INSTRUCTOR'S SOLUTIONS MANUAL

DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

FOURTH EDITION

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Contents

Chapter 1 Solutions

Solutions to Section 1.1

True-False Review:

(a): FALSE. A derivative must involve *some* derivative of the function $y = f(x)$, not necessarily the first derivative.

(b): FALSE. The order of a differential equation is the order of the *highest*, not the lowest, derivative appearing in the differential equation.

(c): FALSE. This differential equation has order two, since the highest order derivative that appears in the equation is the second order expression y''.

(d): FALSE. The carrying capacity refers to the maximum population size that the environment can support in the long run; it is not related to the initial population in any way.

(e): TRUE. The value $y(0)$ is called an initial condition to the differential equation for $y(t)$.

(f): TRUE. According to Newton's Law of Cooling, the rate of cooling is proportional to the *difference* between the object's temperature and the medium's temperature. Since that difference is greater for the object at $100°F$ than the object at $90°F$, the object whose temperature is $100°F$ has a greater rate of cooling.

(g): FALSE. The temperature of the object is given by $T(t) = T_m + ce^{-kt}$, where T_m is the temperature of the medium, and c and k are constants. Since $e^{-kt} \neq 0$, we see that $T(t) \neq T_m$ for all times t. The temperature of the object *approaches* the temperature of the surrounding medium, but never equals it.

(h): TRUE. Since the temperature of the coffee is falling, the temperature *difference* between the coffee and the room is higher initially, during the first hour, than it is later, when the temperature of the coffee has already decreased.

(i): FALSE. The slopes of the two curves are *negative* reciprocals of each other.

(j): TRUE. If the original family of parallel lines have slopes k for $k \neq 0$, then the family of orthogonal trajectories are parallel lines with slope $-\frac{1}{k}$. If the original family of parallel lines are vertical (resp. horizontal), then the family of orthogonal trajectories are horizontal (resp. vertical) parallel lines.

(k): FALSE. The family of orthogonal trajectories for a family of circles centered at the origin is the family of lines passing through the origin.

(l): TRUE. If $v(t)$ denotes the velocity of the object at time t and $a(t)$ denotes the velocity of the object at time t, then we have $a(t) = v'(t)$, which is a differential equation for the unknown function $v(t)$.

(m): FALSE. The restoring force is directed in the direction *opposite* to the displacement from the equilibrium position.

(n): TRUE. The allometric relationship $B = B_0 m^{3/4}$, where B_0 is a constant, relates the metabolic rate and total body mass for any species.

Problems:

1. The order is 2.

2. The order is 1.

3. The order is 3.

4. The order is 2.

5. We compute the first three derivatives of $y(t) = \ln t$:

$$\frac{dy}{dt} = \frac{1}{t}, \qquad \frac{d^2y}{dt^2} = -\frac{1}{t^2}, \qquad \frac{d^3y}{dt^3} = \frac{2}{t^3}.$$

Therefore,

$$2\left(\frac{dy}{dt}\right)^3 = \frac{2}{t^3} = \frac{d^3y}{dt^3},$$

as required.

6. We compute the first two derivatives of $y(x) = x/(x+1)$:

$$\frac{dy}{dx} = \frac{1}{(x+1)^2} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{2}{(x+1)^3}.$$

Then

$$y + \frac{d^2y}{dx^2} = \frac{x}{x+1} - \frac{2}{(x+1)^3} = \frac{x^3+2x^2+x-2}{(x+1)^3} = \frac{(x+1)+(x^3+2x^2-3)}{(x+1)^3} = \frac{1}{(x+1)^2} + \frac{x^3+2x^2-3}{(1+x)^3},$$

as required.

7. We compute the first two derivatives of $y(x) = e^x \sin x$:

$$\frac{dy}{dx} = e^x(\sin x + \cos x) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 2e^x \cos x.$$

Then

$$2y \cot x - \frac{d^2y}{dx^2} = 2(e^x \sin x)\cot x - 2e^x \cos x = 0,$$

as required.

8. $(T - T_m)^{-1}\dfrac{dT}{dt} = -k \Longrightarrow \dfrac{d}{dt}(\ln|T - T_m|) = -k$. The preceding equation can be integrated directly to yield $\ln|T - T_m| = -kt + c_1$. Exponentiating both sides of this equation gives $|T - T_m| = e^{-kt+c_1}$, which can be written as

$$T - T_m = ce^{-kt},$$

where $c = \pm e^{c_1}$. Rearranging yields $T(t) = T_m + ce^{-kt}$.

9. After 4 p.m. In the first two hours after noon, the water temperature increased from 50° F to 55° F, an increase of five degrees. Because the temperature of the water has grown closer to the ambient air temperature, the temperature difference $|T - T_m|$ is smaller, and thus, the rate of change of the temperature of the water grows smaller, according to Newton's Law of Cooling. Thus, it will take longer for the water temperature to increase another five degrees. Therefore, the water temperature will reach 60° F more than two hours later than 2 p.m., or after 4 p.m.

10. The object temperature cools a total of 40° F during the 40 minutes, but according to Newton's Law of Cooling, it cools faster in the beginning (since $|T - T_m|$ is greater at first). Thus, the object cooled half-way from 70° F to 30° F in less than half the total cooling time. Therefore, it took less than 20 minutes for the object to reach 50° F.

11. The given family of curves satisfies: $x^2 + 9y^2 = c \Longrightarrow 2x + 18y\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{x}{9y}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = \frac{9y}{x} \Longrightarrow \frac{1}{y}\frac{dy}{dx} = \frac{9}{x} \Longrightarrow \frac{d}{dx}(\ln|y|) = \frac{9}{x} \Longrightarrow \ln|y| = 9\ln|x| + c_1 \Longrightarrow y = kx^9, \text{where } k = \pm e^{c_1}$$

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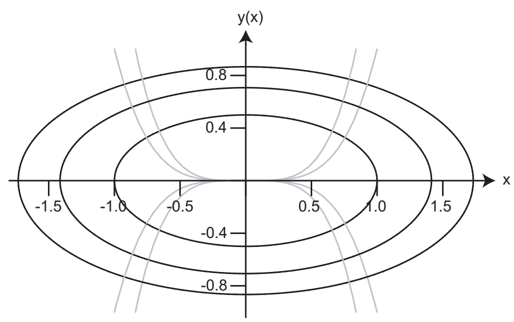

Figure 0.0.1: Figure for Problem 11

12. Given family of curves satisfies: $y = cx^2 \Longrightarrow c = \dfrac{y}{x^2}$. Hence,

$$\frac{dy}{dx} = 2cx = c\left(\frac{y}{x^2}\right)x = \frac{2y}{x}.$$

Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{x}{2y} \Longrightarrow 2y\frac{dy}{dx} = -x \Longrightarrow \frac{d}{dx}(y^2) = -x \Longrightarrow y^2 = -\frac{1}{2}x^2 + c_1 \Longrightarrow 2y^2 + x^2 = c_2,$$

where $c_2 = 2c_1$.

13. Given a family of curves satisfies: $y = \dfrac{c}{x} \Longrightarrow x\dfrac{dy}{dx} + y = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = \frac{x}{y} \Longrightarrow y\frac{dy}{dx} = x \Longrightarrow \frac{d}{dx}\left(\frac{1}{2}y^2\right) = x \Longrightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + c_1 \Longrightarrow y^2 - x^2 = c_2, \text{where } c_2 = 2c_1.$$

14. The given family of curves satisfies: $y = cx^5 \Longrightarrow c = \dfrac{y}{x^5}$. Hence,

$$\frac{dy}{dx} = 5cx^4 = 5\left(\frac{y}{x^5}\right)x^4 = \frac{5y}{x}.$$

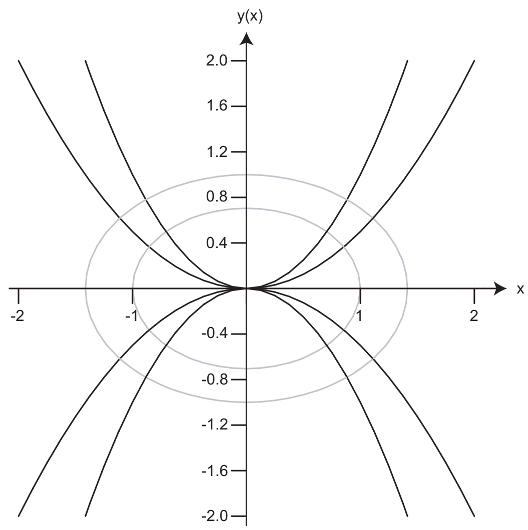

Figure 0.0.2: Figure for Problem 12

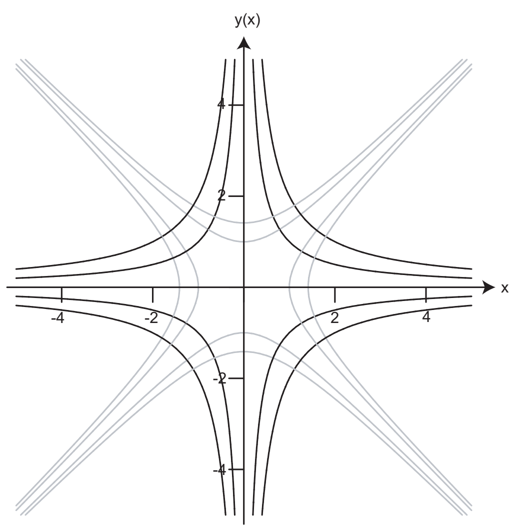

Figure 0.0.3: Figure for Problem 13

Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{x}{5y} \Longrightarrow 5y\frac{dy}{dx} = -x \Longrightarrow \frac{d}{dx}\left(\frac{5}{2}y^2\right) = -x \Longrightarrow \frac{5}{2}y^2 = -\frac{1}{2}x^2 + c_1 \Longrightarrow 5y^2 + x^2 = c_2,$$

where $c_2 = 2c_1$.

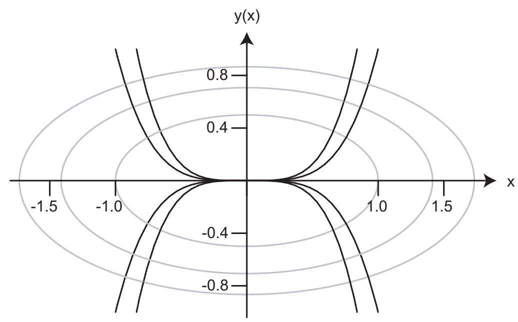

Figure 0.0.4: Figure for Problem 14

15. Given family of curves satisfies: $y = ce^x \Longrightarrow \dfrac{dy}{dx} = ce^x = y$. Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{1}{y} \Longrightarrow y\frac{dy}{dx} = -1 \Longrightarrow \frac{d}{dx}\left(\frac{1}{2}y^2\right) = -1 \Longrightarrow \frac{1}{2}y^2 = -x + c_1 \Longrightarrow y^2 = -2x + c_2.$$

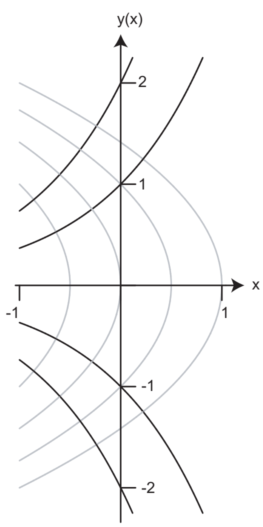

Figure 0.0.5: Figure for Problem 15

16. Given family of curves satisfies: $y^2 = 2x + c \Longrightarrow \dfrac{dy}{dx} = \dfrac{1}{y}$.Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -y \Longrightarrow y^{-1}\frac{dy}{dx} = -1 \Longrightarrow \frac{d}{dx}(\ln|y|) = -1 \Longrightarrow \ln|y| = -x + c_1 \Longrightarrow y = c_2e^{-x}.$$

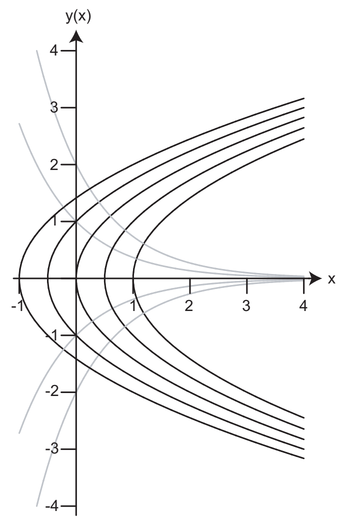

Figure 0.0.6: Figure for Problem 16

17. $y = cx^m \Longrightarrow \dfrac{dy}{dx} = cmx^{m-1}$, but $c = \dfrac{y}{x^m}$ so $\dfrac{dy}{dx} = \dfrac{my}{x}$. Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{x}{my} \Longrightarrow y\frac{dy}{dx} = -\frac{x}{m} \Longrightarrow \frac{d}{dx}\left(\frac{1}{2}y^2\right) = -\frac{x}{m} \Longrightarrow \frac{1}{2}y^2 = -\frac{1}{2m}x^2 + c_1 \Longrightarrow y^2 = -\frac{1}{m}x^2 + c_2.$$

18. $y = mx + c \Longrightarrow \dfrac{dy}{dx} = m$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{1}{m} \Longrightarrow y = -\frac{1}{m}x + c_1.$$

19. $y^2 = mx + c \Longrightarrow 2y\dfrac{dy}{dx} = m \Longrightarrow \dfrac{dy}{dx} = \dfrac{m}{2y}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{2y}{m} \Longrightarrow y^{-1}\frac{dy}{dx} = -\frac{2}{m} \Longrightarrow \frac{d}{dx}(\ln|y|) = -\frac{2}{m} \Longrightarrow \ln|y| = -\frac{2}{m}x + c_1 \Longrightarrow y = c_2e^{-\frac{2x}{m}}.$$

20. $y^2 + mx^2 = c \Longrightarrow 2y\dfrac{dy}{dx} + 2mx = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{mx}{y}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = \frac{y}{mx} \Longrightarrow y^{-1}\frac{dy}{dx} = \frac{1}{mx} \Longrightarrow \frac{d}{dx}(\ln|y|) = \frac{1}{mx} \Longrightarrow m\ln|y| = \ln|x| + c_1 \Longrightarrow y^m = c_2x.$$

21. The given family of curves satisfies: $x^2 + y^2 = 2cx \Longrightarrow c = \dfrac{x^2+y^2}{2x}$. Hence,

$$2x + 2y\frac{dy}{dx} = 2c = \frac{x^2+y^2}{x}.$$

Therefore,

$$2y\frac{dy}{dx} = \frac{x^2+y^2}{x} - 2x = \frac{y^2-x^2}{x},$$

so that

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}.$$

Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{2xy}{y^2-x^2} = \frac{2xy}{x^2-y^2}.$$

22. $u = x^2 + 2y^2 \Longrightarrow 0 = 2x + 4y\dfrac{dy}{dx} \Longrightarrow \dfrac{dy}{dx} = -\dfrac{x}{2y}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = \frac{2y}{x} \Longrightarrow y^{-1}\frac{dy}{dx} = \frac{2}{x} \Longrightarrow \frac{d}{dx}(\ln|y|) = \frac{2}{x} \Longrightarrow \ln|y| = 2\ln|x| + c_1 \Longrightarrow y = c_2x^2.$$

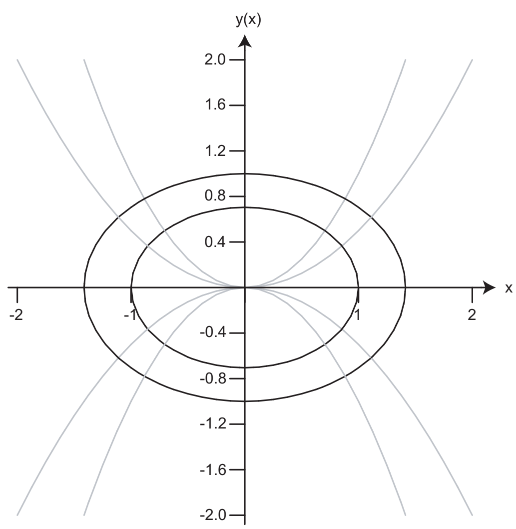

Figure 0.0.7: Figure for Problem 22

23. $m_1 = \tan(a_1) = \tan(a_2 - a) = \dfrac{\tan(a_2) - \tan(a)}{1 + \tan(a_2)\tan(a)} = \dfrac{m_2 - \tan(a)}{1 + m_2\tan(a)}$.

