problem $y' = y^2$, $y(a) = b$.

48. **(b)** Obviously the functions $u(x) = -x^4$ and $v(x) = +x^4$ both satisfy the differential equation $xy' = 4y$. But their derivatives $u'(x) = -4x^3$ and $v'(x) = +4x^3$ match at $x = 0$, where both are zero. Hence the given piecewise-defined function $y(x)$ is differentiable, and therefore satisfies the differential equation because $u(x)$ and $v(x)$ do so (for $x \le 0$ and $x \ge 0$, respectively).

(c) If $a \ge 0$ (for instance), then choose C_+ fixed so that $C_+ a^4 = b$. Then the function

$$y(x) = \begin{cases} C_- x^4 & \text{if } x \le 0 \\ C_+ x^4 & \text{if } x \ge 0 \end{cases}$$

satisfies the given differential equation for every real number value of C_-

SECTION 1.2

INTEGRALS AS GENERAL AND PARTICULAR SOLUTIONS

This section introduces **general solutions** and **particular solutions** in the very simplest situation — a differential equation of the form $y' = f(x)$ — where only direct integration and evaluation of the constant of integration are involved. Students should review carefully the elementary concepts of velocity and acceleration, as well as the fps and mks unit systems.

1. Integration of $y' = 2x+1$ yields $y(x) = \int (2x+1)\,dx = x^2 + x + C$. Then substitution of $x = 0$, $y = 3$ gives $3 = 0+0+C = C$, so $y(x) = x^2 + x + 3$.

2. Integration of $y' = (x-2)^2$ yields $y(x) = \int (x-2)^2\,dx = \frac{1}{3}(x-2)^3 + C$. Then substitution of $x = 2$, $y = 1$ gives $1 = 0 + C = C$, so $y(x) = \frac{1}{3}(x-2)^3 + 1$.

3. Integration of $y' = \sqrt{x}$ yields $y(x) = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$. Then substitution of $x = 4$, $y = 0$ gives $0 = \frac{16}{3} + C$, so $y(x) = \frac{2}{3}\left(x^{3/2} - 8\right)$.

4. Integration of $y' = x^{-2}$ yields $y(x) = \int x^{-2}dx = -1/x + C$. Then substitution of $x = 1$, $y = 5$ gives $5 = -1 + C$, so $y(x) = -1/x + 6$.

5. Integration of $y' = (x+2)^{-1/2}$ yields $y(x) = \int (x+2)^{-1/2}\,dx = 2\sqrt{x+2} + C$. Then substitution of $x = 2$, $y = -1$ gives $-1 = 2\cdot 2 + C$, so $y(x) = 2\sqrt{x+2} - 5$.

6. Integration of $y' = x\left(x^2+9\right)^{1/2}$ yields $y(x) = \int x\left(x^2+9\right)^{1/2}dx = \frac{1}{3}\left(x^2+9\right)^{3/2} + C$. Then substitution of $x = -4$, $y = 0$ gives $0 = \frac{1}{3}(5)^3 + C$, so $y(x) = \frac{1}{3}\left[\left(x^2+9\right)^{3/2} - 125\right]$.

7. Integration of $y' = \dfrac{10}{x^2+1}$ yields $y(x) = \int \dfrac{10}{x^2+1}\,dx = 10\tan^{-1}x + C$. Then substitution of $x = 0$, $y = 0$ gives $0 = 10\cdot 0 + C$, so $y(x) = 10\tan^{-1}x$.

8. Integration of $y' = \cos 2x$ yields $y(x) = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$. Then substitution of $x = 0$, $y = 1$ gives $1 = 0 + C$, so $y(x) = \frac{1}{2}\sin 2x + 1$.

9. Integration of $y' = \dfrac{1}{\sqrt{1-x^2}}$ yields $y(x) = \int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$. Then substitution of $x = 0$, $y = 0$ gives $0 = 0 + C$, so $y(x) = \sin^{-1} x$.

10. Integration of $y' = xe^{-x}$ yields

$$y(x) = \int xe^{-x}\,dx = \int ue^{u}\,du = (u-1)e^{u} = -(x+1)e^{-x} + C,$$

using the substitution $u = -x$ together with Formula #46 inside the back cover of the textbook. Then substituting $x = 0$, $y = 1$ gives $1 = -1 + C$, so $y(x) = -(x+1)e^{-x} + 2$.

11. If $a(t) = 50$, then $v(t) = \int 50dt = 50t + v_0 = 50t + 10$. Hence

$$x(t) = \int (50t + 10)\,dt = 25t^2 + 10t + x_0 = 25t^2 + 10t + 20.$$

12. If $a(t) = -20$, then $v(t) = \int (-20)\,dt = -20t + v_0 = -20t - 15$. Hence

$$x(t) = \int (-20t - 15)\,dt = -10t^2 - 15t + x_0 = -10t^2 - 15t + 5.$$

13. If $a(t) = 3t$, then $v(t) = \int 3t\,dt = \tfrac{3}{2}t^2 + v_0 = \tfrac{3}{2}t^2 + 5$. Hence

$$x(t) = \int \left(\tfrac{3}{2}t^2 + 5\right)dt = \tfrac{1}{2}t^3 + 5t + x_0 = \tfrac{1}{2}t^3 + 5t.$$

14. If $a(t) = 2t + 1$, then $v(t) = \int (2t+1)\,dt = t^2 + t + v_0 = t^2 + t - 7$. Hence

$$x(t) = \int \left(t^2 + t - 7\right)dt = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + x_0 = \tfrac{1}{3}t^3 + \tfrac{1}{2}t - 7t + 4.$$

15. If $a(t) = 4(t+3)^2$, then $v(t) = \int 4(t+3)^2\,dt = \tfrac{4}{3}(t+3)^3 + C = \tfrac{4}{3}(t+3)^3 - 37$ (taking $C = -37$ so that $v(0) = -1$). Hence

$$x(t) = \int \tfrac{4}{3}(t+3)^3 - 37\,dt = \tfrac{1}{3}(t+3)^4 - 37t + C = \tfrac{1}{3}(t+3)^4 - 37t - 26.$$

16. If $a(t) = \dfrac{1}{\sqrt{t+4}}$, then $v(t) = \int \dfrac{1}{\sqrt{t+4}}\,dt = 2\sqrt{t+4} + C = 2\sqrt{t+4} - 5$ (taking $C = -5$ so that $v(0) = -1$). Hence

$$x(t) = \int \left(2\sqrt{t+4} - 5\right)dt = \tfrac{4}{3}(t+4)^{3/2} - 5t + C = \tfrac{4}{3}(t+4)^{3/2} - 5t - \tfrac{29}{3}$$

(taking $C = -29/3$ so that $x(0) = 1$).

17. If $a(t)=(t+1)^{-3}$, then $v(t)=\int (t+1)^{-3}\,dt=-\frac{1}{2}(t+1)^{-2}+C=-\frac{1}{2}(t+1)^{-2}+\frac{1}{2}$ (taking $C=\frac{1}{2}$ so that $v(0)=0$). Hence

$$x(t)=\int -\tfrac{1}{2}(t+1)^{-2}+\tfrac{1}{2}\,dt=\tfrac{1}{2}(t+1)^{-1}+\tfrac{1}{2}t+C=\tfrac{1}{2}\left[(t+1)^{-1}+t-1\right]$$

(taking $C=-\frac{1}{2}$ so that $x(0)=0$).

18. If $a(t)=50\sin 5t$, then $v(t)=\int 50\sin 5t\,dt=-10\cos 5t+C=-10\cos 5t$ (taking $C=0$ so that $v(0)=-10$). Hence

$$x(t)=\int -10\cos 5t\,dt=-2\sin 5t+C=-2\sin 5t+10$$

(taking $C=-10$ so that $x(0)=8$).

Students should understand that Problems 19-22, though different at first glance, are solved in the same way as the preceding ones, that is, by means of the fundamental theorem of calculus in the form $x(t)=x(t_0)+\int_{t_0}^{t} v(s)\,ds$ cited in the text. Actually in these problems $x(t)=\int_0^t v(s)\,ds$, since t_0 and $x(t_0)$ are each given to be zero.

19. The graph of $v(t)$ shows that $v(t)=\begin{cases}5 & \text{if } 0\le t\le 5\\ 10-t & \text{if } 5\le t\le 10\end{cases}$, so that $x(t)=\begin{cases}5t+C_1 & \text{if } 0\le t\le 5\\ 10t-\frac{1}{2}t^2+C_2 & \text{if } 5\le t\le 10\end{cases}$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=5t$ and $x(t)=10t-\frac{1}{2}t^2+C_2$ agree when $t=5$. This implies that $C_2=-\frac{25}{2}$, leading to the graph of $x(t)$ shown.

