

## CHAPTER 2: Analytic Functions

### EXERCISES 2.1: Functions of a Complex Variable

1.
  - a.  $w = (3x^2 - 3y^2 + 5x + 1) + i(6xy + 5y + 1)$
  - b.  $w = \frac{x}{x^2 + y^2} + i \left( -\frac{y}{x^2 + y^2} \right)$
  - c.  $w = \frac{1}{z - i} = \frac{x}{x^2 + (y - 1)^2} + i \frac{-y + 1}{x^2 + (y - 1)^2}$
  - d.  $w = \frac{2x^2 - 2y^2 + 3}{\sqrt{(x - 1)^2 + y^2}} + i \frac{4xy}{\sqrt{(x - 1)^2 + y^2}}$
  - e.  $w = e^{3x} \cos 3y + i e^{3x} \sin 3y$
  - f.  $w = (e^x + e^{-x}) \cos y + i(e^x - e^{-x}) \sin y$   
 $= 2 \cosh x \cos y + i 2 \sinh x \sin y$
2.
  - a.  $\mathbb{C}$
  - b.  $\mathbb{C} \setminus \{0\}$
  - c.  $\mathbb{C} \setminus \{i, -i\}$
  - d.  $\mathbb{C} \setminus \{1\}$
  - e.  $\mathbb{C}$
  - f.  $\mathbb{C}$
3.
  - a.  $\operatorname{Re} w > 5$
  - b.  $\operatorname{Im} w \geq 0$
  - c.  $|w| \geq 1$
  - d. The intersection of  $|w| < 2$  and  $-\pi < \operatorname{Arg} w < \pi/2$
4.
  - a. Taking  $\theta$  from 0 to  $2\pi$ , the points  $z = re^{i\theta}$  traverse the circle  $|z| = r$  exactly once in the counterclockwise direction. For the same values of  $\theta$  the points  $w = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$  traverse the circle  $|w| = \frac{1}{r}$  exactly once in the clockwise direction, hence the mapping is onto.
  - b. For  $z = re^{i\theta_0}$  on the ray  $\operatorname{Arg} z = \theta_0$ ,  $w = \frac{1}{re^{i\theta_0}} = \frac{1}{r}e^{-i\theta_0}$  is on the ray  $\operatorname{Arg} w = -\theta_0$ . Taking values  $0 < r < \infty$  shows that this mapping goes onto the ray  $\operatorname{Arg} w = -\theta_0$ .

- 4 (c)  $|z-1|=1$   $2\pi > \theta \geq 0 \Rightarrow z = 1 + e^{i\theta}$ .  $F(z) = 1/z = 1/(1 + e^{i\theta})$   
 $= (1 + e^{-i\theta}) / \{2(1 + \cos\theta)\} = \frac{1}{2} - i(\frac{1}{2})\sin\theta / (1 + \cos\theta)$   
 which is a vertical line at  $x = \frac{1}{2}$ .

5. a. domain:  $\mathbf{C}$

range:  $\mathbf{C} \setminus \{0\}$

b.  $f(-z) = e^{-z} = \frac{1}{e^z} = \frac{1}{f(z)}$

c. circle  $|w| = e$

d. ray  $\text{Arg } w = \pi/4$

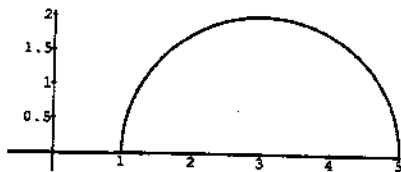
e. infinite sector  $0 \leq \text{Arg } w \leq \pi/4$

6. a.  $J\left(\frac{1}{z}\right) = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{1/z}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) = J(z)$

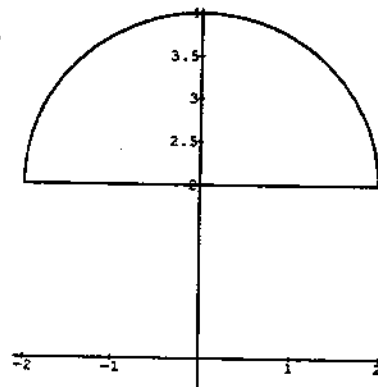
b. For  $z = e^{i\theta}$  on the unit circle  $|z| = 1$ ,  $J(z) = \frac{1}{2}\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right) = \cos \theta$ .  
 For all values of  $\theta$ , this ranges over the real interval  $[-1, 1]$ .

c. For  $z = re^{i\theta}$  on the circle  $|z| = r$ ,  $J(z) = \frac{1}{2}\left(re^{i\theta} + \frac{1}{re^{i\theta}}\right) =$   
 $\frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta + i\frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$ . Setting  $u$  and  $v$  equal to the  
 real and imaginary parts of this expression, respectively, one gets  
 a pair of parametric equations that are equivalent to the ellipse  
 $\frac{u^2}{\left[\frac{1}{2}\left(r + \frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r - \frac{1}{r}\right)\right]^2} = 1$ , which has foci at  $\pm 1$ .

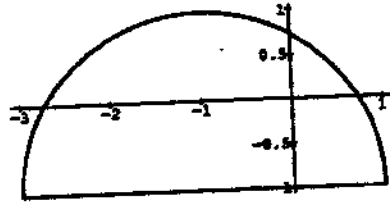
7. a.



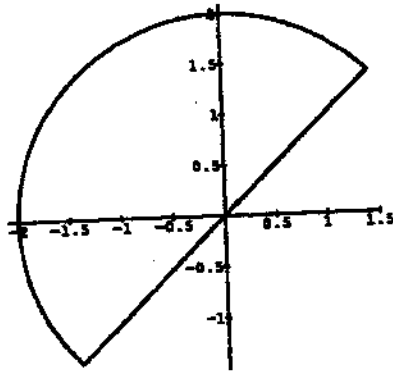
- b.



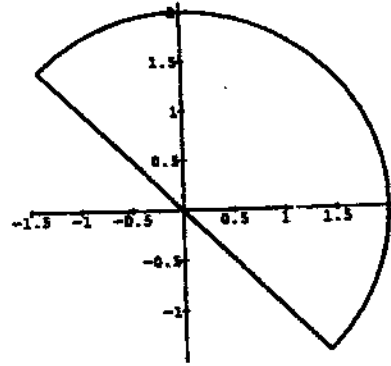
c.



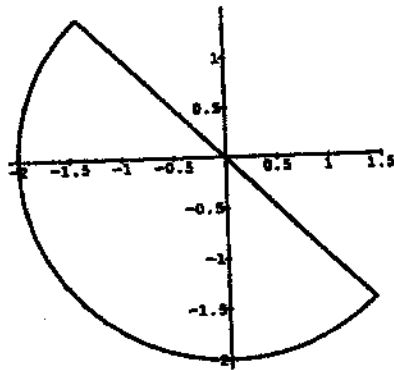
8. a.



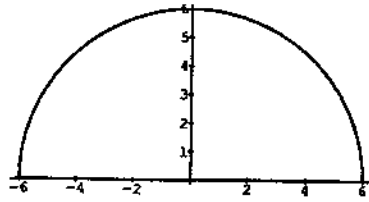
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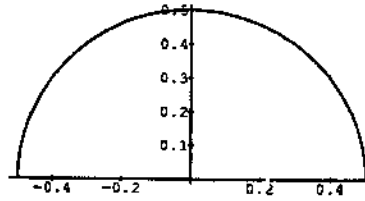
c.



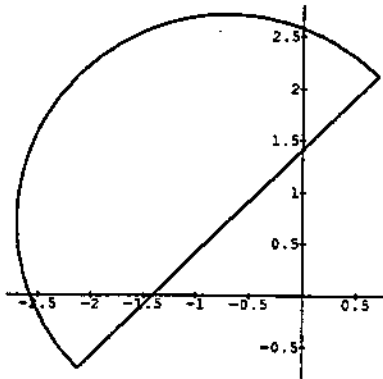
9. a.



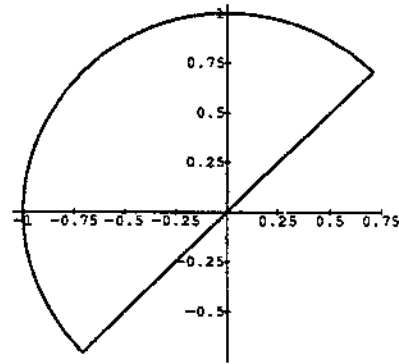
b.