24. $\frac{d^2y}{dt^2} = g \Longrightarrow \frac{dy}{dt} = gt + c_1 \Longrightarrow y(t) = \frac{gt^2}{2} + c_1t + c_2$. Now impose the initial conditions. $y(0) = 0 \Longrightarrow c_2 = 0$. $\frac{dy}{dt}(0) \Longrightarrow c_1 = 0$. Hence, the solution to the initial-value problem is: $y(t) = \frac{gt^2}{2}$. The object hits the ground at time, t_0, when $y(t_0) = 100$. Hence $100 = \frac{gt_0^2}{2}$, so that $t_0 = \sqrt{\frac{200}{g}} \approx 4.52$ s, where we have taken $g = 9.8$ ms^{-2}.

25. From $\dfrac{d^2y}{dt^2} = g$, we integrate twice to obtain the general equations for the velocity and the position of the ball, respectively: $\dfrac{dy}{dt} = gt + c$ and $y(t) = \dfrac{1}{2}gt^2 + ct + d$, where c, d are constants of integration. Setting $y = 0$ to be at the top of the boy's head (and positive direction downward), we know that $y(0) = 0$. Since the object hits the ground 8 seconds later, we have that $y(8) = 5$ (since the ground lies at the position $y = 5$). From the values of $y(0)$ and $y(8)$, we find that $d = 0$ and $5 = 32g + 8c$. Therefore, $c = \dfrac{5 - 32g}{8}$.

(a). The ball reaches its maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore,

$$t = -\frac{c}{g} = \frac{32g - 5}{8g} \approx 3.98 \text{ s.}$$

(b). To find the maximum height of the tennis ball, we compute

$$y(3.98) \approx -253.51 \text{ feet.}$$

So the ball is 253.51 feet *above* the top of the boy's head, which is 258.51 feet above the ground.

26. From $\dfrac{d^2y}{dt^2} = g$, we integrate twice to obtain the general equations for the velocity and the position of the rocket, respectively: $\dfrac{dy}{dt} = gt + c$ and $y(t) = \dfrac{1}{2}gt^2 + ct + d$, where c, d are constants of integration. Setting $y = 0$ to be at ground level, we know that $y(0) = 0$. Thus, $d = 0$.

(a). The rocket reaches maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore, the time that the rocket achieves its maximum height is $t = -\dfrac{c}{g}$. At this time, $y(t) = -90$ (the negative sign accounts for the fact that the positive direction is chosen to be downward). Hence,

$$-90 = y\left(-\frac{c}{g}\right) = \frac{1}{2}g\left(-\frac{c}{g}\right)^2 + c\left(-\frac{c}{g}\right) = \frac{c^2}{2g} - \frac{c^2}{g} = -\frac{c^2}{2g}.$$

Solving this for c, we find that $c = \pm\sqrt{180g}$. However, since c represents the initial velocity of the rocket, and the initial velocity is negative (relative to the fact that the positive direction is downward), we choose $c = -\sqrt{180g} \approx -42.02 \text{ ms}^{-1}$, and thus the initial speed at which the rocket must be launched for optimal viewing is approximately 42.02 ms^{-1}.

(b). The time that the rocket reaches its maximum height is $t = -\dfrac{c}{g} \approx -\dfrac{-42.02}{9.81} = 4.28$ s.

27. From $\dfrac{d^2y}{dt^2} = g$, we integrate twice to obtain the general equations for the velocity and the position of the rocket, respectively: $\dfrac{dy}{dt} = gt + c$ and $y(t) = \dfrac{1}{2}gt^2 + ct + d$, where c, d are constants of integration. Setting $y = 0$ to be at the level of the platform (with positive direction downward), we know that $y(0) = 0$. Thus, $d = 0$.

(a). The rocket reaches maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore, the time that the rocket achieves its maximum height is $t = -\dfrac{c}{g}$. At this time, $y(t) = -85$ (this is 85 m above the platform, or 90 m above the ground). Hence,

$$-85 = y\left(-\frac{c}{g}\right) = \frac{1}{2}g\left(-\frac{c}{g}\right)^2 + c\left(-\frac{c}{g}\right) = \frac{c^2}{2g} - \frac{c^2}{g} = -\frac{c^2}{2g}.$$

Solving this for c, we find that $c = \pm\sqrt{170g}$. However, since c represents the initial velocity of the rocket, and the initial velocity is negative (relative to the fact that the positive direction is downward), we choose $c = -\sqrt{170g} \approx -40.84$ ms^{-1}, and thus the initial speed at which the rocket must be launched for optimal viewing is approximately 40.84 ms^{-1}.

(b). The time that the rocket reaches its maximum height is $t = -\dfrac{c}{g} \approx -\dfrac{-40.84}{9.81} = 4.16$ s.

28. If $y(t)$ denotes the displacement of the object from its initial position at time t, the motion of the object can be described by the initial-value problem

$$\frac{d^2y}{dt^2} = g, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = -2.$$

We first integrate this differential equation: $\dfrac{d^2y}{dt^2} = g \Longrightarrow \dfrac{dy}{dt} = gt + c_1 \Longrightarrow y(t) = \dfrac{gt^2}{2} + c_1t + c_2$. Now impose the initial conditions. $y(0) = 0 \Longrightarrow c_2 = 0$. $\dfrac{dy}{dt}(0) = -2 \Longrightarrow c_1 = -2$. Hence the solution to the initial-value problem is $y(t) = \dfrac{gt^2}{2} - 2t$. We are given that $y(10) = h$. Consequently, $h = \dfrac{g(10)^2}{2} - 2 \cdot 10 \Longrightarrow h = 10(5g - 2) \approx 470$ m where we have taken $g = 9.8$ ms^{-2}.

29. If $y(t)$ denotes the displacement of the object from its initial position at time t, the motion of the object can be described by the initial-value problem

$$\frac{d^2y}{dt^2} = g, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = v_0.$$

We first integrate the differential equation: $\dfrac{d^2y}{dt^2} = g \Longrightarrow \dfrac{dy}{dt} = gt + c_1 \Longrightarrow y(t) = \dfrac{gt^2}{2} + c_1t + c_2$. Now impose the initial conditions. $y(0) = 0 \Longrightarrow c_2 = 0$. $\dfrac{dy}{dt}(0) = v_0 \Longrightarrow c_1 = v_0$. Hence the solution to the initial-value problem is $y(t) = \dfrac{gt^2}{2} + v_0t$. We are given that $y(t_0) = h$. Consequently, $h = gt_0^2 + v_0t_0$. Solving for v_0 yields $v_0 = \dfrac{2h - gt_0^2}{2t_0}$.

30. From $y(t) = A\cos(\omega t - \phi)$, we obtain

$$\frac{dy}{dt} = -A\omega \sin(\omega t - \phi) \quad \text{and} \quad \frac{d^2y}{dt^2} = -A\omega^2 \cos(\omega t - \phi).$$

Hence,

$$\frac{d^2y}{dt^2} + \omega^2 y = -A\omega^2 \cos(\omega t - \phi) + A\omega^2 \cos(\omega t - \phi) = 0.$$

Substituting $y(0) = a$, we obtain $a = A\cos(-\phi) = A\cos(\phi)$. Also, from $\frac{dy}{dt}(0) = 0$, we obtain $0 = -A\omega\sin(-\phi) = A\omega\sin(\phi)$. Since $A \neq 0$ and $\omega \neq 0$ and $|\phi| < \pi$, we have $\phi = 0$. It follows that $a = A$.

31. $y(t) = c_1\cos(\omega t) + c_2\sin(\omega t) \Longrightarrow \dfrac{dy}{dt} = -c_1\omega\sin(\omega t) + c_2\omega\cos(\omega t) \Longrightarrow \dfrac{d^2y}{dt^2} = -c_1\omega^2\cos(\omega t) - c_2\omega^2\sin(\omega t) = -\omega^2[c_1\cos(\omega t) + c_2\cos(\omega t)] = -\omega^2 y$. Consequently, $\dfrac{d^2y}{dt^2} + \omega^2 y = 0$. To determine the amplitude of the motion we write the solution to the differential equation in the equivalent form:

$$y(t) = \sqrt{c_1^2 + c_2^2}\left[\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\cos(\omega t) + \frac{c_2}{\sqrt{c_1^2 + c_2^2}}\sin(\omega t)\right].$$

We can now define an angle ϕ by

$$\cos\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \sin\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

Then the expression for the solution to the differential equation is

$$y(t) = \sqrt{c_1^2 + c_2^2}[\cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi] = \sqrt{c_1^2 + c_2^2}\cos(\omega t + \phi).$$

Consequently the motion corresponds to an oscillation with amplitude $A = \sqrt{c_1^2 + c_2^2}$.

32. In this problem we have $m_0 = 3\text{g}$, $M = 2700\text{g}$, $a = 1.5$. Substituting these values into Equation (1.1.26) yields

$$m(t) = 2700\left\{1 - \left[1 - \left(\frac{1}{900}\right)^{1/4}\right]e^{-1.5t/(4(2700)^{1/4})}\right\}^4.$$

Therefore the mass of the heron after 30 days is

$$m(30) = 2700\left\{1 - \left[1 - \left(\frac{1}{900}\right)^{1/4}\right]e^{-45/(4(2700)^{1/4})}\right\}^4 \approx 1271.18\,\text{g}.$$

33. In this problem we have $m_0 = 8\text{g}$, $M = 280\text{g}$, $a = 0.25$. Substituting these values into Equation (1.1.26) yields

$$m(t) = 280\left\{1 - \left[1 - \left(\frac{1}{35}\right)^{1/4}\right]e^{-t/(16(280)^{1/4})}\right\}^4.$$

We need to find the time, t when the mass of the rat reaches 75% of its fully grown size. Therefore we need to find t such that

$$\frac{75}{100}\cdot 280 = 280\left\{1 - \left[1 - \left(\frac{1}{35}\right)^{1/4}\right]e^{-t/(16(280)^{1/4})}\right\}^4.$$

Solving algebraically for t yields

$$t = 16\cdot(280)^{1/4}\cdot\ln\left[\frac{1-(1/35)^{1/4}}{1-(75/100))^{1/4}}\right] \approx 140 \text{ days}.$$

Solutions to Section 1.2

True-False Review:

(a): TRUE. This is condition 1 in Definition 1.2.8.

(b): TRUE. This is the content of Theorem 1.2.12.

(c): FALSE. There are solutions to $y'' + y = 0$ that do not have the form $c_1 \cos x + 5c_2 \cos x$, such as $y(x) = \sin x$. Therefore, $c_1 \cos x + 5c_2 \cos x$ does not meet the second requirement set forth in Definition 1.2.8 for the general solution.

(d): FALSE. There are solutions to $y'' + y = 0$ that do not have the form $c_1 \cos x + 5c_1 \sin x$, such as $y(x) = \cos x + \sin x$. Therefore, $c_1 \cos x + 5c_1 \sin x$ does not meet the second requirement set form in Definition 1.2.8 for the general solution.

(e): TRUE. Since the right-hand side of the differential equation is a function of x only, we can integrate both sides n times to obtain the formula for the solution $y(x)$.

Problems:

1. Linear.

2. Non-linear, because of the y^2 expression on the right side of the equation.

3. Non-linear, because of the term yy'' on tthe left side of the equation.

4. Non-linear, because of the expression $\tan y$ appearing on the left side of the equation.

5. Linear.

6. Non-linear, because of the expression $\dfrac{1}{y'}$ on the left side of the equation.

7. $y(x) = c_1e^{-5x} + c_2e^{5x} \implies y' = -5c_1e^{-5x} + 5c_2e^{5x} \implies y'' = 25c_1e^{-5x} + 25c_2e^{5x} \implies y'' - 25y = (25c_1e^{-5x} + 25c_2e^{5x}) - 25(c_1e^{-5x} + c_2e^{5x}) = 0$. Thus $y(x) = c_1e^{-5x} + c_2e^{5x}$ is a solution of the given differential equation for all $x \in \mathbb{R}$.

8. $y(x) = c_1 \cos 2x + c_2 \sin 2x \implies y' = -2c_1 \sin 2x + 2c_2 \cos 2x \implies y'' = -4c_1 \cos 2x - 4c_2 \sin 2x \implies y'' + 4y = (-4c_1 \cos 2x - 4c_2 \sin 2x) + 4(c_1 \cos 2x + c_2 \sin 2x) = 0$. Thus $y(x) = c_1 \cos 2x + c_2 \sin 2x$ is a solution of the given differential equation for all $x \in \mathbb{R}$.

9. $y(x) = c_1e^{x} + c_2e^{-2x} \implies y' = c_1e^{x} - 2c_2e^{-2x} \implies y'' = c_1e^{x} + 4c_2e^{-2x} \implies y'' + y' - 2y = (c_1e^{x} + 4c_2e^{-2x}) + (c_1e^{x} - 2c_2e^{-2x}) - 2(c_1e^{x} + c_2e^{-2x}) = 0$. Thus $y(x) = c_1e^{x} + c_2e^{-2x}$ is a solution of the given differential equation for all $x \in \mathbb{R}$.

10. $y(x) = \dfrac{1}{x+4} \implies y' = -\dfrac{1}{(x+4)^2} = -y^2$. Thus $y(x) = \dfrac{1}{x+4}$ is a solution of the given differential equation for $x \in (-\infty, -4)$ or $x \in (-4, \infty)$.

11. $y(x) = c_1\sqrt{x} \implies y' = \dfrac{c_1}{2\sqrt{x}} = \dfrac{y}{2x}$. Thus $y(x) = c_1\sqrt{x}$ is a solution of the given differential equation for all $x \in \{x : x > 0\}$.