Alternate solution for Problem 19 (and similarly for 20-22): The graph of $v(t)$ shows that $v(t)=\begin{cases}5 & \text{if } 0\le t\le 5\\ 10-t & \text{if } 5\le t\le 10\end{cases}$. Thus for $0\le t\le 5$, $x(t)=\int_0^t v(s)\,ds$ is given by $\int_0^t 5ds=5t$, whereas for $5\le t\le 10$ we have

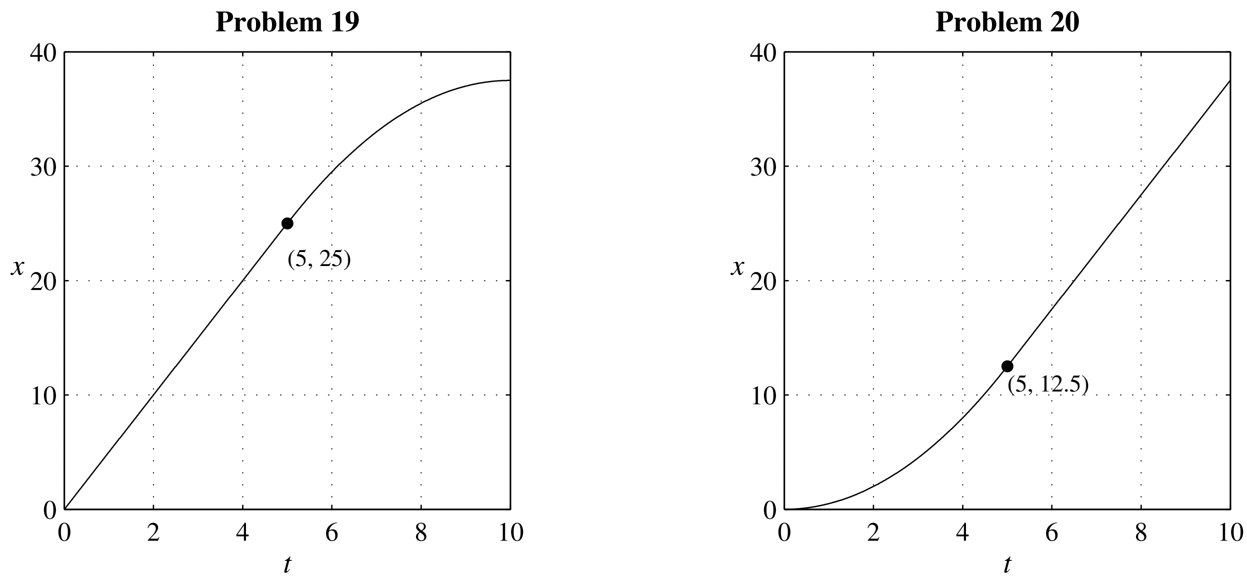
$$x(t)=\int_0^t v(s)\,ds=\int_0^5 5\,ds+\int_5^t 10-s\,ds$$

$$=25+\left(10s-\frac{s^2}{2}\Bigg|_{s=5}^{s=t}\right)=25+10t-\frac{t^2}{2}-\frac{75}{2}=10t-\frac{t^2}{2}-\frac{25}{2}.$$

The graph of $x(t)$ is shown.

20. The graph of $v(t)$ shows that $v(t)=\begin{cases} t & \text{if } 0\le t\le 5 \\ 5 & \text{if } 5\le t\le 10 \end{cases}$, so that

$x(t)=\begin{cases} \frac{1}{2}t^2+C_1 & \text{if } 0\le t\le 5 \\ 5t+C_2 & \text{if } 5\le t\le 10 \end{cases}$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=\frac{1}{2}t^2$ and $x(t)=5t+C_2$ agree when $t=5$. This implies that $C_2=-\frac{25}{2}$, leading to the graph of $x(t)$ shown.



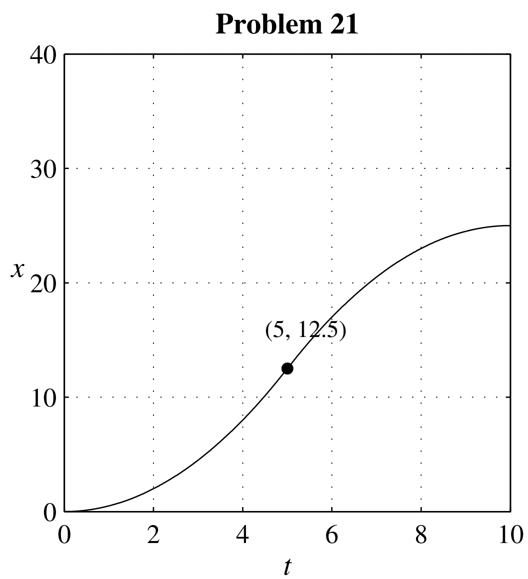
21. The graph of $v(t)$ shows that $v(t)=\begin{cases} t & \text{if } 0\le t\le 5 \\ 10-t & \text{if } 5\le t\le 10 \end{cases}$, so that

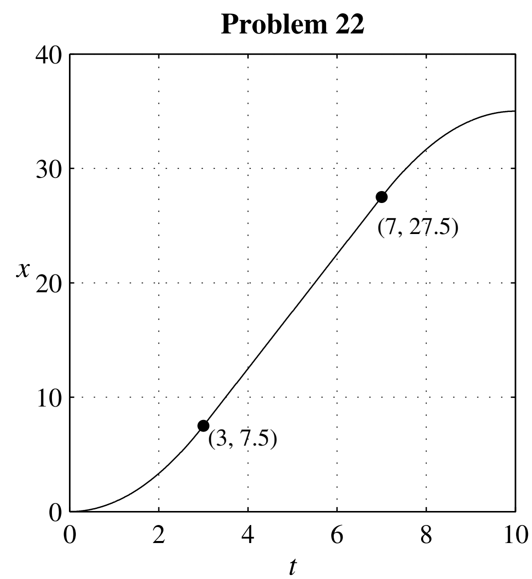
$x(t)=\begin{cases} \frac{1}{2}t^2+C_1 & \text{if } 0\le t\le 5 \\ 10t-\frac{1}{2}t^2+C_2 & \text{if } 5\le t\le 10 \end{cases}$. Now $C_1=0$ because $x(0)=0$, and continuity of $x(t)$ requires that $x(t)=\frac{1}{2}t^2$ and $x(t)=10t-\frac{1}{2}t^2+C_2$ agree when $t=5$. This implies that $C_2=-25$, leading to the graph of $x(t)$ shown.

22. For $0\le t\le 3$, $v(t)=\frac{5}{3}t$, so $x(t)=\frac{5}{6}t^2+C_1$. Now $C_1=0$ because $x(0)=0$, so $x(t)=\frac{5}{6}t^2$ on this first interval, and its right-endpoint value is $x(3)=\frac{15}{2}$.

For $3\le t\le 7$, $v(t)=5$, so $x(t)=5t+C_2$ Now $x(3)=\frac{15}{2}$ implies that $C_2=-\frac{15}{2}$, so $x(t)=5t-\frac{15}{2}$ on this second interval, and its right-endpoint value is $x(7)=\frac{55}{2}$.

For $7\le t\le 10$, $v-5=-\frac{5}{3}(t-7)$, so $v(t)=-\frac{5}{3}t+\frac{50}{3}$. Hence $x(t)=-\frac{5}{6}t^2+\frac{50}{3}t+C_3$, and $x(7)=\frac{55}{2}$ implies that $C_3=-\frac{290}{6}$. Finally, $x(t)=\frac{1}{6}(-5t^2+100t-290)$ on this third interval, leading to the graph of $x(t)$ shown.





23. $v(t) = -9.8t + 49$, so the ball reaches its maximum height ($v = 0$) after $t = 5$ seconds. Its maximum height then is $y(5) = -4.9(5)^2 + 49(5) = 122.5$ meters.

24. $v = -32t$ and $y = -16t^2 + 400$, so the ball hits the ground ($y = 0$) when $t = 5$ sec, and then $v = -32(5) = -160$ ft/sec.

25. $a = -10$ m/s^2 and $v_0 = 100$ km/h ≈ 27.78 m/s, so $v = -10t + 27.78$, and hence $x(t) = -5t^2 + 27.78t$. The car stops when $v = 0$, that is $t \approx 2.78$ s, and thus the distance traveled before stopping is $x(2.78) \approx 38.59$ meters.

26. $v = -9.8t + 100$ and $y = -4.9t^2 + 100t + 20$.

(a) $v = 0$ when $t = 100/9.8$ s, so the projectile's maximum height is $y(100/9.8) = -4.9(100/9.8)^2 + 100(100/9.8) + 20 \approx 530$ meters.

(b) It passes the top of the building when $y(t) = -4.9t^2 + 100t + 20 = 20$, and hence after $t = 100/4.9 \approx 20.41$ seconds.

(c) The roots of the quadratic equation $y(t) = -4.9t^2 + 100t + 20 = 0$ are $t = -0.20, 20.61$. Hence the projectile is in the air 20.61 seconds.

27. $a = -9.8$ m/s^2, so $v = -9.8t - 10$ and $y = -4.9t^2 - 10t + y_0$. The ball hits the ground when $y = 0$ and $v = -9.8t - 10 = -60$ m/s, so $t \approx 5.10$ s. Hence the height of the building is

$$y_0 = 4.9(5.10)^2 + 10(5.10) \approx 178.57 \text{ m}.$$

28. $v = -32t - 40$ and $y = -16t^2 - 40t + 555$. The ball hits the ground ($y = 0$) when $t \approx 4.77$ s, with velocity $v = v(4.77) \approx -192.64$ ft/s, an impact speed of about 131 mph.

