Seminars on Continuous Time Finance

Raquel M. Gaspar

Fall 2003

Contents

3 Stochastic Integrals

Exercise 3.1

(a) Since $Z(t)$ is determinist, we have

$$\begin{aligned} dZ(t) &= \alpha e^{\alpha t} dt \\ &= \alpha Z(t) dt. \end{aligned}$$

(b) By definition of a stochastic differential

$$dZ(t) = g(t) dW(t)$$

(c) Using Itô's formula

$$\begin{aligned} dZ(t) &= \frac{\alpha^2}{2} e^{\alpha W(t)} dt + \alpha e^{\alpha W(t)} dW(t) \\ &= \frac{\alpha^2}{2} Z(t) dt + \alpha Z(t) dW \end{aligned}$$

(d) Using Itô's formula and considering the dynamics of $X(t)$ we have

$$\begin{aligned} dZ(t) &= \alpha e^{\alpha x} dX(t) + \frac{\alpha^2}{2} e^{\alpha x} (dX(t))^2 \\ &= Z(t) \left[\alpha\mu + \frac{1}{2}\alpha^2\sigma^2 \right] dt + \alpha\sigma Z(t) dW(t). \end{aligned}$$

(e) Using Itô's formula and considering the dynamics of $X(t)$ we have

$$\begin{aligned} dZ(t) &= 2X(t)dX(t) + (d(X(t))^2 \\ &= Z(t) \left[2\alpha + \sigma^2 \right] dt + 2Z\sigma dW(t). \end{aligned}$$

Exercise 3.3 By definition we have that the dynamics of $X(t)$ are given by $dX(t) = \sigma(t) dW(t)$.

Consider $Z(t) = e^{iuX(t)}$. Then using the Itô's formula we have that the dynamic of $Z(t)$ can be described by

$$dZ(t) = \left[-\frac{u^2}{2}\sigma^2(t) \right] Z(t) dt + \left[iu\sigma(t) \right] Z(t) dW(t)$$

From $Z(0) = 1$ we get,

$$Z(t) = 1 - \frac{u^2}{2} \int_0^t \sigma^2(s) Z(s) ds + iu \int_0^t \sigma(s) Z(s) dW(s).$$

Taking expectations we have,

$$\begin{aligned} E[Z(t)] &= 1 - \frac{u^2}{2} E\left[\int_0^t \sigma^2(s)Z(s)ds\right] + iuE\left[\int_0^t \sigma(s)Z(s)dW(s)\right] \\ &= 1 - \frac{u^2}{2}\left[\int_0^t \sigma^2(s)E[Z(s)]\,ds\right] + 0 \end{aligned}$$

By setting $E[Z(t)] = m(t)$ and differentiating with respect to t we find an ordinary differential equation,

$$\frac{\partial m(t)}{\partial t} = -\frac{u^2}{2} m(t)\sigma^2(t)$$

with the initial condition $m(0) = 1$ and whose solution is

$$\begin{aligned} m(t) &= \exp\left\{-\frac{u^2}{2}\int_0^t \sigma 2(s)ds\right\} \\ &= E[Z(t)] \\ &= E\left[e^{iuX(t)}\right] \end{aligned}$$

So, $X(t)$ is normally distributed. By the properties of the normal distribution the following relation

$$E\left[e^{iuX(t)}\right] = e^{iuE[X(t)] - \frac{u^2}{2}V[X(t)]}$$

where $V[X(t)]$ is the variance of $X(t)$, so it must be that $E[X(t)] = 0$ and $V[X(t)] = \int_0^t \sigma^2(s)ds$.

Exercise 3.5 We have a sub martingale if $E[X(t)|\mathcal{F}_s] \geq X(s)\forall, t \geq s$. From the dynamics of X we can write

$$X(t) = X(s) + \int_s^t \mu(z)dz + \int_s^t \sigma(z)dW(z).$$

By taking expectation, conditioned at time s, from both sides we get

$$\begin{aligned} E[X(t)|\mathcal{F}_s] &= E[X(s)|\mathcal{F}_s] + E\left[\left.\int_s^t \mu(z)dz\right|\mathcal{F}_s\right] \\ &= X(s) + E^s\left[\left.\underbrace{\int_s^t \mu(z)dz}_{\geq 0}\right|\mathcal{F}_s\right] \\ &\geq X(s) \end{aligned}$$

so X is a sub martingale.