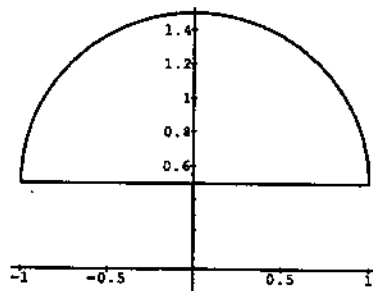
10. a. translate by  $i$ , rotate  $\pi/4$



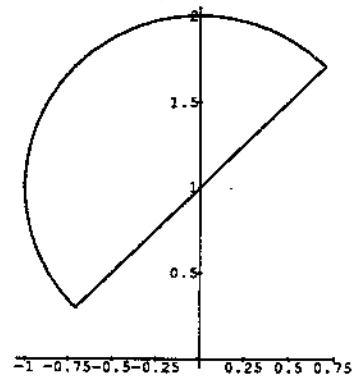
b. reduce by  $1/2$ , rotate  $\pi/4$



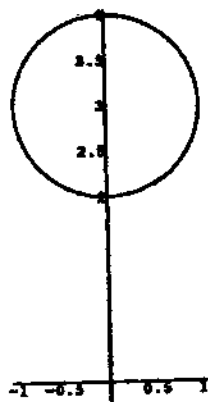
c. translate by  $i$ , reduce by  $1/2$



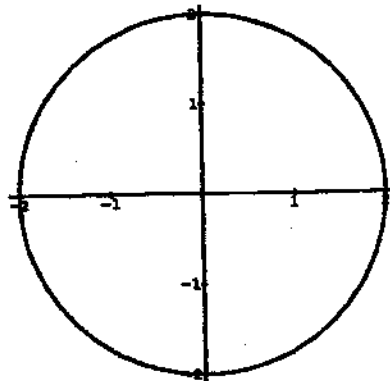
d. reduce by  $1/2$ , rotate  $\pi/4$ , translate by  $i$



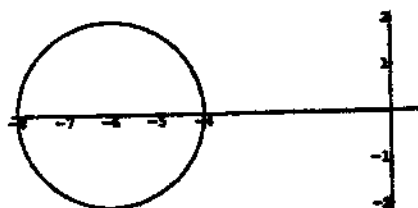
11. a. translate by  $-3$ ,  
rotate  $-\pi/2$



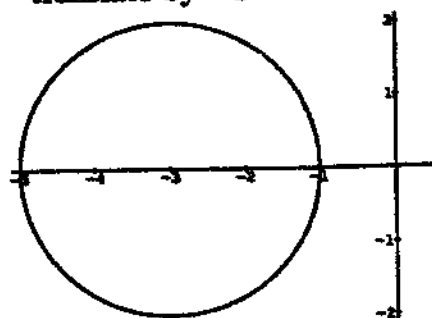
- b. magnify by 2,  
rotate  $-\pi/2$



- c. translate by  $-3$ ,  
magnify by 2



- d. magnify by 2, rotate  $-\pi/2$ ,  
translate by  $-3$



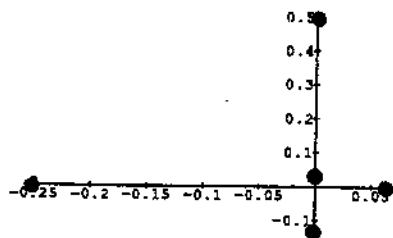
12. Let  $a = \rho e^{i\phi}$ ,  $F(z) = \rho z$ ,  $G(z) = e^{i\phi} z$ , and  $H(z) = z + b$ . Then  $H(G(F(z))) = az + b$ .

13. (a)  $w = u + iv = z^2 = (1 + iy)^2 = 1 - y^2 + i2y$   
 $u = 1 - y^2$ ,  $v = 2y \Rightarrow y = v/2 \Rightarrow u = 1 - v^2/4$  a parabola in the  $w$ -plane.
- (b)  $w = u + iv = z^2 = (x + iy)^2 = (x + i/x)^2 = x^2 - 1/x^2 + 2i$   
 $u = x^2 - 1/x^2$ ,  $v = 2$  a straight line in the  $w$ -plane.
- (c)  $w = u + iv = z^2 = (1 + e^{i\theta})^2 = (1 + 2e^{i\theta} + e^{i2\theta}) = (e^{-i\theta} + 2 + e^{i\theta})e^{i\theta}$   
 $= (2 + 2\cos\theta)e^{i\theta} = 2(1 + \cos\theta)e^{i\theta}$  a cardioid in the  $w$ -plane.
14. (a)  $x_1 = 2x/(|z|^2 + 1)$ ,  $x_2 = 2y/(|z|^2 + 1)$ ,  $x_3 = (|z|^2 - 1)/(|z|^2 + 1)$   
 $w = e^{i\phi} z = x\cos\phi - y\sin\phi + i(x\sin\phi + y\cos\phi)$ ,  $|w| = |z|$   
 $\underline{x}_1 = (x\cos\phi - y\sin\phi)/(|z|^2 + 1)$ ,  $\underline{x}_2 = (x\sin\phi + y\cos\phi)/(|z|^2 + 1)$ ,  $\underline{x}_3 = x_3$   
 $\underline{x}_1 = (x_1\cos\phi - x_2\sin\phi)$ ,  $\underline{x}_2 = (x_1\sin\phi + x_2\cos\phi)$ ,  $\underline{x}_3 = x_3$  which corresponds to a rotation of an angle  $\phi$  about the  $x_3$  axis.
- (b)  $w = -1/z$ .  $|w| = 1/|z|$ .  $w = -1/(x+iy) = -x/|z| + iy/|z|$   
 $\underline{x}_1 = -x_1$ ,  $\underline{x}_2 = x_2$ ,  $\underline{x}_3 = -x_3$  so that  $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$  is obtained from  $(x_1, x_2, x_3)$  by a  $180^\circ$  rotation about the  $x_2$  axis.
15.  $w = (1+z)/(1-z) = (1+x+iy)/(1-x-iy) = (1-|z|^2 + i2y)/(1-2x+|z|^2)$   
 $|w|^2 = (1+2x+|z|^2)/(1-2x+|z|^2)$   
 $(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, x_2, x_1)$  so that  $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$  is obtained by a  $90^\circ$  counterclockwise rotation about the  $x_2$  axis.

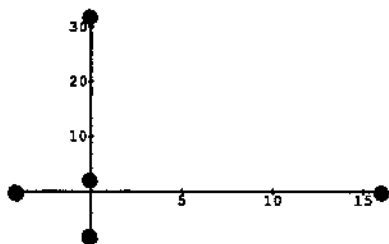
16.  $w = (1 - iz)/(1 + iz) = (1 - ix + y)/(1 + ix - y) = (1 - |z|^2 + i2x)/(1 - 2y + |z|^2)$   
 $|w|^2 = (1 + 2y + |z|^2)/(1 - 2y + |z|^2)$ .  
 $(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, -x_1, x_2)$  so that  $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$  is obtained as a  $90^\circ$  counterclockwise rotation about the  $x_2$  axis followed by a  $90^\circ$  counterclockwise rotation about the  $x_3$  axis.
17. Any circle or line in the  $z$ -plane corresponds to a line or circle on the stereographic projection onto the Riemann sphere. The function  $w=1/z$  rotates the Riemann sphere  $180^\circ$  about the  $x_1$  axis. Lines and circles on the rotated sphere project to lines and circles in the  $w$ -plane. As a result lines and circles in the  $z$ -plane map to lines and circles in the  $w$ -plane.

## EXERCISES 2.2: Limits and Continuity

1. The first five terms are, respectively,  $\frac{i}{2}$ ,  $-\frac{1}{4}$ ,  $-\frac{i}{8}$ ,  $\frac{1}{16}$ , and  $\frac{i}{32}$ . The sequence converges to 0 in a spiral-like fashion.



2.  $2i, -4, -8i, 16, 32i$ ; divergent because terms grow in modulus without bound.