29. Integration of $dv/dt = 0.12t^2 + 0.6t$ with $v(0) = 0$ gives $v(t) = 0.04t^3 + 0.3t^2$. Hence $v(10) = 70$ ft/s. Then integration of $dx/dt = 0.04t^3 + 0.3t^2$ with $x(0) = 0$ gives $x(t) = 0.01t^4 + 0.1t^3$, so $x(10) = 200$ ft. Thus after 10 seconds the car has gone 200 ft and is traveling at 70 ft/s.

30. Taking $x_0 = 0$ and $v_0 = 60$ mph $= 88$ ft/s, we get $v = -at + 88$, and $v = 0$ yields $t = 88/a$. Substituting this value of *t*, as well as $x = 176$ ft, into $x = -at^2/2 + 88t$ leads to $a = 22$ ft/s^2. Hence the car skids for $t = 88/22 = 4$ s.

31. If $a = -20$ m/s^2 and $x_0 = 0$, then the car's velocity and position at time *t* are given by $v = -20t + v_0$ and $x = -10t^2 + v_0 t$. It stops when $v = 0$ (so $v_0 = 20t$), and hence when $x = 75 = -10t^2 + (20t)t = 10t^2$. Thus $t = \sqrt{7.5}$ s, so

$$v_0 = 20\sqrt{7.5} \approx 54.77 \text{ m/s} \approx 197 \text{ km/hr}.$$

32. Starting with $x_0 = 0$ and $v_0 = 50$ km/h $= 5 \times 10^4$ m/h, we find by the method of Problem 30 that the car's deceleration is $a = (25/3) \times 10^7$ m/h^2. Then, starting with $x_0 = 0$ and $v_0 = 100$ km/h $= 10^5$ m/h, we substitute $t = v_0/a$ into $x = -\frac{1}{2}at^2 + v_0 t$ and find that $x = 60$ m when $v = 0$. Thus doubling the initial velocity quadruples the distance the car skids.

33. If $v_0 = 0$ and $y_0 = 20$, then $v = -at$ and $y = -\frac{1}{2}at^2 + 20$. Substitution of $t = 2$, $y = 0$ yields $a = 10$ ft/s^2. If $v_0 = 0$ and $y_0 = 200$, then $v = -10t$ and $y = -5t^2 + 200$. Hence $y = 0$ when $t = \sqrt{40} = 2\sqrt{10}$ s and $v = -20\sqrt{10} \approx -63.25$ ft/s.

34. **On Earth:** $v = -32t + v_0$, so $t = v_0/32$ at maximum height (when $v = 0$). Substituting this value of *t* and $y = 144$ in $y = -16t^2 + v_0 t$, we solve for $v_0 = 96$ ft/s as the initial speed with which the person can throw a ball straight upward.
On Planet Gzyx: From Problem 33, the surface gravitational acceleration on planet Gzyx is $a = 10$ ft/s^2, so $v = -10t + 96$ and $y = -5t^2 + 96t$. Therefore $v = 0$ yields $t = 9.6$ s and so $y_{max} = y(9.6) = 460.8$ ft is the height a ball will reach if its initial velocity is 96 ft/s.

35. If $v_0 = 0$ and $y_0 = h$, then the stone's velocity and height are given by $v = -gt$ and $y = -0.5gt^2 + h$, respectively. Hence $y = 0$ when $t = \sqrt{2h/g}$, so $v = -g\sqrt{2h/g} = -\sqrt{2gh}$.

36. The method of solution is precisely the same as that in Problem 30. We find first that, on Earth, the woman must jump straight upward with initial velocity $v_0 = 12$ ft/s to reach a maximum height of 2.25 ft. Then we find that, on the Moon, this initial velocity yields a maximum height of about 13.58 ft.

37. We use units of miles and hours. If $x_0 = v_0 = 0$, then the car's velocity and position after t hours are given by $v = at$ and $x = \frac{1}{2}at^2$, respectively. Since $v = 60$ when $t = 5/6$, the velocity equation yields . Hence the distance traveled by 12:50 pm is $x = \frac{1}{2} \cdot 72 \cdot (5/6)^2 = 25$ miles.

38. Again we have $v = at$ and $x = \frac{1}{2}at^2$. But now $v = 60$ when $x = 35$. Substitution of $a = 60/t$ (from the velocity equation) into the position equation yields $35 = \frac{1}{2}(60/t)t^2 = 30t$, whence $t = 7/6$ h, that is, 1:10 pm.

39. Integration of $y' = (9/v_S)(1 - 4x^2)$ yields $y = (3/v_S)(3x - 4x^3) + C$, and the initial condition $y(-1/2) = 0$ gives $C = 3/v_S$. Hence the swimmer's trajectory is $y(x) = (3/v_S)(3x - 4x^3 + 1)$. Substitution of $y(1/2) = 1$ now gives $v_S = 6$ mph.

40. Integration of $y' = 3(1 - 16x^4)$ yields $y = 3x - (48/5)x^5 + C$, and the initial condition $y(-1/2) = 0$ gives $C = 6/5$. Hence the swimmer's trajectory is

$$y(x) = (1/5)(15x - 48x^5 + 6),$$

and so his downstream drift is $y(1/2) = 2.4$ miles.

41. The bomb equations are $a = -32$, $v = -32t$, and $s_B = s = -16t^2 + 800$ with $t = 0$ at the instant the bomb is dropped. The projectile is fired at time $t = 2$, so its corresponding equations are $a = -32$, $v = -32(t-2) + v_0$, and $s_P = s = -16(t-2)^2 + v_0(t-2)$ for $t \geq 2$ (the arbitrary constant vanishing because $s_P(2) = 0$). Now the condition $s_B(t) = -16t^2 + 800 = 400$ gives $t = 5$, and then the further requirement that $s_P(5) = 400$ yields $v_0 = 544/3 \approx 181.33$ ft/s for the projectile's needed initial velocity.

42. Let $x(t)$ be the (positive) altitude (in miles) of the spacecraft at time t (hours), with $t = 0$ corresponding to the time at which its retrorockets are fired; let $v(t) = x'(t)$ be the velocity of the spacecraft at time t. Then $v_0 = -1000$ and $x_0 = x(0)$ is unknown. But the (constant) acceleration is $a = +20000$, so $v(t) = 20000t - 1000$ and $x(t) = 10000t^2 - 1000t + x_0$. Now $v(t) = 20000t - 1000 = 0$ (soft touchdown) when $t = \frac{1}{20}\,\text{h}$ (that is, after exactly 3 minutes of descent). Finally, the condition $0 = x\left(\frac{1}{20}\right) = 10000\left(\frac{1}{20}\right)^2 - 1000\left(\frac{1}{20}\right) + x_0$ yields $x_0 = 25\,\text{miles}$ for the altitude at which the retrorockets should be fired.

43. The velocity and position functions for the spacecraft are $v_S(t) = 0.0098t$ and $x_S(t) = 0.0049t^2$, and the corresponding functions for the projectile are $v_P(t) = \frac{1}{10}c = 3\times10^7$ and $x_P(t) = 3\times10^7 t$. The condition that $x_S = x_P$ when the spacecraft overtakes the projectile gives $0.0049t^2 = 3\times10^7 t$, whence

$$t = \frac{3\times10^7}{0.0049} \approx 6.12245\times10^9\ \text{s} \approx \frac{6.12245\times10^9}{(3600)(24)(365.25)} \approx 194\ \text{years}.$$

Since the projectile is traveling at $\frac{1}{10}$ the speed of light, it has then traveled a distance of about 19.4 light years, which is about 1.8367×10^{17} meters.

44. Let $a > 0$ denote the constant deceleration of the car when braking, and take $x_0 = 0$ for the car's position at time $t = 0$ when the brakes are applied. In the police experiment with $v_0 = 25$ ft/s, the distance the car travels in t seconds is given by

$$x(t) = -\frac{1}{2}at^2 + \frac{88}{60}\cdot 25t,$$

with the factor $\frac{88}{60}$ used to convert the velocity units from mi/h to ft/s. When we solve simultaneously the equations $x(t) = 45$ and $x'(t) = 0$ we find that $a = \frac{1210}{81} \approx 14.94\,\text{ft/s}^2$. With this value of the deceleration and the (as yet) unknown velocity v_0 of the car involved in the accident, its position function is

$$x(t) = -\frac{1}{2}\cdot\frac{1210}{81}t^2 + v_0 t.$$

The simultaneous equations $x(t) = 210$ and $x'(t) = 0$ finally yield $v_0 = \frac{110}{9}\sqrt{42} \approx 79.21\,\text{ft/s}$, that is, almost exactly 54 miles per hour.

45. Equation (10) gives

$$v(t)^2 - v_0^2 = (at + v_0)^2 - v_0^2 = a^2t^2 + 2atv_0 + \cancel{v_0^2} - \cancel{v_0^2} = a^2t^2 + 2atv_0,$$

whereas by Eq. (11),

$$2a\left[x(t)-x_0\right]=2a\left(\frac{1}{2}at^2+v_0t+\not{x_0}-\not{x_0}\right)=a^2t^2+2av_0t,$$

proving the formula.