Exercise 3.6 Set $X(t) = h(W_1(t), \cdots, W_n(t))$.

We have by Itô that

$$dX(t) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(t) dW_j(t)$$

where $\frac{\partial h}{\partial x_i}$ denotes the first derivative with respect to the i-th variable, $\frac{\partial^2 h}{\partial x_i \partial x_j}$ denotes the second order cross-derivative between the i-th and j-th variable and all derivatives should be evaluated at $(W_1(s), \cdots, W_n(s))$.

Since we are dealing with independent Wiener processes we know

$$\forall u: \quad dW_i(u) dW_j(u) = 0 \text{ for } i \neq j \quad \text{and} \quad dW_i(u) dW_j(u) = du \text{ for } i = j,$$

so, integrating we get

$$\begin{aligned} X(t) &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} dW_i(u) dW_j(u) \\ &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} \left[dW_i(u) \right]^2 \\ &= \int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du. \end{aligned}$$

Taking expectations

$$\begin{aligned} E\left[X(t) \middle| \mathcal{F}_s \right] &= E\left[\int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) \middle| \mathcal{F}_s \right] + E\left[\frac{1}{2} \int_0^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right] \\ &= \underbrace{\int_0^s \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) + \frac{1}{2} \int_0^s \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du}_{X(s)} \\ &\quad + \underbrace{E\left[\int_0^t \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dW_i(u) \middle| \mathcal{F}_s \right]}_{0} + E\left[\frac{1}{2} \int_s^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right] \\ &= X(s) + E\left[\frac{1}{2} \int_s^t \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} du \middle| \mathcal{F}_s \right]. \end{aligned}$$

- If h is *harmonic* the last term is zero, since $\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} = 0$, we have

$$E\left[X(t) \middle| \mathcal{F}_s \right] = X(s) \quad \text{so } X \text{ is a martingale.}$$

- If h is *subharmonic* the last term is always nonnegative, since $\sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} \geq 0$ we have

$$E\left[X(t)|\mathcal{F}_s\right] \geq X(s) \quad \text{so } X \text{ is a submartingale.}$$

Exercise 3.8

(a) Using the Itô's formula we find the dynamics of $R(t)$,

$$\begin{aligned} dR(t) &= 2X(t)(dX(t)) + 2Y(t)(dY(t)) + \frac{1}{2}\left[2(dX(t))^2 + 2(dY(t))^2\right] \\ &= (2\alpha+1)\left[X^2(t) + Y^2(t)\right] dt \\ &= (2\alpha+1)R(t)dt \end{aligned}$$

From the dynamics we can see immediately that $R(t)$ is deterministic (it has no stochastic component!).

(b) Integrating the SDE for $X(t)$ and taking expectations we have

$$X(t) = x_0 + \alpha \int_0^t E\left[X(s)\right] ds$$

Which once more can be solve setting $m(t) = E\left[X(t)\right]$,taking the derivative with respect to t and using ODE methods, to get the answer

$$E\left[X(t)\right] = x_0 e^{\alpha t}$$

4 Differential Equations

Exercise 4.1 We have:

$$dY(t) = \alpha e^{\alpha t} x_0 dt, \quad dZ(t) = \alpha e^{\alpha t} \sigma dt, \quad dR(t) = e^{-\alpha t} dW(t).$$

Itô's formula then gives us (the cross term $dZ(t) \cdot dR(t)$ vanishes)

$$\begin{aligned} dX(t) &= dY(t) + Z(t) \cdot dR(t) + R(t) \cdot dZ(t) \\ &= \alpha e^{\alpha t} x_0 dt + e^{\alpha t} \cdot \sigma \cdot e^{-\alpha t} dW(t) + \int_0^t e^{-\alpha s} dW(s) \cdot \alpha e^{\alpha t} \sigma dt \\ &= \alpha \left[e^{\alpha t} x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW(s)\right] dt + \sigma dW(t) \\ &= \alpha X(t) dt + \sigma dW(t). \end{aligned}$$

Exercise 4.5 Using the dynamics of $X(t)$ and the Itô formula we get

$$\begin{aligned} dY(t) &= \left[\alpha\beta + \frac{1}{2}\beta(\beta-1)\sigma^2\right] Y(t)dt + \sigma\beta Y(t)dW(t) \\ &= \mu Y(t)dt + \delta Y(t)dW(t) \end{aligned}$$

where $\mu = \alpha\beta + \frac{1}{2}\beta(\beta-1)\sigma^2$ and $\delta = \sigma\beta$ so Y is also a GBM.