3. If  $\lim_{n \rightarrow \infty} z_n = z_0$ , then for any  $\epsilon > 0$ , there is an integer  $N$  such that  $|z_n - z_0| < \epsilon$  for all  $n > N$ . For the same integer  $N$  we have  $|x_n - x_0| \leq |z_n - z_0| < \epsilon$  and  $|y_n - y_0| \leq |z_n - z_0| < \epsilon$  for all  $n > N$ . Therefore,  $\lim_{n \rightarrow \infty} x_n = x_0$  and  $\lim_{n \rightarrow \infty} y_n = y_0$ .

If  $\lim_{n \rightarrow \infty} x_n = x_0$  and  $\lim_{n \rightarrow \infty} y_n = y_0$ , then for any  $\epsilon_1 > 0$  and  $\epsilon_2 > 0$  there are integers  $N_1$  and  $N_2$  such

$|x_n - x_0| < \epsilon_1$  for all  $n > N_1$  and  $|y_n - y_0| < \epsilon_2$  for all  $n > N_2$ . Given any  $\epsilon > 0$ ;

let  $\epsilon_1 = \epsilon/2$  and  $\epsilon_2 = \epsilon/2$ . Then

$|z_n - z_0| \leq |x_n - x_0| + |y_n - y_0| < \epsilon_1 + \epsilon_2 = \epsilon$  for all  $n > \text{maximum}(N_1, N_2)$ .

Thus  $\lim_{n \rightarrow \infty} z_n = z_0$ .

4. If  $z_n = x_n + iy_n \rightarrow z_0 = x_0 + iy_0$ , then  $x_n \rightarrow x_0$  and  $y_n \rightarrow y_0$  (see Problem 3).  
 $\underline{z}_n = x_n - iy_n \rightarrow x_0 - iy_0 = \underline{z}_0$ .

If  $\underline{z}_n = x_n - iy_n \rightarrow \underline{z}_0 = x_0 - iy_0$ , then  $x_n \rightarrow x_0$  and  $y_n \rightarrow y_0$  (see Problem 3)..

$z_n = x_n + iy_n \rightarrow x_0 + iy_0 = z_0$ . Thus  $z_n \rightarrow z_0$  if and only if  $\underline{z}_n \rightarrow \underline{z}_0$ .

5.  $\lim_{n \rightarrow \infty} |z_n| = 0 \implies$  There exists an integer  $N$  such that  
 $||z_n| - 0| = |z_n| < \epsilon$  whenever  $n > N. \implies |z_n - 0| < \epsilon$  whenever  $n > N. \implies \lim_{n \rightarrow \infty} z_n = 0$ , and conversely.

6.  $z_0^n \rightarrow 0$  as  $n \rightarrow \infty$  by problem 3, since the real-valued sequence  $|z_0^n| \rightarrow 0$  as  $n \rightarrow \infty$ . On the other hand, if  $|z_0| > 1$ , then  $|z_0^n| \rightarrow \infty$  as  $n \rightarrow \infty$  so  $z_0^n$  diverges.

7. a. converges to 0  
 b. does not converge  
 c. converges to  $\pi$   
 d. converges to  $2 + i$   
 e. converges to 0  
 f. does not converge

8. Given  $\epsilon > 0$ , choose  $\delta = \epsilon/6$ . Then whenever  $0 < |z - (1 + i)| < \delta$ ,

$$|6z - 4 - (2 + 6i)| = 6|z - (1 + i)| < 6(\epsilon/6) = \epsilon$$

9. Given  $\epsilon > 0$ , choose  $\delta = \frac{\epsilon}{1 + \epsilon}$ . Whenever  $0 < |z - (-i)| < \delta$  notice that  $|z| > 1 - \delta$  and

$$\left| \frac{1}{z} - i \right| = \left| \left( -\frac{i}{z} \right) (i + z) \right| = \frac{1}{|z|} |z - (-i)| < \left( \frac{1}{1 - \delta} \right) \delta = \epsilon$$

10. Given that  $f$  and  $g$  are continuous at  $z_0$ ,

$$\lim_{z \rightarrow z_0} f(z) \pm g(z) = \lim_{z \rightarrow z_0} f(z) \pm \lim_{z \rightarrow z_0} g(z) = f(z_0) \pm g(z_0)$$

$\implies f(z) \pm g(z)$  is continuous at  $z_0$ .

$$\lim_{z \rightarrow z_0} f(z)g(z) = \lim_{z \rightarrow z_0} f(z) \lim_{z \rightarrow z_0} g(z) = f(z_0)g(z_0)$$

$\implies f(z)g(z)$  is continuous at  $z_0$ .

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \rightarrow z_0} f(z)}{\lim_{z \rightarrow z_0} g(z)} = \frac{f(z_0)}{g(z_0)}, \text{ provided } g(z_0) \neq 0$$

$\implies \frac{f(z)}{g(z)}$  is continuous at  $z_0$ .

- 11.
- a.  $-8i$
  - b.  $-\frac{7}{2}i$
  - c.  $6i$
  - d.  $-1/2$
  - e.  $2z_0$
  - f.  $4\sqrt{2}$

12. Clearly  $\text{Arg } z$  is discontinuous at  $z = 0$ . Let  $a > 0$  be any real number and consider the sequence

$$z_n = -a - i/n \quad n = 1, 2, \dots, \text{ which converges to } -a.$$

For each  $n$ ,  $-\pi < \text{Arg } z_n < -\pi/2$ , but  $\text{Arg } (-a) = \pi$ .

13.  $\lim_{z \rightarrow z_0} f(z)$  exists for all  $z \neq -1$ ;  $f$  is continuous for all  $z \neq 0, -1$ ;  $f$  has a removable discontinuity at  $z = 0$ .

14. Let  $z_0$  be any complex number. Given  $\varepsilon > 0$  choose  $\delta = \varepsilon$ . Then whenever  $|z - z_0| < \delta$ ,

$$|g(z) - g(z_0)| = |\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \varepsilon.$$

15. Given  $\varepsilon > 0$  choose  $\delta$  so that  $|f(z) - f(z_0)| < \varepsilon$  whenever  $|z - z_0| < \delta$ . Then, whenever  $|z - z_0| < \delta$ :

- a.  $|\overline{f(z)} - \overline{f(z_0)}| = |\overline{f(z) - f(z_0)}| = |f(z) - f(z_0)| < \varepsilon$
- b.  $|\text{Re } f(z) - \text{Re } f(z_0)| = |\text{Re}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$
- c.  $|\text{Im } f(z) - \text{Im } f(z_0)| = |\text{Im}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$
- d.  $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| < \varepsilon$



16. Given  $\varepsilon > 0$ , choose  $\delta_0 > 0$  such that  $|f(g(z)) - f(g(z_0))| < \varepsilon$  whenever  $|g(z) - g(z_0)| < \delta_0$ . Now choose  $\delta > 0$  such that  $|g(z) - g(z_0)| < \delta_0$  whenever  $|z - z_0| < \delta$ . Then  $|f(g(z)) - f(g(z_0))| < \varepsilon$  whenever  $|z - z_0| < \delta$ ; hence  $f(g(z))$  is continuous at  $z_0$ .

17. No: Observe that although  $\frac{1}{n} \rightarrow 0$  and  $\frac{i}{n} \rightarrow 0$  as  $n \rightarrow \infty$ ,

$f\left(\frac{1}{n}\right) \rightarrow 1 + 2i$  and  $f\left(\frac{i}{n}\right) \rightarrow 2i$ ; thus  $\lim_{z \rightarrow 0} f(z)$  does not exist.

18. If  $\lim_{z \rightarrow z_0} f(z) = w_0$ , then given  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$|f(z) - w_0| < \varepsilon$  for all  $|z - z_0| < \delta$ . Notice that

$|f(z) - w_0| = |f(z) - w_0| = |f(z) - w_0| < \varepsilon$  for all  $|z - z_0| < \delta$ . So that  $\lim_{z \rightarrow z_0} \underline{f(z)} = \underline{w_0}$ .

$\lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) = \lim_{z \rightarrow z_0} ((f(z) + \underline{f(z)})/2) = (w_0 + \underline{w_0})/2 = \mu_0$ .