To apply this formula to Example 2, let x_0 denote (as in the example) the height of the lander above the lunar surface at the moment when the retrorockets should be activated. Thus $v_0=-450$. We further take $x(t)=0$ and $v(t)=0$, corresponding to the lander's touch down on the planet's surface. Because $a=+2.5$, our formula gives $(-450)^2-0^2=2\cdot 2.5\cdot(x_0-0)$, or $x_0=\dfrac{(-450)^2}{2\cdot 2.5}=40{,}500\,\text{m}$, in agreement with the example.

SECTION 1.3

SLOPE FIELDS AND SOLUTION CURVES

The instructor may choose to delay covering Section 1.3 until later in Chapter 1. However, before proceeding to Chapter 2, it is important that students come to grips at some point with the question of the existence of a unique solution of a differential equation — and realize that it makes no sense to look for the solution without knowing in advance that it exists. It may help some students to simplify the statement of the existence-uniqueness theorem as follows:

> Suppose that the function $f(x,y)$ and the partial derivative $\partial f/\partial y$ are both continuous in some neighborhood of the point (a,b). Then the initial value problem
>
> $$\frac{dy}{dx}=f(x,y),\; y(a)=b$$
>
> has a unique solution in some neighborhood of the point a.

Slope fields and geometrical solution curves are introduced in this section as a concrete aid in visualizing solutions and existence-uniqueness questions. Instead, we provide some details of the construction of the figure for the Problem 1 answer, and then include without further comment the similarly constructed figures for Problems 2 through 9.

1. The following sequence of *Mathematica* 7 commands generates the slope field and the solution curves through the given points. Begin with the differential equation $dy/dx=f(x,y)$, where

```
f[x_, y_] := -y - Sin[x]
```

Then set up the viewing window

```
a = -3; b = 3; c = -3; d = 3;
```

The slope field is then constructed by the command

```
dfield = VectorPlot[{1, f[x, y]}, {x, a, b}, {y, c, d},
   PlotRange -> {{a, b}, {c, d}}, Axes -> True, Frame -> True,
   FrameLabel -> {TraditionalForm[x], TraditionalForm[y]},
   AspectRatio -> 1, VectorStyle -> {Gray, "Segment"},
   VectorScale -> {0.02, Small, None},
   FrameStyle -> (FontSize -> 12), VectorPoints -> 21,
   RotateLabel -> False]
```

The original curve shown in Fig. 1.3.15 of the text (and its initial point not shown there) are plotted by the commands

```
x0 = -1.9; y0 = 0;
point0 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{y'[x] == f[x, y[x]], y[x0] == y0}, y[x],
   {x, a, b}];
curve0 = Plot[soln[[1, 1, 2]], {x, a, b}, PlotStyle ->
   {Thickness[0.0065], Blue}];
Show[curve0, point0]
```

(The *Mathematica* **NDSolve** command carries out an approximate numerical solution of the given differential equation. Numerical solution techniques are discussed in Sections 2.4–2.6 of the textbook.)

The coordinates of the 12 points are marked in Fig. 1.3.15 in the textbook. For instance the 7th point is $(-2.5,1)$. It and the corresponding solution curve are plotted by the commands

```
x0 = -2.5; y0 = 1;
point7 = Graphics[{PointSize[0.025], Point[{x0, y0}]}];
soln = NDSolve[{y'[x] == f[x, y[x]], y[x0] == y0}, y[x],
   {x, a, b}];
curve7 = Plot[soln[[1, 1, 2]], {x, a, b},
   PlotStyle -> {Thickness[0.0065], Blue}];
Show[curve7, point7]
```

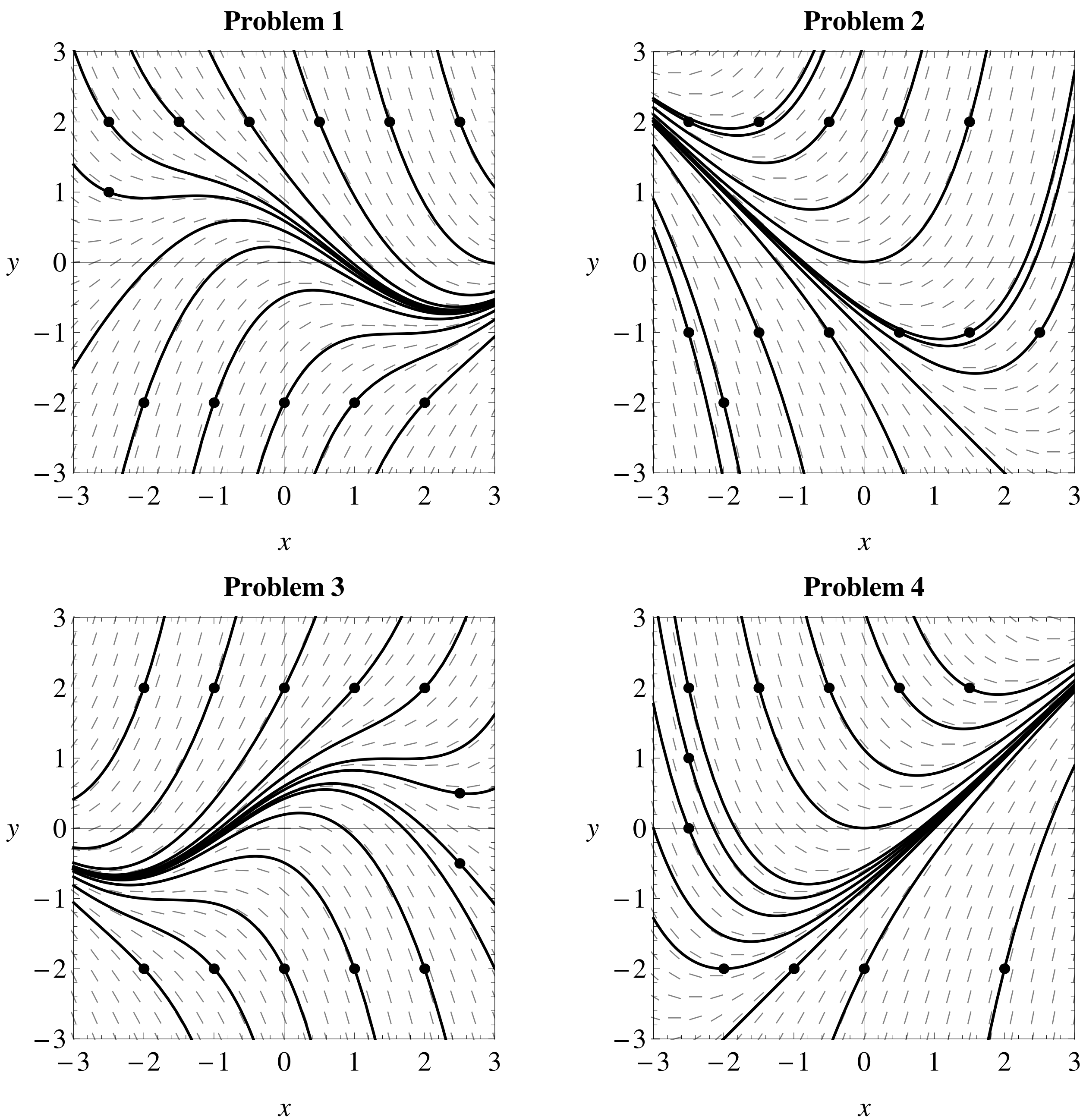
The following command superimposes the two solution curves and starting points found so far upon the slope field:

```
Show[dfield, point0, curve0, point7, curve7]
```

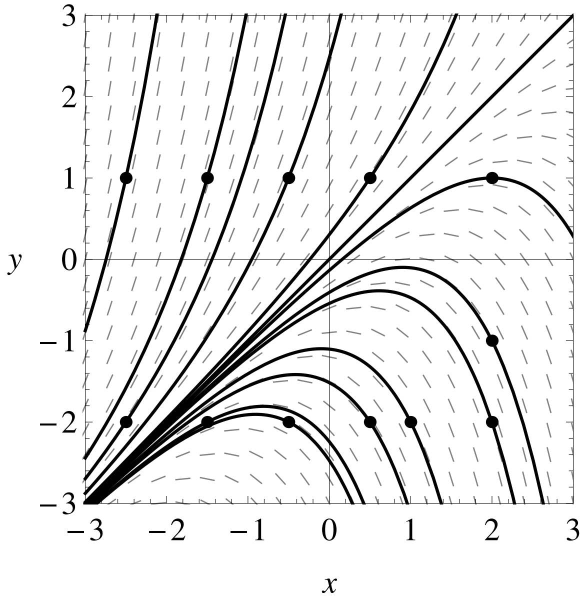
We could continue in this way to build up the entire graphic called for in the problem. Here is an alternative looping approach, variations of which were used to generate the graphics below for Problems 1-10:

```
points = {{-2.5,2}, {-1.5,2}, {-0.5,2}, {0.5,2}, {1.5,2},
   {2.5,2}, {-2,-2}, {-1,-2}, {0,-2}, {1,-2}, {2,-2}, {-2.5,1}};
curves = {}; (* start with null lists *)
dots = {};
Do [
   x0 = points[[i, 1]];
   y0 = points[[i, 2]];
   newdot = Graphics[{PointSize[0.025],Point[{x0, y0}]}];
```

