Solutions to Selected Exercises

Section 1.1

2. $\{2, 4\}$ 3. $\{7, 10\}$ 5. $\{2, 3, 5, 6, 8, 9\}$ 6. $\{1, 3, 5, 7, 9, 10\}$

8. A 9. $\varnothing$ 11. B 12. $\{1, 4\}$ 14. $\{1\}$

15. $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ 18. $\{n \in \mathbf{Z}^+ \mid n \geq 6\}$ 19. $\{2n - 1 \mid n \in \mathbf{Z}^+\}$

21. $\{n \in \mathbf{Z}^+ \mid n \leq 5 \text{ or } n = 2m,\ m \geq 3\}$ 22. $\{2n \mid n \geq 3\}$ 24. $\{1, 3, 5\}$

25. $\{n \in \mathbf{Z}^+ \mid n \leq 5 \text{ or } n = 2m + 1,\ m \geq 3\}$ 27. $\{n \in \mathbf{Z}^+ \mid n \geq 6 \text{ or } n = 2 \text{ or } n = 4\}$

29. 1 30. 3

33. We find that $B = \{2, 3\}$. Since A and B have the same elements, they are equal.

34. Let $x \in A$. Then $x = 1, 2, 3$. If $x = 1$, since $1 \in \mathbf{Z}^+$ and $1^2 < 10$, then $x \in B$. If $x = 2$, since $2 \in \mathbf{Z}^+$ and $2^2 < 10$, then $x \in B$. If $x = 3$, since $3 \in \mathbf{Z}^+$ and $3^2 < 10$, then $x \in B$. Thus if $x \in A$, then $x \in B$.

    Now suppose that $x \in B$. Then $x \in \mathbf{Z}^+$ and $x^2 < 10$. If $x \geq 4$, then $x^2 > 10$ and, for these values of x, $x \notin B$. Therefore $x = 1, 2, 3$. For each of these values, $x^2 < 10$ and x is indeed in B. Also, for each of the values $x = 1, 2, 3$, $x \in A$. Thus if $x \in B$, then $x \in A$. Therefore $A = B$.

37. Since $(-1)^3 - 2(-1)^2 - (-1) + 2 = 0$, $-1 \in B$. Since $-1 \notin A$, $A \neq B$.

38. Since $3^2 - 1 > 3$, $3 \notin B$. Since $3 \in A$, $A \neq B$. 41. Equal 42. Not equal

45. Let $x \in A$. Then $x = 1, 2$. If $x = 1$,

$$x^3 - 6x^2 + 11x = 1^3 - 6 \cdot 1^2 + 11 \cdot 1 = 6.$$

    Thus $x \in B$. If $x = 2$,

$$x^3 - 6x^2 + 11x = 2^3 - 6 \cdot 2^2 + 11 \cdot 2 = 6.$$

    Again $x \in B$. Therefore $A \subseteq B$.

46. Let $x \in A$. Then $x = (1, 1)$ or $x = (1, 2)$. In either case, $x \in B$. Therefore $A \subseteq B$.

49. Since $(-1)^3 - 2(-1)^2 - (-1) + 2 = 0$, $-1 \in A$. However, $-1 \notin B$. Therefore A is not a subset of B.

50. Consider 4, which is in A. If $4 \in B$, then $4 \in A$ and $4 + m = 8$ for some $m \in C$. However, the only value of m for which $4 + m = 8$ is $m = 4$ and $4 \notin C$. Therefore $4 \notin B$. Since $4 \in A$ and $4 \notin B$, A is not a subset of B.

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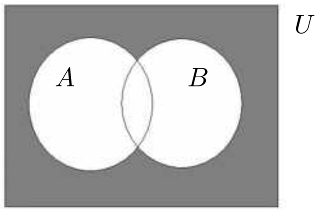
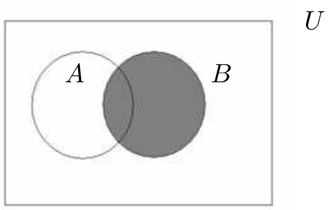
53.



54.



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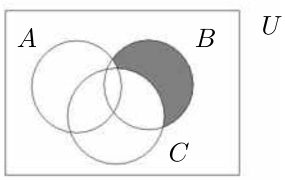

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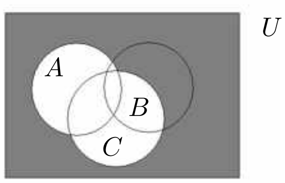
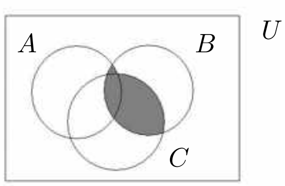


59.



62. 32

63. 105

65. 51

67. Suppose that n students are taking both a mathematics course and a computer science course. Then $4n$ students are taking a mathematics course, but not a computer science course, and $7n$ students are taking a computer science course, but not a mathematics course. The following Venn diagram depicts the situation:

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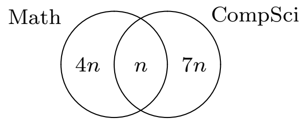

Thus, the total number of students is

$$4n + n + 7n = 12n.$$

The proportion taking a mathematics course is

$$\frac{5n}{12n} = \frac{5}{12},$$

which is greater than one-third.

69. $\{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$

70. $\{(1,1),(1,2),(2,1),(2,2)\}$ 73. $\{(1,a,a),(2,a,a)\}$

74. $\{(1,1,1),(1,2,1),(2,1,1),(2,2,1),(1,1,2),(1,2,2),(2,1,2),(2,2,2)\}$

77. Vertical lines (parallel) spaced one unit apart extending infinitely to the left and right.

79. Consider all points on a horizontal line one unit apart. Now copy these points by moving the horizontal line n units straight up and straight down for all integers $n > 0$. The set of all points obtained in this way is the set $\mathbf{Z} \times \mathbf{Z}$.

80. Ordinary 3-space

82. Take the lines described in the instructions for this set of exercises and copy them by moving n units out and back for all $n > 0$. The set of all points obtained in this way is the set $\mathbf{R} \times \mathbf{Z} \times \mathbf{Z}$.

84. $\{1,2\}$
$\{1\},\{2\}$

85. $\{a,b,c\}$
$\{a,b\},\{c\}$
$\{a,c\},\{b\}$
$\{b,c\},\{a\}$
$\{a\},\{b\},\{c\}$

88. False 89. True 91. False 92. True

94. $\varnothing,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},$
$\{a,c,d\},\{b,c,d\},\{a,b,c,d\}$. All except $\{a,b,c,d\}$ are proper subsets.