$\lim_{x \rightarrow x_0, y \rightarrow y_0} \nu(x, y) = \lim_{z \rightarrow z_0} ((f(z) - \underline{f(z)})/2i) = (w_0 - \underline{w_0})/2i = \nu_0$ .

Thus,  $\lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) = \mu_0$  and  $\lim_{x \rightarrow x_0, y \rightarrow y_0} \nu(x, y) = \nu_0$ .

Conversely, if  $\lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) = \mu_0$  and  $\lim_{x \rightarrow x_0, y \rightarrow y_0} \nu(x, y) = \nu_0$ , then

(by Theorem 1.)  $\mu_0 + i\nu_0 = \lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) + i \lim_{x \rightarrow x_0, y \rightarrow y_0} \nu(x, y) =$

$\lim_{z \rightarrow z_0} ((f(z) + \underline{f(z)})/2) + \lim_{z \rightarrow z_0} ((f(z) - \underline{f(z)})/2) = \lim_{z \rightarrow z_0} f(z) = w_0$ .

Also  $\mu_0 - i\nu_0 = \lim_{x \rightarrow x_0, y \rightarrow y_0} \mu(x, y) - i \lim_{x \rightarrow x_0, y \rightarrow y_0} \nu(x, y) = \lim_{z \rightarrow z_0}$

$((f(z) + \underline{f(z)})/2) - \lim_{z \rightarrow z_0} ((f(z) - \underline{f(z)})/2) = \lim_{z \rightarrow z_0} f(z) = \underline{w_0}$ .

Thus,  $\lim_{z \rightarrow z_0} f(z) = w_0$ .

19.  $-\frac{1}{2} - i$ , since  $\lim_{\substack{x \rightarrow -1 \\ y \rightarrow -1}} \frac{x}{x^2 + 3y} = -\frac{1}{2}$  and  $\lim_{\substack{x \rightarrow -1 \\ y \rightarrow -1}} xy = -1$ .

20. For any  $z_0$  in the complex plane,

$$\lim_{z \rightarrow z_0} e^z = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} e^x \cos y + i \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} e^x \sin y = e^{x_0} \cos y_0 + i e^{x_0} \sin y_0 = e^{z_0}.$$

21. a. 1

b. 0

c.  $-\pi/2 + i$

d. 1

22. By contradiction: Suppose  $\lim_{z \rightarrow z_0} f(z) \neq w_0$ . Then there is an  $\varepsilon > 0$

for which there exists a sequence  $\{z_n\}$  such that  $|z_n - z_0| < \frac{1}{n}$  but  $|f(z_n) - w_0| > \varepsilon$ . For this sequence,  $\lim_{n \rightarrow \infty} z_n = z_0$  but  $\lim_{n \rightarrow \infty} f(z_n) \neq w_0$ , contrary to hypothesis.

23. If  $z_n \rightarrow \infty$ , then for any  $M > 0$  there exist an integer  $N$  such  $|z_n| > M$  for all  $n > N$ . Consider the chordal distance  $\chi(z_n, \infty) = 2/\sqrt{|z_n|^2 + 1} < 2/\sqrt{|z_n|^2} = 2/|z_n| < 2/M < \epsilon$  for all  $n > N$ . Thus  $z_n \rightarrow \infty$  as  $n \rightarrow \infty$  is equivalent to  $\chi(z_n, \infty) \rightarrow 0$  as  $n \rightarrow \infty$ .
24. If  $\lim_{z \rightarrow z_0} f(z) = \infty$ , then for any  $M > 0$  there exist  $\delta > 0$  such that  $|f(z)| > M$  for all  $|z - z_0| < \delta$ . Consider  $\chi(f(z), \infty) = 2/\sqrt{|f(z)|^2 + 1} < 2/\sqrt{|f(z)|^2} = 2/|f(z)| < 2/M < \epsilon$  for all  $|z - z_0| < \delta$ . Thus  $\lim_{z \rightarrow z_0} f(z) = \infty$ , is equivalent to  $\lim_{z \rightarrow \infty} \chi(f(z), \infty) = 0$ .
25. (a)  $\infty$       (b) 3      (c)  $\infty$       (d)  $\infty$       (f) the limit does not exist.

### EXERCISES 2.3: Analyticity

1. Let  $\Delta z = z - z_0$  so that  $\Delta z \rightarrow 0 \iff z \rightarrow z_0$ . Then

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = L \iff$$

given  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$\left| \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - L \right| < \epsilon \text{ whenever } |\Delta z - 0| < \delta \iff$$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - L \right| < \epsilon \text{ whenever } |z - z_0| < \delta \iff$$

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = L.$$

2. If  $\lambda(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$ , then  $\lambda(z) \rightarrow 0$  as  $z \rightarrow z_0$  and  $f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0) = f(z)$ .
3.  $\lim_{z \rightarrow z_0} f(z) = \lim_{z \rightarrow z_0} [f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0)] = f(z_0) + 0 + 0 = f(z_0)$ .

4. a.  $\lim_{\Delta z \rightarrow 0} \frac{\operatorname{Re}(z + \Delta z) - \operatorname{Re}(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\operatorname{Re}(\Delta z)}{\Delta z} = \begin{cases} 1, & \text{if } \Delta z = \Delta x \\ 0, & \text{if } \Delta z = i\Delta y \end{cases}$   
 b.  $\lim_{\Delta z \rightarrow 0} \frac{\operatorname{Im}(z + \Delta z) - \operatorname{Im}(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\operatorname{Im}(\Delta z)}{\Delta z} = \begin{cases} 0, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = i\Delta y \end{cases}$   
 c. Case 1,  $z = 0$ .

$$\lim_{\Delta z \rightarrow 0} \frac{|0 + \Delta z| - |0|}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta x + i\Delta y} = \begin{cases} \pm 1, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = \pm i\Delta y \end{cases}$$

Case 2,  $z \neq 0$ .

$$\begin{aligned} & \lim_{\Delta z \rightarrow 0} \frac{|z + \Delta z| - |z|}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} - \sqrt{x^2 + y^2}}{\Delta x + i\Delta y} \\ &= \lim_{\Delta z \rightarrow 0} \frac{(x + \Delta x)^2 + (y + \Delta y)^2 - (x^2 + y^2)}{(\Delta x + i\Delta y)(\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} + \sqrt{x^2 + y^2})} \\ &= \lim_{\Delta z \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2 + 2y\Delta y + (\Delta y)^2}{(\Delta x + i\Delta y)(\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} + \sqrt{x^2 + y^2})} \\ &= \begin{cases} \frac{x}{\sqrt{x^2 + y^2}}, & \text{if } \Delta z = \Delta x, z \neq 0 \\ \frac{y}{i\sqrt{x^2 + y^2}}, & \text{if } \Delta z = i\Delta y, z \neq 0 \end{cases} \end{aligned}$$

5. Rule 5:  $(f \pm g)'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{(f \pm g)(z_0 + \Delta z) - (f \pm g)(z_0)}{\Delta z}$   
 $= \lim_{\Delta z \rightarrow 0} \left[ \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \pm \frac{g(z_0 + \Delta z) - g(z_0)}{\Delta z} \right]$   
 $= f'(z_0) \pm g'(z_0)$

Rule 7:  $(fg)'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{fg(z_0 + \Delta z) - fg(z_0)}{\Delta z}$

$$\begin{aligned}
&= \lim_{\Delta z \rightarrow 0} \left\{ \frac{f(z_0 + \Delta z)g(z_0 + \Delta z) - f(z_0 + \Delta z)g(z_0)}{\Delta z} \right. \\
&\quad \left. + \frac{f(z_0 + \Delta z)g(z_0) - f(z_0)g(z_0)}{\Delta z} \right\} \\
&= \lim_{\Delta z \rightarrow 0} \left\{ f(z_0 + \Delta z) \frac{[g(z_0 + \Delta z) - g(z_0)]}{\Delta z} \right. \\
&\quad \left. + g(z_0) \frac{[f(z_0 + \Delta z) - f(z_0)]}{\Delta z} \right\} \\
&= f(z_0)g'(z_0) + g(z_0)f'(z_0)
\end{aligned}$$