Exercise 4.6 From the Itô formula and using the dynamics of X and Y

$$\begin{aligned} dZ(t) &= \frac{1}{Y(t)}dX(t) - \frac{X(t)}{Y(t)^2}dY(t) - \frac{1}{Y(t)^2}dX(t)dY(t) + \frac{X(t)}{Y(t)^3}(dY(t))^2 \\ &= Z(t)\left[\alpha - \gamma + \delta^2\right] dt + \sigma Z(t)dW(t) - \delta Z(t)dV(t). \end{aligned}$$

Exercise 4.9 *From Feyman-Kac we have* We have

$$F(t,x) = E^{t,x}\left[2\ln[X(T)]\right],$$

and

$$\begin{aligned} dX(s) &= \mu X(s)ds + \sigma X dW(s), \\ X(t) &= x. \end{aligned}$$

Solving the SDE, we obtain (check the solution of the GBM in th extra exercises if you do not remmeber)

$$X(T) = \exp\left\{\ln x + (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma[W(T) - W(t)]\right\},$$

and thus

$$F(t,x) = 2\ln(x) + 2(\mu - \frac{1}{2}\sigma^2)(T-t).$$

Exercise 4.10 Given the dynamics of $X(t)$ any $F(t,x)$ that solves the problem has the dynamics given by

$$\begin{aligned} dF(t,x) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX(t))^2 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}\left[\mu(t,x)dt + \sigma(t,x)dW(t)\right] + k(t,x)dt - k(t,x)dt \end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left[\sigma^2(t,x)dW(t)\right] \\
= & \left\{\underbrace{\frac{\partial F}{\partial t}+\mu(t,x)\frac{\partial F}{\partial x}+\frac{1}{2}\sigma^2(t,x)+k(t,x)}_{0}\right\}dt-k(t,x)dt \\
& +\frac{\partial F}{\partial x}\sigma(t,x)dW(t) \\
= & -k(t,x)dt+\frac{\partial F}{\partial x}\sigma(t,x)dW(t)
\end{aligned}
$$

We now write $F(T,X(T))$ in terms of $F(t,x)$ and the dynamics of F during the time period $t\ldots T$(recall that we defined $X(t)=x$)

$$
\begin{aligned}
F(t,X(T)) &= F(t,x)-\int_t^T k(s,X(s)ds+\int_t^T\frac{\partial F}{\partial x}\sigma(s,X(s))dW(s) \\
&\Leftrightarrow \\
F(t,x) &= F(T,X(T))+\int_t^T k(s,X(s)ds-\int_t^T\frac{\partial F}{\partial x}\sigma(s,X(s))dW(s)
\end{aligned}
$$

Taking expectations $E_{t,x}[.]$ from both sides

$$
\begin{aligned}
F(t,x) &= E_{t,x}\left[F(T,X(T))\right]+E_{t,x}\left[\int_t^T k(s,X(s)ds\right] \\
&= E_{t,x}\left[\Phi(T)\right]+\int_t^T E_{t,x}\left[k(s,X(s)\right]ds
\end{aligned}
$$

Exercise 4.11 Using the representation formula from Exercise 4.10 we get

$$
F(t,x)=E_{t,x}\left[2\ln[X^2(T)]\right]+\int_t^T E_{t,x}\left[X(s)\right]ds,
$$

Given

$$
dX(s)=X(s)dW(s).
$$

The first term is easily computed as in the exercise 4.9 above. Furthermore it is easily seen directly from the SDE (how?)that $E_{t,x}[X(s)]=x$. Thus we have the result

$$
\begin{aligned}
F(t,x) &= 2\ln(x)-(T-t)+x(T-t) \\
&= \ln(x^2)+(x-1)(T-t)
\end{aligned}
$$

6 Arbitrage Pricing

Exercise 6.1

(a) From standard theory we have

$\Pi(t) = F(t, S(t))$, where F solves the Black-Scholes equation.

