Chapter 2

Problems

1. (a) $S = \{(r, r), (r, g), (r, b), (g, r), (g, g), (g, b), (b, r), b, g), (b, b)\}$
   (b) $S = \{(r, g), (r, b), (g, r), (g, b), (b, r), (b, g)\}$

2. $S = \{(n, x_1, \ldots, x_{n-1}), n \geq 1, x_i \neq 6, i = 1, \ldots, n-1\}$, with the interpretation that the outcome is $(n, x_1, \ldots, x_{n-1})$ if the first 6 appears on roll n, and x_i appears on roll, i, $i = 1, \ldots, n-1$. The event $(\cup_{n=1}^{\infty} E_n)^c$ is the event that 6 never appears.

3. $EF = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (6, 1)\}$.
   $E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1. $FG = \{(1, 4), (4, 1)\}$.
   EF^c is the event that neither of the dice lands on 1 and the sum is odd. $EFG = FG$.

4. $A = \{1{,}0001{,}0000001, \ldots\}$ $B = \{01, 00001, 00000001, \ldots\}$
   $(A \cup B)^c = \{00000 \ldots, 001, 000001, \ldots\}$

5. (a) $2^5 = 32$
   (b)
   $W = \{(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0), (1, 1, 0, 1, 0)$
   $(1, 1, 0, 0, 1), (1, 1, 0, 0, 0), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1), (1, 0, 1, 1, 0), (0, 1, 1, 1, 0), (0, 0, 1, 1, 1)$
   $(0, 0, 1, 1, 0), (1, 0, 1, 0, 1)\}$

   (c) 8
   (d) $AW = \{(1, 1, 1, 0, 0), (1, 1, 0, 0, 0)\}$

6. (a) $S = \{(1, g), (0, g), (1, f), (0, f), (1, s), (0, s)\}$
   (b) $A = \{(1, s), (0, s)\}$
   (c) $B = \{(0, g), (0, f), (0, s)\}$
   (d) $\{(1, s), (0, s), (1, g), (1, f)\}$

7. (a) 6^{15}
   (b) $6^{15} - 3^{15}$
   (c) 4^{15}

8. (a) .8
   (b) .3
   (c) 0

9. Choose a customer at random. Let A denote the event that this customer carries an American Express card and V the event that he or she carries a VISA card.

   $$P(A \cup V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = .74.$$

   Therefore, 74 percent of the establishment's customers carry at least one of the two types of credit cards that it accepts.

10. Let R and N denote the events, respectively, that the student wears a ring and wears a necklace.

(a) $P(R \cup N) = 1 - .6 = .4$

(b) $.4 = P(R \cup N) = P(R) + P(N) - P(RN) = .2 + .3 - P(RN)$
Thus, $P(RN) = .1$

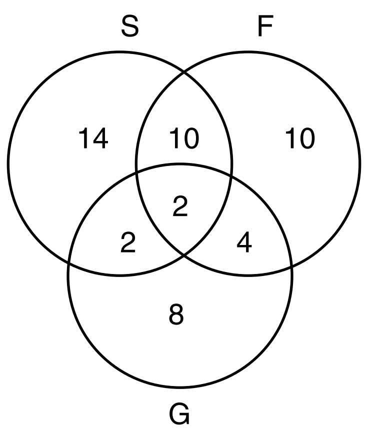
11. Let A be the event that a randomly chosen person is a cigarette smoker and let B be the event that she or he is a cigar smoker.

(a) $1 - P(A \cup B) = 1 - (.07 + .28 - .05) = .7$. Hence, 70 percent smoke neither.

(b) $P(A^cB) = P(B) - P(AB) = .07 - .05 = .02$. Hence, 2 percent smoke cigars but not cigarettes.

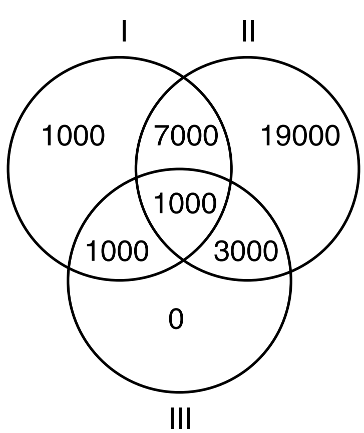
12. (a) $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$
The desired probability is $1 - 1/2 = 1/2$.

(b) Use the Venn diagram below to obtain the answer 32/100.



(c) Since 50 students are not taking any of the courses, the probability that neither one is taking a course is $\binom{50}{2}\Big/\binom{100}{2}$ = 49/198 and so the probability that at least one is taking a course is 149/198.

13.