95. $2^{10} = 1024$; $2^{10} - 1 = 1023$ 98. $B \subseteq A$ 99. $A = U$

102. The symmetric difference of two sets consists of the elements in one or the other but not both.

103. $A \triangle A = \varnothing$, $A \triangle \overline{A} = U$, $U \triangle A = \overline{A}$, $\varnothing \triangle A = A$

105. The set of primes

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Section 1.2

2. Is a proposition. Negation: $6 + 9 \neq 15$.

3. Not a proposition

4. Is a proposition. Negation: $\pi \neq 3.14$.

6. Is a proposition. Negation: For every positive integer n, $19340 \neq n \cdot 17$.

7. Is a proposition. Negation: Audrey Meadows was not the original "Alice" in the "Honeymooners."

9. Is a proposition. Negation: The line "Play it again, Sam" does not occur in the movie *Casablanca*.

10. Is a proposition. Some even integer greater than 4 is not the sum of two primes.

12. Not a proposition. The statement is neither true nor false.

13. No heads were obtained.

15. No heads or no tails were obtained.

18. True

19. True

21. False

22. False

24.

p	q	$(\neg p \vee \neg q) \vee p$
T	T	T
T	F	T
F	T	T
F	F	T

25.

p	q	$(p \vee q) \wedge \neg p$
T	T	F
T	F	F
F	T	T
F	F	F

27.

p	q	$(p \wedge q) \vee (\neg p \vee q)$
T	T	T
T	F	F
F	T	T
F	F	T

28.

p	q	r	$\neg(p \wedge q) \vee (r \wedge \neg p)$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

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30.

p	q	r	$\neg(p \wedge q) \vee (\neg q \vee r)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

32. $\neg(p \wedge q)$. True.

33. $p \vee \neg(q \wedge r)$. True.

35. Lee takes computer science and mathematics.

36. Lee takes computer science or mathematics.

38. Lee takes computer science but not mathematics.

39. Lee takes neither computer science nor mathematics.

41. You do not miss the midterm exam and you pass the course.

42. You play football or you miss the midterm exam or you pass the course.

44. Either you play football and you miss the midterm exam or you do not miss the midterm exam and you pass the course.

46. It is not Monday and either it is raining or it is hot.

47. It is not the case that today is Monday or it is raining, and it is hot.

50. Today is Monday and either it is raining or it is hot, and it is hot or either it is raining or today is Monday.

51. $p \wedge q$

52. $p \wedge \neg q$

54. $p \vee q$

55. $(p \vee q) \wedge \neg p$

57. $p \wedge r \wedge q$

58. $(p \vee r) \wedge q$

60. $(q \vee \neg p) \wedge \neg r$

62. $p \wedge \neg r$

63. $p \wedge q \wedge r$

65. $\neg p \wedge \neg q \wedge r$

66. $\neg(p \vee q \vee \neg r)$

67.

p	q	p *exor* q
T	T	F
T	F	T
F	T	T
F	F	F

69. Inclusive-or

70. Inclusive-or

72. Exclusive-or

73. Exclusive-or

77. "lung disease" -cancer

78. "minor league" baseball team illinois -"midwest league"

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Section 1.3

2. If Rosa has 160 quarter-hours of credits, then she may graduate.
3. If Fernando buys a computer, then he obtains $2000.
5. If a person gets that job, then that person knows someone who knows the boss.
6. If you go to the Super Bowl, then you can afford the ticket.
8. If a better car is built, then Buick will build it.
9. If the chairperson gives the lecture, then the audience will go to sleep.
11. If the switch is not turned properly, then the light will not be on.
13. Contrapositive of Exercise 2: If Rosa does not graduate, then she does not have 160 quarter-hours of credits.

15. False 16. False 18. False 19. True 21. True 22. True

24. Unknown 25. Unknown 27. True 28. Unknown 30. Unknown

31. Unknown 34. True 35. True 37. True 38. False

40. True 41. False 44. $(p \wedge r) \to q$ 45. $\neg((r \wedge \neg q) \to r)$

48. $(\neg p \vee \neg r) \to \neg q$ 49. $r \to q$ 51. $q \to (p \vee r)$ 52. $(q \wedge p) \to \neg r$

54. If it is not raining, then it is hot and today is Monday.
55. If today is not Monday, then either it is raining or it is hot.
57. If today is Monday and either it is raining or it is hot, then either it is hot, it is raining, or today is Monday.
58. If today is Monday or (it is not Monday and it is not the case that (it is raining or it is hot)), then either today is Monday or it is not the case that (it is hot or it is raining).
60. Let p: $4 > 6$ and q: $9 > 12$. Given statement: $p \to q$; true. Converse: $q \to p$; if $9 > 12$, then $4 > 6$; true. Contrapositive: $\neg q \to \neg p$; if $9 \leq 12$, then $4 \leq 6$; true.
61. Let p: $|1| < 3$ and q: $-3 < 1 < 3$. Given statement: $q \to p$; true. Converse: $p \to q$; if $|1| < 3$, then $-3 < 1 < 3$; true. Contrapositive: $\neg p \to \neg q$; if $|1| \geq 3$, then either $-3 \geq 1$ or $1 \geq 3$; true.

64. $P \not\equiv Q$ 65. $P \equiv Q$ 67. $P \not\equiv Q$ 68. $P \equiv Q$ 70. $P \not\equiv Q$

71. $P \not\equiv Q$ 74. Either Dale is not smart or not funny.

75. Shirley will not take the bus and not catch a ride to school.
78. (a) If p and q are both false, $(p\ \mathit{imp2}\ q) \wedge (q\ \mathit{imp2}\ p)$ is false, but $p \leftrightarrow q$ is true.
    (b) Making the suggested change does not alter the last line of the *imp2* table.
79.

p	q	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

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Section 1.4

2. Invalid

$$\begin{array}{l} p \to q \\ \underline{\neg r \to \neg q} \\ \therefore r \end{array}$$

3. Valid

$$\begin{array}{l} p \leftrightarrow r \\ \underline{r} \\ \therefore p \end{array}$$

5. Valid

$$\begin{array}{l} p \to (q \vee r) \\ \underline{\neg q \wedge \neg r} \\ \therefore \neg p \end{array}$$

7. Valid

$$\begin{array}{l} (p \vee q) \to (r \vee s) \\ \underline{p \wedge \neg r} \\ \therefore s \end{array}$$

8. Invalid

$$\begin{array}{l} p \to r \\ q \to s \\ \neg(q \wedge p) \\ \underline{\neg p} \\ \therefore s \end{array}$$

11. If 4 megabytes of memory is better than no memory at all, then either we will buy a new computer or we will buy more memory. If we will buy a new computer, then we will not buy more memory. Therefore if 4 megabytes of memory is better than no memory at all, then we will buy a new computer. Invalid.

12. If 4 megabytes of memory is better than no memory at all, then we will buy a new computer. If we will buy a new computer, then we will buy more memory. Therefore, we will buy more memory. Invalid.