Using Itô we obtain

$$d\Pi(t) = \left[\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{\partial s^2}\right] dt + \sigma S(t)\frac{\partial F}{\partial s} dW(t).$$

Using the fact that F satisfies the Black-Scholes equation, and that $F(t, S(t)) = \Pi(t)$ we obtain

$$d\Pi(t) = r\Pi(t)\, dt + \sigma S(t)\frac{\partial F}{\partial s} dW(t)$$

and so $g(t) = \sigma S(t)\frac{\partial F}{\partial s}$.

(b) Apply Itô's formula to the process $Z(t) = \frac{\Pi(t)}{B(t)}$ and use the result in (a).

$$\begin{aligned} dZ(t) &= \frac{1}{B(t)}(d\Pi(t)) - \frac{\Pi(t)}{B^2(t)}(d(B(t)) \\ &= \frac{g(t)}{B(t)} dW(t) \\ &= Z(t)\frac{\sigma S(t)}{\Pi(t)}\frac{\partial F}{\partial s} dW(t) \end{aligned}$$

Z is a martingale since $E_t[Z(T)] = Z(t)$ for all $t < T$ and its diffusion coefficient is given by $\sigma_Z(t) = \frac{\sigma S(t)}{\Pi(t)}\frac{\partial F}{\partial s}$.

Exercise 6.4 We have as usual

$$\Pi(t) = e^{-r(T-t)} E^Q_{t,s}\left[S^\beta(T)\right].$$

We know from earlier exercises (check exercises 3.4 and 4.5) that $Y(t) = S^\beta(t)$ satisfies the SDE under Q

$$dY(t) = \left[r\beta + \frac{1}{2}\beta(\beta - 1)\sigma^2\right] Y(t)dt + \sigma\beta Y(t)dW(t).$$

Using the standard technique, we can integrate, take expectations, differentiate with respect to time and solve by ODE techniques, to obtain

$$E^Q_{t,s}\left[S^\beta(T)\right] = s^\beta e^{\left[r\beta + \frac{1}{2}\beta(\beta-1)\sigma^2\right](T-t)},$$

So,

$$\Pi(t) = s^{\beta} e^{\left[r(\beta-1)+\frac{1}{2}\beta(\beta-1)\sigma^2\right](T-t)}.$$

Exercise 6.6 We consider only the case when $t < T_0$. The other case is handled in very much the same way. We have to compute $E^Q_{t,s}\left[\frac{S(T_1)}{S(T_0)}\right]$. Define the process X on the time interval $[T_0, T_1]$ by

$$X(u) = \frac{S(u)}{S(T_0)}.$$

We now want to compute $E^Q_{t,s}\left[X(T_1)\right]$. The stochastic differential (under Q) of X is easily seen to be

$$\begin{aligned} dX(u) &= rXdu + \sigma X dW(u), \\ X(T_0) &= 1. \end{aligned}$$

From this SDE it follows at once (the same technique of integrating, taking expectations, differentiate with respect to time and solve by ODE techniques) that

$$E^Q_{t,s}\left[X(T_1)\right] = e^{r(T_1-T_0)},$$

and thus the price, at t of the contract is given by

$$\Pi\left(t\right) = e^{-r(T_0-t)}.$$

Exercise 6.7 The price in SEK of the ACME INC., Z, is defined as $Z(t) = S(t)Y(t)$ and by Itô has the following dynamics under Q

$$dZ(t) = rZ(t)dt + \sigma Z(t)dW_1(t) + \delta Z(t)dW_2(t)$$

We also have, by using Itô once more, that the dynamics of $\ln Z^2$ are

$$d\ln Z^2(t) = \left[2r - \sigma^2 - \delta^2\right]dt + 2\sigma dW_1(t) + 2\delta dW_2(t)$$

which integrating and taking conditioned expectations give us

$$E^Q_{t,z}\left[\ln[Z^2(T)]\right] = \ln z^2 + \left[2r - \sigma^2 - \delta^2\right](T-t)$$

Since we know that

$$\Pi(t) = F(t,s) = e^{-r(T-t)}E^Q_{t,z}\left[\ln[Z^2(T)]\right],$$

the arbitrage free pricing function Π is

$$\begin{aligned}\Pi(t) &= e^{-r(T-t)}\left\{\ln z^2 + \left[2r-\sigma^2-\delta^2\right](T-t)\right\}\\ &= e^{-r(T-t)}\left\{2\ln(sy) + \left[2r-\sigma^2-\delta^2\right](T-t)\right\},\end{aligned}$$

where, as usual, $z = Z(t)$, $s = S(t)$ and $y = Y(t)$.