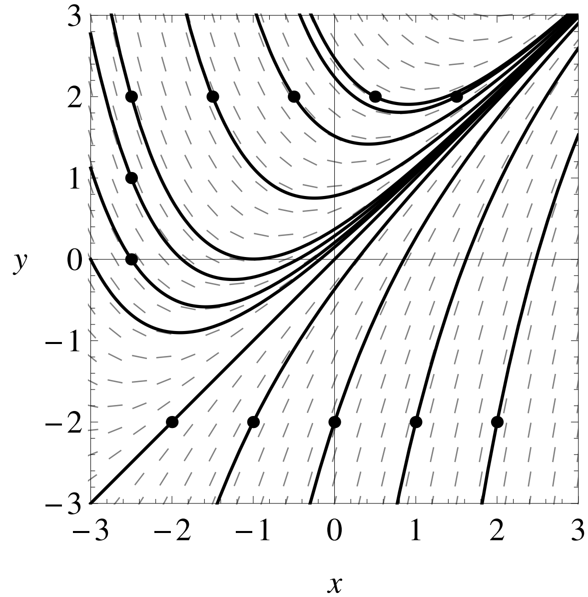
```
    dots = AppendTo[dots, newdot];
    soln = NDSolve[{y'[x] == f[x, y[x]],y[x0] == y0}, y[x],
       {x, a, b}];
    newcurve = Plot[soln[[1, 1, 2]], {x, a, b},
       PlotStyle -> {Thickness[0.0065], Black}];
    AppendTo[curves, newcurve],
    {i, 1, Length[points]}];
 Show[dfield, curves, dots, PlotLabel -> Style["Problem 1", Bold,
    11]]
```



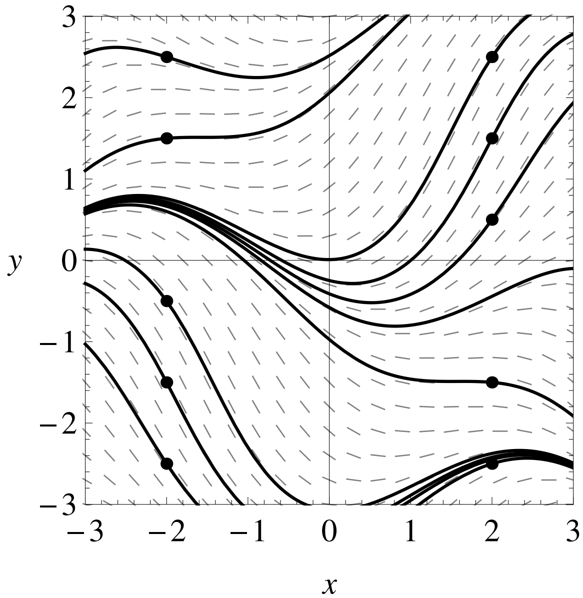
Problem 5



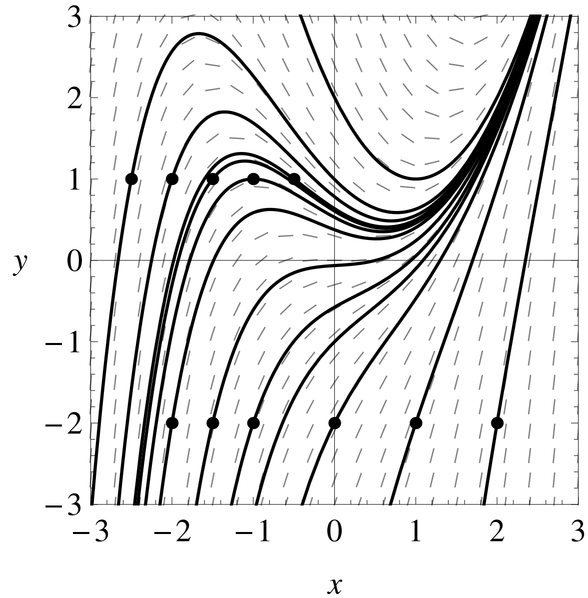
Problem 6

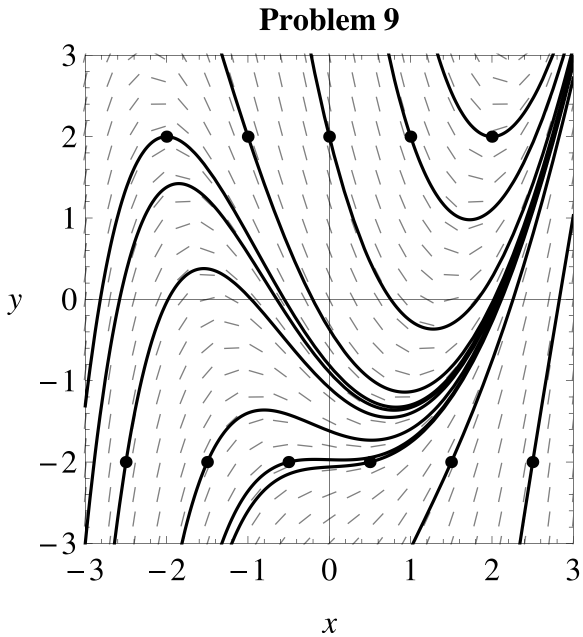


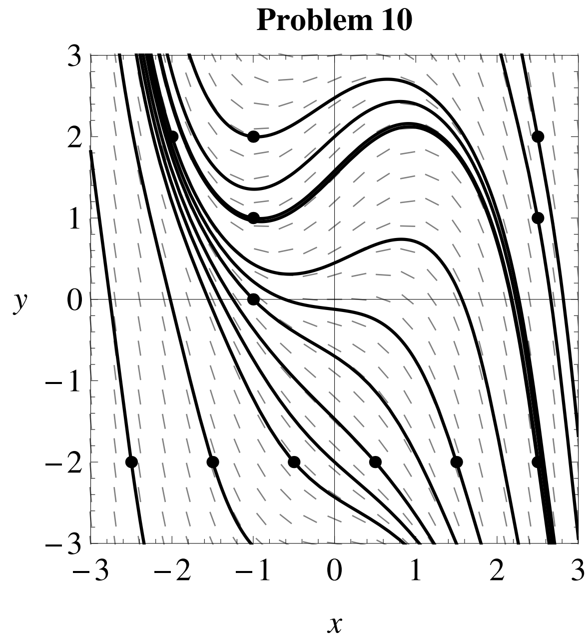
Problem 7



Problem 8







11. Because both $f(x,y)=2x^2y^2$ and $D_y f(x,y)=4x^2y$ are continuous everywhere, the existence-uniqueness theorem of Section 1.3 in the textbook guarantees the existence of a unique solution in some neighborhood of $x=1$.

12. Both $f(x,y)=x\ln y$ and $\partial f/\partial y = x/y$ are continuous in a neighborhood of $(1,1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x=1$.

13. Both $f(x,y)=y^{1/3}$ and $\partial f/\partial y=\frac{1}{3}y^{-2/3}$ are continuous near $(0,1)$, so the theorem guarantees the existence of a unique solution in some neighborhood of $x=0$.

14. The function $f(x,y)=y^{1/3}$ is continuous in a neighborhood of $(0,0)$, but $\partial f/\partial y=\frac{1}{3}y^{-2/3}$ is not, so the theorem guarantees existence but not uniqueness in some neighborhood of $x=0$. (See Remark 2 following the theorem.)

15. The function $f(x,y)=(x-y)^{1/2}$ is not continuous at $(2,2)$ because it is not even defined if $y>x$. Hence the theorem guarantees neither existence nor uniqueness in any neighborhood of the point $x=2$.

16. The function $f(x,y)=(x-y)^{1/2}$ and $\partial f/\partial y=-\frac{1}{2}(x-y)^{-1/2}$ are continuous in a neighborhood of $(2,1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x=2$.

17. Both $f(x,y)=(x-1)/y$ and $\partial f/\partial y=-(x-1)/y^2$ are continuous near $(0,1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x=0$.

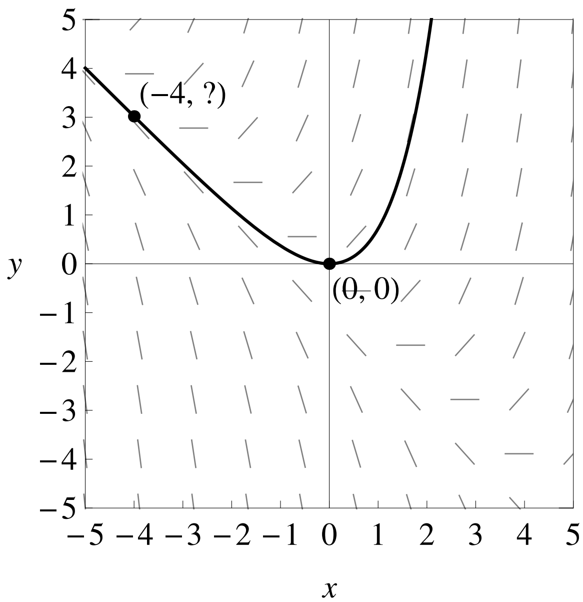
18. Neither $f(x,y)=(x-1)/y$ nor $\partial f/\partial y=-(x-1)/y^2$ is continuous near $(1,0)$, so the existence-uniqueness theorem guarantees nothing.