(a) 20,000
(b) 12,000
(c) 11,000
(d) 68,000
(e) 10,000

14. $P(M) + P(W) + P(G) - P(MW) - P(MG) - P(WG) + P(MWG) = .312 + .470 + .525 - .086 - .042 - .147 + .025 = 1.057$

15. (a) $4\binom{13}{5}\Big/\binom{52}{5}$

(b) $13\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}\Big/\binom{52}{5}$

(c) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}\Big/\binom{52}{5}$

(d) $13\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}\Big/\binom{52}{5}$

(e) $13\binom{4}{4}\binom{48}{1}\Big/\binom{52}{5}$

16. (a) $\dfrac{6\cdot5\cdot4\cdot3\cdot2}{6^5}$ (b) $\dfrac{6\binom{5}{2}5\cdot4\cdot3}{6^5}$ (c) $\dfrac{\binom{6}{2}4\binom{5}{2}\binom{3}{2}}{6^5}$

(d) $\dfrac{6\cdot5\cdot4\binom{5}{3}}{21}$ (e) $\dfrac{6\cdot5\binom{5}{3}}{6^5}$ (f) $\dfrac{6\cdot5\binom{5}{4}}{6^5}$

(g) $\dfrac{6}{6^5}$

17. $\dfrac{\prod_{i=1}^{8} i^2}{64\cdot63\cdots58}$

18. $\dfrac{2\cdot4\cdot16}{52\cdot51}$

19. 4/36 + 4/36 +1/36 + 1/36 = 5/18

20. Let A be the event that you are dealt blackjack and let B be the event that the dealer is dealt blackjack. Then,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= \frac{4\cdot4\cdot16}{52\cdot51} + \frac{4\cdot4\cdot16\cdot3\cdot15}{52\cdot51\cdot50\cdot49} \\ &= .0983 \end{aligned}$$

where the preceding used that $P(A) = P(B) = 2 \times \dfrac{4\cdot16}{52\cdot51}$. Hence, the probability that neither is dealt blackjack is .9017.

21. (a) $p_1 = 4/20$, $p_2 = 8/20$, $p_3 = 5/20$, $p_4 = 2/20$, $p_5 = 1/20$

(b) There are a total of $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$ children. Hence,

$$q_1 = 4/48,\ q_2 = 16/48,\ q_3 = 15/48,\ q_4 = 8/48,\ q_5 = 5/48$$

22. The ordering will be unchanged if for some k, $0 \le k \le n$, the first k coin tosses land heads and the last $n - k$ land tails. Hence, the desired probability is $(n + 1/2^n$

23. The answer is 5/12, which can be seen as follows:

$$\begin{aligned}1 &= P\{\text{first higher}\} + P\{\text{second higher}\} + p\{\text{same}\}\\ &= 2P\{\text{second higher}\} + p\{\text{same}\}\\ &= 2P\{\text{second higher}\} + 1/6\end{aligned}$$

Another way of solving is to list all the outcomes for which the second is higher. There is 1 outcome when the second die lands on two, 2 when it lands on three, 3 when it lands on four, 4 when it lands on five, and 5 when it lands on six. Hence, the probability is $(1 + 2 + 3 + 4 + 5)/36 = 5/12$.

25. $P(E_n) = \left(\dfrac{26}{36}\right)^{n-1} \dfrac{6}{36}, \quad \displaystyle\sum_{n=1}^{\infty} P(E_n) = \frac{2}{5}$

27. Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3 \cdot 9! + 7 \cdot 6 \cdot 3 \cdot 7! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 5! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 3!}{10!}$$

28. $P\{\text{same}\} = \dfrac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$

$$P\{\text{different}\} = \binom{5}{1}\binom{6}{1}\binom{8}{1} \Big/ \binom{19}{3}$$

If sampling is with replacement

$$P\{\text{same}\} = \frac{5^3 + 6^3 + 8^3}{(19)^3}$$

$$\begin{aligned}P\{\text{different}\} &= P(RBG) + P\{BRG) + P(\mathrm{RGB}) + \ldots + P(GBR)\\ &= \frac{6 \cdot 5 \cdot 6 \cdot 8}{(19)^3}\end{aligned}$$

29. (a) $\dfrac{n(n-1)+m(m-1)}{(n+m)(n+m-1)}$

(b) Putting all terms over the common denominator $(n + m)^2(n + m - 1)$ shows that we must prove that

$$n^2(n + m - 1) + m^2(n + m - 1) \geq n(n - 1)(n + m) + m(m - 1)(n + m)$$

which is immediate upon multiplying through and simplifying.