14. If 4 megabytes of memory is better than no memory at all, then we will buy a new computer. If we will buy a new computer, then we will buy more memory. 4 megabytes of memory is better than no memory at all. Therefore we will buy more memory. Valid.

16. If the hardware is unreliable or the output is correct, then the while loop is not faulty. If the output is correct, then the while loop is faulty. Either the for loop is faulty or the output is correct. Therefore the hardware is unreliable. Invalid.

17. If, if the for loop is faulty, then the hardware is unreliable, then the while loop is faulty. If, if the while loop is faulty, then the output is correct, then the for loop is faulty. The hardware is unreliable and the output is correct. Either the for loop is faulty or the while loop is faulty. Therefore the for loop is faulty and the while loop is faulty. Invalid.

19. If the for loop is faulty, then the while loop is faulty or the hardware is unreliable. If the while loop is faulty, then the for loop is faulty or the output is correct. Either the for loop is faulty or the while loop is not faulty. The output is not correct. Therefore the for loop is faulty or the hardware is unreliable. Invalid.

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21. Valid 22. Valid 24. Valid

25. Suppose that $p_1, p_2, \ldots, p_n$ are all true. Since the argument $p_1, p_2 \ / \therefore\ p$ is valid, p is true. Since $p, p_3, \ldots, p_n$ are all true and the argument

$$p, p_3, \ldots, p_n \ / \therefore\ c$$

is valid, c is true. Therefore the argument

$$p_1, p_2, \ldots, p_n \ / \therefore\ c$$

is valid.

28. Modus ponens 29. Disjunctive syllogism

31. Let p denote the proposition "there is gas in the car," let q denote the proposition "I go to the store," let r denote the proposition "I get a soda," and let s denote the proposition "the car transmission is defective." Then the hypotheses are:

$$p \to q, \quad q \to r, \quad \neg r.$$

From $p \to q$ and $q \to r$, we may use the hypothetical syllogism to conclude $p \to r$. From $p \to r$ and $\neg r$, we may use modus tollens to conclude $\neg p$. From $\neg p$, we may use addition to conclude $\neg p \vee s$. Since $\neg p \vee s$ represents the proposition "there is not gas in the car or the car transmission is defective," we conclude that the conclusion does follow from the hypotheses.

32. Let p denote the proposition "Jill can sing," let q denote the proposition "Dweezle can play," let r denote the proposition "I'll buy the compact disk," and let s denote the proposition "I'll buy the compact disk player." Then the hypotheses are:

$$(p \vee q) \to r, \quad p, \quad s.$$

From p, we may use addition to conclude $p \vee q$. From $p \vee q$ and $(p \vee q) \to r$, we may use modus ponens to conclude r. From r and s, we may use conjunction to conclude $r \wedge s$. Since $r \wedge s$ represents the proposition "I'll buy the compact disk and the compact disk player," we conclude that the conclusion does follow from the hypotheses.

34. The truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

shows that whenever p is true, $p \vee q$ is also true. Therefore addition is a valid argument.

35. The truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

shows that whenever $p \wedge q$ is true, p is also true. Therefore simplification is a valid argument.

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37. The truth table

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

shows that whenever $p \to q$ and $q \to r$ are true, $p \to r$ is also true. Therefore hypothetical syllogism is a valid argument.

38. The truth table

p	q	$p \vee q$	$\neg p$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	T

shows that whenever $p \vee q$ and $\neg p$ are true, q is also true. Therefore disjunctive syllogism is a valid argument.

Section 1.5

2. The statement is a command, not a propositional function.

3. The statement is a command, not a propositional function.

5. The statement is not a propositional function since it has no variables.

6. The statement is a propositional function. The domain of discourse is the set of real numbers.

8. 1 divides 77. True.

9. 3 divides 77. False.

11. For some n, n divides 77. True.

12. For every n, n does not divide 77. False.

14. It is false that for every n, n divides 77. True.

15. It is false that for some n, n divides 77. False.

17. False

18. True

20. True

21. True

23. False

24. True

26. $\neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$

27. $\neg(P(1) \wedge P(2) \wedge P(3) \wedge P(4))$

29. $\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$

30. $\neg(P(1) \vee P(2) \vee P(3) \vee P(4))$

33. Some student is taking a math course.

34. Every student is not taking a math course.

36. It is not the case that every student is taking a math course.

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37. It is not the case that some student is taking a math course.

40. There is some person such that if the person is a professional athlete, then the person plays soccer. True.

41. Every soccer player is a professional athlete. False.

43. Every person is either a professional athlete or a soccer player. False.

44. Someone is either a professional athlete or a soccer player. True.

46. Someone is a professional athlete and a soccer player. True.

49. $\exists x(P(x) \wedge Q(x))$

50. $\forall x(Q(x) \rightarrow P(x))$

54. True 55. True 57. False 58. True

60. No. The suggested replacement returns false if $\neg P(d_1)$ is true, and true if $\neg P(d_1)$ is false.

62. Literal meaning: Every old thing does not covet a twenty-something. Intended meaning: Some old thing does not covet a twenty-something. Let $P(x)$ denote the statement "x is an old thing" and $Q(x)$ denote the statement "x covets a twenty-something." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

63. Literal meaning: Every hospital did not report every month. (Domain of discourse: the 74 hospitals.) Intended meaning (most likely): Some hospital did not report every month. Let $P(x)$ denote the statement "x is a hospital" and $Q(x)$ denote the statement "x reports every month." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

65. Literal meaning: Everyone does not have a degree. (Domain of discourse: People in Door County.) Intended meaning: Someone does not have a degree. Let $P(x)$ denote the statement "x has a degree." The intended statement is $\exists x \neg P(x)$.

66. Literal meaning: No lampshade can be cleaned. Intended meaning: Some lampshade cannot be cleaned. Let $P(x)$ denote the statement "x is a lampshade" and $Q(x)$ denote the statement "x can be cleaned." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

68. Literal meaning: No person can afford a home. Intended meaning: Some person cannot afford a home. Let $P(x)$ denote the statement "x is a person" and $Q(x)$ denote the statement "x can afford a home." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

69. The literal meaning is as Mr. Bush spoke. He probably meant: Someone in this country doesn't agree with the decisions I've made. Let $P(x)$ denote the statement "x agrees with the decisions I've made." Symbolically, the clarified statement is $\exists x \, \neg P(x)$.

71. Literal meaning: Every move does not work out. Intended meaning: Some move does not work out. Let $P(x)$ denote the statement "x is a move" and $Q(x)$ denote the statement "x works out ." The intended statement is $\exists x(P(x) \wedge \neg Q(x))$.