19. Both $f(x,y)=\ln(1+y^2)$ and $\partial f/\partial y=2y/(1+y^2)$ are continuous near $(0,0)$, so the theorem guarantees the existence of a unique solution near $x=0$.

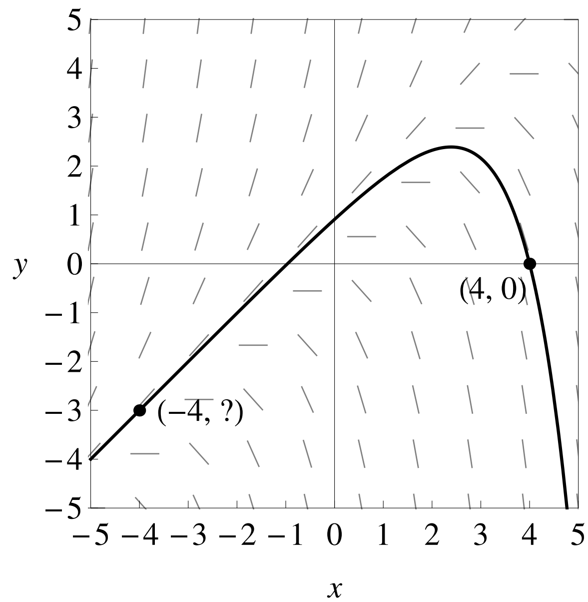
20. Both $f(x,y)=x^2-y^2$ and $\partial f/\partial y=-2y$ are continuous near $(0,1)$, so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of $x=0$.

21. The figure shown can be constructed using commands similar to those in Problem 1, above. Tracing this solution curve, we see that $y(-4)\approx 3$. (An exact solution of the differential equation yields the more accurate approximation $y(-4)=3+e^{-4}\approx 3.0183$.)

Problem 21

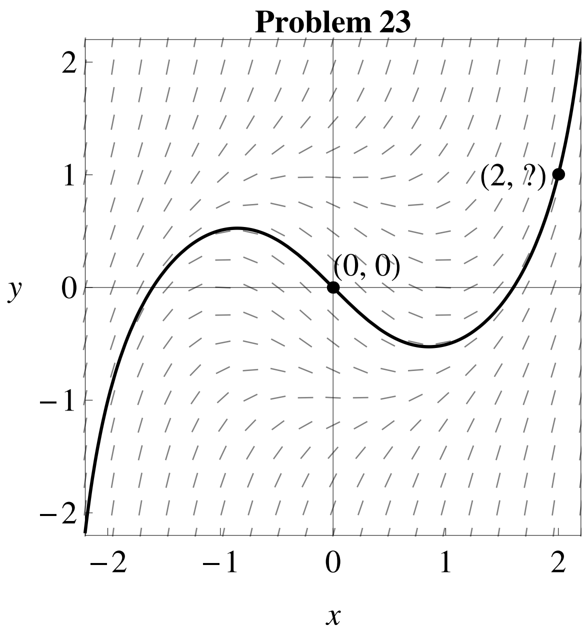


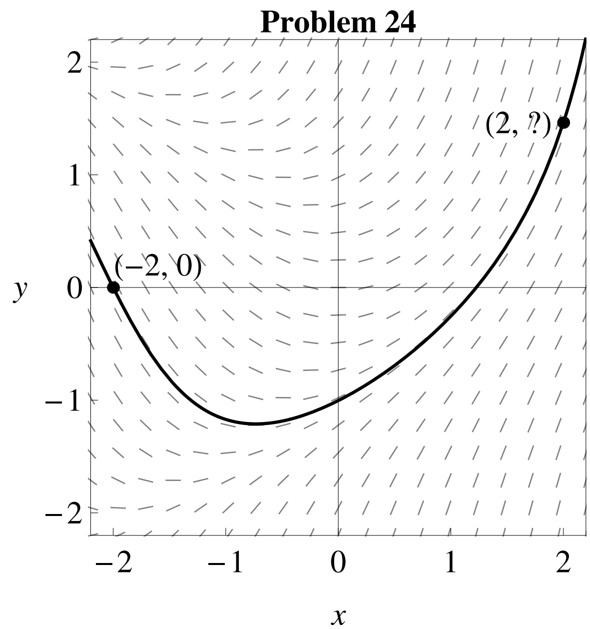
Problem 22



22. Tracing the curve in the figure shown, we see that $y(-4)\approx -3$. An exact solution of the differential equation yields the more accurate approximation $y(-4)\approx -3.0017$.

23. Tracing the curve in the figure shown, we see that $y(2)\approx 1$. A more accurate approximation is $y(2)\approx 1.0044$.



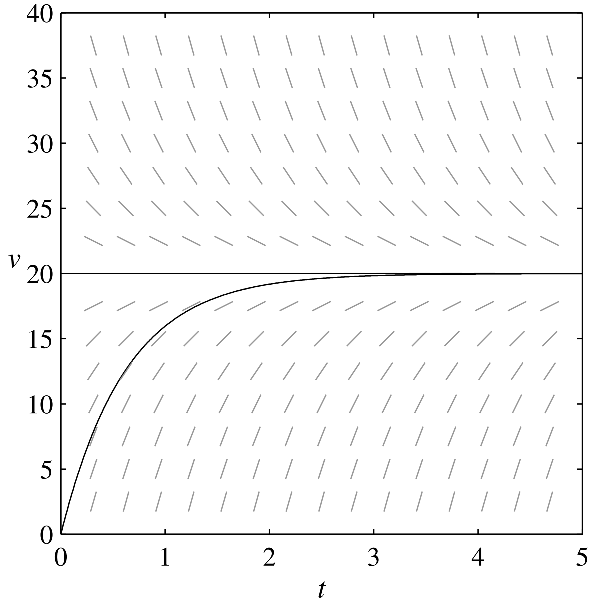


24. Tracing the curve in the figure shown, we see that $y(2) \approx 1.5$. A more accurate approximation is $y(2) \approx 1.4633$.

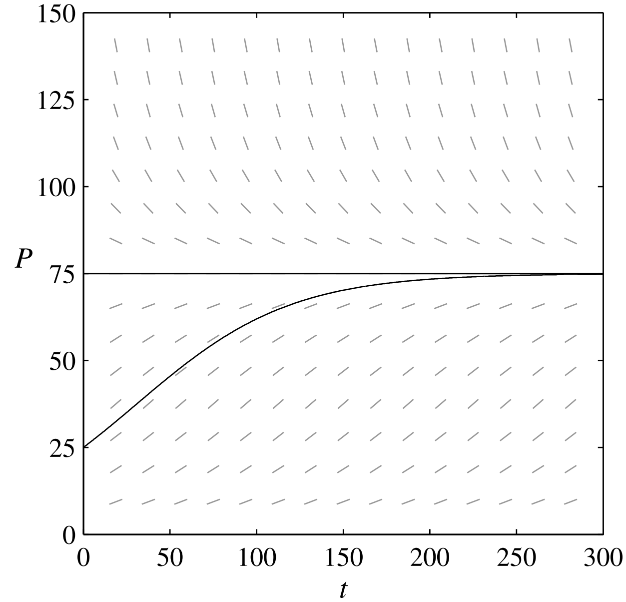
25. The figure indicates a limiting velocity of 20 ft/sec — about the same as jumping off a $6\frac{1}{4}$-foot wall, and hence quite survivable. Tracing the curve suggests that $v(t) = 19$ ft/sec when t is a bit less than 2 seconds. An exact solution gives $t \approx 1.8723$ then.

26. The figure suggests that there are 40 deer after about 60 months; a more accurate value is $t \approx 61.61$. And it's pretty clear that the limiting population is 75 deer.

Problem 25



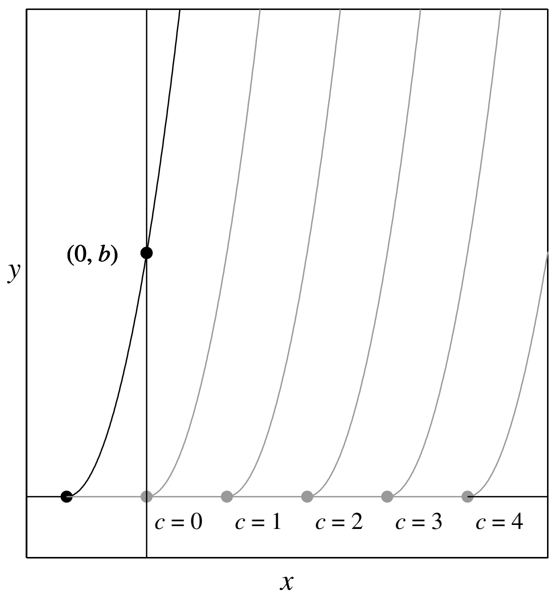
Problem 26



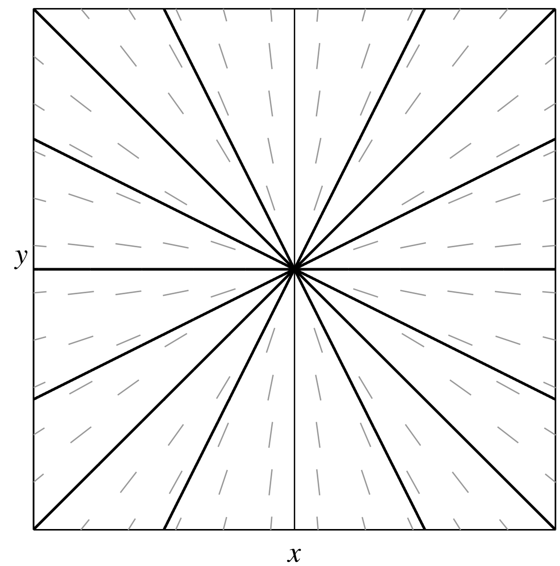
27. **a)** It is clear that $y(x)$ satisfies the differential equation at each *x* with $x<c$ or $x>c$,and by examining left- and right-hand derivatives we see that the same is true at $x=c$. Thus $y(x)$ not only satisfies the differential equation for all *x*, it also satisfies the given initial value problem whenever $c\ge 0$. The infinitely many solutions of the initial value problem are illustrated in the figure. Note that $f(x,y)=2\sqrt{y}$ is not continuous in any neighborhood of the origin, and so Theorem 1 guarantees neither existence nor uniqueness of solution to the given initial value problem. As it happens, existence occurs, but not uniqueness.