12. $y(x) = c_1e^{-x} \sin(2x) \implies y' = 2c_1e^{-x} \cos(2x) - c_1e^{-x} \sin(2x) \implies y'' = -3c_1e^{-x} \sin(2x) - 4c_1e^{-x} \cos(2x) \implies y'' + 2y' + 5y = -3c_1e^{-x} \sin(2x) - 4c_1e^{-x} \cos(2x) + 2[2c_1e^{-x} \cos(2x) - c_1e^{-x} \sin(2x)] + 5[c_1e^{-x} \sin(2x)] = 0$. Thus $y(x) = c_1e^{-x} \sin(2x)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

13. $y(x) = c_1 \cosh(3x) + c_2 \sinh(3x) \implies y' = 3c_1 \sinh(3x) + 3c_2 \cosh(3x) \implies y'' = 9c_1 \cosh(3x) +$

$9c_2 \sinh(3x) \implies y'' - 9y = [9c_1 \cosh(3x) + 9c_2 \sinh(3x)] - 9[c_1 \cosh(3x) + c_2 \sinh(3x)] = 0$. Thus $y(x) = c_1 \cosh(3x) + c_2 \sinh(3x)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

14. $y(x) = \dfrac{c_1}{x^3} + \dfrac{c_2}{x} \implies y' = -\dfrac{3c_1}{x^4} - \dfrac{c_2}{x^2} \implies y'' = \dfrac{12c_1}{x^5} + \dfrac{2c_2}{x^3} \implies x^2y'' + 5xy' + 3y = x^2\left(\dfrac{12c_1}{x^5} + \dfrac{2c_2}{x^3}\right) + 5x\left(-\dfrac{3c_1}{x^4} - \dfrac{c_2}{x^2}\right) + 3\left(\dfrac{c_1}{x^3} + \dfrac{c_2}{x}\right) = 0$. Thus $y(x) = \dfrac{c_1}{x^3} + \dfrac{c_2}{x}$ is a solution to the given differential equation for all $x \in (-\infty, 0)$ or $x \in (0, \infty)$.

15. $y(x) = c_1 x^2 \ln x \implies y' = c_1(2x \ln x + x) \implies y'' = c_1(2\ln x + 3) \implies x^2y'' - 3xy' + 4y = x^2 \cdot c_1(2\ln x + 3) - 3x \cdot c_1(2x\ln x + x) + 4c_1x^2 \ln x = c_1x^2\left[(2\ln x + 3) - 3(2x \ln x + 1) + 4\ln x\right] = 0$. Thus $y(x) = c_1x^2 \ln x$ is a solution of the given differential equation for all $x > 0$.

16. $y(x) = c_1x^2 \cos(3\ln x) \implies y' = c_1[2x\cos(3\ln x) - 3x\sin(3\ln x)] \implies y'' = c_1[-7\cos(3\ln x) - 6\sin(3\ln x)] \implies x^2y'' - 3xy' + 13y = x^2 \cdot c_1[-7\cos(3\ln x) - 9\sin(3\ln x)] - 3x \cdot c_1[2x\cos(3\ln x) - 3x\sin(3\ln x)] + 13c_1x^2\cos(3\ln x) = c_1x^2\left\{[-7\cos(3\ln x) - 9\sin(3\ln x)] - 3[2\cos(3\ln x) - 3\sin(3\ln x)] + 13\cos(3\ln x)\right\} = 0$. Thus $y(x) = c_1x^2\cos(3\ln x)$ is a solution of the given differential equation for all $x > 0$.

17. $y(x) = c_1\sqrt{x} + 3x^2 \implies y' = \dfrac{c_1}{2\sqrt{x}} + 6x \implies y'' = -\dfrac{c_1}{4\sqrt{x^3}} + 6 \implies 2x^2y'' - xy' + y = 2x^2\left(-\dfrac{c_1}{4\sqrt{x^3}} + 6\right) - x\left(\dfrac{c_1}{2\sqrt{x}} + 6x\right) + (c_1\sqrt{x} + 3x^2) = 9x^2$. Thus $y(x) = c_1\sqrt{x} + 3x^2$ is a solution to the given differential equation for all $x \in \{x : x > 0\}$.

18. $y(x) = c_1x^2 + c_2x^3 - x^2\sin x \implies y' = 2c_1x + 3c_2x^2 - x^2\cos x - 2x\sin x \implies y'' = 2c_1 + 6c_2x + x^2\sin x - 2x\cos x - 2x\cos - 2\sin x$. Substituting these results into the given differential equation yields

$$\begin{aligned}
x^2y'' - 4xy' + 6y &= x^2(2c_1 + 6c_2x + x^2\sin x - 4x\cos x - 2\sin x) - 4x(2c_1x + 3c_2x^2 - x^2\cos x - 2x\sin x) \\
&\qquad + 6(c_1x^2 + c_2x^3 - x^2\sin x) \\
&= 2c_1x^2 + 6c_2x^3 + x^4\sin x - 4x^3\cos x - 2x^2\sin x - 8c_1x^2 - 12c_2x^3 + 4x^3\cos x + 8x^2\sin x \\
&\qquad + 6c_1x^2 + 6c_2x^3 - 6x^2\sin x \\
&= x^4\sin x.
\end{aligned}$$

Hence, $y(x) = c_1x^2 + c_2x^3 - x^2\sin x$ is a solution to the differential equation for all $x \in \mathbb{R}$.

19. $y(x) = c_1e^{ax} + c_2e^{bx} \implies y' = ac_1e^{ax} + bc_2e^{bx} \implies y'' = a^2c_1e^{ax} + b^2c_2e^{bx}$. Substituting these results into the differential equation yields

$$\begin{aligned}
y'' - (a+b)y' + aby &= a^2c_1e^{ax} + b^2c_2e^{bx} - (a+b)(ac_1e^{ax} + bc_2e^{bx}) + ab(c_1e^{ax} + c_2e^{bx}) \\
&= (a^2c_1 - a^2c_1 - abc_1 + abc_1)e^{ax} + (b^2c_2 - abc_2 - b^2c_2 + abc_2)e^{bx} \\
&= 0.
\end{aligned}$$

Hence, $y(x) = c_1e^{ax} + c_2e^{bx}$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

20. $y(x) = e^{ax}(c_1 + c_2x) \implies y' = e^{ax}(c_2) + ae^{ax}(c_1 + c_2x) = e^{ax}(c_2 + ac_1 + ac_2x) \implies y'' = ea^{ax}(ac_2) + ae^{ax}(c_2 + ac_1 + ac_2x) = ae^{ax}(2c_2 + ac_1 + ac_2x)$. Substituting these into the differential equation yields

$$\begin{aligned}
y'' - 2ay' + a^2y &= ae^{ax}(2c_2 + ac_1 + ac_2x) - 2ae^{ax}(c_2 + ac_1 + ac_2x) + a^2e^{ax}(c_1 + c_2x) \\
&= ae^{ax}(2c_2 + ac_1 + ac_2x - 2c_2 - 2ac_1 - 2ac_2x + ac_1 + ac_2x) \\
&= 0.
\end{aligned}$$

Thus, $y(x) = e^{ax}(c_1 + c_2x)$ is a solution to the given differential eqaution for all $x \in \mathbb{R}$.

21. $y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$ so,

$$\begin{aligned} y' &= e^{ax}(-bc_1 \sin bx + bc_2 \cos bx) + ae^{ax}(c_1 \cos bx + c_2 \sin bx) \\ &= e^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx] \text{ so,} \\ y'' &= e^{ax}[-b(bc_2 + ac_1)\sin bx + b(ac_2 + bc_1)\cos bx] + ae^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 + bc_1)\sin bx] \\ &= e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2)\cos bx + (a^2c_2 - b^2c_2 - abc_1)\sin bx]. \end{aligned}$$

Substituting these results into the differential equation yields

$$\begin{aligned} y'' - 2ay' + (a^2 + b^2)y &= (e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2)\cos bx + (a^2c_2 - b^2c_2 - abc_1)\sin bx]) \\ &\quad - 2a(e^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx]) + (a^2 + b^2)(e^{ax}(c_1 \cos bx + c_2 \sin bx)) \\ &= e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2 - 2abc_2 - 2a^2c_1 + a^2c_1 + b^2c_1)\cos bx \\ &\quad + (a^2c_2 - b^2c_2 - 2abc_1 + 2abc_1 - 2a^2c_2 + a^2c_2 + b^2c_2)\sin bx] \\ &= 0 \end{aligned}.$$

Thus, $y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

22. $y(x) = e^{rx} \Longrightarrow y' = re^{rx} \Longrightarrow y'' = r^2e^{rx}$. Substituting these results into the given differential equation yields $e^{rx}(r^2 - r - 6) = 0$, so that r must satisfy $r^2 - r - 6 = 0$, or $(r - 3)(r + 2) = 0$. Consequently $r = 3$ and $r = -2$ are the only values of r for which $y(x) = e^{rx}$ is a solution to the given differential equation. The corresponding solutions are $y(x) = e^{3x}$ and $y(x) = e^{-2x}$.

23. $y(x) = e^{rx} \Longrightarrow y' = re^{rx} \Longrightarrow y'' = r^2e^{rx}$. Substituting these results into the given differential equation yields $e^{rx}(r^2 + 6r + 9) = 0$, so that r must satisfy $r^2 + 6r + 9 = 0$, or $(r + 3)^2 = 0$. Consequently $r = -3$ is the only value of r for which $y(x) = e^{rx}$ is a solution to the given differential equation. The corresponding solution are $y(x) = e^{-3x}$.

24. $y(x) = x^r \Longrightarrow y' = rx^{r-1} \Longrightarrow y'' = r(r-1)x^{r-2}$. Substitution into the given differential equation yields $x^r[r(r - 1) + r - 1] = 0$, so that r must satisfy $r^2 - 1 = 0$. Consequently $r = -1$ and $r = 1$ are the only values of r for which $y(x) = x^r$ is a solution to the given differential equation. The corresponding solutions are $y(x) = x^{-1}$ and $y(x) = x$.

25. $y(x) = x^r \Longrightarrow y' = rx^{r-1} \Longrightarrow y'' = r(r-1)x^{r-2}$. Substitution into the given differential equation yields $x^r[r(r - 1) + 5r + 4] = 0$, so that r must satisfy $r^2 + 4r + 4 = 0$, or equivalently $(r + 2)^2 = 0$. Consequently $r = -2$ is the only value of r for which $y(x) = x^r$ is a solution to the given differential equation. The corresponding solution is $y(x) = x^{-2}$.

26. $y(x) = \frac{1}{2}x(5x^2 - 3) = \frac{1}{2}(5x^3 - 3x) \Longrightarrow y' = \frac{1}{2}(15x^2 - 3) \Longrightarrow y'' = 15x$. Substitution into the Legendre equation with $N = 3$ yields $(1 - x^2)y'' - 2xy' + 12y = (1 - x^2)(15x) + x(15x^2 - 3) + 6x(5x^2 - 3) = 0$. Consequently the given function is a solution to the Legendre equation with $N = 3$.

27. $y(x) = a_0 + a_1x + a_2x^2 \Longrightarrow y' = a_1 + 2a_2x \Longrightarrow y'' = 4a_2$. Substitution into the given differential equation yields $(1 - x^2)(2a_2) - x(a_1 + 2a_2x) + 4(a_0 + a_1x + a_2x^2) = 0 \Longrightarrow 3a_1x + 2a_2 + 4a_0 = 0$. For this equation to hold for all x we require $3a_1 = 0$, and $2a_2 + 4a_0 = 0$. Consequently $a_1 = 0$, and $a_2 = -2a_0$. The corresponding solution to the differential equation is $y(x) = a_0(1 - 2x^2)$. Imposing the normalization condition $y(1) = 1$ requires that $a_0 = -1$. Hence, the required solution to the differential equation is $y(x) = 2x^2 - 1$.

28. $x \sin y - e^x = c \Longrightarrow x \cos y \dfrac{dy}{dx} + \sin y - e^x = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{e^x - \sin y}{x \cos y}$.

29. $xy^2 + 2y - x = c \Longrightarrow 2xy\dfrac{dy}{dx} + y^2 + 2\dfrac{dy}{dx} - 1 = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{1-y^2}{2(xy+1)}$.

30. $e^{xy} + x = c \Longrightarrow e^{xy}[x\dfrac{dy}{dx} + y] - 1 = 0 \Longrightarrow xe^{xy}\dfrac{dy}{dx} + ye^{xy} = 1 \Longrightarrow \dfrac{1-ye^{xy}}{xe^{xy}}$. Given $y(1) = 0 \Longrightarrow$ $e^{0(1)} - 1 = c \Longrightarrow c = 0$. Therefore, $e^{xy} - x = 0$, so that $y = \dfrac{\ln x}{x}$.

31. $e^{y/x} + xy^2 - x = c \Longrightarrow e^{y/x}\dfrac{x\dfrac{dy}{dx} - y}{x^2} + 2xy\dfrac{dy}{dx} + y^2 - 1 = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{x^2(1-y^2) + ye^{y/x}}{x(e^{y/x} + 2x^2y)}$.

32. $x^2y^2 - \sin x = c \Longrightarrow 2x^2y\dfrac{dy}{dx} + 2xy^2 - \cos x = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{\cos x - 2xy^2}{2x^2y}$. Since $y(\pi) = \dfrac{1}{\pi}$, then $\pi^2\left(\dfrac{1}{\pi}\right)^2 - \sin\pi = c \Longrightarrow c = 1$. Hence, $x^2y^2 - \sin x = 1$ so that $y^2 = \dfrac{1+\sin x}{x^2}$. Since $y(\pi) = \dfrac{1}{\pi}$, take the branch of y where $x < 0$ so $y(x) = \dfrac{\sqrt{1+\sin x}}{x}$.

33. $\dfrac{dy}{dx} = \sin x \Longrightarrow y(x) = -\cos x + c$ for all $x \in \mathbb{R}$.

34. $\dfrac{dy}{dx} = x^{-2/3} \Longrightarrow y(x) = 3x^{1/3} + c$ for all $x \neq 0$.

35. $\dfrac{d^2y}{dx^2} = xe^x \Longrightarrow \dfrac{dy}{dx} = xe^x - e^x + c_1 \Longrightarrow y(x) = xe^x - 2e^x + c_1x + c_2$ for all $x \in \mathbb{R}$.

36. $\dfrac{d^2y}{dx^2} = x^n$, where n is an integer.
If $n = -1$ then $\dfrac{dy}{dx} = \ln|x| + c_1 \Longrightarrow y(x) = x\ln|x| + c_1x + c_2$ for all $x \in (-\infty, 0)$ or $x \in (0,\infty)$.
If $n = -2$ then $\dfrac{dy}{dx} = -x^{-1} + c_1 \Longrightarrow y(x) = c_1x + c_2 - \ln|x|$ for all $x \in (-\infty, 0)$ or $x \in (0,\infty)$.
If $n \neq -1$ and $n \neq -2$ then $\dfrac{dy}{dx} = \dfrac{x^{n+1}}{n+1} + c_1 \Longrightarrow y = \dfrac{x^{n+2}}{(n+1)(n+2)} + c_1x + c_2$ for all $x \in \mathbb{R}$.

37. $\dfrac{dy}{dx} = x^2\ln x \Longrightarrow y(x) = \dfrac{1}{3}x^3\ln x - \dfrac{1}{9}x^3 + c_1 = \dfrac{1}{9}x^3(3\ln x - 1) + c_1$. $y(1) = 2 \Longrightarrow 2 = \dfrac{1}{9}(0-1) + c_1 \Longrightarrow$ $c_1 = \dfrac{19}{9}$. Therefore, $y(x) = \dfrac{1}{9}x^3(3\ln x - 1) + \dfrac{19}{9} = \dfrac{1}{9}\left[x^3(3\ln x - 1) + 19\right]$.