74. Let

$$\begin{aligned} p(x)&: \quad x \text{ is good.} \\ q(x)&: \quad x \text{ is too long.} \\ r(x)&: \quad x \text{ is short enough.} \end{aligned}$$

The domain of discourse is the set of movies. The assertions are

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$$\forall x(p(x) \to \neg q(x))$$
$$\forall x(\neg p(x) \to \neg r(x))$$
$$p(\textit{Love Actually})$$
$$q(\textit{Love Actually}).$$

By universal instantiation,

$$p(\textit{Love Actually}) \to \neg q(\textit{Love Actually}).$$

Since $p(\textit{Love Actually})$ is true, then $\neg q(\textit{Love Actually})$ is also true. But this contradicts, $q(\textit{Love Actually})$.

77. Let $P(x)$ denote the propositional function "x is a member of the Titans," let $Q(x)$ denote the propositional function "x can hit the ball a long way," and let $R(x)$ denote the propositional function "x can make a lot of money." The hypotheses are

$$P(\text{Ken}),\ \ Q(\text{Ken}),\ \ \forall x\ Q(x) \to R(x).$$

By universal instantiation, we have $Q(\text{Ken}) \to R(\text{Ken})$. From $Q(\text{Ken})$ and $Q(\text{Ken}) \to R(\text{Ken})$, we may use modus ponens to conclude $R(\text{Ken})$. From $P(\text{Ken})$ and $R(\text{Ken})$, we may use conjunction to conclude $P(\text{Ken}) \wedge R(\text{Ken})$. By existential generalization, we have $\exists x\ P(x) \wedge R(x)$ or, in words, someone is a member of the Titans and can make a lot of money. We conclude that the conclusion does follow from the hypotheses.

78. Let $P(x)$ denote the propositional function "x is in the discrete mathematics class," let $Q(x)$ denote the propositional function "x loves proofs," and let $R(x)$ denote the propositional function "x has taken calculus." The hypotheses are

$$\forall x\ P(x) \to Q(x),\ \ \exists x\ P(x) \wedge \neg R(x).$$

By existential instantiation, we have $P(d) \wedge \neg R(d)$ for some d in the domain of discourse. From $P(d) \wedge \neg R(d)$, we may use simplification to conclude $P(d)$ and $\neg R(d)$. By universal instantiation, we have $P(d) \to Q(d)$. From $P(d) \to Q(d)$ and $P(d)$, we may use modus ponens to conclude $Q(d)$. From $Q(d)$ and $\neg R(d)$, we may use conjunction to conclude $Q(d) \wedge \neg R(d)$. By existential generalization, we have $\exists\ Q(x) \wedge \neg R(x)$ or, in words, someone who loves proofs has never taken calculus. We conclude that the conclusion does follow from the hypotheses.

80. By definition, the proposition $\exists x \in D\ P(x)$ is true when $P(x)$ is true for some x in the domain of discourse. Taking x equal to a $d \in D$ for which $P(d)$ is true, we find that $P(d)$ is true for some $d \in D$.

81. By definition, the proposition $\exists x \in D\ P(x)$ is true when $P(x)$ is true for some x in the domain of discourse. Since $P(d)$ is true for some $d \in D$, $\exists x \in D\ P(x)$ is true.

Section 1.6

2. Everyone is taller than someone.

3. Someone is taller than everyone.

The solutions for 7–20 are for Exercise 2.

7. False 8. False 10. False 11. False 13. False 14. False

16. False 17. True 19. True 20. False

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23. Everyone is taller than or the same height as someone.

24. Someone is taller than or the same height as everyone.

29. For every person, there is a person such that if the persons are distinct, the first is taller than the second.

30. There is a person such that, for every person, if the persons are distinct, the first is taller than the second.

35. $\forall x \forall y L(x, y)$. False.

36. $\exists x \exists y L(x, y)$. True.

40. $\forall x \neg A(x, \text{Profesor Sandwich})$

41. $\forall x \exists y E(x) \rightarrow A(x, y)$

44. True

45. False

49. True

50. False

52. False

53. False

55. False

56. False

58. True

59. True

61. True

62. False

64. True

65. True

67.
```
for i = 1 to n
    if (forall_dj(i))
        return true
return false

forall_dj(i) {
    for j = 1 to n
        if (¬P(d_i, d_j))
            return false
    return true
}
```

68.
```
for i = 1 to n
    for j = 1 to n
        if (P(d_i, d_j))
            return true
return false
```

70. Since the first two quantifiers are universal and the last quantifier is existential, Farley chooses x and y, after which, you choose z. If Farley chooses values that make $x \geq y$, say $x = y = 0$, whatever value you choose for z,

$$(z > x) \wedge (z < y)$$

is false. Since Farley can always win the game, the quantified propositional function is false.

71. Since the first two quantifiers are universal and the last quantifier is existential, Farley chooses x and y, after which, you choose z. Whatever values Farley chooses, you can choose z to be one less than the minimum of x and y; thus making

$$(z < x) \wedge (z < y)$$

true. Since you can always win the game, the quantified propositional function is true.

73. Since the first two quantifiers are universal and the last quantifier is existential, Farley chooses x and y, after which, you choose z. If Farley chooses values such that $x \geq y$, the proposition

$$(x < y) \rightarrow ((z > x) \wedge (z < y))$$

is true by default (i.e., it is true regardless of what value you choose for z). If Farley chooses values such that $x < y$, you can choose $z = (x + y)/2$ and again the proposition

$$(x < y) \rightarrow ((z > x) \wedge (z < y))$$

is true. Since you can always win the game, the quantified propositional function is true.

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75. The proposition must be true. $P(x,y)$ is true for all x and y; therefore, no matter which value for x we choose, the proposition $\forall y P(x,y)$ is true.

76. The proposition must be true. Since $P(x,y)$ is true for all x and y, we may choose *any* values for x and y to make $P(x,y)$ true.

78. The proposition can be false. Let N denote the set of persons James James, Terry James, and Lee James; let the domain of discourse be $N \times N$; and let $P(x,y)$ be the statement "x's first name is the same as y's last name." Then $\exists x \forall y P(x,y)$ is true, but $\forall x \exists y P(x,y)$ is false.

79. The proposition must be true. Since $\exists x \forall y P(x,y)$ is true, there is some value for x for which $\forall y P(x,y)$ is true. Choosing any value for y whatsoever makes $P(x,y)$ true. Therefore $\exists x \exists y P(x,y)$ is true.

81. The proposition can be false. Let $P(x,y)$ be the statement $x > y$ and let the domain of discourse be $\mathbf{Z}^+ \times \mathbf{Z}^+$. Then $\exists x \exists y P(x,y)$ is true, but $\forall x \exists y P(x,y)$ is false.

82. The proposition can be false. Let $P(x,y)$ be the statement $x > y$ and let the domain of discourse be $\mathbf{Z}^+ \times \mathbf{Z}^+$. Then $\exists x \exists y P(x,y)$ is true, but $\exists x \forall y P(x,y)$ is false.