b) If $b<0$, then the initial value problem $y'=2\sqrt{y}$, $y(0)=b$ has no solution, because the square root of a negative number would be involved. If $b>0$, then we get a unique solution curve through $(0,b)$ defined for all *x* by following a parabola (as in the figure, in black) — down (and leftward) to the *x*-axis and then following the *x*-axis to the left. Finally if $b=0$, then starting at $(0,0)$ we can follow the positive *x*-axis to the point $(c,0)$ and then branch off on the parabola $y=(x-c)^2$, as shown in gray. Thus there are infinitely many solutions in this case.

Problem 27a



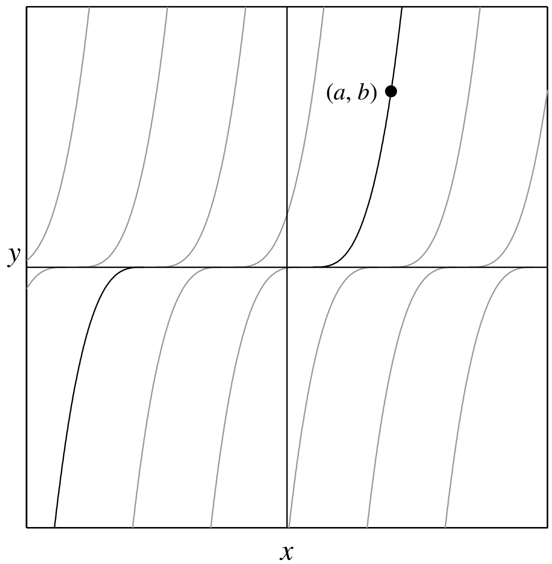
Problem 28



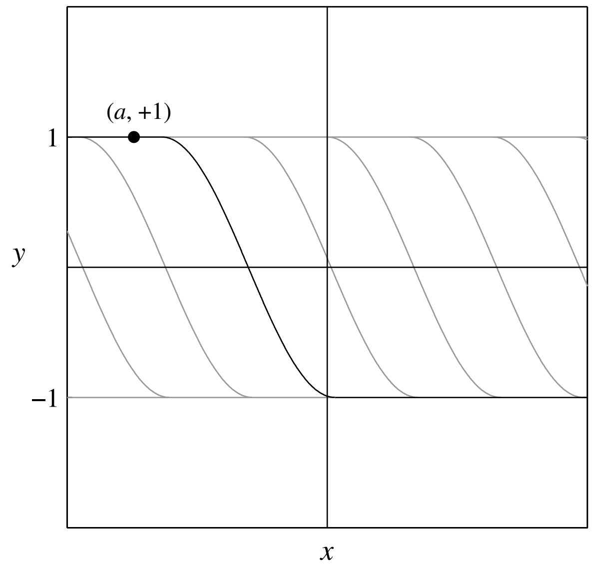
28. The figure makes it clear that the initial value problem $xy' = y$, $y(a) = b$ has a unique solution if $a \neq 0$, infinitely many solutions if $a = b = 0$, and no solution if $a = 0$ but $b \neq 0$ (so that the point (a,b) lies on the positive or negative *y*-axis). Each of these conclusions is consistent with Theorem 1.

29. As with Problem 27, it is clear that $y(x)$ satisfies the differential equation at each *x* with $x < c$ or $x > c$, and by examining left- and right-hand derivatives we see that the same is true at $x = c$. Looking at the figure on the left below, we see that if, for instance, $b > 0$, then we can start at the point (a,b) and follow a branch of a cubic down to the *x*-axis, then follow the *x*-axis an arbitrary distance before branching down on another cubic. This gives infinitely many solutions of the initial value problem $y' = 3y^{2/3}$, $y(a) = b$ that are defined for all *x*. However, if $b \neq 0$, then there is only a single cubic $y = (x - c)^3$ passing through (a,b), so the solution is unique near $x = a$ (as Theorem 1 would predict).

Problem 29



Problem 30



30. The function $y(x)$ satisfies the given differential equation on the interval $c < x < c + \pi$, since $y'(x) = -\sin(x - c) < 0$ there and thus

$$-\sqrt{1-y^2} = -\sqrt{1-\cos^2(x-c)} = -\sqrt{\sin^2(x-c)} = -\sin(x-c) = y'.$$

Moreover, the same is true for $x < c$ and $x > c + \pi$ (since $y^2 \equiv 1$ and $y' \equiv 0$ there), and at $x = c, c + \pi$ by examining one-sided derivatives. Thus $y(x)$ satisfies the given differential equation for all *x*.

If $|b| > 1$, then the initial value problem $y' = -\sqrt{1-y^2}$, $y(a) = b$ has no solution, because the square root of a negative number would be involved. If $|b| < 1$, then there is only one curve of the form $y = \cos(x - c)$ through the point (a, b), giving a unique solution. But if $b = \pm 1$, then we can combine a left ray of the line $y = +1$, a cosine curve from the line $y = +1$ to the line $y = -1$, and then a right ray of the line $y = -1$. Looking at the figure, we see that this gives infinitely many solutions (defined for all *x*) through any point of the form $(a, \pm 1)$.

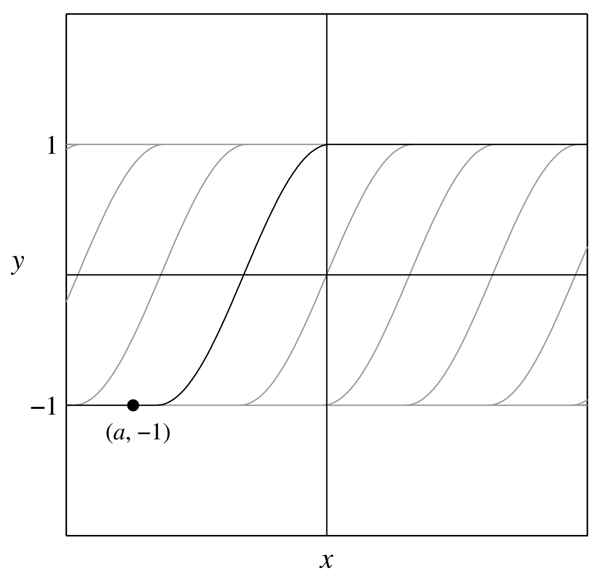
31. The function $y(x) = \begin{cases} -1 & \text{if } x < c - \pi/2 \\ \sin(x-c) & \text{if } c - \pi/2 < x < c + \pi/2 \\ +1 & \text{if } x < c + \pi/2 \end{cases}$ satisfies the given differential equation on the interval $c - \dfrac{\pi}{2} < x < c + \dfrac{\pi}{2}$, since $y'(x) = \cos(x - c) > 0$ there and thus

$$\sqrt{1-y^2} = \sqrt{1-\sin^2(x-c)} = \sqrt{\cos^2(x-c)} = \cos(x-c) = y'.$$

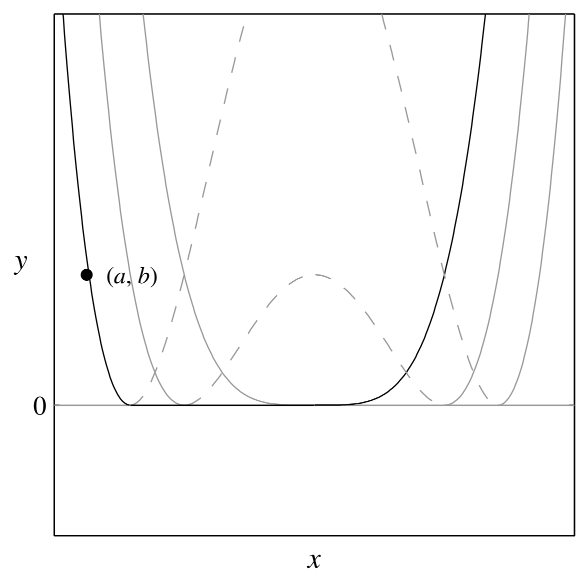
Moreover, the same is true for $x < \frac{\pi}{2}$ and $x > c + \frac{\pi}{2}$ (since $y^2 \equiv 1$ and $y' \equiv 0$ there), and at $x = \frac{\pi}{2}, c + \frac{\pi}{2}$ by examining one-sided derivatives. Thus $y(x)$ satisfies the given differential equation for all *x*.