38. $\dfrac{d^2y}{dx^2} = \cos x \Longrightarrow \dfrac{dy}{dx} = \sin x + c_1 \Longrightarrow y(x) = -\cos x + c_1x + c_2$.
Thus, $y'(0) = 1 \Longrightarrow c_1 = 1$, and $y(0) = 2 \Longrightarrow c_2 = 3$. Thus, $y(x) = 3 + x - \cos x$.

39. $\dfrac{d^3y}{dx^3} = 6x \Longrightarrow \dfrac{d^2y}{dx^2} = 3x^2 + c_1 \Longrightarrow \dfrac{dy}{dx} = x^3 + c_1x + c_2 \Longrightarrow y = \frac{1}{4}x^4 + \frac{1}{2}c_1x^2 + c_2x + c_3$.
Thus, $y''(0) = 4 \Longrightarrow c_1 = 4$, and $y'(0) = -1 \Longrightarrow c_2 = -1$, and $y(0) = 1 \Longrightarrow c_3 = 1$. Thus, $y(x) = \frac{1}{4}x^4 + 2x^2 - x + 1$.

40. $y'' = xe^x \Longrightarrow y' = xe^x - e^x + c_1 \Longrightarrow y = xe^x - 2e^x + c_1x + c_2$.

Thus, $y'(0) = 4 \Longrightarrow c_1 = 5$, and $y(0) = 3 \Longrightarrow c_2 = 5$. Thus, $y(x) = xe^x - 2e^x + 5x + 5$.

41. Starting with $y(x) = c_1e^x + c_2e^{-x}$, we find that $y'(x) = c_1e^x - c_2e^{-x}$ and $y''(x) = c_1e^x + c_2e^{-x}$. Thus, $y'' - y = 0$, so $y(x) = c_1e^x + c_2e^{-x}$ is a solution to the differential equation on $(-\infty, \infty)$. Next we establish that every solution to the differential equation has the form $c_1e^x + c_2e^{-x}$. Suppose that $y = f(x)$ is any solution to the differential equation. Then according to Theorem 1.2.12, $y = f(x)$ is the unique solution to the initial-value problem

$$y'' - y = 0, \qquad y(0) = f(0), \qquad y'(0) = f'(0).$$

However, consider the function

$$y(x) = \frac{f(0) + f'(0)}{2}e^x + \frac{f(0) - f'(0)}{2}e^{-x}.$$

This is of the form $y(x) = c_1e^x + c_2e^{-x}$, where $c_1 = \frac{f(0)+f'(0)}{2}$ and $c_2 = \frac{f(0)-f'(0)}{2}$, and therefore solves the differential equation $y'' - y = 0$. Furthermore, evaluation this function at $x = 0$ yields

$$y(0) = f(0) \qquad \text{and} \qquad y'(0) = f'(0).$$

Consequently, this function solves the initial-value problem above. However, by assumption, $y(x) = f(x)$ solves the same initial-value problem. Owing to the uniqueness of the solution to this initial-value problem, it follows that these two solutions are the same:

$$f(x) = c_1e^x + c_2e^{-x}.$$

Consequently, every solution to the differential equation has the form $y(x) = c_1e^x + c_2e^{-x}$, and therefore this is the general solution on any interval I.

42. $\dfrac{d^2y}{dx^2} = e^{-x} \Longrightarrow \dfrac{dy}{dx} = -e^{-x} + c_1 \Longrightarrow y(x) = e^{-x} + c_1x + c_2$. Thus, $y(0) = 1 \Longrightarrow c_2 = 0$, and $y(1) = 0 \Longrightarrow c_1 = -\frac{1}{e}$. Hence, $y(x) = e^{-x} - \frac{1}{e}x$.

43. $\dfrac{d^2y}{dx^2} = -6 - 4\ln x \Longrightarrow \dfrac{dy}{dx} = -2x - 4x\ln x + c_1 \Longrightarrow y(x) = -2x^2\ln x + c_1x + c_2$. Since, $y(1) = 0 \Longrightarrow$ $c_1 + c_2 = 0$, and since, $y(e) = 0 \Longrightarrow ec_1 + c_2 = 2e^2$. Solving this system yields $c_1 = \dfrac{2e^2}{e-1}, c_2 = -\dfrac{2e^2}{e-1}$. Thus, $y(x) = \dfrac{2e^2}{e-1}(x-1) - 2x^2\ln x$.

44. $y(x) = c_1\cos x + c_2\sin x$

(a). $y(0) = 0 \Longrightarrow 0 = c_1(1) + c_2(0) \Longrightarrow c_1 = 0$. $y(\pi) = 1 \Longrightarrow 1 = c_2(0)$, which is impossible. No solutions.

(b). $y(0) = 0 \Longrightarrow 0 = c_1(1) + c_2(0) \Longrightarrow c_1 = 0$. $y(\pi) = 0 \Longrightarrow 0 = c_2(0)$, so c_2 can be anything. Infinitely many solutions.

45-50. Use some kind of technology to define each of the given functions. Then use the technology to simplify the expression given on the left-hand side of each differential equation and verify that the result corresponds to the expression on the right-hand side.

51. (a). Use some form of technology to substitute $y(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5$ where a, b, c, d, e, f are constants, into the given Legendre equation and set the coefficients of each power of x in the resulting equation to zero. The result is:

$$e = 0, \ \ 20f + 18d = 0, \ \ e + 2c = 0, \ \ 3d + 14b = 0, \ \ c + 15a = 0.$$

Now solve for the constants to find: $a = c = e = 0$, $d = -\frac{14}{3}b$, $f = -\frac{9}{10}d = \frac{21}{5}b$. Consequently the corresponding solution to the Legendre equation is:

$$y(x) = bx\left(1 - \frac{14}{3}x^2 + \frac{21}{5}x^4\right).$$

Imposing the normalization condition $y(1) = 1$ requires $1 = b(1 - \frac{14}{3} + \frac{21}{5}) \Longrightarrow b = \frac{15}{8}$. Consequently the required solution is $y(x) = \frac{1}{8}x(15 - 70x^2 + 63x^4)$.

52. (a). $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + ...$

(b). A Maple plot of $J(0, x, 4)$ is given in the accompanying figure.

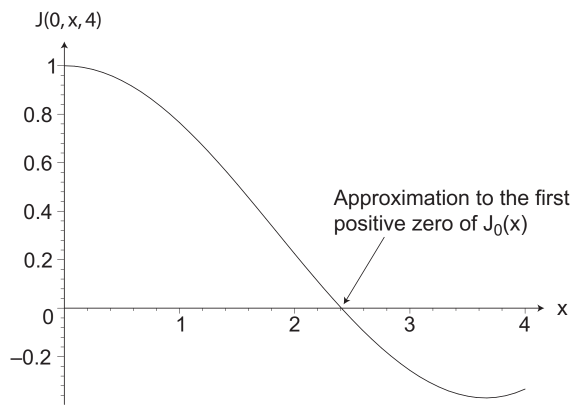

Figure 0.0.8: Figure for Problem 52(b)

(c). From this graph, an approximation to the first positive zero of $J_0(x)$ is 2.4. Using the Maple internal function BesselJZeros gives the approximation 2.404825558.

(c) A Maple plot of the functions $J_0(x)$ and $J(0, x, 4)$ on the interval [0,2] is given in the accompanying figure. We see that to the printer resolution, these graphs are indistinguishable. On a larger interval, for example, [0,3], the two graphs would begin to differ dramatically from one another.

(d). By trial and error, we find the smallest value of m to be $m = 11$. A plot of the functions $J(0, x)$ and $J(0, x, 11)$ is given in the accompanying figure.

Solutions to Section 1.3

True-False Review:

(a): TRUE. This is precisely the remark after Theorem 1.3.2.

(b): FALSE. For instance, the differential equation in Example 1.3.7 has no equilibrium solutions.

(c): FALSE. This differential equation has equilibrium solutions $y(x) = 2$ and $y(x) = -2$.

(d): TRUE. For this differential equation, we have $f(x, y) = x^2 + y^2$. Therefore, any equation of the form $x^2 + y^2 = k$ is an isocline, by definition.

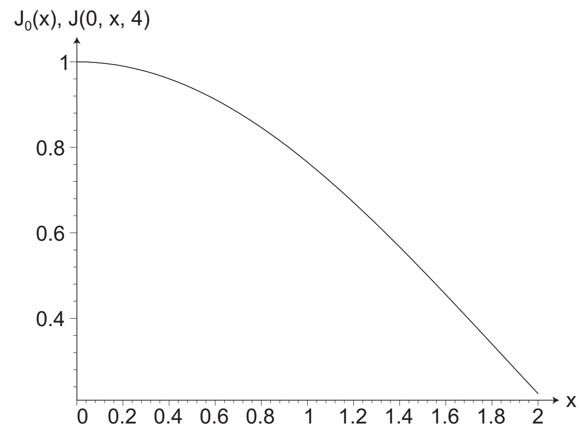

Figure 0.0.9: Figure for Problem 52(c)

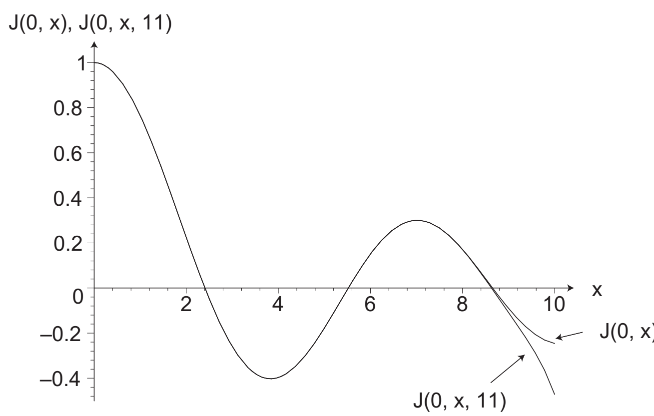

Figure 0.0.10: Figure for Problem 52(d)

(e): TRUE. Equilibrium solutions are always horizontal lines. These are always parallel to each other.

(f): TRUE. The isoclines have the form $\frac{x^2+y^2}{2y} = k$, or $x^2+y^2 = 2ky$, or $x^2+(y-k)^2 = k^2$, so the statement is valid.

(g): TRUE. An equilibrium solution *is* a solution, and two solution curves to the differential equation $\frac{dy}{dx} = f(x,y)$ do not intersect.

Problems:

1. $y = ce^{2x} \Longrightarrow c = ye^{-2x}$. Hence, $\dfrac{dy}{dx} = 2ce^{2x} = 2y$.

2. $y = e^{cx} \Longrightarrow \ln y = cx \Longrightarrow c = \dfrac{\ln y}{x}, x \neq 0$. Hence, $\dfrac{dy}{dx} = ce^{cx} = \dfrac{y}{x}\ln y, x \neq 0$.

3. $y = cx^2 \Longrightarrow c = \dfrac{y}{x^2}$. Hence, $\dfrac{dy}{dx} = 2cx = 2\dfrac{y}{x^2}x = \dfrac{2y}{x}$.

4. $y = cx^{-1} \Longrightarrow c = xy$. Hence, $\dfrac{dy}{dx} = -cx^{-2} = -(xy)x^{-2} = -\dfrac{y}{x}$.

5. $y^2 = cx \Longrightarrow c = \dfrac{y^2}{x}$. Hence, $2y\dfrac{dy}{dx} = c$, so that, $\dfrac{dy}{dx} = \dfrac{c}{2y} = \dfrac{y}{2x}$.

6. $x^2 + y^2 = 2cx \Longrightarrow \dfrac{x^2+y^2}{2x} = c$. Hence, $2x + 2y\dfrac{dy}{dx} = 2c = \dfrac{x^2+y^2}{x}$, so that, $y\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2x} - x$. Consequently, $\dfrac{dy}{dx} = \dfrac{y^2-x^2}{2xy}$.

7. $(x-c)^2 + (y-c)^2 = 2c^2 \Longrightarrow x^2 - 2cx + y^2 - 2cy = 0 \Longrightarrow c = \dfrac{x^2+y^2}{2(x+y)}$. Differentiating the given equation yields $2(x-c) + 2(y-c)\dfrac{dy}{dx} = 0$, so that $2\left[x - \dfrac{x^2+y^2}{2(x+y)}\right] + 2\left[y - \dfrac{x^2+y^2}{2(x+y)}\right]\dfrac{dy}{dx} = 0$, that is $\dfrac{dy}{dx} = -\dfrac{x^2+2xy-y^2}{y^2+2xy-x^2}$.

8. $2cy = x^2 - c^2 \Longrightarrow c^2 + 2cy - x^2 = 0 \Longrightarrow c = \dfrac{-2y \pm \sqrt{4y^2+4x^2}}{2} = -y \pm \sqrt{x^2+y^2}$. Hence, $2c\frac{dy}{dx} = 2x$, so that $\frac{dy}{dx} = \frac{x}{c} = \frac{x}{-y\pm\sqrt{x^2+y^2}}$.

9. $x^2 + y^2 = c \Longrightarrow 2x + 2y\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$.

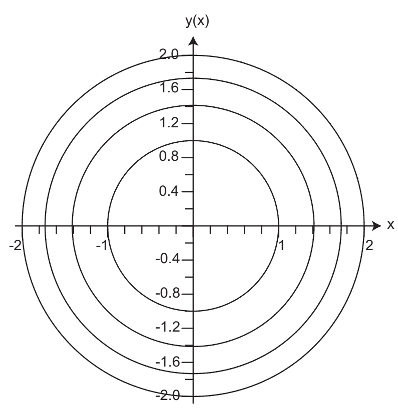

Figure 0.0.11: Figure for Problem 9

10. $y = cx^3 \Longrightarrow \dfrac{dy}{dx} = 3cx^2 = 3\dfrac{y}{x^3}x^2 = \dfrac{3y}{x}$. The initial condition $y(2) = 8 \Longrightarrow 8 = c(2)^3 \Longrightarrow c = 1$. Thus the unique solution to the initial value problem is $y = x^3$.

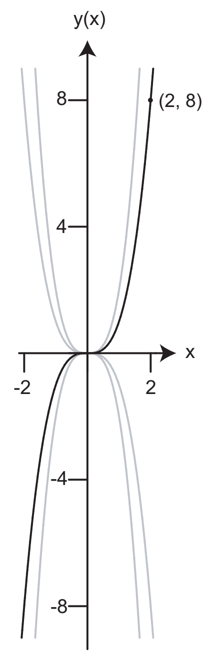

Figure 0.0.12: Figure for Problem 10

11. $y^2 = cx \Longrightarrow 2y\dfrac{dy}{dx} = c \Longrightarrow 2y\dfrac{dy}{dx} = \dfrac{y^2}{x} \Longrightarrow \dfrac{dy}{dx} = y2x \Longrightarrow 2x \cdot dy - y \cdot dx = 0$. The initial condition $y(1) = 2 \Longrightarrow c = 4$, so that the unique solution to the initial value problem is $y^2 = 4x$.