Exercise 6.9 The *forward price*, i.e. the amount of money to be payed out at time T, but decided at the time t is

$$F(t,T) = E_t^Q\left[\mathcal{X}\right].$$

Note that the forward price *is not* the *price of the forward contract* on the T-claim $\mathcal{X}$ which is what we are looking for.

Take for instance the long position: at time T, the buyer of a forward contract receives $\mathcal{X}$ and pays $F(t,T)$. Hence, the price at time t of that position is

$$\Pi(t;\mathcal{X}-F(t,T)) = E_t^Q\left[e^{-r(T-t)}\left(\mathcal{X} - \underbrace{F(t,T)}_{E_t^Q[\mathcal{X}]}\right)\right] = 0.$$

At time $s > t$, however, the underlying asset may have changed in value, in a way different from expectations, so then the price of a forward contract can be defined as

$$\begin{aligned}\Pi(s;\mathcal{X}-F(t,T)) &= E_s^Q\left[e^{-r(T-s)}\left(\mathcal{X}-F(t,T)\right)\right]\\ &= e^{-r(T-s)}\left[E_s^Q\left[\mathcal{X}\right] - \overbrace{E_t^Q\left[\mathcal{X}\right]}^{F(t,T)}\right].\end{aligned}$$

Remark: For the special case where the contract is on one share S we get:

$$\Pi(s) = e^{-r(T-s)}\left[E_s^Q\left[S(T)\right] - \underbrace{S(t)e^{r(T-t)}}_{E_t^Q[S(T)]}\right].$$

We can also easily calculate $E_s^Q\left[S(T)\right]$ since

$$E_s^Q\left[S(T)\right] = \underbrace{S(t) + r\int_t^s S(u)du}_{S(s)} + r\int_s^T E_s^Q\left[S(u)\right]du$$

so,

$$E_s^Q\left[S(T)\right] = S(s)e^{r(T-s)}$$

and, therefore, the free arbitrage pricing function at time $s > t$ is

$$\Pi(s) = S(s) - S(t)e^{r(s-t)}.$$

7 Completeness and Hedging

Exercise 7.2 We have $F(t, s, z)$ be defined by

$$\begin{aligned} F_t + r \cdot s \cdot F_s + \frac{1}{2}\sigma^2 s^2 F_{ss} + gF_z &= rF \\ F(T, s, z) &= \Phi(s, z) \end{aligned}$$

and the dynamics under Q for S and Z

$$\begin{aligned} dS(u) &= rS(u)du + \sigma S(u)dW(u) \\ dZ(u) &= g(u, S(u))du \end{aligned}$$

We want to show that $F(t, S(t), Z(t)) = e^{-r(T-t)}E_{t,s,z}^Q\left[\Phi(S(T), Z(T))\right]$.

For that we find , by Itô, the dynamics of $\Pi(t) = F(t, S(t), Z(t))$, the arbitrage free pricing process

$$\begin{aligned} d\Pi(t) &= F_t dt + F_s\left[(rS(t)dt + \sigma S(t)dW(t)\right] + F_z \cdot g(t, S(t))dt + \frac{1}{2}F_{ss}\sigma^2 S^2(t)dt \\ &= \underbrace{\left[F_t + r \cdot S(t) \cdot F_s + \frac{1}{2}\sigma^2 S^2(t)F_{ss} + g(t, S(t))F_z\right]}_{r\Pi(t)} + \sigma S(t)F_s dW(t) \end{aligned}$$

Integrating we have

$$\Pi(T) = \Pi(t) + r\int_t^T \Pi(u)du + \sigma\int_t^T S(u)F_s dW(u)$$

Hence

$$E_{t,z,s}^Q\left[\Pi(T)\right] = \Pi(t) + r\int_t^T E_{t,z,s}^Q\left[\Pi(u)\right]du$$

So, using the usual "trick" of setting $m(u) = E_{t,z,s}^Q\left[\Pi(u)\right]$ and using techniques of ODE we finally get

$$\Pi(t) = F(t, S(t), Z(t)) = e^{-r(T-t)}E_{t,s,z}^Q\left[\Phi(S(T), Z(T))\right].$$

(Remember that $\Pi(T) = F(T, S(T), Z(T)) = \Phi(S(T), Z(T))$.)