6. Let  $n > 0$  be an integer.

$$\text{Then } \frac{d}{dz} z^{-n} = \frac{d}{dz} \left( \frac{1}{z^n} \right) = \frac{-nz^{n-1}}{z^{2n}} \text{ (using Rule 8)} = -nz^{-n-1}.$$

7. a.  $18z^2 + 16z + i$

b.  $-12z(z^2 - 3i)^{-7}$

c.  $\frac{-iz^4 + (2 + 27i)z^2 + 2\pi z + 18}{(iz^3 + 2z + \pi)^2}$

d.  $\frac{-(z+2)^2(5z^2 + (16+i)z - 3 + 8i)}{(z^2 + iz + 1)^5}$

e.  $24i(z^3 - 1)^3(z^2 + iz)^{99}(53z^4 + 28iz^3 - 50z - 25i)$

8. Let  $z = z_0 + \Delta z$ . Then

$$\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \left| \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right| = |f'(z_0)|.$$

$$\lim_{z \rightarrow z_0} \arg[f(z) - f(z_0)] - \arg(z - z_0) = \lim_{z \rightarrow z_0} \arg \left[ \frac{f(z) - f(z_0)}{z - z_0} \right]$$

$$\arg \left[ \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right] = \arg[f'(z_0)]$$

9. a.  $2 - 3i$

b.  $\pm i$

c.  $\frac{-1 \pm i\sqrt{15}}{2}$

d.  $\frac{1}{2}, 1$

10.  $\lim_{\Delta z \rightarrow 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$

$$= \lim_{\Delta z \rightarrow 0} \frac{(z_0 + \Delta z)(\bar{z}_0 + \overline{\Delta z}) - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \left( \bar{z}_0 + \frac{\overline{\Delta z}}{\Delta z} z_0 + \overline{\Delta z} \right) = \begin{cases} \bar{z}_0 + z_0 & \text{if } \Delta z = \Delta x \\ \bar{z}_0 - z_0 & \text{if } \Delta z = i\Delta y \end{cases}$$

If  $z_0 = 0$ , then the difference quotient is

$$\lim_{\Delta z \rightarrow 0} (0 + 0 + \overline{\Delta z}) = 0.$$

11. a. nowhere analytic

b. nowhere analytic

c. analytic except at  $z = 5$

d. everywhere analytic

e. nowhere analytic

f. analytic except at  $z = 0$

g. nowhere analytic

h. nowhere analytic

12. The case when  $n = 1$  is trivial. Assume that the result holds for all positive integers less than or equal to  $n$  and define

$Q(z) = P(z)(z - z_{n+1})$ . Since  $Q'(z) = P'(z)(z - z_{n+1}) + P(z)$ , it follows that

$$\frac{Q'(z)}{Q(z)} = \frac{P'(z)}{P(z)} + \frac{1}{z - z_{n+1}} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_{n+1}}$$

13. a, b, d, f, and g are always true

$$14. \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow z_0} \frac{[f(z) - f(z_0)]/(z - z_0)}{[g(z) - g(z_0)]/(z - z_0)} = \frac{f'(z_0)}{g'(z_0)}$$

$$15. \frac{3}{5}$$

16. Any point on the line through  $z_1$  and  $z_2$  has the form

$$z = -\frac{1}{2} + i\sqrt{3}\left(\frac{1}{2} - t\right), \quad t \text{ real (see Section 1.3, Exercise 18). However, } f(z_2) - f(z_1) = 0 \text{ but } f'(w) = 3w^2 \neq 0 \text{ on the line in question.}$$

$$\begin{aligned} 17. F'(z_0) &= f(z_0)(gh)'(z_0) + f'(z_0)gh(z_0) \\ &= f(z_0)[g(z_0)h'(z_0) + g'(z_0)h(z_0)] + f'(z_0)g(z_0)h(z_0) \\ &= f'(z_0)g(z_0)h(z_0) + f(z_0)g'(z_0)h(z_0) + f(z_0)g(z_0)h'(z_0) \end{aligned}$$

#### EXERCISES 2.4: The Cauchy-Riemann Equations

$$1. \quad a. \quad \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

$$b. \quad \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$$

$$c. \quad \frac{\partial u}{\partial y} = 2 \neq -\frac{\partial v}{\partial x} = 1$$

2.  $\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3 = \frac{\partial v}{\partial y}$ , but  $\frac{\partial u}{\partial y} = 6xy = \frac{\partial v}{\partial x}$ . Therefore  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$  only when  $x = 0$  or  $y = 0$ . This means  $h$  is differentiable on the axes but  $h$  is nowhere analytic since lines are not open sets in the complex plane.

3.  $\frac{\partial u}{\partial x} = 6x + 2 = \frac{\partial v}{\partial y}$ ,  $\frac{\partial u}{\partial y} = -6y = -\frac{\partial v}{\partial x}$ . Since these partial derivatives exist and are continuous for all  $x$  and  $y$ ,  $g$  is analytic.  $g$  can be written as  $g(z) = 3z^2 + 2z - 1$ .

$$4. \frac{\partial u}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{u(\Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{u(0, \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0}{\Delta y} = 0.$$

Similarly  $\frac{\partial v}{\partial x} = 0$  and  $\frac{\partial v}{\partial y} = 0$ .

However, when  $\Delta z \rightarrow 0$  through real values ( $\Delta z = \Delta x$ )

$$\lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = 0,$$

while along the real line  $y = x$  ( $\Delta z = \Delta x + i\Delta x$ )

$$\lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta x \rightarrow 0} \frac{\frac{(\Delta x)^{4/3}(\Delta x)^{5/3} + i(\Delta x)^{5/3}(\Delta x)^{4/3}}{2(\Delta x)^2}}{\Delta x(1 + i)}$$

$$= \frac{1}{2}.$$

Therefore  $f$  is not differentiable at  $z = 0$ .

$$5. \frac{\partial u}{\partial x} = 2e^{x^2-y^2} [x \cos(2xy) - y \sin(2xy)] = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2} [y \cos(2xy) + x \sin(2xy)] = -\frac{\partial v}{\partial x}$$

$f$  is entire because these first partials exist and are continuous for all  $x$  and  $y$ .

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 2e^{x^2-y^2} (x + iy) [\cos(2xy) + i \sin(2xy)]$$

$$= 2e^{(x^2-y^2)} e^{i2xy} (x + iy)$$

$$= 2ze^{z^2}$$

(This derivative could have been obtained directly, since  $f(z) = e^{z^2}$ .)

6.  $z = re^{i\theta} \implies x = r \cos \theta$  and  $y = r \sin \theta$  and

$$f(z) = u(x(r, \theta), y(r, \theta)) + iv(x(r, \theta), y(r, \theta))$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta$$

Similar applications of the chain rule yield

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}(-r \sin \theta) + \frac{\partial u}{\partial y} r \cos \theta \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta \\ \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}(-r \sin \theta) + \frac{\partial v}{\partial y} r \cos \theta\end{aligned}$$

Replace the partial derivatives on the right sides of the equations for  $\frac{\partial u}{\partial r}$  and  $\frac{\partial v}{\partial r}$  by their Cauchy-Riemann counterparts to obtain:

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta = \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial r} &= -\frac{\partial u}{\partial y} \cos \theta + \frac{\partial u}{\partial x} \sin \theta = -\frac{1}{r} \frac{\partial u}{\partial \theta}\end{aligned}$$

7. Let  $h(z) = f(z) - g(z)$ . Then  $h$  is analytic in  $D$  and  $h'(z) = 0$  so  $h$  is a constant function.

$$h(z) = c = f(z) - g(z) \implies f(z) = g(z) + c$$

8.  $u(x, y) = c$  in  $D \implies \frac{\partial u}{\partial x} = 0$  and  $\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$ . Hence

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0 \text{ so } f \text{ is constant in } D.$$

9. By contradiction. If  $f$  is analytic in a domain  $D$  then  $v(x, y) = 0$  (a constant)  $\implies f$  is constant (by condition 8)  $\implies u$  is constant. (However, there is no open set in which  $u(x, y) = |z^2 - z|$  is constant).

10.  $\text{Im } f(z) = 0$  in  $D \implies \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \implies \frac{\partial u}{\partial x} = 0$   
 $\implies f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0 \implies f$  is constant in  $D$ .