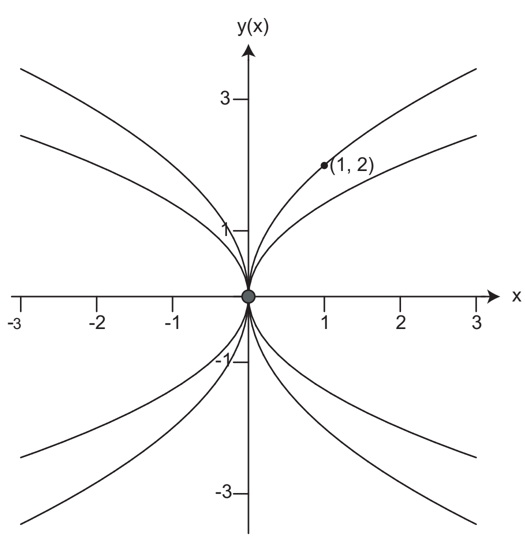

Figure 0.0.13: Figure for Problem 11

12. $(x-c)^2+y^2=c^2 \Longrightarrow x^2-2cx+c^2+y^2=c^2$, so that

$$x^2-2cx+y^2=0. \tag{0.0.1}$$

Differentiating with respect to x yields

$$2x-2c+2y\frac{dy}{dx}=0. \tag{0.0.2}$$

But from (0.0.1), $c=\dfrac{x^2+y^2}{2x}$ which, when substituted into (0.0.2), yields $2x-\left(\dfrac{x^2+y^2}{x}\right)+2y\dfrac{dy}{dx}=0$, that is, $\dfrac{dy}{dx}=\dfrac{y^2-x^2}{2xy}$. Imposing the initial condition $y(2)=2 \Longrightarrow$ from (0.0.1) $c=2$, so that the unique solution to the initial value problem is $y=+\sqrt{x(4-x)}$.

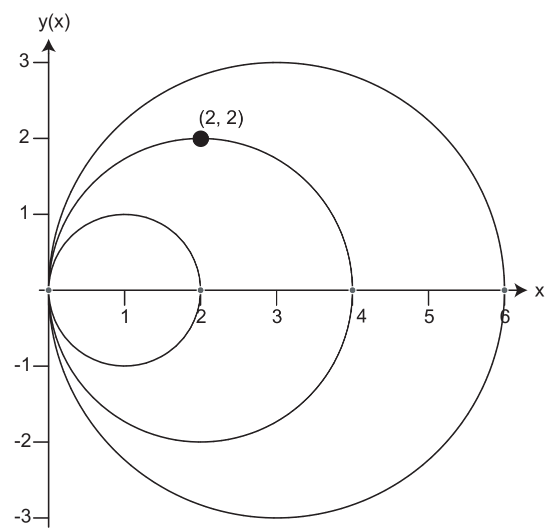

Figure 0.0.14: Figure for Problem 12

13. Let $f(x,y)=x\sin(x+y)$, which is continuous for all $x,y\in\mathbb{R}$.
$\dfrac{\partial f}{\partial y}=x\cos(x+y)$, which is continuous for all $x,y\in\mathbb{R}$.
By Theorem 1.3.2, $\dfrac{dy}{dx}=x\sin(x+y),\;\; y(x_0)=y_0$ has a unique solution for some interval $I\in\mathbb{R}$.

14. $\dfrac{dy}{dx}=\dfrac{x}{x^2+1}(y^2-9),\;\; y(0)=3.$
$f(x,y)=\dfrac{x}{x^2+1}(y^2-9)$, which is continuous for all $x,y\in\mathbb{R}$.
$\dfrac{\partial f}{\partial y}=\dfrac{2xy}{x^2+1}$, which is continuous for all $x,y\in\mathbb{R}$.
So the initial value problem stated above has a unique solution on any interval containing $(0,3)$. By inspection we see that $y(x)=3$ is the unique solution.

15. The initial-value problem does not necessarily have a unique solution since the hypothesis of the existence and uniqueness theorem are not satisfied at (0,0). This follows since $f(x, y) = xy^{1/2}$, so that $\frac{\partial f}{\partial y} = \frac{1}{2}xy^{-1/2}$ which is not continuous at $(0, 0)$.

16. (a). $f(x, y) = -2xy^2 \Longrightarrow \frac{\partial f}{\partial y} = -4xy$. Both of these functions are continuous for all (x, y), and therefore the hypothesis of the uniqueness and existence theorem are satisfied for any (x_0, y_0).

(b). $y(x) = \frac{1}{x^2 + c} \Longrightarrow y' = -\frac{2x}{(x^2 + c)^2} = -2xy^2$.

(c). $y(x) = \frac{1}{x^2 + c}$.

(i). $y(0) = 1 \Longrightarrow 1 = \frac{1}{c} \Longrightarrow c = 1$. Hence, $y(x) = \frac{1}{x^2 + 1}$. The solution is valid on the interval $(-\infty, \infty)$.

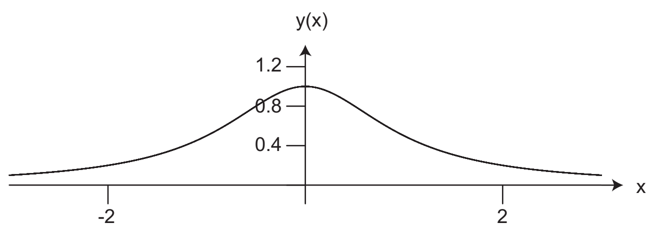

Figure 0.0.15: Figure for Problem 16c(i)

(ii). $y(1) = 1 \Longrightarrow 1 = \frac{1}{1 + c} \Longrightarrow c = 0$. Hence, $y(x) = \frac{1}{x^2}$. This solution is valid on the interval $(0, \infty)$.

(iii). $y(0) = -1 \Longrightarrow -1 = \frac{1}{c} \Longrightarrow c = -1$. Hence, $y(x) = \frac{1}{x^2 - 1}$. This solution is valid on the interval $(-1, 1)$.

(d). Since, by inspection, $y(x) = 0$ satisfies the given initial-value problem, it must be the unique solution to the initial-value problem.

17. (a). Both $f(x, y) = y(y - 1)$ and $\frac{\partial f}{\partial y} = 2y - 1$ are continuous at all points (x, y). Consequently, the hypothesis of the existence and uniqueness theorem are satisfied by the given initial-value problem for any x_0, y_0.

(b). Equilibrium solutions: $y(x) = 0, y(x) = 1$.

(c). Differentiating the given differential equation yields $\frac{d^2y}{dx^2} = (2y - 1)\frac{dy}{dx} = (2y - 1)y(y - 1)$. Hence the solution curves are concave up for $0 < y < \frac{1}{2}$, and $y > 1$, and concave down for $y < 0$, and $\frac{1}{2} < y < 1$.

(d). The solutions will be bounded provided $0 \leq y_0 \leq 1$.

18. (a). Equilibrium solutions: $y(x) = -2$, $y(x) = 1$.

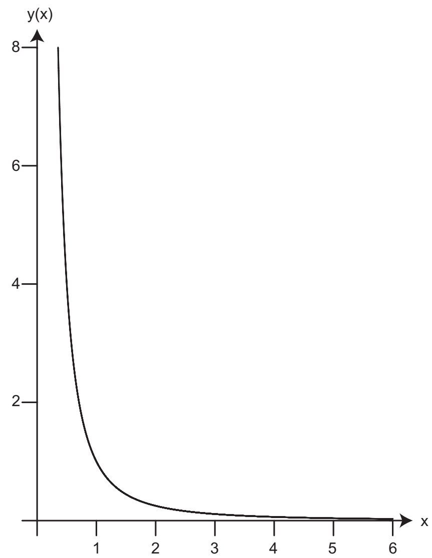

Figure 0.0.16: Figure for Problem 16c(ii)

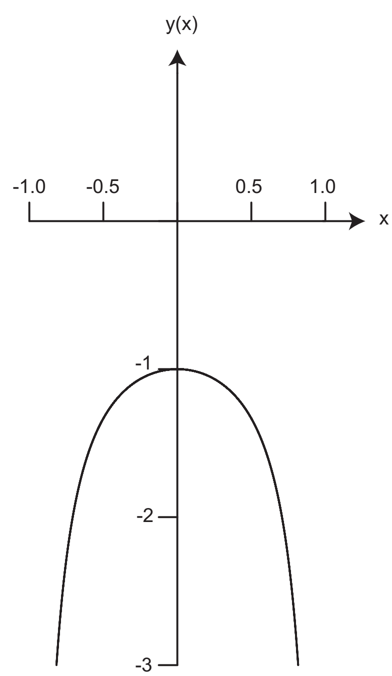

Figure 0.0.17: Figure for Problem 16c(iii)

(b). $\frac{dy}{dx} = (y+2)(y-1) \Longrightarrow$ the solutions are increasing when $y < -2$ and $y > 1$, and the solutions are

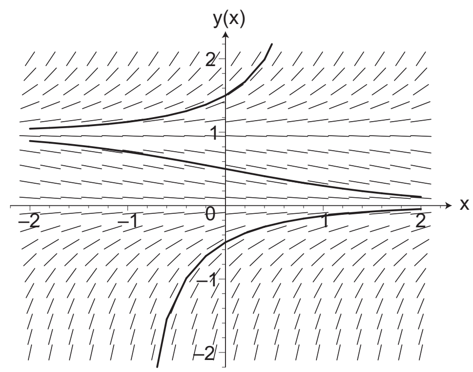

Figure 0.0.18: Figure for Problem 17(d)

decreasing when $-2 < y < 1$.

(c). Differentiating the given differential equation yields $\frac{d^2y}{dx^2} = (2y+1)\frac{dy}{dx} = (2y+1)(y+2)(y-1)$. Hence the solution curves are concave up for $-2 < y < -\frac{1}{2}$, and $y > 1$, and concave down for $y < -2$, and $-\frac{1}{2} < y < 1$.

19. (a). Equilibrium solution: $y(x) = 2$.

(b). $\frac{dy}{dx} = (y-2)^2 \Longrightarrow$ the solutions are increasing when $y < 2$ and $y > 2$.

(c). Differentiating the given differential equation yields $\frac{d^2y}{dx^2} = 2(y-2)\frac{dy}{dx} = 2(y-2)^3$. Hence the solution curves are concave up for $y > 2$, and concave down for $y < 2$.

20. (a). Equilibrium solutions: $y(x) = 0$, $y(x) = 1$.

(b). $\frac{dy}{dx} = y^2(y-1) \Longrightarrow$ the solutions are increasing when $y < 1$, and the solutions are decreasing when $y < 1$.

(c). Differentiating the given differential equation yields $\frac{d^2y}{dx^2} = (3y^2-2y)\frac{dy}{dx} = y^3(3y-2)(y-1)$. Hence the solution curves are concave up for $0 < y < -\frac{2}{3}$, and $y > 1$, and concave down for $y < 0$, and $2/3 < y < 1$.

21. (a). Equilibrium solutions: $y(x) = 0$, $y(x) = 1$, $y(x) = -1$.

(b). $\frac{dy}{dx} = (y+2)(y-1) \Longrightarrow$ the solutions are increasing when $-1 < y < 0$ and $y > 1$, and the solutions are decreasing when $y < -1$, and $0 < y < 1$.

(c). Differentiating the given differential equation yields $\frac{d^2y}{dx^2} = (3y^2-1)\frac{dy}{dx} = (3y^2-1)y(y-1)(y+1)$. Hence the solution curves are concave up for $-1 < y < -\frac{1}{\sqrt{3}}$, and $0 < y < \frac{1}{\sqrt{3}}$, and $y > 1$, and concave down for $y < -1$, and $-\frac{1}{\sqrt{3}} < y < 0$, and $\frac{1}{\sqrt{3}} < y < 1$.

22. $y' = 4x$. There are no equilibrium solutions. The slope of the solution curves is positive for $x > 0$ and is negative for $x < 0$. The isoclines are the lines $x = \frac{k}{4}$.

Slope of Solution Curve	Equation of Isocline
-4	$x = -1$
-2	$x = -1/2$
0	$x = 0$
2	$x = 1/2$
4	$x = 1$

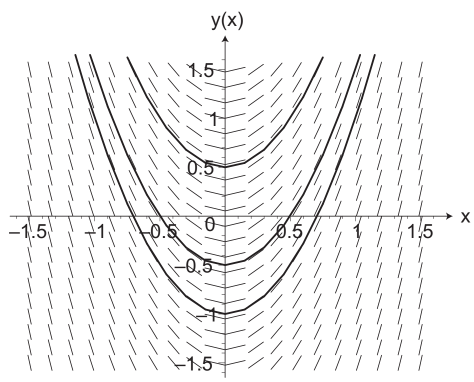

Figure 0.0.19: Figure for Problem 22

23. $y' = \frac{1}{x}$. There are no equilibrium solutions. The slope of the solution curves is positive for $x > 0$ and increases without bound as $x \to 0^+$. The slope of the curve is negative for $x < 0$ and decreases without bound as $x \to 0^-$. The isoclines are the lines $\frac{1}{x} = k$.

Slope of Solution Curve	Equation of Isocline
± 4	$x = \pm 1/4$
± 2	$x = \pm 1/2$
$\pm 1/2$	$x = \pm 2$
$\pm 1/4$	$x = \pm 4$
$\pm 1/10$	$x = \pm 10$

24. $y' = x + y$. There are no equilibrium solutions. The slope of the solution curves is positive for $y > -x$, and negative for $y < -x$. The isoclines are the lines $y + x = k$.

Slope of Solution Curve	Equation of Isocline
-2	$y = -x - 2$
-1	$y = -x - 1$
0	$y = -x$
1	$y = -x + 1$
2	$y = -x + 2$

Since the slope of the solution curve along the isocline $y = -x - 1$ coincides with the slope of the isocline, it follows that $y = -x - 1$ is a solution to the differential equation. Differentiating the given differential

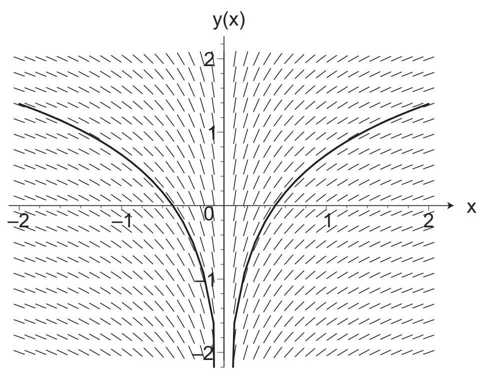

Figure 0.0.20: Figure for Problem 23

equation yields: $y'' = 1 + y' = 1 + x + y$. Hence the solution curves are concave up for $y > -x - 1$, and concave down for $y < -x - 1$. Putting this information together leads to the slope field in the accompanying figure.

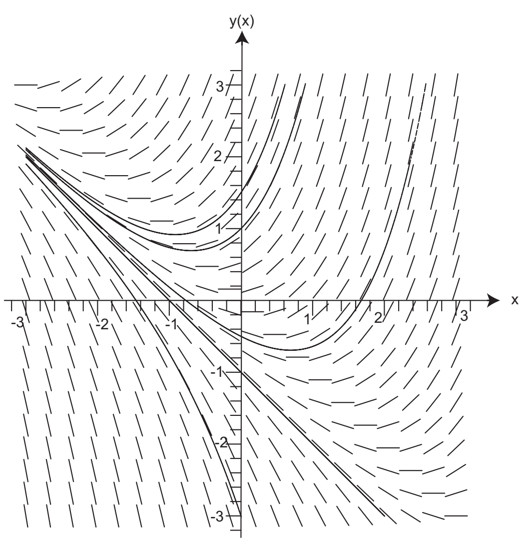

Figure 0.0.21: Figure for Problem 24

25. $y' = \frac{x}{y}$. There are no equilibrium solutions. The slope of the solution curves is zero when $x = 0$. The solution has a vertical tangent line at all points along the x-axis (except the origin). Differentiating the differential equation yields: $y' = \dfrac{1}{y} - \dfrac{x}{y^2}y' = \dfrac{1}{y} - \dfrac{x^2}{y^3} = \dfrac{1}{y^3}(y^2 - x^2)$. Hence the solution curves are concave up for $y > 0$ and $y^2 > x^2$; $y < 0$ and $y^2 < x^2$ and concave down for $y > 0$ and $y^2 < x^2$; $y < 0$ and $y^2 > x^2$. The isoclines are the lines $\frac{x}{y} = k$.