84. The proposition can be true. Let $P(x,y)$ be the statement $x \leq y$ and let the domain of discourse be $\mathbf{Z}^+ \times \mathbf{Z}^+$. Then $\forall x \forall y P(x,y)$ is false, but $\exists x \forall y P(x,y)$ is true.

85. The proposition can be true. Let $P(x,y)$ be the statement $x \leq y$ and let the domain of discourse be $\mathbf{Z}^+ \times \mathbf{Z}^+$. Then $\forall x \forall y P(x,y)$ is false, but $\exists x \exists y P(x,y)$ is true.

87. The proposition can be true. Let N denote the set of persons James James, Terry James, and Lee James; let the domain of discourse be $N \times N$; and let $P(x,y)$ be the statement "x's first name is different from y's last name." Then $\forall x \exists y P(x,y)$ is false, but $\exists x \forall y P(x,y)$ is true.

88. The proposition can be true. Let $P(x,y)$ be the statement $x > y$ and let the domain of discourse be $\mathbf{Z}^+ \times \mathbf{Z}^+$. Then $\forall x \exists y P(x,y)$ is false, but $\exists x \exists y P(x,y)$ is true.

90. The proposition can be true. Let $P(x,y)$ be the statement $x < y$ and let the domain of discourse be $\mathbf{Z}^+ \times \mathbf{Z}^+$. Then $\exists x \forall y P(x,y)$ is false, but $\forall x \exists y P(x,y)$ is true.

91. The proposition can be true. Let $P(x,y)$ be the statement $x \leq y$ and let the domain of discourse be $\mathbf{Z} \times \mathbf{Z}$. Then $\exists x \forall y P(x,y)$ is false, but $\exists x \exists y P(x,y)$ is true.

93. $\forall x\, \exists y\, P(x,y)$ must be false. Since $\exists x\, \exists y\, P(x,y)$ is false, for every x and for every y, $P(x,y)$ is false. Choose $x = x'$ in the domain of discourse. For this choice of x, $P(x,y)$ is false for every y. Therefore $\forall x\, \exists y\, P(x,y)$ is false.

94. $\exists x\, \forall y\, P(x,y)$ must be false. Since $\exists x\, \exists y\, P(x,y)$ is false, for every x and for every y, $P(x,y)$ is false. Choose $y = y'$ in the domain of discourse. Now, for any choice of x, $P(x,y)$ is false for $y = y'$. Therefore $\exists x\, \forall y\, P(x,y)$ is false.

96. Not equivalent. Let $P(x,y)$ be the statement $x > y$ and let the domain of discourse be $\mathbf{Z}^+ \times \mathbf{Z}^+$. Then $\neg(\forall x \exists y P(x,y))$ is true, but $\forall x \neg(\exists y P(x,y))$ is false.

97. Equivalent by De Morgan's law

100. $\exists \varepsilon > 0\, \forall \delta > 0\, \exists x\, ((0 < |x-a| < \delta) \wedge (|f(x) - L| \geq \varepsilon))$

101. $\forall L\, \exists \varepsilon > 0\, \forall \delta > 0\, \exists x\, ((0 < |x-a| < \delta) \wedge (|f(x) - L| \geq \varepsilon))$

102. Literal meaning: No school may be right for every child. Intended meaning: Some school may not be right for some child. Let $P(x,y)$ denote the statement "school x is right for child y." The intended statement is $\exists x \exists y \neg P(x,y)$.

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Problem-Solving Corner: Quantifiers

1. The statement of Example 1.6.6 is
$$\forall x \exists y (x + y = 0).$$
As was pointed out in Example 1.6.6, this statement is true. Now
$$\forall x \forall y (x + y = 0)$$
is false; a counterexample is $x = y = 1$. Also
$$\exists x \forall y (x + y = 0)$$
is false since, given any x, if $y = 1 - x$, then $x + y \neq 0$.

2. Yes; the statement $\forall m \exists n (m < n)$ with domain of discourse $\mathbf{Z} \times \mathbf{Z}$ of Example 1.6.1 also solves problems (a) and (b).

Section 2.1

2. For all x, for all y, $x + y = y + x$.

3. An *isosceles trapezoid* is a trapezoid with equal legs.

5. The medians of any triangle intersect at a single point.

6. If $0 < x < 1$ and $\varepsilon > 0$, there exists a positive integer n satisfying $x^n < \varepsilon$.

8. Let m and n be odd integers. Then there exist k_1 and k_2 such that $m = 2k_1 + 1$ and $n = 2k_2 + 1$. Now
$$m + n = (2k_1 + 1) + (2k_2 + 1) = 2(k_1 + k_2 + 1).$$
Therefore, $m + n$ is even.

9. Let m and n be even integers. Then there exist k_1 and k_2 such that $m = 2k_1$ and $n = 2k_2$. Now
$$mn = (2k_1)(2k_2) = 2(2k_1k_2).$$
Therefore, mn is even.

11. Let m be an odd integer and n be an even integer. Then there exist k_1 and k_2 such that $m = 2k_1 + 1$ and $n = 2k_2$. Now
$$mn = (2k_1 + 1)(2k_2) = 2(2k_1k_2 + k_2).$$
Therefore, mn is even.

12. Let m and n be integers such that m and $m + n$ are even. Then there exist k_1 and k_2 such that $m = 2k_1$ and $m + n = 2k_2$. Now
$$n = (m + n) - m = 2k_2 - 2k_1 = 2(k_2 - k_1).$$
Therefore, n is even.

14. Let x and y be rational numbers. Then there exist integers m_1, n_1, m_2, n_2 such that $x = m_1/n_1$ and $y = m_2/n_2$. Now $xy = (m_1m_2)/(n_1n_2)$. Therefore xy is rational.

15. Let x be a nonzero rational number. Then there exist integers $m \neq 0$ and $n \neq 0$ such that $x = m/n$. Now $1/x = n/m$. Therefore $1/x$ is rational.

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17. Let $m = 3k_1 + 2$ and $n = 3k_2 + 2$ be integers of the prescribed from. Then

$$mn = 9k_1k_2 + 6k_1 + 6k_2 + 4 = 3(3k_1k_2 + 2k_1 + 2k_2 + 1) + 1$$

is of the form $3k_3 + 1$, where $k_3 = 3k_1k_2 + 2k_1 + 2k_2 + 1$.

19. $$\begin{aligned} x \cdot 0 + 0 &= x \cdot 0 && \text{because } b + 0 = b \text{ for all real numbers } b \\ &= x \cdot (0 + 0) && \text{because } b + 0 = b \text{ for all real numbers } b \\ &= x \cdot 0 + x \cdot 0 && \text{because } a(b + c) = ab + ac \text{ for all real numbers } a, b, c \end{aligned}$$

Taking $a = c = x \cdot 0$ and $b = 0$, the preceding equation becomes $a + b = a + c$; therefore, $0 = b = c = x \cdot 0$.