30. (a) $\dfrac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} = 1/18$

(b) $\dfrac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} - 1/18 = 1/6$

(c) $\dfrac{\binom{7}{3}\binom{8}{4}+\binom{7}{4}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} = 1/2$

31. $P(\{\text{complete}\} =$
$P\{\text{same}\} =$

32. $\dfrac{g(b+g-1)!}{(b+g)!} = \dfrac{g}{b+g}$

33. $\dfrac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} = \dfrac{70}{323}$

34. $\binom{32}{13}\Big/\binom{52}{13}$

35. (a) $$\frac{\binom{12}{3}\binom{16}{2}\binom{18}{2}}{\binom{46}{7}}$$

(b) $$1-\frac{\binom{34}{7}}{\binom{46}{7}}-\frac{\binom{12}{1}\binom{34}{6}}{\binom{46}{7}}$$

(c) $$\frac{\binom{12}{7}+\binom{16}{7}+\binom{18}{7}}{\binom{46}{7}}$$

(d) $$P(R_3 \cup B_3) = P(R_3) + P(B_3) - P(R_3B_3) = \frac{\binom{12}{3}\binom{34}{4}}{\binom{46}{7}} + \frac{\binom{16}{3}\binom{30}{4}}{\binom{46}{7}} - \frac{\binom{12}{3}\binom{16}{3}\binom{18}{1}}{\binom{46}{7}}$$

36. (a) $\binom{4}{2}\Big/\binom{52}{2} \approx .0045$,

(b) $13\binom{4}{2}\Big/\binom{52}{2} = 1/17 \approx .0588$

37. (a) $\binom{7}{5}\Big/\binom{10}{5} = 1/12 \approx .0833$

(b) $\binom{7}{4}\binom{3}{1}\Big/\binom{10}{5} + 1/12 = 1/2$

38. $1/2 = \binom{3}{2}\Big/\binom{n}{2}$ or $n(n-1) = 12$ or $n = 4$.

39. $$\frac{5\cdot 4\cdot 3}{5\cdot 5\cdot 5} = \frac{12}{25}$$

40. $P\{1\} = \dfrac{4}{44} = \dfrac{1}{64}$

$P\{2\} = \binom{4}{2}\left[4 + \binom{4}{2} + 4\right] \Big/ 4^4 = \dfrac{84}{256}$

$P\{3\} = \binom{4}{3}\binom{3}{1}\dfrac{4!}{2!} \Big/ 4^4 = \dfrac{36}{64}$

$P\{4\} = \dfrac{4!}{4^4} = \dfrac{6}{64}$

41. $1 - \dfrac{5^4}{6^4}$

42. $1 - \left(\dfrac{35}{36}\right)^n$

43. $\dfrac{2(n-1)(n-2)}{n!} = \dfrac{2}{n}$ in a line

$\dfrac{2n(n-2)!}{n!} = \dfrac{2}{n-1}$ if in a circle, $n \geq 2$

44. (a) If A is first, then A can be in any one of 3 places and B's place is determined, and the others can be arranged in any of 3! ways. As a similar result is true, when B is first, we see that the probability in this case is $2 \cdot 3 \cdot 3!/5! = 3/10$

(b) $2 \cdot 2 \cdot 3!/5! = 1/5$

(c) $2 \cdot 3!/5! = 1/10$

45. $1/n$ if discard, $\dfrac{(n-1)^{k-1}}{n^k}$ if do not discard

46. If n in the room,

$$P\{\text{all different}\} = \frac{12 \cdot 11 \cdot \quad \cdot (13-n)}{12 \cdot 12 \cdot \quad \cdot 12}$$

When $n = 5$ this falls below 1/2. (Its value when $n = 5$ is .3819)

47. $12!/(12)^{12}$

48. $\binom{12}{4}\binom{8}{4}\dfrac{(20)!}{(3!)^4(2!)^4} \Big/ (12)^{20}$

49. $\binom{6}{3}\binom{6}{3} \Big/ \binom{12}{6}$

50. $\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}\Big/\binom{52}{13}\binom{39}{13}$

51. $\binom{n}{m}(n-1)^{n-m}\ /\ N^n$

52. (a) $\dfrac{20\cdot18\cdot16\cdot14\cdot12\cdot10\cdot8\cdot6}{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13}$

(b) $\dfrac{\binom{10}{1}\binom{9}{6}\frac{8!}{2!}2^6}{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13}$

53. Let A_i be the event that couple i sit next to each other. Then

$$P(\cup_{i=1}^{4} A_i) = 4\frac{2\cdot7!}{8!} - 6\frac{2^2\cdot6!}{8!} + 4\frac{2^3\cdot5!}{8!} - \frac{2^4\cdot4!}{8!}$$

and the desired probability is 1 minus the preceding.