Slope of Solution Curve	Equation of Isocline
± 2	$y = \pm x/2$
± 1	$y = \pm x$
$\pm 1/2$	$y = \pm 2x$
$\pm 1/4$	$y = \pm 4x$
$\pm 1/10$	$y = \pm 10x$

Note that $y = \pm x$ are solutions to the differential equation.

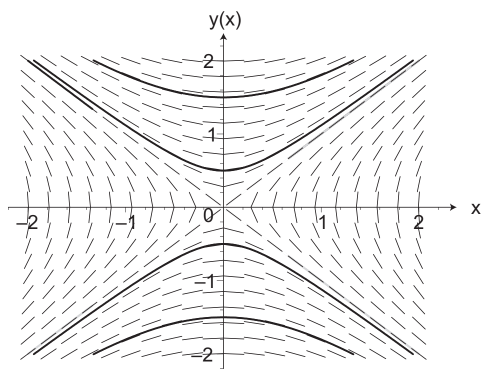

Figure 0.0.22: Figure for Problem 25

26. $y' = -\frac{4x}{y}$. Slope is zero when $x = 0\,(y \neq 0)$. The solutions have a vertical tangent line at all points along the x-axis(except the origin). The isoclines are the lines $-\frac{4x}{y} = k$. Some values are given in the table below.

Slope of Solution Curve	Equation of Isocline
± 1	$y = \pm 4x$
± 2	$y = \pm 2x$
± 3	$y = \pm 4x/3$

Differentiating the given differential equation yields: $y' = -\dfrac{4}{y} + \dfrac{4xy'}{y^2} = -\dfrac{4}{y} - \dfrac{16x^2}{y^3} = -\dfrac{4(y^2+4x^2)}{y}$. Consequently the solution curves are concave up for $y < 0$, and concave down for $y > 0$. Putting this information together leads to the slope field in the accompanying figure.

27. $y' = x^2 y$. Equilibrium solution: $y(x) = 0 \Longrightarrow$ no solution curve can cross the x-axis. Slope: zero when $x = 0$ or $y = 0$. Positive when $y > 0\,(x \neq 0)$, negative when $y < 0\,(x \neq 0)$. Differentiating the given differential equation yields: $\dfrac{d^2y}{dx^2} = 2xy + x^2\dfrac{dy}{dx} = 2xy + x^4 y = xy(2+x^3)$. So, when $y > 0$, the solution curves are concave up for $x \in (-\infty, (-2)^{1/3})$, and for $x > 0$, and are concave down for $x \in ((-2)^{1/3}, 0)$. When $y < 0$, the solution curves are concave up for $x \in ((-2)^{1/3}, 0)$, and concave down for $x \in (-\infty, (-2)^{1/3})$ and for $x > 0$. The isoclines are the hyperbolas $x^2 y = k$.

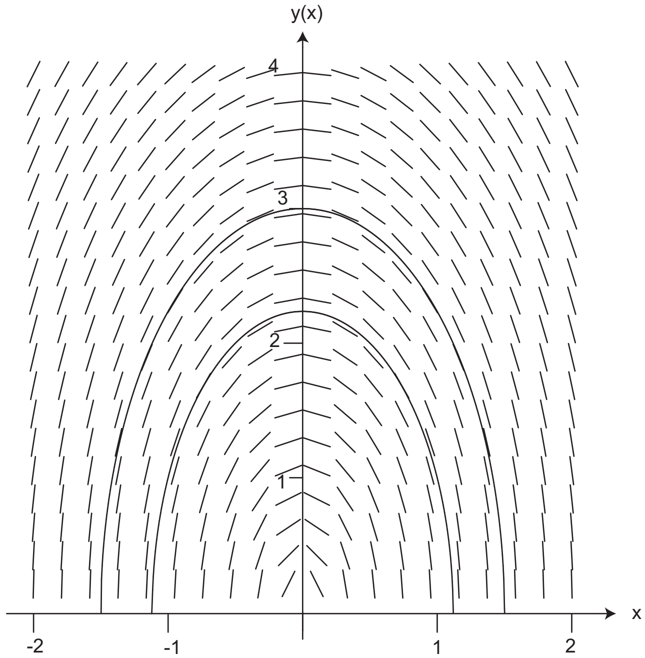

Figure 0.0.23: Figure for Problem 26

Slope of Solution Curve	Equation of Isocline
± 2	$y = \pm 2/x^2$
± 1	$y = \pm 1/x^2$
$\pm 1/2$	$y = \pm 1/(2x)^2$
$\pm 1/4$	$y = \pm 1/(4x)^2$
$\pm 1/10$	$y = \pm 1/(10x)^2$
0	$y = 0$

28. $y' = x^2 \cos y$. The slope is zero when $x = 0$. There are equilibrium solutions when $y = (2k+1)\frac{\pi}{2}$. The slope field is best sketched using technology. The accompanying figure gives the slope field for $-\frac{\pi}{2} < y < \frac{3\pi}{2}$.

29. $y' = x^2 + y^2$. The slope of the solution curves is zero at the origin, and positive at all the other points. There are no equilibrium solutions. The isoclines are the circles $x^2 + y^2 = k$.

Slope of Solution Curve	Equation of Isocline
1	$x = \pm 1/4$
2	$x = \pm 1/2$
3	$x = \pm 2$
4	$x = \pm 4$
5	$x = \pm 10$

30. $\frac{dT}{dt} = -\frac{1}{80}(T - 70)$. Equilibrium solution: $T(t) = 70$. The slope of the solution curves is positive for

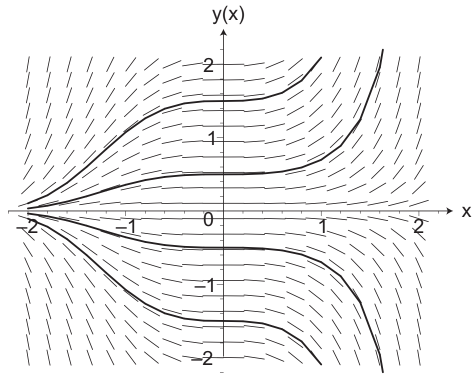

Figure 0.0.24: Figure for Problem 27

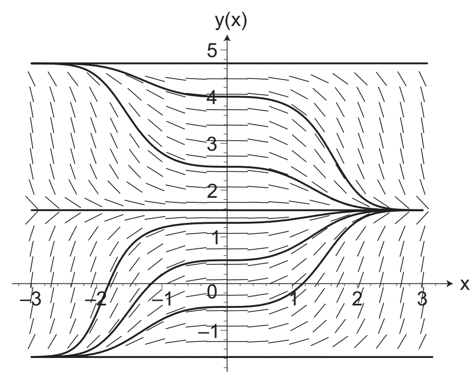

Figure 0.0.25: Figure for Problem 28

$T > 70$, and negative for $T < 70$. $\dfrac{d^2T}{dt^2} = -\dfrac{1}{80}\dfrac{dT}{dt} = \dfrac{1}{6400}(T-70)$. Hence the solution curves are concave up for $T > 70$, and concave down for $T < 70$. The isoclines are the horizontal lines $-\frac{1}{80}(T-70) = k$.

Slope of Solution Curve	Equation of Isocline
$-1/4$	$T = 90$
$1/5$	$T = 86$
0	$T = 70$
$1/5$	$T = 54$
$1/4$	$T = 50$

31. $y' = -2xy$.

32. $y' = \dfrac{x \sin x}{1+y^2}$.

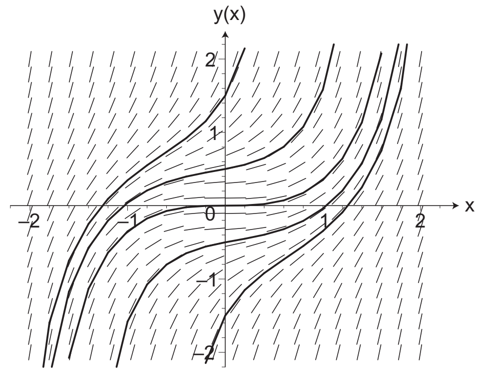

Figure 0.0.26: Figure for Problem 29

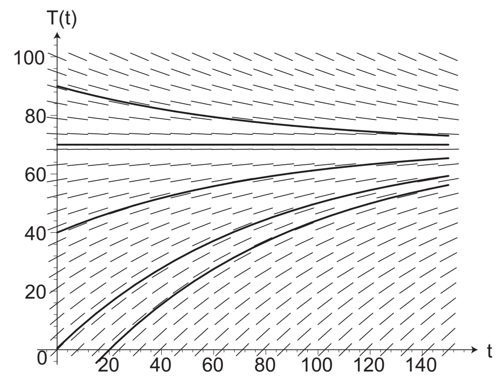

Figure 0.0.27: Figure for Problem 30

33. $y' = 3x - y.$

34. $y' = 2x^2 \sin y.$

35. $y' = \dfrac{2 + y^2}{3 + 0.5x^2}.$

36. $y' = \dfrac{1 - y^2}{2 + 0.5x^2}.$

37. (a). Slope field for the differential equation $y' = x^{-1}(3 \sin x - y)$.

(b). Slope field with solution curves included.

The figure suggests that the solution to the differential equation are unbounded as $x \to 0^+$.

(c). Slope field with solution curve corresponding to the initial condition $y(\frac{\pi}{2}) = \frac{6}{\pi}$.

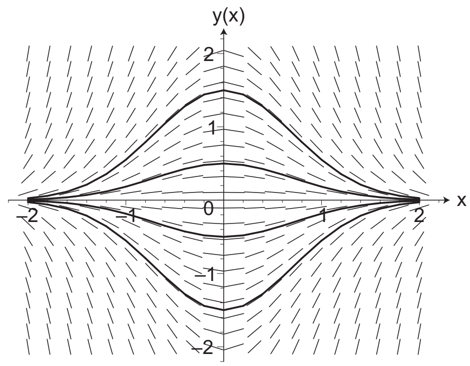

Figure 0.0.28: Figure for Problem 31

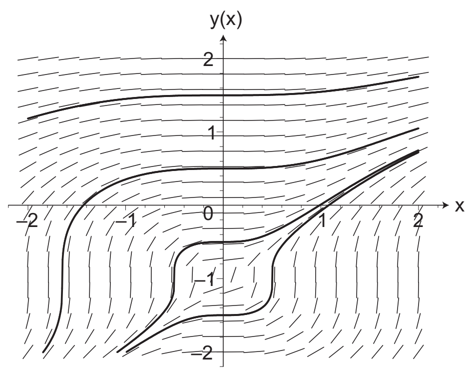

Figure 0.0.29: Figure for Problem 32

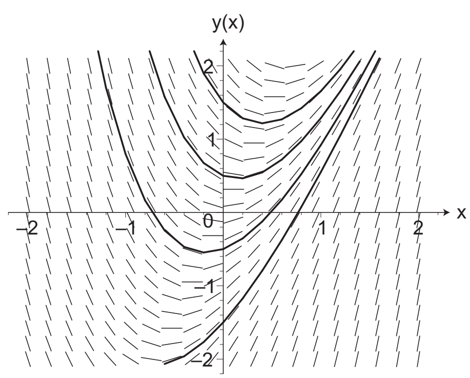

Figure 0.0.30: Figure for Problem 33

This solution curve is bounded as $x \to 0^+$.

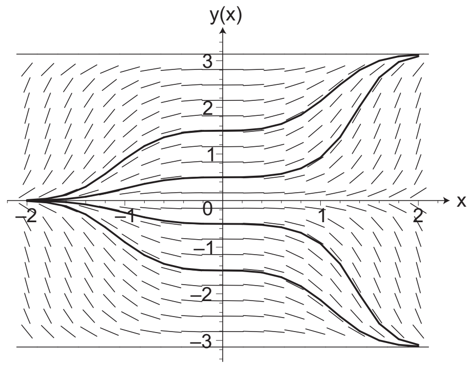

Figure 0.0.31: Figure for Problem 34

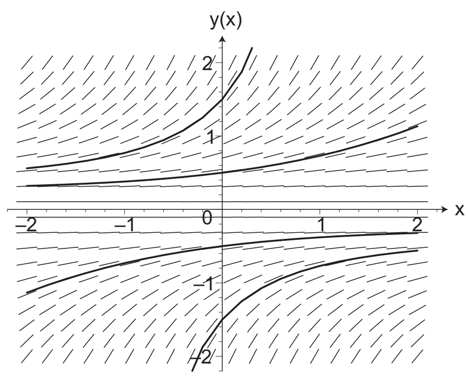

Figure 0.0.32: Figure for Problem 35

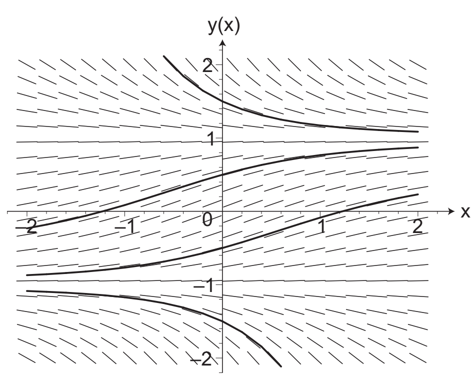

Figure 0.0.33: Figure for Problem 36

(d). In the accompanying figure we have sketched several solution curves on the interval (0,15].

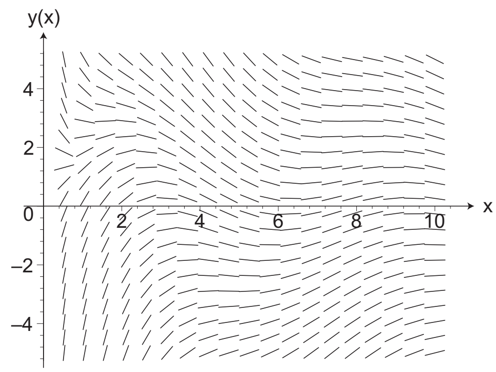

Figure 0.0.34: Figure for Problem 37(a)

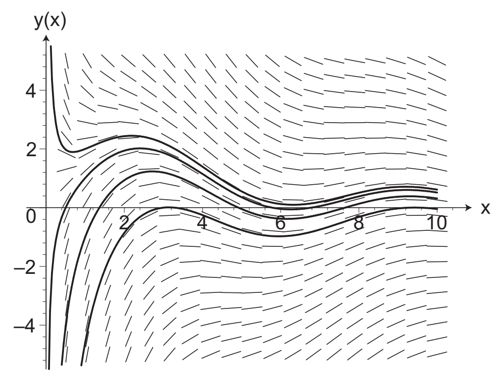

Figure 0.0.35: Figure for Problem 37(b)

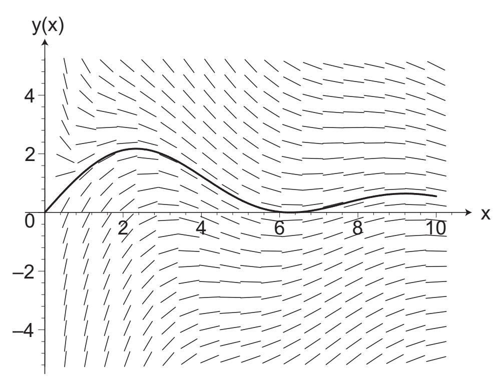

Figure 0.0.36: Figure for Problem 37(c)

The figure suggests that the solution curves approach the x-axis as $x \to \infty$.