11.  $\text{Re } f(z) = \frac{1}{2}[f(z) + \overline{f(z)}]$  is real valued and analytic if both  $f$  and  $\overline{f}$  are analytic. Hence  $\text{Re } f(z)$  is constant by Exercise 10. It follows that  $f(z)$  is constant by Exercise 8.



12.  $|f(z)|$  constant in  $D \implies |f(z)|^2 = u^2 + v^2$  is constant in  $D$ . If  $u = 0$  or  $v = 0$  in  $D$ , then  $f$  is constant by Exercises 8 and 10. Otherwise,

$$\begin{aligned}\frac{\partial |f|^2}{\partial x} &= 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0 \\ \frac{\partial |f|^2}{\partial y} &= 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = -2u \frac{\partial v}{\partial x} + 2v \frac{\partial u}{\partial x} = 0\end{aligned}$$

$$\implies \frac{1}{2}v \frac{\partial |f|^2}{\partial x} - \frac{1}{2}u \frac{\partial |f|^2}{\partial y} = 0 = (u^2 + v^2) \frac{\partial v}{\partial x}$$

$$\implies \frac{\partial v}{\partial x} = 0 \implies \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0$$

$$\implies f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0$$

$$\implies f \text{ is constant in } D.$$

13.  $|f(z)|$  is analytic and real-valued, so the result follows from Exercises 10 and 12.
14. If the line is vertical then  $\operatorname{Re} f(z)$  is constant and this reduces to Problem 8. If the line is not vertical, then  $v(x, y) = mu(x, y) + b$ , and

$$\begin{aligned}\frac{\partial v}{\partial x} &= m \frac{\partial u}{\partial x} = m \frac{\partial v}{\partial y}, \\ \frac{\partial v}{\partial y} &= m \frac{\partial u}{\partial y} = -m \frac{\partial v}{\partial x} = -m^2 \frac{\partial v}{\partial y}.\end{aligned}$$

It follows that

$$\frac{\partial v}{\partial y} = 0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ and } f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0.$$

Hence  $f(z)$  is constant.

$$\begin{aligned}15. J(x_0, y_0) &= \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \Big|_{(x_0, y_0)} \\ &= \left[ \frac{\partial u}{\partial x}(x_0, y_0) \right]^2 + \left[ \frac{\partial v}{\partial x}(x_0, y_0) \right]^2 \\ &= |f'(z_0)|^2 \quad (\text{using Equation (1)})\end{aligned}$$

$$\begin{aligned}
16. \quad a. \quad \frac{\partial \tilde{f}}{\partial \xi} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi} \\
&= \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \frac{1}{2} + \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \frac{1}{2i} \\
&= \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\
\frac{\partial \tilde{f}}{\partial \eta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta} \\
&= \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \frac{1}{2} + \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \left( \frac{-1}{2i} \right) \\
&= \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
b. \quad \frac{\partial \tilde{f}}{\partial \eta} = 0 &\Leftrightarrow 0 = \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \text{ and } 0 = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
&\Leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\end{aligned}$$

### EXERCISES 2.5: Harmonic Functions

$$\begin{aligned}
1. \quad a. \quad u(x, y) &= x^2 - y^2 + 2x + 1, \quad \frac{\partial^2 u}{\partial x^2} = 2 = -\frac{\partial^2 u}{\partial y^2} \implies \Delta u = 0 \\
v(x, y) &= 2xy + 2y, \quad \frac{\partial^2 v}{\partial x^2} = 0 = -\frac{\partial^2 v}{\partial y^2} \implies \Delta v = 0 \\
b. \quad u(x, y) &= \frac{x}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3} = -\frac{\partial^2 u}{\partial y^2} \implies \Delta u = 0 \\
v(x, y) &= -\frac{y}{x^2 + y^2}, \quad \frac{\partial^2 v}{\partial x^2} = \frac{-2y(3x^2 - y^2)}{(x^2 + y^2)^3} = -\frac{\partial^2 v}{\partial y^2} \implies \Delta v = 0 \\
c. \quad u(x, y) &= e^x \cos y, \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y = -\frac{\partial^2 u}{\partial y^2} \implies \Delta u = 0 \\
v(x, y) &= e^x \sin y, \quad \frac{\partial^2 v}{\partial x^2} = e^x \sin y = -\frac{\partial^2 v}{\partial y^2} \implies \Delta v = 0 \\
2. \quad h(x, y) &= ax^2 + bxy - ay^2
\end{aligned}$$

3. a.  $u = \operatorname{Re}(-iz)$ ,  $v = -x + a$ , where  $a$  is a constant  
 b.  $u = \operatorname{Re}(-ie^x)$ ,  $v = -e^x \cos y + a$   
 c.  $u = \operatorname{Re}\left(\frac{-i}{2}z^2 - iz - z\right)$ ,  $v = -\frac{1}{2}(x^2 - y^2) - (x + y) + a$

d. It is straightforward to verify that  $\Delta u = 0$ .

$$\frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}$$

$$\Rightarrow v(x, y) = \int \cos x \cosh y dy = \cos x \sinh y + \psi(x)$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y = -\frac{\partial v}{\partial x} = \sin x \sinh y + \psi'(x) \Rightarrow \psi(x) = a$$

Thus,  $v(x, y) = \cos x \sinh y + a$ .

e. It is straightforward to verify that  $\Delta u = 0$ .

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y} \Rightarrow$$

$$v(x, y) = \int \frac{x}{x^2 + y^2} dy = \tan^{-1}\left(\frac{y}{x}\right) + \psi(x)$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2} - \psi'(x) \Rightarrow \psi(x) = a$$

Thus,  $v(x, y) = \tan^{-1}\left(\frac{y}{x}\right) + a$ .

f.  $u = \operatorname{Re}(-ie^{x^2})$ ,  $v = -e^{x^2 - y^2} \cos(2xy) + a$ .

4. Suppose  $v$  and  $w$  are both harmonic conjugates of  $u$ , and consider  $\phi(x, y) = w(x, y) - v(x, y)$ . Then (using the Cauchy-Riemann equations for  $v$  and  $w$ ),

$$\frac{\partial \phi}{\partial x} = \frac{\partial w}{\partial x} - \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} - \left(-\frac{\partial u}{\partial y}\right) = 0$$

and similarly  $\frac{\partial \phi}{\partial y} = 0$ . Hence  $\phi(x, y) = a$ , from which it follows that

$$w(x, y) = v(x, y) + a.$$

5. If  $f(z) = u(x, y) + iv(x, y)$  is analytic then  $-if(z) = v(x, y) - iu(x, y)$  is analytic. Thus  $-u$  is a harmonic conjugate of  $v$ .

6. Since  $f(z) = u + iv$  is analytic,  $\frac{1}{2}[f(z)]^2 = \frac{1}{2}(u^2 - v^2) + iuv$  is analytic.

Thus  $uv = \text{Im} \frac{1}{2}[f(z)]^2$  is harmonic.

7.  $\phi(x, y) = x + 1$

8. a. Yes, because  $\Delta(u + v) = \Delta u + \Delta v = 0$ .

b. No. Take  $u = x, v = x^2 - y^2$  as an example.

c. Yes, because  $\Delta(u_x) = u_{xxx} + u_{xyy} = u_{xxx} + u_{yyx}$

$$= \frac{\partial}{\partial x}(\Delta u) = \frac{\partial}{\partial x}(0) = 0.$$

9.  $\phi(x, y) = xy - 1$  (this is  $\text{Im} \left( \frac{1}{2}z^2 - i \right)$ )