20. We must have $X = Y$. To prove this, suppose that $x \in X$. Since Y is nonempty, choose $y \in Y$. Then $(x, y) \in X \times Y$. Since $X \times Y = Y \times X$, $(x, y) \in Y \times X$. Therefore $x \in Y$. Similarly, if $x \in Y$, then $x \in X$. Thus $X = Y$.

22. Let $x \in X$. Then $x \in X \cup Y$. Therefore $X \subseteq X \cup Y$.

23. Let $x \in X \cup Z$. Then $x \in X$ or $x \in Z$. If $x \in X$, since $X \subseteq Y$, $x \in Y$. Therefore $x \in Y \cup Z$. If $x \in Z$, then $x \in Y \cup Z$. In either case, $x \in Y \cup Z$. Therefore $X \cup Z \subseteq Y \cup Z$.

25. Let $x \in Z - Y$. Then $x \in Z$ and $x \notin Y$. Now x cannot be in X, for if $x \in X$, since $X \subseteq Y$, then $x \in Y$, which is not the case. Since $x \in Z$ and $x \notin X$, $x \in Z - X$. Therefore $Z - Y \subseteq Z - X$.

26. Let $x \in Y - (Y - X)$. Then $x \in Y$ and $x \notin Y - X$. Since $x \in Y$, we must have $x \in X$ (if $x \notin X$, we would have $x \in Y - X$). Therefore $Y - (Y - X) \subseteq X$.

Now let $x \in X$. Then $x \notin Y - X$. Since $X \subseteq Y$, $x \in Y$. Thus $x \in Y - (Y - X)$. Therefore $X \subseteq Y - (Y - X)$. We have shown that $Y - (Y - X) = X$.

28. Let $Z \in \mathcal{P}(X) \cup \mathcal{P}(Y)$. Then $Z \in \mathcal{P}(X)$ or $Z \in \mathcal{P}(Y)$. If $Z \in \mathcal{P}(X)$, then Z is a subset of X and, thus, Z is also a subset of $X \cup Y$. Therefore $Z \in \mathcal{P}(X \cup Y)$. Similarly, if $Z \in \mathcal{P}(Y)$, $Z \in \mathcal{P}(X \cup Y)$. In either case, $Z \in \mathcal{P}(X \cup Y)$. Therefore $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$.

29. Let $Z \in \mathcal{P}(X \cap Y)$. Then Z is a subset of $X \cap Y$. Therefore Z is a subset of X and a subset of Y. Thus $Z \in \mathcal{P}(X) \cap \mathcal{P}(Y)$. We have proved that $\mathcal{P}(X \cap Y) \subseteq \mathcal{P}(X) \cap \mathcal{P}(Y)$.

Let $Z \in \mathcal{P}(X) \cap \mathcal{P}(Y)$. Then $Z \in \mathcal{P}(X)$ and $Z \in \mathcal{P}(Y)$. Since $Z \in \mathcal{P}(X)$, Z is a subset of X. Since $Z \in \mathcal{P}(Y)$, Z is a subset of Y. Since Z is a subset of X and Y, Z is a subset of $X \cap Y$. Thus $Z \in \mathcal{P}(X \cap Y)$. Therefore $\mathcal{P}(X) \cap \mathcal{P}(Y) \subseteq \mathcal{P}(X \cap Y)$. It follows that $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$.

31. Let $X = \{a\}$ and $Y = \{b\}$. Then

$$\mathcal{P}(X) = \{\varnothing, \{a\}\}, \qquad \mathcal{P}(Y) = \{\varnothing, \{b\}\},$$

so

$$\mathcal{P}(X) \cup \mathcal{P}(Y) = \{\varnothing, \{a\}, \{b\}\}.$$

Since $X \cup Y = \{a, b\}$,

$$\mathcal{P}(X \cup Y) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$

Now $\{a, b\} \in \mathcal{P}(X \cup Y)$, but $\{a, b\} \notin \mathcal{P}(X) \cup \mathcal{P}(Y)$. Therefore $\mathcal{P}(X \cup Y) \subseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$ is false in general.

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32.
$$\begin{aligned}
(X \cap Y) - (X \cap Z) &= (X \cap Y) \cap \overline{(X \cap Z)} && [A - B = A \cap \overline{B}]\\
&= (X \cap Y) \cap (\overline{X} \cup \overline{Z}) && \text{[De Morgan's law; Theorem 1.1.21, part (k)]}\\
&= ((X \cap Y) \cap \overline{X}) \cup ((X \cap Y) \cap \overline{Z}) && \text{[Distributive law; Theorem 1.1.21, part (c)]}\\
&= ((Y \cap X) \cap \overline{X}) \cup ((X \cap Y) \cap \overline{Z}) && \text{[Commutative law; Theorem 1.1.21, part (b)]}\\
&= (Y \cap (X \cap \overline{X})) \cup (X \cap (Y \cap \overline{Z})) && \text{[Associative law; Theorem 1.1.21, part (a)]}\\
&= (Y \cap \varnothing) \cup (X \cap (Y \cap \overline{Z})) && \text{[Complement law; Theorem 1.1.21, part (e)]}\\
&= \varnothing \cup (X \cap (Y \cap \overline{Z})) && \text{[Bound law; Theorem 1.1.21, part (g)]}\\
&= (X \cap (Y \cap \overline{Z})) \cup \varnothing && \text{[Commutative law; Theorem 1.1.21, part (b)]}\\
&= X \cap (Y \cap \overline{Z}) && \text{[Identity law; Theorem 1.1.21, part (d)]}\\
&= X \cap (Y - Z) && [A - B = A \cap \overline{B}]
\end{aligned}$$

34. False. Let $X = \{a\}$ and $Y = Z = \{b\}$. Then

$$X \cup (Y - Z) = \{a\}, \qquad (X \cup Y) - (X \cup Z) = \varnothing.$$

35. True. $\overline{Y - X} = \overline{Y \cap \overline{X}} = \overline{Y} \cup \overline{\overline{X}} = \overline{Y} \cup X = X \cup \overline{Y}$.

36. False. Let $X = \{a\}$ and $Y = Z = \{b\}$. Then

$$X - (Y \cup Z) = \{a\}, \qquad (X - Y) \cup Z = \{a, b\}.$$

38. False. Let $X = \{a\}$, $Y = \{b\}$, and $U = \{a, b\}$. Then

$$\overline{X - Y} = \{b\}, \qquad \overline{Y - X} = \{a\}.$$

40. True. Let $x \in (X \cap Y) \cup (Y - X)$. Now either $x \in X \cap Y$ or $x \in Y - X$. In either case, $x \in Y$. Therefore $(X \cap Y) \cup (Y - X) \subseteq Y$.