If $|b| > 1$, then the initial value problem $y' = \sqrt{1 - y^2}$, $y(a) = b$ has no solution because the square root of a negative number would be involved. If $|b| < 1$, then there is only one curve of the form $y = \sin(x - c)$ through the point (a, b); this gives a unique solution. But if $b = \pm 1$, then we can combine a left ray of the line $y = -1$, a sine curve from the line $y = -1$ to the line $y = +1$, and then a right ray of the line $y = +1$. Looking at the figure, we see that this gives infinitely many solutions (defined for all *x*) through any point of the form $(a, \pm 1)$.

Problem 31



Problem 32



32. The function $y(x)$ satisfies the given differential equation for $x^2 > c$, since $y'(x) = 4x(x^2 - c) = 4x\sqrt{y}$ there. Moreover, the same is true for $x^2 < c$ (since $y = y' \equiv 0$ there), and at $x = \pm\sqrt{c}$ by examining one-sided derivatives. Thus $y(x)$ satisfies the given differential equation for all *x*.

Looking at the figure, we see that we can piece together a "left half" of a quartic for *x* negative, an interval along the *x*-axis, and a "right half" of a quartic curve for *x* positive. This makes it clear that the initial value problem $y' = 4x\sqrt{y}$, $y(a) = b$ has infinitely many solutions (defined for all *x*) if $b \geq 0$. There is no solution if $b < 0$ because this would involve the square root of a negative number.

33. Looking at the figure provided in the answers section of the textbook, it suffices to observe that, among the pictured curves $y = x/(cx-1)$ for all possible values of c,

- there is a unique one of these curves through any point not on either coordinate axis;
- there is no such curve through any point on the y-axis other than the origin; and
- there are infinitely many such curves through the origin (0,0).

But in addition we have the constant-valued solution $y(x) \equiv 0$ that "covers" the x-axis. It follows that the given differential equation has near (a,b)

- a unique solution if $a \neq 0$;
- no solution if $a = 0$ but $b \neq 0$;
- infinitely many different solutions if $a = b = 0$.

Once again these findings are consistent with Theorem 1.

34. **(a)** With a computer algebra system we find that the solution of the initial value problem $y' = y - x + 1$, $y(-1) = -1.2$ is $y(x) = x - 0.2e^{x+1}$, whence $y(1) \approx -0.4778$. With the same differential equation but with initial condition $y(-1) = -0.8$ the solution is $y(x) = x + 0.2e^{x+1}$, whence $y(1) \approx 2.4778$

(b) Similarly, the solution of the initial value problem $y' = y - x + 1$, $y(-3) = -3.01$ is $y(x) = x - 0.01e^{x+3}$, whence $y(3) \approx -1.0343$. With the same differential equation but with initial condition $y(-3) = -2.99$ the solution is $y(x) = x + 0.01e^{x+3}$, whence $y(3) \approx 7.0343$. Thus close initial values $y(-3) = -3 \pm 0.01$ yield $y(3)$ values that are far apart.

35. **(a)** With a computer algebra system we find that the solution of the initial value problem $y' = x - y + 1$, $y(-3) = -0.2$ is $y(x) = x + 2.8e^{-x-3}$, whence $y(2) \approx 2.0189$. With the same differential equation but with initial condition $y(-3) = +0.2$ the solution is $y(x) = x + 3.2e^{-x-3}$, whence $y(2) \approx 2.0216$.

(b) Similarly, the solution of the initial value problem $y' = x - y + 1$, $y(-3) = -0.5$ is $y(x) = x + 2.5e^{-x-3}$, whence $y(2) \approx 2.0168$. With the same differential equation but with initial condition $y(-3) = +0.5$ the solution is $y(x) = x + 3.5e^{-x-3}$, whence $y(2) \approx 2.0236$. Thus the initial values $y(-3) = \pm 0.5$ that are not close both yield $y(2) \approx 2.02$.

SECTION 1.4

SEPARABLE EQUATIONS AND APPLICATIONS

Of course it should be emphasized to students that the possibility of separating the variables is the first one you look for. The general concept of natural growth and decay is important for all differential equations students, but the particular applications in this section are optional. Torricelli's law in the form of Equation (24) in the text leads to some nice concrete examples and problems.

Also, in the solutions below, we make free use of the fact that if C is an arbitrary constant, then so is $5-3C$, for example, which we can (and usually do) replace simply with C itself. In the same way we typically replace e^C by C, with the understanding that C is then an arbitrary nonzero constant.

1. For $y \neq 0$ separating variables gives $\int \frac{dy}{y} = -\int 2x\,dx$, so that $\ln|y| = -x^2 + C$, or $y(x) = \pm e^{-x^2+C} = Ce^{-x^2}$, where C is an arbitrary nonzero constant. (The equation also has the singular solution $y \equiv 0$.)

2. For $y \neq 0$ separating variables gives $\int \frac{dy}{y^2} = -\int 2x\,dx$, so that $-\frac{1}{y} = -x^2 + C$, or $y(x) = \frac{1}{x^2 + C}$. (The equation also has the singular solution $y \equiv 0$.)

3. For $y \neq 0$ separating variables gives $\int \frac{dy}{y} = \int \sin x\,dx$, so that $\ln|y| = -\cos x + C$, or $y(x) = \pm e^{-\cos x + C} = Ce^{-\cos x}$, where C is an arbitrary nonzero constant. (The equation also has the singular solution $y \equiv 0$.)

4. For $y \neq 0$ separating variables gives $\int \frac{dy}{y} = \int \frac{4}{1+x}\,dx$, so that $\ln|y| = 4\ln(1+x) + C$, or $y(x) = C(1+x)^4$, where C is an arbitrary nonzero constant. (The equation also has the singular solution $y \equiv 0$.)

5. For $-1 < y < 1$ and $x > 0$ separating variables gives $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{1}{2\sqrt{x}}\,dx$, so that $\sin^{-1} y = \sqrt{x} + C$, or $y(x) = \sin\left(\sqrt{x} + C\right)$. (The equation also has the singular solutions $y \equiv 1$ and $y \equiv -1$.)

6. For $x, y > 0$ separating variables gives $\int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\,dx$, so that $2\sqrt{y} = 2x^{3/2} + C$, or $y(x) = \left(x^{3/2} + C\right)^2$. For $x, y < 0$ we write $\frac{dy}{dx} = 3\sqrt{(-x)(-y)}$, leading to $\int \frac{dy}{\sqrt{-y}} = \int 3\sqrt{-x}\,dx$, or $-2\sqrt{-y} = -2(-x)^{3/2} + C$, or $y(x) = -\left[(-x)^{3/2} + C\right]^2$.

7. For $y \neq 0$ separating variables gives $\int \frac{dy}{y^{1/3}} = \int 4x^{1/3}\,dx$, so that $\frac{3}{2}y^{2/3} = 3x^{4/3} + C$, or $y(x) = \left(2x^{4/3} + C\right)^{3/2}$. (The equation also has the singular solution $y \equiv 0$.)

8. For $y \neq \frac{\pi}{2} + k\pi$, k integer, separating variables gives $\int \cos y\,dy = \int 2x\,dx$, so that $\sin y = x^2 + C$, or $y(x) = \sin^{-1}\left(x^2 + C\right)$.

9. For $y \neq 0$ separating variables and decomposing into partial fractions give $\int \frac{dy}{y} = \int \frac{2}{1-x^2}\,dx = \int \frac{1}{1+x} + \frac{1}{1-x}\,dx$, so that $\ln|y| = \ln|1+x| - \ln|1-x| + C$, or $|y| = C\left|\frac{1+x}{1-x}\right|$, where C is an arbitrary positive constant, or $y(x) = C\frac{1+x}{1-x}$, where C is an arbitrary nonzero constant. (The equation also has the singular solution $y \equiv 0$.)

10. For $y \neq -1$ and $x \neq -1$ separating variables gives $\int \frac{1}{(1+y)^2}\,dy = \int \frac{1}{(1+x)^2}\,dx$, so that $\frac{-1}{1+y} = \frac{-1}{1+x} + C$, or $1 + y = \frac{1}{\frac{1}{1+x} + C} = \frac{1+x}{1+C(1+x)}$, or finally

$$y(x) = \frac{1+x}{1+C(1+x)} - 1 = \frac{1+x-\left[1+C(1+x)\right]}{1+C(1+x)} = \frac{x-C(1+x)}{1+C(1+x)},$$

where C is an arbitrary constant. (The equation also has the singular solution $y \equiv -1$.)

11. For $y > 0$ separating variables gives $\int \frac{dy}{y^3} = \int x\,dx$, so that $-\frac{1}{2y^2} = \frac{x^2}{2} + C$, or $y(x) = \left(C - x^2\right)^{-1/2}$, where C is an arbitrary constant. Likewise $y(x) = -\left(C - x^2\right)^{-1/2}$ for $y < 0$. (The equation also has the singular solution $y \equiv 0$.)