10. Let  $x = r \cos \theta$  and  $y = r \sin \theta$ .

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial \phi}{\partial x} \cos \theta + \frac{\partial \phi}{\partial y} \sin \theta \\ \frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial^2 \phi}{\partial x^2} \frac{\partial x}{\partial r} \cos \theta + \frac{\partial^2 \phi}{\partial y \partial x} \frac{\partial y}{\partial r} \cos \theta \\ &\quad + \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial x}{\partial r} \sin \theta + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial y}{\partial r} \sin \theta \\ &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \theta + \frac{\partial^2 \phi}{\partial y \partial x} 2 \sin \theta \cos \theta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \theta \\ \frac{\partial \phi}{\partial \theta} &= \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial \phi}{\partial x} (-r \sin \theta) + \frac{\partial \phi}{\partial y} r \cos \theta \\ \frac{\partial^2 \phi}{\partial \theta^2} &= \frac{\partial^2 \phi}{\partial x^2} \frac{\partial x}{\partial \theta} (-r \sin \theta) + \frac{\partial^2 \phi}{\partial y \partial x} \frac{\partial y}{\partial \theta} (-r \sin \theta) + \frac{\partial \phi}{\partial x} (-r \cos \theta) \\ &\quad + \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial x}{\partial \theta} (r \cos \theta) + \frac{\partial^2 \phi}{\partial y^2} \frac{\partial y}{\partial \theta} (r \cos \theta) + \frac{\partial \phi}{\partial y} (-r \sin \theta) \\ &= \frac{\partial^2 \phi}{\partial x^2} r^2 \sin^2 \theta + \frac{\partial^2 \phi}{\partial y \partial x} (-2r^2 \sin \theta \cos \theta) + \frac{\partial^2 \phi}{\partial y^2} r^2 \cos^2 \theta \\ &\quad + \frac{\partial \phi}{\partial x} (-r \cos \theta) + \frac{\partial \phi}{\partial y} (-r \sin \theta). \end{aligned}$$

Combining these partial derivatives, one gets

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

11.  $\text{Im } f(z) = y - \frac{y}{x^2 + y^2} = 0 \implies yx^2 + y^3 - y = y(x^2 + y^2 - 1) = 0.$

The points satisfying  $x^2 + y^2 - 1 = 0$  lie on the circle  $|z| = 1$ . The points (other than  $z = 0$ ) satisfying  $y = 0$  lie on the real axis.

12.  $f(z) = z^n = r^n(\cos \theta + i \sin \theta)^n = r^n(\cos n\theta + i \sin n\theta) \implies$   
 $\text{Re } f(z) = r^n \cos n\theta$  and  $\text{Im } f(z) = r^n \sin n\theta$  are harmonic since  $f$  is analytic.

13.  $\phi(x, y) = \text{Im } z^4 = r^4 \sin 4\theta = -4xy^3 + 4x^3y$

14. Let  $\phi(x, y) = \ln |f(z)| = \frac{1}{2} \ln(u^2 + v^2)$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} = \frac{u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x}}{u^2 + v^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{(v^2 - u^2) \left[ \left( \frac{\partial u}{\partial x} \right)^2 - \left( \frac{\partial v}{\partial x} \right)^2 \right] - 4uv \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}}{(u^2 + v^2)^2} + \frac{u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 v}{\partial x^2}}{u^2 + v^2}$$

A similar calculation yields  $\frac{\partial^2 \phi}{\partial y^2}$ . By applying Laplace's equation and the Cauchy-Riemann equations of  $u$  and  $v$  to  $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$ , the sum simplifies to reveal that  $\Delta \phi = 0$ .

15. Consider  $\varphi(z) = \text{Re}(Az^n + Bz^{-n}) + C$  which is harmonic for  $1 \leq |z| \leq 2$ . Consider the polar form for  $z$ .  $z = re^{i\theta}$  and select  $n=3$  to agree with the cosine argument.  $\varphi(re^{i\theta}) = Ar^3 \text{Re}(e^{i3\theta}) + Br^{-3} \text{Re}(e^{-i3\theta}) + C$ .  
 $\varphi(re^{i\theta}) = Ar^3 \cos 3\theta + Br^{-3} \cos 3\theta + C = (Ar^3 + Br^{-3}) \cos 3\theta + C$ .

$r=1 \implies (A+B) \cos 3\theta + C = 0 \implies A + B = 0, C = 0.$

$r=2 \implies (A \cdot 8 + B/8) \cos 3\theta = 5 \cos 3\theta. A = 40/63, B = -40/63$

$\varphi(re^{i\theta}) = (40/63)(r^3 - r^{-3}) \cos 3\theta = (40/63) \text{Re}(z^3 - z^{-3}).$

16.  $\phi(x, y) = \frac{1}{\ln 3} \ln |z| - 1$  or  $\phi(x, y) = \ln \left| \frac{z}{3} \right|$  are two possibilities.

17. a.  $\phi(x, y) = \operatorname{Re}(z^2 + 5z + 1) = x^2 - y^2 + 5x + 1$

b.  $\phi(x, y) = 2\operatorname{Re}\left(\frac{z^2}{z + 2i}\right) = \frac{2x(x^2 + 4y + y^2)}{x^2 + y^2 + 4y + 4}$

18. Let  $u = \phi_x$ ,  $v = -\phi_y$ . Then

$$\frac{\partial u}{\partial x} = \phi_{xx} = -\phi_{yy} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \phi_{xy} = -\frac{\partial v}{\partial x}$$

19.  $\cos^2\theta = (\frac{1}{2})\cos 2\theta + \frac{1}{2} = \varphi(z) = A\operatorname{Re}(r^{-2}e^{-i2\theta}) + B = Ar^{-2}\cos 2\theta + B$ . In the limit as  $r \rightarrow \infty$   $\varphi(z) = \frac{1}{2} \Rightarrow B = \frac{1}{2}$ . On the circle  $|z|=1$ ,  $r=1$ .  $\Rightarrow A = \frac{1}{2}$ .  
 $\varphi(z) = (\frac{1}{2})r^{-2}\cos 2\theta + \frac{1}{2} = \operatorname{Re}[1/(2z^2)] + \frac{1}{2}$ .

20. In order that  $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ , let  $v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, \eta) d\eta + \psi(x)$ . Then

$$\begin{aligned} \frac{\partial v}{\partial x} &= \int_0^y \frac{\partial^2 u}{\partial x^2}(x, \eta) d\eta + \psi'(x) \\ &= -\int_0^y \frac{\partial^2 u}{\partial y^2}(x, \eta) d\eta + \psi'(x) \quad (\text{because } u \text{ is harmonic}) \\ &= -\frac{\partial u}{\partial y}(x, y) + \frac{\partial u}{\partial y}(x, 0) + \psi'(x). \end{aligned}$$

In order that  $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ , it must be true that  $\psi'(x) = -\frac{\partial u}{\partial y}(x, 0)$ .

Thus,

$$\psi(x) = -\int_0^x \frac{\partial u}{\partial y}(\zeta, 0) d\zeta + a$$

and

$$v(x, y) = \int_0^y \frac{\partial u}{\partial x}(x, \eta) d\eta - \int_0^x \frac{\partial u}{\partial y}(\zeta, 0) d\zeta + a.$$

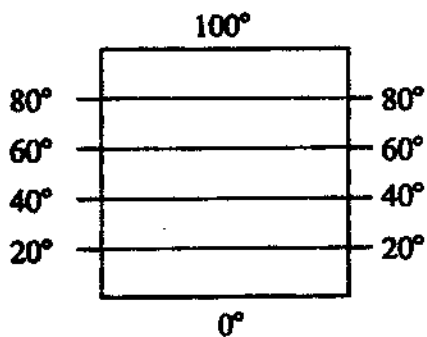
21. It is easily verified that  $u = \ln|z|$  satisfies Laplace's equation on  $\mathbb{C} \setminus \{0\}$  and that  $u + iv = \ln|z| + i\operatorname{Arg}(z)$  satisfies the Cauchy-Riemann equations on the domain  $D = \mathbb{C} \setminus \{\text{nonpositive real axis}\}$ , so that

**$\operatorname{Arg}(z)$  is a harmonic conjugate of  $u$  on  $D$ . By Problem 4, any harmonic conjugate of  $u$  has to be of the form  $\operatorname{Arg}(z) + a$  in  $D$ . It is impossible to have a harmonic conjugate of this form that is continuous on  $\mathbb{C} \setminus \{0\}$ .**

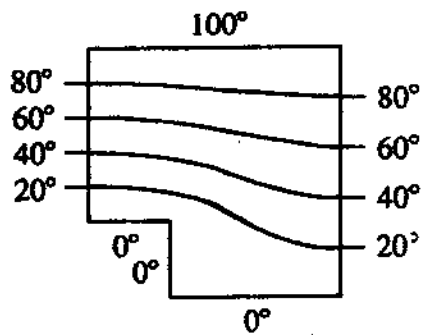
$$\begin{aligned} 22. \frac{\partial u}{\partial x} &= \phi_{xx}\phi_y + \phi_x\phi_{yx} + \psi_{xx}\psi_y + \psi_x\psi_{yx} \\ &= -\phi_{yy}\phi_y + \phi_x\phi_{yx} - \psi_{yy}\psi_y + \psi_x\psi_{yx} = \frac{\partial v}{\partial y} \end{aligned}$$

**EXERCISES 2.6: Steady-State Temperature as a Harmonic Function.**

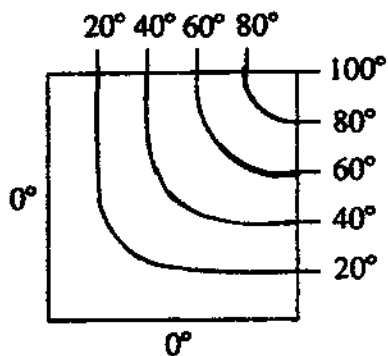
1. a.