Now suppose that $x \in Y$. Either $x \in X$ or $x \notin X$. If $x \in X$, then $x \in X \cap Y$. Thus $x \in (X \cap Y) \cup (Y - X)$. If $x \notin X$, then $x \in Y - X$. Again $x \in (X \cap Y) \cup (Y - X)$. Thus $Y \subseteq (X \cap Y) \cup (Y - X)$. Therefore $(X \cap Y) \cup (Y - X) = Y$.

41. True. Let $a \in X \times (Y \cup Z)$. Then $a = (x, y)$ where $x \in X$ and $y \in Y \cup Z$. Now $y \in Y$ or $y \in Z$. If $y \in Y$, then $a = (x, y) \in X \times Y$. Thus $a \in (X \times Y) \cup (X \times Z)$. If $y \in Z$, then $a = (x, y) \in X \times Z$. Again $a \in (X \times Y) \cup (X \times Z)$. Therefore $X \times (Y \cup Z) \subseteq (X \times Y) \cup (X \times Z)$.

Now suppose that $a \in (X \times Y) \cup (X \times Z)$. Then either $a \in X \times Y$ or $a \in X \times Z$. If $a \in X \times Y$, then $a = (x, y)$ where $x \in X$ and $y \in Y$. In particular, $y \in Y \cup Z$. Thus $a = (x, y) \in X \times (Y \cup Z)$. If $a \in X \times Z$, then $a = (x, z)$ where $x \in X$ and $z \in Z$. In particular, $z \in Y \cup Z$. Thus $a = (x, z) \in X \times (Y \cup Z)$. Therefore $(X \times Y) \cup (X \times Z) \subseteq X \times (Y \cup Z)$. We have proved that $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.

43. True. Let $a \in X \times (Y - Z)$. Then $a = (x, y)$, where $x \in X$ and $y \in Y - Z$. Thus $y \in Y$ and $y \notin Z$ and, so, $(x, y) \in X \times Y$ and $(x, y) \notin X \times Z$. Therefore $a = (x, y) \in (X \times Y) - (X \times Z)$. We have shown that $X \times (Y - Z) \subseteq (X \times Y) - (X \times Z)$.

Now suppose that $a \in (X \times Y) - (X \times Z)$. Then $a \in X \times Y$ and $a \notin X \times Z$. Thus $a = (x, y)$, where $x \in X$, $y \in Y$, and $y \notin Z$. Therefore $a = (x, y) \in X \times (Y - Z)$. We have shown that $(X \times Y) - (X \times Z) \subseteq X \times (Y - Z)$. It follows that $X \times (Y - Z) = (X \times Y) - (X \times Z)$.

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44. False. Take $X = \{1, 2\}$, $Y = \{1\}$, $Z = \{2\}$. Then

$$Y \times Z = \{(1, 2)\}, \quad X - Y = \{2\}, \quad X - Z = \{1\}.$$

Thus

$$X - (Y \times Z) = \{1, 2\} \quad \text{and} \quad (X - Y) \times (X - Z) = \{(2, 1)\}.$$

47–56. Argue as in the proof given in the book of the first associative law [Theorem 1.1.21, part (a)].

58. By definition

$$(A \triangle B) \triangle A = [(A \triangle B) \cup A] - [(A \triangle B) \cap A].$$

Show that

$$(A \triangle B) \cup A = A \cup B \quad \text{and} \quad (A \triangle B) \cap A = A \cap \overline{B}.$$

The statement then follows easily.

59. The statement is true. We first prove that $A \subseteq B$. Let $x \in A$.

We divide the proof into two cases. First, we consider the case that $x \in C$. Then $x \notin A \triangle C$. Therefore $x \notin B \triangle C$. Thus $x \in B$ (since if $x \notin B$, then $x \in B \triangle C$).

Next, we consider the case that $x \notin C$. Then $x \in A \triangle C$. Therefore $x \in B \triangle C$. Thus $x \in B$.

In either case, $x \in B$, and so $A \subseteq B$. Similarly, $B \subseteq A$, and so $A = B$.

61. The statement is false. Let

$$A = \{1, 2\}, \quad B = \{2, 3\}, \quad C = \{1, 3\}.$$

Since $B \cap C = \{3\}$,

$$A \triangle (B \cap C) = \{1, 2, 3\}.$$

Now

$$A \triangle B = \{1, 3\} \quad \text{and} \quad A \triangle C = \{2, 3\},$$

thus

$$(A \triangle B) \cap (A \triangle C) = \{3\}.$$

62. The statement is false. Let

$$A = \{1, 2\}, \quad B = \{2, 3\}, \quad C = \{1, 3\}.$$

Since $B \triangle C = \{1, 2\}$,

$$A \cup (B \triangle C) = \{1, 2\}.$$

Since $A \cup B = A \cup C = \{1, 2, 3\}$,

$$(A \cup B) \triangle (A \cup C) = \varnothing.$$

64. Yes, $\triangle$ is commutative:

$$A \triangle B = (A \cup B) - (A \cap B) = (B \cup A) - (B \cap A) = B \triangle A.$$

65. Yes, $\triangle$ is associative. We first prove that

$$(A \triangle B) \triangle C = (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap C). \qquad (1)$$

[For the motivation of this formula, draw the Venn diagram of $(A \triangle B) \triangle C$.] By Exercise 57,

$$(A \triangle B) \triangle C = [(A \triangle B) - C] \cup [C - (A \triangle B)].$$

Again using Exercise 57 and the fact that $X - Y = X \cap \overline{Y}$, we have

$$(A \triangle B) - C = [(A - B) \cup (B - A)] - C = [(A \cap \overline{B}) \cup (B \cap \overline{A})] \cap \overline{C}.$$

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Using the definition of $\triangle$, the fact that $X - Y = X \cap \overline{Y}$, and De Morgan's laws, we have

$$\overline{A \triangle B} = \overline{(A \cup B) - (A \cap B)} = \overline{(A \cup B) \cap \overline{(A \cap B)}} = \overline{(A \cup B)} \cup \overline{\overline{(A \cap B)}} = (\overline{A} \cap \overline{B}) \cup (A \cap B).$$

Thus

$$C - (A \triangle B) = C \cap \overline{(A \triangle B)} = C \cap [(\overline{A} \cap \overline{B}) \cup (A \cap B)].$$

Combining the preceding equations and using Theorem 1.1.21, we obtain equation (1)

$$\begin{aligned} (A \triangle B) \triangle C &= [(A \triangle B) - C] \cup [C - (A \triangle B)] \\ &= \{[(A \cap \overline{B}) \cup (B \cap \overline{A})] \cap \overline{C}\} \cup \{C \cap [(\overline{A} \cap \overline{B}) \cup (A \cap B)]\} \\ &= (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap C). \end{aligned}$$

By Exercise 64, $\triangle$ is commutative. Thus

$$A \triangle (B \triangle C) = (B \triangle C) \triangle A.$$

We can obtain a formula for $(B \triangle C) \triangle A$ using equation (1) with A replaced by B, B replaced by C, and C replaced by A. However, noting that the right-hand side of equation (1) is symmetric in A, B, and C, we see that the two expressions

$$(A \triangle B) \triangle C \quad \text{and} \quad A \triangle (B \triangle C)$$

are equal. Therefore, $\triangle$ is associative.