54. $P(S \cup H \cup D \cup C) = P(S) + P(H) + P(D) + P(\mathrm{C}) - P(SH) - \ldots - P(SHDC)$

$$= \frac{4\binom{39}{13}}{\binom{52}{13}} - \frac{6\binom{26}{13}}{\binom{52}{13}} + \frac{4\binom{13}{13}}{\binom{52}{13}}$$

$$= \frac{4\binom{39}{13} - 6\binom{26}{13} + 4}{\binom{52}{13}}$$

55. (a) $P(S \cup H \cup D \cup C) = P(S) + \ldots - P(SHDC)$

$$= \frac{4\binom{2}{2}}{\binom{52}{13}} - \frac{6\binom{2}{2}\binom{2}{2}\binom{48}{9}}{\binom{52}{13}} + \frac{4\binom{2}{2}^3\binom{46}{7}}{\binom{52}{13}} - \frac{\binom{2}{2}^4\binom{44}{5}}{\binom{52}{13}}$$

$$= \frac{4\binom{50}{11} - 6\binom{48}{9} + 4\binom{46}{7} - \binom{44}{5}}{\binom{52}{13}}$$

(b) $P(1 \cup 2 \cup \ldots \cup 13) = \dfrac{13\binom{48}{9}}{\binom{52}{13}} - \dfrac{\binom{13}{2}\binom{44}{5}}{\binom{52}{13}} + \dfrac{\binom{13}{3}\binom{40}{1}}{\binom{52}{13}}$

56. Player B. If Player A chooses spinner (a) then B can choose spinner (c). If A chooses (b) then B chooses (a). If A chooses (c) then B chooses (b). In each case B wins probability 5/9.

Theoretical Exercises

5. $F_i = E_i \bigcap_{j=1}^{i=1} E_j^c$

6. (a) EF^cG^c

(b) EF^cG

(c) $E \cup F \cup G$

(d) $EF \cup EG \cup FG$

(e) EFG

(f) $E^cF^cG^c$

(g) $E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$

(h) $(EFG)^c$

(i) $EFG^c \cup EF^cG \cup E^cFG$

(j) S

8. The number of partitions that has $n + 1$ and a fixed set of i of the elements 1, 2, …, n as a subset is T_{n-i}. Hence, (where $T_0 = 1$). Hence, as there are $\binom{n}{i}$ such subsets.

$$T_{n+1} = \sum_{i=0}^{n}\binom{n}{i}T_{n-i} = 1+\sum_{i=0}^{n-1}\binom{n}{i}T_{n-i} = 1+\sum_{k=1}^{n}\binom{n}{k}T_k .$$

11. $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$

12. $P(EF^c \cup E^cF) = P(EF^c) + P(E^cF)$
$= P(E) - P(EF) + P(F) - P(EF)$

13. $E = EF \cup EF^c$

15. $$\frac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}}$$

16. $P(E_1 \ldots E_n) \geq P(E_1 \ldots E_{n-1}) + P(E_n) - 1$ by Bonferonni's Ineq.

$$\geq \sum_{1}^{n-1} P(E_i) - (n-2) + P(E_n) - 1 \text{ by induction hypothesis}$$

19. $$\frac{\binom{n}{r-1}\binom{m}{k-r}(n-r+1)}{\binom{n+m}{k-1}(n+m-k+1)}$$

21. Let $y_1, y_2, \ldots, y_k$ denote the successive runs of losses and $x_1, \ldots, x_k$ the successive runs of wins. There will be $2k$ runs if the outcome is either of the form $y_1, x_1, \ldots, y_k\, x_k$ or $x_1 y_1, \ldots x_k, y_k$ where all x_i, y_i are positive, with $x_1 + \ldots + x_k = n$, $y_1 + \ldots + y_k = m$. By Proposition 6.1 there are $2\binom{n-1}{k-1}\binom{m-1}{k-1}$ number of outcomes and so

$$P\{2k \text{ runs}\} = 2\binom{n-1}{k-1}\binom{m-1}{k-1}\Big/\binom{m+n}{n}.$$

There will be $2k + 1$ runs if the outcome is either of the form $x_1, y_1, \ldots, x_k, y_k, x_{k+1}$ or $y_1, x_1, \ldots, y_k, x_k\, y_{k+1}$ where all are positive and $\sum x_i = n$, $\sum y_i = m$. By Proposition 6.1 there are $\binom{n-1}{k}\binom{m-1}{k-1}$ outcomes of the first type and $\binom{n-1}{k-1}\binom{m-1}{k}$ of the second.