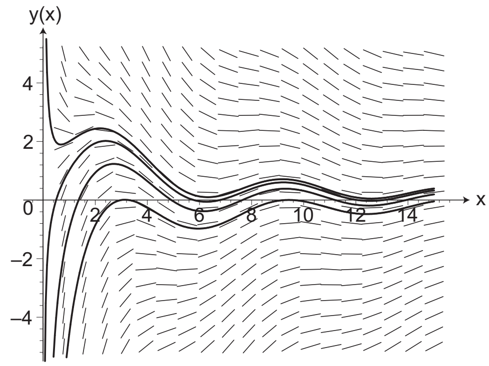

Figure 0.0.37: Figure for Problem 37(d)

38. (a). Differentiating the given equation gives $\dfrac{dy}{dx} = 2kx = 2\dfrac{y}{x}$. Hence the differential equation of the orthogonal trajectories is $\dfrac{dy}{dx} = -\dfrac{x}{2y}$.

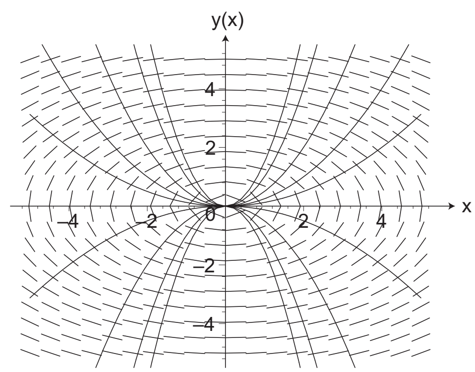

Figure 0.0.38: Figure for Problem 38(a)

(b). The orthogonal trajectories appear to be ellipses. This can be verified by integrating the differential equation derived in (a).

39. If $a > 0$, then as illustrated in the following slope field ($a = 0.5, b = 1$), it appears that $\lim_{t\to\infty} i(t) = \frac{b}{a}$.

If $a < 0$, then as illustrated in the following slope field ($a = -0.5, b = 1$) it appears that $i(t)$ diverges as $t \to \infty$.

If $a = 0$ and $b \neq 0$, then once more $i(t)$ diverges as $t \to \infty$. The accompanying figure shows a representative case when $b > 0$. Here we see that $\lim_{t\to\infty} i(t) = +\infty$. If $b < 0$, then $\lim_{t\to\infty} i(t) = -\infty$.

If $a = b = 0$, then the general solution to the differential equation is $i(t) = i_0$ where i_0 is a constant.

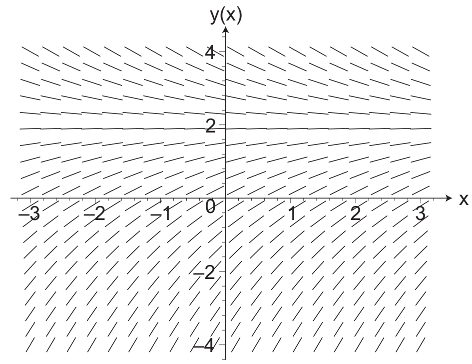

Figure 0.0.39: Figure for Problem 39 when $a > 0$

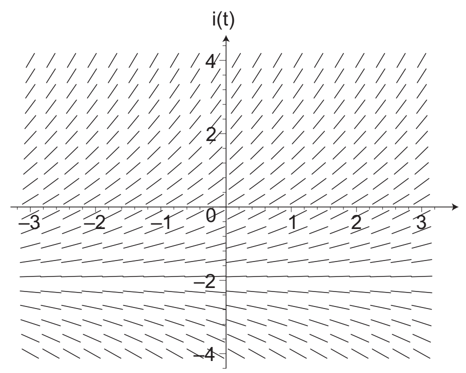

Figure 0.0.40: Figure for Problem 39 when $a < 0$

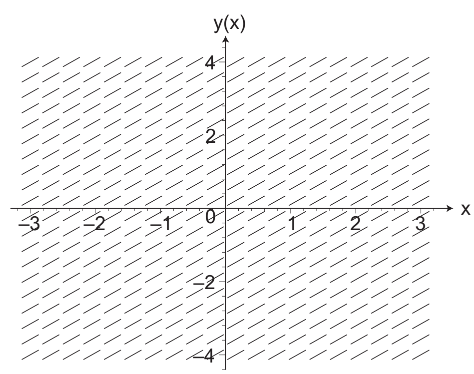

Figure 0.0.41: Figure for Problem 39 when $a = 0$

Solutions to Section 1.4

True-False Review:

(a): TRUE. The differential equation $\frac{dy}{dx} = f(x)g(y)$ can be written $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$, which is the proper form, according to Definition 1.4.1, for a separable differential equation.

(b): TRUE. A separable differential equation is a first-order differential equation, so the general solution contains one constant. The value of that constant can be determined from an initial condition, as usual.

(c): TRUE. Newton's Law of Cooling is usually expressed as $\frac{dT}{dt} = -k(T - T_m)$, and this can be rewritten as

$$\frac{1}{T - T_m}\frac{dT}{dt} = -k,$$

and this form shows that the equation is separable.

(d): FALSE. The expression $x^2 + y^2$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.

(e): FALSE. The expression $x\sin(xy)$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.

(f): TRUE. We can write the given equation as $e^{-y}\frac{dy}{dx} = e^x$, which is the proper form for a separable equation.

(g): TRUE. We can write the given equation as $(1 + y^2)\frac{dy}{dx} = \frac{1}{x^2}$, which is the proper form for a separable equation.

(h): FALSE. The expression $\frac{x+4y}{4x+y}$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.

(i): TRUE. We can write $\frac{x^3y+x^2y^2}{x^2+xy} = xy$, so we can write the given differential equation as $\frac{1}{y}\frac{dy}{dx} = x$, which is the proper form for a separable equation.

Problems:

1. Separating the variables and integrating yields

$$\int \frac{dy}{y} = 2\int x dx \Longrightarrow \ln|y| = x^2 + c_1 \Longrightarrow y(x) = ce^{x^2}.$$

2. Separating the variables and integrating yields

$$\int y^{-2}dy = \int \frac{dx}{x^2+1} \Longrightarrow y(x) = -\frac{1}{\tan^{-1}x + c}.$$

3. Separating the variables and integrating yields

$$\int e^y dy = \int e^{-x}dx = 0 \Longrightarrow e^y + e^{-x} = c \Longrightarrow y(x) = \ln(c - e^{-x}).$$

4. Separating the variables and integrating yields

$$\int \frac{dy}{y} = \int \frac{(\ln x)^{-1}}{x}dx \Longrightarrow y(x) = c\ln x.$$

5. Separating the variables and integrating yields

$$\int \frac{dx}{x-2} = \int \frac{dy}{y} \Longrightarrow \ln|x-2| - \ln|y| = c_1 \Longrightarrow y(x) = c(x-2).$$

6. Separating the variables and integrating yields

$$\int \frac{dy}{y-1} = \int \frac{2x}{x^2+3}dx \Longrightarrow \ln|y-1| = \ln|x^2+3| + c_1 \Longrightarrow y(x) = c(x^2+3)+1.$$

7. $y - x\dfrac{dy}{dx} = 3 - 2x^2\dfrac{dy}{dx} \Longrightarrow x(2x-1)\dfrac{dy}{dx} = (3-y)$. Separating the variables and integrating yields

$$\begin{aligned} -\int \frac{dy}{y-3} = \int \frac{dx}{x(2x-1)} &\Longrightarrow -\ln|y-3| = -\int \frac{dx}{x} + \int \frac{2}{2x-1}dx \\ &\Longrightarrow -\ln|y-3| = -\ln|x| + \ln|2x-1| + c_1 \\ &\Longrightarrow \frac{x}{(y-3)(2x-1)} = c_2 \Longrightarrow y(x) = \frac{cx-3}{2x-1}. \end{aligned}$$

8. $\dfrac{dy}{dx} = \dfrac{\cos(x-y)}{\sin x \sin y} - 1 \Longrightarrow \dfrac{dy}{dx} = \dfrac{\cos x \cos y}{\sin x \sin y} \Longrightarrow \int \dfrac{\sin y}{\cos y}dy = \int \dfrac{\cos x}{\cos y}dx \Longrightarrow -\ln|\cos y| = \ln|\sin x| + c_1 \Longrightarrow$ $\cos y = c\csc x$.

9. $\dfrac{dy}{dx} = \dfrac{x(y^2-1)}{2(x-2)(x-1)} \Longrightarrow \int \dfrac{dy}{(y+1)(y-1)} = \dfrac{1}{2}\int \dfrac{xdx}{(x-2)(x-1)}, y \neq \pm 1$. Thus,

$-\dfrac{1}{2}\int \dfrac{dy}{y+1} + \dfrac{1}{2}\int \dfrac{dy}{y-1} = \dfrac{1}{2}\left(2\int \dfrac{dx}{x-2} - \int \dfrac{dx}{x-1}\right) \Longrightarrow -\ln|y+1| + \ln|y-1| = 2\ln|x-2| - \ln|x-1| + c_1$

$\Longrightarrow \dfrac{y-1}{y+1} = c\dfrac{(x-2)^2}{x-1} \Longrightarrow y(x) = \dfrac{(x-1)+c(x-2)^2}{(x-1)-c(x-2)^2}$. By inspection we see that $y(x) = 1$, and $y(x) = -1$ are solutions of the given differential equation. The former is included in the above solution when $c = 0$.

10. $\dfrac{dy}{dx} = \dfrac{x^2y-32}{16-x^2} + 2 \Longrightarrow \int \dfrac{dy}{y-2} = \int \dfrac{x^2}{16-x^2}dx \Longrightarrow \ln|y-2| = -\int \left(1 + \dfrac{16}{x^2-16}\right)dx \Longrightarrow \ln|y-2| =$ $-x - 16\int \dfrac{dx}{x^2-16} \Longrightarrow \ln|y-2| = -x - 16\left(-\frac{1}{8}\int \dfrac{dx}{x+4} + \frac{1}{8}\int \dfrac{dx}{x-4}\right) \Longrightarrow \ln|y-2| = -x + 2\ln|x+4| -$ $2\ln|x-4| + c_1 \Longrightarrow y(x) = 2 + c\left(\dfrac{x+4}{x-4}\right)^2 e^{-x}$.

11. $(x-a)(x-b)\dfrac{dy}{dx} - (y-c) = 0 \Longrightarrow \int \dfrac{dy}{y-c} = \int \dfrac{dx}{(x-a)(x-b)} \Longrightarrow \int \dfrac{dy}{y-c} = \dfrac{1}{a-b}\int \left(\dfrac{1}{x-a} - \dfrac{1}{x-b}\right)dx \Longrightarrow$ $\ln|y-c| = \ln\left[c_1\left|\dfrac{x-a}{x-b}\right|^{1/(a-b)}\right] \Longrightarrow \left|(y-c)\left(\dfrac{x-b}{x-a}\right)^{1/(a-b)}\right| = c_1 \Longrightarrow y - c = c_2\left(\dfrac{x-a}{x-b}\right)^{1/(a-b)} \Longrightarrow$ $y(x) = c + c_2\left(\dfrac{x-a}{x-b}\right)^{1/(a-b)}$.

12. $(x^2+1)\dfrac{dy}{dx}+y^2=-1 \Longrightarrow \int \dfrac{dy}{1+y^2}=-\int \dfrac{dx}{1+x^2} \Longrightarrow \tan^{-1} y=\tan^{-1} x+c$, but $y(0)=1$ so $c=\dfrac{\pi}{4}$. Thus, $\tan^{-1} y=\tan^{-1} x+\dfrac{\pi}{4}$ or $y(x)=\dfrac{1-x}{1+x}$.

13. $(1-x^2)\dfrac{dy}{dx}+xy=ax \Longrightarrow \int \dfrac{dy}{a-y}=-\frac{1}{2}\int -\dfrac{2x}{1-x^2}dx \Longrightarrow -\ln|a-y|=-\frac{1}{2}\ln|1-x^2|+c_1 \Longrightarrow y(x)=a+c\sqrt{1-x^2}$, but $y(0)=2a$ so $c=a$ and therefore, $y(x)=a(1+\sqrt{1-x^2})$.

14. $\dfrac{dy}{dx}=1-\dfrac{\sin(x+y)}{\sin x\sin y} \Longrightarrow \dfrac{dy}{dx}=-\tan x\cot y \Longrightarrow -\int \dfrac{\sin y}{\cos y}dy=\int \dfrac{\sin x}{\cos x}dx \Longrightarrow -\ln|\cos x\cos y|=c$, but $y(\dfrac{\pi}{4})=\dfrac{\pi}{4}$ so $c=\ln(2)$. Hence, $-\ln|\cos x\cos y|=\ln(2) \Longrightarrow y(x)=\cos^{-1}\left(\frac{1}{2}\sec x\right)$.

15. $\dfrac{dy}{dx}=y^3\sin x \Longrightarrow \int \dfrac{dy}{y^3}=\int \sin x dx$ for $y\neq 0$. Thus $-\dfrac{1}{2y^2}=-\cos x+c$. However, we cannot impose the initial condition $y(0)=0$ on the last equation since it is not defined at $y=0$. But, by inspection, $y(x)=0$ is a solution to the given differential equation and further, $y(0)=0$; thus, the unique solution to the initial value problem is $y(x)=0$.

16. $\dfrac{dy}{dx}=\frac{2}{3}(y-1)^{1/2} \Longrightarrow \int \dfrac{dy}{(y-1)^{1/2}}=\frac{2}{3}\int dx$ if $y\neq 1 \Longrightarrow 2(y-1)^{1/2}=\frac{2}{3}x+c$ but $y(1)=1$ so $c=-\frac{2}{3} \Longrightarrow 2\sqrt{y-1}=\frac{2}{3}x-\frac{2}{3} \Longrightarrow \sqrt{y-1}=\frac{1}{3}(x-1)$. This does not contradict the Existence-Uniqueness theorem because the hypothesis of the theorem is not satisfied when $x=1$.

17. (a). $m\dfrac{dv}{dt}=mg-kv^2 \Longrightarrow \dfrac{m}{k\left[(mg/k)-v^2\right]}dv=dt$. If we let $a=\sqrt{\frac{mg}{k}}$ then the preceding equation can be written as $\dfrac{m}{k}\int \dfrac{1}{a^2-v^2}dv=\int dt$ which can be integrated directly to obtain

$$\frac{m}{2ak}\ln\left(\frac{a+v}{a-v}\right)=t+c,$$

that is, upon exponentiating both sides,

$$\frac{a+v}{a-v}=c_1e^{\frac{2ak}{m}t}.$$

Imposing the initial condition $v(0)=0$, yields $c=0$ so that

$$\frac{a+v}{a-v}=e^{\frac{2ak}{m}t}.$$

Therefore,

$$v(t)=a\left(\frac{e^{\frac{2akt}{m}}-1}{e^{\frac{2akt}{m}}+1}\right)$$

which can be written in the equivalent form

$$v(t)=a\tanh\left(\frac{gt}{a}\right).$$

(b). No. As $t\to\infty, v\to a$ and as $t\to 0^+, v\to 0$.