Exercise 7.3 The price arbitrage free price is given by (note that this time our claim is *not* simple, i.e. it is not of the form $\mathcal{X} = \Phi(S(T))$).

$$\begin{aligned}\Pi(t) &= e^{-r(T_2-t)} E_t^Q[\mathcal{X}] \\ &= e^{-r(T_2-t)} \frac{1}{T_2-T_1} \int_{T_1}^{T_2} E_t^Q[S(u)]\, du\end{aligned}$$

We know that under Q

$$\begin{aligned}dS(u) &= rS(u)du + \sigma S(u)dW(u) \\ S(t) &= s\end{aligned}$$

So,

$$\Rightarrow E_t^Q[S(u)] = se^{r(u-t)}$$

$$\frac{1}{T_2-T_1} \int_{T_1}^{T_2} se^{r(u-t)} du = \frac{1}{T_2-T_1} \frac{s}{r} \left[e^{r(T_2-t)} - e^{r(T_1-t)}\right]$$

The price to the "mean" contract is thus

$$\Pi(t) = \frac{s}{r(T_2-T_1)} \left[1 - e^{-r(T_2-T_1)}\right].$$

8 Parity Relations and Delta Hedging

Exercise 8.1 The T-claim $\mathcal{X}$ given by:

$$\mathcal{X} = \begin{cases} K, & \text{if } S(T) \le A \\ K + A - S(T), & \text{if } A < S(T) < K + A \\ 0, & \text{otherwise.} \end{cases},$$

has then following contract function (recall that $\mathcal{X} = \Phi S(T))$

$$\Phi(x) = \begin{cases} K, & \text{if } x \le A \\ K + A - x, & \text{if } A < x < K + A \\ 0, & \text{otherwise.} \end{cases},$$

which can be decomposed into the following "basic" contract functions written

$$\Phi(x) = K \cdot \underbrace{1}_{\Phi_B(x)} - \underbrace{\max[0, x - A]}_{\Phi_{c,A}(x)} + \underbrace{\max[0, x - A - K]}_{\Phi_{c,A+K}(x)}.$$

Having this T-claim $\mathcal{X}$ is then equivalent to having the following (replicating) portfolio at time T:

* K in monetary units

* short (position in) a call with strike A

* long (position in) a call with strike $A + K$

Given the decomposition of the contract function Φ into basic contract functions, we immediately have that the arbitrage free pricing process Π is

$$\Pi(t) = K \cdot \overbrace{e^{-r(T-t)}}^{B(t)} - c(s, A, T) + c(s, A + K, T)$$

where $c(s, A, T)$ and $c(s, A+K, T)$ stand for the prices of European call options on S and maturity T with strike prices A and $A + K$, respectively. The Black-Scholes formula give us both $c(s, A, T)$ and $c(s, A + K, T)$.

The hedge portfolio thus consists of a reverse position in the above components, i.e., borrow $e^{-r(T-t)}K$, buy a call with strike K and sell a call with strike $A+K$.

Exercise 8.4 We apply, once again, the exact same technique. The T-claim $\mathcal{X}$ given by:

$$\mathcal{X} = \begin{cases} 0, & \text{if } S(T) < A \\ S(T) - A, & \text{if } A \leq S(T) \leq B \\ C - S(T), & \text{if } B < S(T) \leq C \\ 0, & \text{if } S(T) > C. \end{cases}$$

where $B = \frac{A+C}{2}$, has a contract function Φ that can be written as

$$\Phi(x) = \underbrace{\max[0, x - A]}_{\Phi_{c,A}(x)} + \underbrace{\max[0, x - C]}_{\Phi_{c,C}(x)} - 2\underbrace{\max[0, x - B]}_{\Phi_{c,B}(x)}$$

Having this *butterfly* is then equivalent to having the following constant(replicating) portfolio at time T:

* long (position in) a call option with strike A

* long (position in) a call option with strike C

* short (position in) a call option with strike B