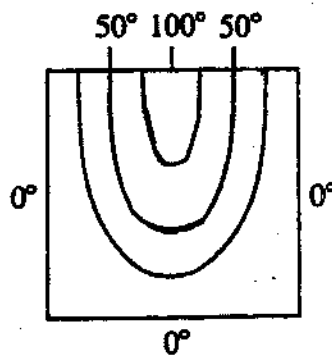
b.



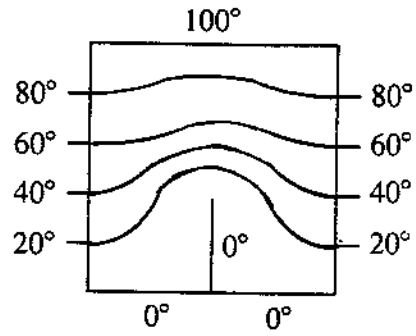
c.



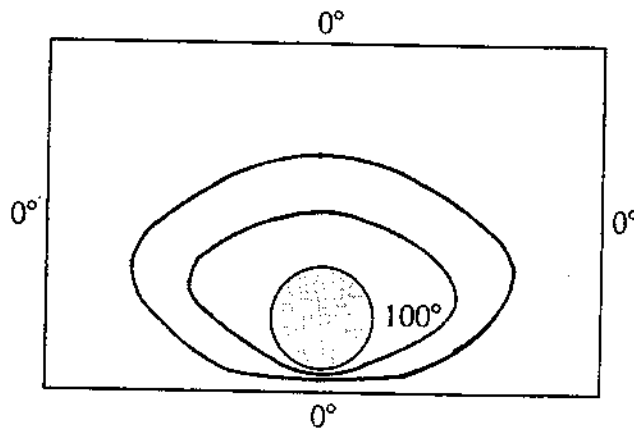
d.



e.

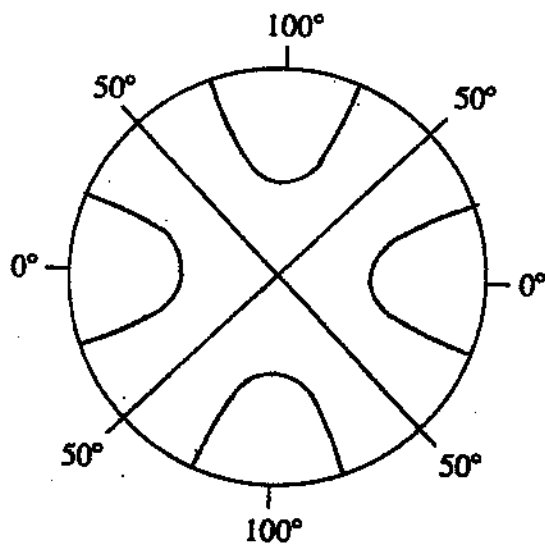


2. This does not violate the maximum principle.





3. This does not violate the maximum principle.



## Exercises 2.7

1.  $f(z) = z^2 + c$  where  $c$  is a real constant.  
 $\zeta_1 = (1 + \sqrt{1-4c})/2$ ,  $\zeta_2 = (1 - \sqrt{1-4c})/2$   
 Only  $\zeta_2$  is an attractor for  $-3/4 < c < 1/4$ .
2.  $f(\zeta) = \zeta$  and  $f'(\zeta) > 1$  Therefore we can pick a real number  $\rho$  between 1 and  $|f'(\zeta)|$  such that  $|f(z) - \zeta| = \rho|z - \zeta|$  for all  $z$  in a sufficiently small disk around  $\zeta$ . If any point  $z_0$  in this disk is the seed for an orbit  $z_1 = f(z_0)$ ,  $z_2 = f(z_1)$ , ...  $z_n = f(z_{n-1})$ , then we have  $|z_n - \zeta| \geq \rho|z_{n-1} - \zeta| \geq \dots \geq \rho^n|z_0 - \zeta|$ . Because  $\rho > 1$ , the point  $z_n$  moves away from  $\zeta$  until the magnitude of the derivative becomes 1 or less. The orbit is out of the disk.
3. (a) Fixed points are  $\zeta_1 = i$ ,  $\zeta_2 = -i$ . Both are repellers.  
 (b) Fixed points are  $\zeta_1 = 1/2$ ,  $\zeta_2 = -1/2$ ,  $\zeta_3 = -1$ . Fixed points  $\zeta_1$  and  $\zeta_3$  are repellers, but fixed point  $\zeta_2$  is an attractor.
4.  $z_0 = e^{i2\pi\alpha}$  with  $\alpha$  an irrational real number.  $z_n = e^{i2\pi\alpha 2^n}$ . Because  $|z_n| = 1$ , the trajectory will follow the unit circle. If iterations  $p$  and  $q$  coincide,  $2\pi\alpha 2^p - 2\pi\alpha 2^q = 2\pi\alpha(2^p - 2^q) = 2\pi k$  for some integer  $k$ . But because  $(2^p - 2^q)$  is an integer that can be represented by  $m$ , the equation  $2\pi\alpha m = 2\pi k$  is satisfied only if  $k = \alpha m$  or  $\alpha = k/m$ . Because  $\alpha$  is irrational it cannot be represented by a rational number and no iterations repeat.
5. Fixed points are  $\zeta_1 = -1/2 + i\sqrt{5}/2$  (an attractor) and  $\zeta_2 = -1/2 - i\sqrt{5}/2$  (a repeller).
6.  $f(z) = z^2$ . The seed is  $z_0$ .  $z_1 = z_0^2$ ,  $z_2 = z_0^4$ , ...  $z_n = z_0^{2^n}$ . To have an  $n$  cycle  $z_n = z_0 = z_0^{2^n}$ . Or  $z_n/z_0 = z_0^{2^n-1} = 1 = e^{i2\pi}$ . Solving gives  $z_0 = e^{i2\pi/(2^n-1)}$ .
7. The cycle is 4.  $2^4(2\pi/p) = 2\pi \pmod p \Rightarrow 2^4 = 1 \pmod p$ .  $p=3,5,15$ . 3 will give repeated cycles of length 2. 5 and 15 will give the desired cycles of length 4.
8. Student Matlab: `n=100;c=.253; z0=0;y(1)=z0; for k=1:n-1,y(k+1)=y(k)^2+c;end plot(y)`
9. If  $|\alpha| \leq 1$  the whole complex plane is the filled Julia set. If  $|\alpha| \geq 1$  the origin is the filled Julia set.
10.  $f(z) = z - F(z)/F'(z)$ .  $f(\zeta) = \zeta - F(\zeta)/F'(\zeta) = \zeta \Rightarrow F(\zeta)/F'(\zeta) = 0 \Rightarrow F(\zeta) = 0$  with the possible exception of the points where  $F'(\zeta) = 0$ .  
 $f'(z) = 1 - F'(z)/F'(z) + F(z)F''(z)/(F'(z))^2 = F(z)F''(z)/(F'(z))^2$   
 $f'(\zeta) = F(\zeta)F''(\zeta)/(F'(\zeta))^2 = 0$  where  $F'(\zeta) \neq 0$  and every zero of  $F(z)$  is an attractor as long as  $F'(\zeta) \neq 0$ .