(c). $v(t) = a\tanh\left(\frac{gt}{a}\right) \Longrightarrow \dfrac{dy}{dt} = a\tanh\left(\frac{gt}{a}\right) \Longrightarrow a\int \tanh\left(\frac{gt}{a}\right)dt \Longrightarrow y(t) = \dfrac{a^2}{g}\ln(\cosh\left(\frac{gt}{a}\right)) + c_1$ and if $y(0) = 0$ then $y(t) = \dfrac{a^2}{g}\ln\left[\cosh\left(\frac{gt}{a}\right)\right]$.

18. The required curve is the solution curve to the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{4y}, y(0) = \frac{1}{2}$. Separating the variables in the differential equation yields $4y^{-1}dy = -1dx$, which can be integrated directly to obtain $2y^2 = -\dfrac{x^2}{2} + c$. Imposing the initial condition we obtain $c = \frac{1}{2}$, so that the solution curve has the equation $2y^2 = -x^2 + \frac{1}{2}$, or equivalently, $4y^2 + 2x^2 = 1$.

19. The required curve is the solution curve to the initial-value problem $\dfrac{dy}{dx} = e^{x-y}, y(3) = 1$. Separating the variables in the differential equation yields $e^y dy = e^x dx$, which can be integrated directly to obtain $e^y = e^x + c$. Imposing the initial condition we obtain $c = e - e^3$, so that the solution curve has the equation $e^y = e^x + e - e^3$, or equivalently, $y = \ln(e^x + e - e^3)$.

20. The required curve is the solution curve to the initial-value problem $\dfrac{dy}{dx} = x^2y^2, y(-1) = 1$. Separating the variables in the differential equation yields $\frac{1}{y^2}dy = x^2dx$, which can be integrated directly to obtain $-\frac{1}{y} = \frac{1}{3}x^3 + c$. Imposing the initial condition we obtain $c = -\frac{2}{3}$, so that the solution curve has the equation $y = -\frac{1}{\frac{1}{3}x^3 - \frac{2}{3}}$, or equivalently, $y = \frac{3}{2-x^3}$.

21. (a). Separating the variables in the given differential equation yields $\dfrac{1}{1+v^2}dv = -dt$. Integrating we obtain $\tan^{-1}(v) = -t + c$. The initial condition $v(0) = v_0$ implies that $c = \tan^{-1}(v_0)$, so that $\tan^{-1}(v) = -t + \tan^{-1}(v_0)$. The object will come to rest if there is time t, at which the velocity is zero. To determine t_r, we set $v = 0$ in the previous equation which yields $\tan^{-1}(0) = t_r + \tan^{-1}(v_0)$. Consequently, $t_r = \tan^{-1}(v_0)$. The object does not remain at rest since we see from the given differential equation that $\dfrac{dv}{dt} < 0$ at $t = t_r$, and so v is decreasing with time. Consequently v passes through zero and becomes negative for $t < t_r$.

(b). From the chain rule we have $\dfrac{dv}{dt} = \dfrac{dx}{dt}$. Then $\dfrac{dv}{dx} = v\dfrac{dv}{dx}$. Substituting this result into the differential equation (1.4.22) yields $v\dfrac{dv}{dx} = -(1+v^2)$. We now separate the variables: $\dfrac{v}{1+v^2}dv = -dx$. Integrating we obtain $\ln(1+v^2) = -2x + c$. Imposing the initial condition $v(0) = v_0, x(0) = 0$ implies that $c = \ln(1+v_0^2)$, so that $\ln(1+v^2) = -2x + \ln(1+v_0^2)$. When the object comes to rest the distance travelled by the object is $x = \frac{1}{2}\ln(1+v_0^2)$.

22. (a). $\dfrac{dv}{dt} = -kv^n \Longrightarrow v^{-n}dv = -kdt$.
$n \neq 1 :\Longrightarrow \dfrac{1}{1-n}v^{1-n} = -kt + c$. Imposing the initial condition $v(0) + v_0$ yields $c = \dfrac{1}{1-n}v_0^{1-n}$, so that $v = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}$. The object comes to rest in a finite time if there is a positive value of t for which $v = 0$.
$n = 1 :\Longrightarrow$ Integrating$v^{-n}dv = -kdt$ and imposing the initial conditions yields $v = v_0e^{-kt}$, and the object does not come to rest in a finite amount of time.

(b). If $n \neq 1, 2$, then $\dfrac{dx}{dt} = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}$, where $x(t)$ denotes the distanced travelled by the

object. Consequently, $x(t) = -\frac{1}{k(2-n)}[v_0^{1-n} + (n-1)kt]^{(2-n)/(1-n)} + c$. Imposing the initial condition $x(0) = 0$ yields $c = \frac{1}{k(2-n)}v_0^{2-n}$, so that $x(t) = -\frac{1}{k(2-n)}[v_o^{1-n} + n(n-1)kt]^{(2-n)/(1-n)} + \frac{1}{k(2-n)}v_0^{2-n}$. For $1 < n < 2$, we have $\frac{2-n}{1-n} < 0$, so that $\lim_{t\to\infty} x(t) = \frac{1}{k(2-n)}$. Hence the maximum distance that the object can travel in a finite time is less than $\frac{1}{k(2-n)}$.
If $n = 1$, then we can integrate to obtain $x(t) = \frac{v_0}{k}(1 - e^{-kt})$, where we have imposed the initial condition $x(0) = 0$. Consequently, $\lim_{t\to\infty} x(t) = \frac{v_0}{k}$. Thus in this case the maximum distance that the object can travel in a finite time is less than $\frac{v_0}{k}$.

(c). If $n > 2$, then $x(t) = -\frac{1}{k(2-n)}[v_o^{1-n} + n(n-1)kt]^{(2-n)/(1-n)} + \frac{1}{k(2-n)}v_0^{2-n}$ is still valid. However, in this case $\frac{2-n}{1-n} > 0$, and so $\lim_{t\to\infty} x(t) = +\infty$. Consequently, there is no limit to the distance that the object can travel.
If $n = 2$, then we return to $v = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}$. In this case $\frac{dx}{dt} = (v_0^{-1} + kt)^{-1}$, which can be integrated directly to obtain $x(t) = \frac{1}{k}\ln(1 + v_0kt)$, where we have imposed the initial condition that $x(0) = 0$. Once more we see that $\lim_{t\to\infty} x(t) = +\infty$, so that there is no limit to the distance that the object can travel.

23. Solving $p = p_0(\frac{\rho}{\rho_0})^{1/\gamma}$. Consequently the given differential equation can be written as $dp = -g\rho_0(\frac{p}{p_0})^{1/\gamma}dy$, or equivalently, $p^{-1/\gamma}dp = -\frac{g\rho_0}{p_0^{1/\gamma}}dy$. This can be integrated directly to obtain $\frac{\gamma p^{(\gamma-1)/\gamma}}{\gamma - 1} = -\frac{g\rho_0 y}{p_0^{1/\gamma}} + c$. At the center of the Earth we have $p = p_0$. Imposing this initial condition on the preceding solution gives $c = \frac{\gamma p_0^{(\gamma-1)/\gamma}}{\gamma - 1}$. Substituting this value of c into the general solution to the differential equation we find, after some simplification, $p^{(\gamma-1)/\gamma} = p_0^{(\gamma-1)/\gamma}\left[1 - \frac{(\gamma-1)\rho_0 gy}{\gamma p_0}\right]$, so that $p = p_0\left[1 - \frac{(\gamma-1)\rho_0 gy}{\gamma p_0}\right]^{(\gamma-1)/\gamma}$.

24. $\frac{dT}{dt} = -k(T - T_m) \Longrightarrow \frac{dT}{dt} = -k(T - 75) \Longrightarrow \frac{dT}{T - 75} = -kdt \Longrightarrow \ln|T - 75| = -kt + c_1 \Longrightarrow T(t) = 75 + ce^{-kt}$. $T(0) = 135 \Longrightarrow c = 60$ so $T = 75 + 60e^{-kt}$. $T(1) = 95 \Longrightarrow 95 = 75 + 60e^{-k} \Longrightarrow k = \ln 3 \Longrightarrow T(t) = 75 + 60e^{-t\ln 3}$. Now if $T(t) = 615$ then $615 = 75 + 60^{-t\ln 3} \Longrightarrow t = -2$h. Thus the object was placed in the room at 2p.m.

25. $\frac{dT}{dt} = -k(T - 450) \Longrightarrow T(t) = 450 + Ce^{-kt}.T(0) = 50 \Longrightarrow C = -400$ so $T(t) = 450 - 400e^{-kt}$ and $T(20) = 150 \Longrightarrow k = \frac{1}{20}\ln\frac{4}{3}$; hence, $T(t) = 450 - 400(\frac{3}{4})^{t/20}$.
(i) $T(40) = 450 - 400(\frac{3}{4})^2 = 225°$F.
(ii) $T(t) = 350 = 450 - 400(\frac{3}{4})^{t/20} \Longrightarrow (\frac{3}{4})^{t/20} = \frac{1}{4} \Longrightarrow t = \frac{20\ln 4}{\ln(4/3)} \approx 96.4$ minutes.

26. $\frac{dT}{dt} = -k(T-34) \Longrightarrow \frac{dT}{T-34} = -kdt \Longrightarrow T(t) = 34 + ce^{-kt}$. $T(0) = 38 \Longrightarrow c = 4$ so that $T(t) = 34 + 4e^{-kt}$.$T(1) = 36 \Longrightarrow k = \ln 2$; hence, $T(t) = 34 + 4e^{-t\ln 2}$. Now $T(t) = 98 \Longrightarrow T(t) = 34 + 4e^{-kt} = 98 \Longrightarrow 2^{-t} = 16 \Longrightarrow t = -4$h. Thus $T(-4) = 98$ and Holmes was right, the time of death was 10 a.m.

27. $T(t) = 75 + ce^{-kt}$. $T(10) = 415 \Longrightarrow 75 + ce^{-10k} = 415 \Longrightarrow 340 = ce^{-10k}$ and $T(20) = 347 \Longrightarrow 75 + ce^{-20k} = 347 \Longrightarrow 272 = ce^{-20k}$. Solving these two equations yields $k = \frac{1}{10}\ln\frac{5}{4}$ and $c = 425$; hence, $T = 75 + 425(\frac{4}{5})^{t/10}$

(a) Furnace temperature: $T(0) = 500$°F.

(b) If $T(t) = 100$ then $100 = 75 + 425(\frac{4}{5})^{t/10} \Longrightarrow t = \frac{10\ln 17}{\ln\frac{5}{4}} \approx 126.96$ minutes. Thus the temperature of the coal was 100°F at 6:07 p.m.

28. $\frac{dT}{dt} = -k(T-72) \Longrightarrow \frac{dT}{T-72} = -kdt \Longrightarrow T(t) = 72 + ce^{-kt}$. Since $\frac{dT}{dt} = -20, -k(T-72) = -20$ or $k = \frac{10}{39}$. Since $T(1) = 150 \Longrightarrow 150 = 72 + ce^{-10/39} \Longrightarrow c = 78e^{10/39}$; consequently, $T(t) = 72 + 78e^{10(1-t)/39}$.

(i). Initial temperature of the object: $t = 0 \Longrightarrow T(t) = 72 + 78e^{10/30} \approx 173$°F

(ii). Rate of change of the temperature after 10 minutes: $T(10) = 72 + 78e^{-30/13}$ so after 10 minutes, $\frac{dT}{dt} = -\frac{10}{39}(72 + 78e^{-30/13} - 72) \Longrightarrow \frac{dT}{dt} = -\frac{260}{13}e^{-30/13} \approx 2$°F per minute.

29. Substituting $a = 0.5$, $M = 2000$ g, and $m_0 = 4$ g into the initial-value problem (1.4.17) yields

$$\frac{dm}{dt} = 0.5m^{3/4}\left[1 - \left(\frac{m}{2000}\right)^{1/4}\right], \quad m(0) = 4.$$

Separating the variables in the preceding differential equation gives

$$\frac{1}{m^{3/4}\left[1 - \left(\frac{m}{2000}\right)^{1/4}\right]}dm = 0.5\,dt$$

so that

$$\int \frac{1}{m^{3/4}\left[1 - \left(\frac{m}{2000}\right)^{1/4}\right]}dm = 0.5t + c.$$

To evaluate the integral on the left-hand-side of the preceding equation, we make the change of variable

$$w = \left(\frac{m}{2000}\right)^{1/4}, \quad dw = \frac{1}{4}\cdot\frac{1}{2000}\left(\frac{m}{2000}\right)^{-3/4}dm$$

and simplify to obtain

$$4\cdot(2000)^{1/4}\int\frac{1}{1-w}dw = 0.5t + c$$

which can be integrated directly to obtain

$$-4\cdot(2000)^{1/4}\ln(1-w) = 0.5t + c.$$

Exponentiating both sides of the preceding equation, and solving for w yields

$$w = 1 - c_1e^{-0.125t/(2000)^{1/4}}$$

or equivalently,

$$\left(\frac{m}{2000}\right)^{1/4} = 1 - c_1 e^{-0.125t/(2000)^{1/4}}.$$

Consequently,

$$m(t) = 2000\left[1 - c_1 e^{-0.125t/(2000)^{1/4}}\right]^4. \tag{0.0.3}$$

Imposing the initial condition $m(0) = 4$ yields

$$4 = 2000\,(1 - c_1)^4$$

so that

$$c_1 = 1 - \left(\frac{1}{500}\right)^{1/4} \approx 0.7885.$$

Inserting this expression for c_1 into Equation (0.0.3) gives

$$m(t) = 2000\left[1 - 0.7885e^{-0.125t/(2000)^{1/4}}\right]^4.$$

Consequently,

$$m(100) = 2000\left[1 - 0.7885e^{-12.5/(2000)^{1/4}}\right]^4 \approx 1190.5 \text{ g}.$$

30. Substituting $a = 0.10$, $M = 0.15$ g, and $m_0 = 0.008$ g into the initial-value problem (1.4.17) yields

$$\frac{dm}{dt} = 0.1m^{3/4}\left[1 - \left(\frac{m}{0.15}\right)^{1/4}\right], \quad m(0) = 0.008.$$

Separating the variables in the preceding differential equation gives

$$\frac{1}{m^{3/4}\left[1 - \left(\frac{m}{0.15}\right)^{1/4}\right]}\,dm = 0.1\,dt$$

so that

$$\int \frac{1}{m^{3/4}\left[1 - \left(\frac{m}{0.15}\right)^{1/4}\right]}\,dm = 0.1t + c.$$

To evaluate the integral on the left-hand-side of the preceding equation, we make the change of variable

$$w = \left(\frac{m}{0.15}\right)^{1/4}, \quad dw = \frac{1}{4}\cdot\frac{1}{0.15}\left(\frac{m}{0.15}\right)^{-3/4} dm$$

and simplify to obtain

$$4\cdot(0.15)^{1/4}\int \frac{1}{1-w}\,dw = 0.1t + c$$

which can be integrated directly to obtain

$$-4\cdot(0.15)^{1/4}\ln(1-w) = 0.1t + c.$$