Section 2.2

2. False; $x = \sqrt{2}$ is a counterexample.

3. We prove the contrapositive: If x is rational, then x^3 is rational.

   Suppose that x is rational. Then there exist integers p and q such that $x = p/q$. Now $x^3 = p^3/q^3$. Thus x^3 is rational.

5. Suppose, by way of contradiction, that $x < 1$ and $y < 1$ and $z < 1$. Adding these inequalities gives $x + y + z < 3$, which is a contradiction.

6. Suppose, by way of contradiction, that $x > \sqrt{2}$ and $y > \sqrt{2}$. Multiplying these inequalities gives $xy > 2$, which is a contradiction.

8. Suppose, by way of contradiction, that $x + y$ is rational. Since x and $x + y$ are rational, there exist integers p_1, p_2, q_1, q_2 such that $x = p_1/q_1$ and $x + y = p_2/q_2$. Now

$$y = (x + y) - x = \frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{p_2 q_1 - p_1 q_2}{q_1 q_2}.$$

   Therefore y is rational, which is a contradiction.

9. False; a counterexample is $x = 0, y = \sqrt{2}$.

11. Since the integers increase without bound, there exists $n \in \mathbf{Z}$ such that $\sqrt{2}/(b - a) < n$. Therefore $\sqrt{2}/n < b - a$. Choose $m \in \mathbf{Z}$ as large as possible satisfying $m\sqrt{2}/n \leq a$. Then, by the choice of m, $a < (m + 1)\sqrt{2}/n$. Also

$$\frac{(m+1)\sqrt{2}}{n} = \frac{m\sqrt{2}}{n} + \frac{\sqrt{2}}{n} < a + (b - a) = b.$$

   Therefore $x = (m + 1)\sqrt{2}/n$ is an irrational number satisfying $a < x < b$. (If $(m + 1)\sqrt{2}/n$ is rational, say $(m + 1)\sqrt{2}/n = p/q$ where p and q are integers, then $\sqrt{2} = np/[(m + 1)q]$ is rational, which is not the case.)

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12. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we have found irrational numbers a and b (namely $a = b = \sqrt{2}$) such that a^b is rational. Suppose that $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Now $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2$ is rational. We have found irrational numbers a and b such that a^b is rational.

    This proof is nonconstructive since it does not show whether the desired pair is $a = b = \sqrt{2}$ or $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$.

14. Let $a = 2$ and $b = 1/2$. Then a and b are rational. Now $a^b = 2^{1/2} = \sqrt{2}$ is irrational. This proof is a constructive existence proof.

15. Suppose, by way of contradiction, that $x > y$. Let $\varepsilon = (x - y)/2$. Then

$$y + \varepsilon = y + \frac{x - y}{2} = \frac{x + y}{2} < \frac{x + x}{2} = x,$$

    which is a contradiction.

17. First prove that if b is a rational number, then b^n is rational for every positive integer n. To this end, let $b = p/q$, where p and q are integers. Then $b^n = p^n/q^n$. Since b is the quotient of integers, b is rational.

    Now suppose, but way of contradiction, that a^r is rational for some positive rational number $r = p/q$, where p and q are positive integers. Because a^r is rational, $(a^r)^q$ is rational by the result of the first paragraph. Since $(a^r)^q = a^p$, a^p is rational. This contradicts the hypothesis that a^n is irrational for every positive integer n.

18. We show that Abby, Cary, Dale, and Edie went to the concert, but not Bosco. Suppose that Bosco went. Then Cary and Dale also went. Since Cary went, Edie went; and since Dale went, Abby went. But this contradicts the hypothesis that exactly four went to the concert. Therefore, Bosco did not go. This means Abby, Cary, Dale, and Edie went to the concert.

20. Suppose, by way of contradiction, that $X \times \varnothing$ is not empty. Then there exists $(x, y) \in X \times \varnothing$. Now $y \in \varnothing$, which is a contradiction.

21. Suppose that every box contains less than 12 balls. Then each box contains at most 11 balls and the maximum number of balls contained by the nine boxes is $9 \cdot 11 = 99$. Contradiction.

23. Suppose, by way of contradiction, that each of the other three suits contains at most six cards. Then these three suits together contain at most $3 \cdot 6 = 18$ cards. Together with the other suit, which contains exactly seven cards, we can account for at most 25 cards. Since S contains 26 cards, we have a contradiction. Therefore there is another suit in which S has at least seven cards.

24. Since there is a suit in which S_1 has at least nine cards, S_2 has at most four cards in this suit. Now suppose, by way of contradiction, that S_2 has at most seven cards in the other three suits. These three suits contain at most $3 \cdot 7 = 21$ cards. Together with at most four cards in the other suit, we can account for at most 25 cards in S_2. Since S_2 contains 26 cards, we have a contradiction. Therefore there is a suit in which S_2 has at least eight cards.

26. For $n = 3$, we have $n^2 > 2^n$.

28. The statement is false. Let $s_1 = s_2 = 3$. Then $A = 3$. For no i do we have $s_i > A$. The proof is by counterexample.

29. The statement is true and we prove it using proof by contradiction. Suppose that for every j, $s_j \leq A$. Since $s_j \leq A$ for all j and $s_i < A$,

$$s_1 + \cdots + s_i + \cdots + s_n < A + \cdots + A + \cdots + A = nA.$$

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Dividing by n, we obtain
$$\frac{s_1 + \cdots + s_n}{n} < A,$$
which is a contradiction.

31. Since $s_i \neq s_j$, either $s_i \neq A$ or $s_j \neq A$. By changing the notation, if necessary, we may assume that $s_i \neq A$. Either $s_i < A$ or $s_i > A$. If $s_i > A$, the proof is complete; so assume that $s_i < A$. We show that there exists k such that $s_k > A$. Suppose, by way of contradiction, that $s_m \leq A$ for all m, that is,
$$\begin{aligned} s_1 &\leq A \\ s_2 &\leq A \\ &\vdots \\ s_n &\leq A. \end{aligned}$$
Adding these inequalities yields
$$s_1 + s_2 + \cdots + s_i + \cdots + s_n < nA$$
since $s_i < A$. Dividing by n gives
$$\frac{s_1 + s_2 + \cdots + s_n}{n} < A,$$
which is a contradiction. Therefore there exists k such that $s_k > A$.

33. If m and n are positive integers and $m > 3$, then $m^3 + 2n^2 > 36$. If m and n are positive integers and $n > 4$, then $m^3 + 2n^2 > 36$. Thus it suffices to consider the cases $1 \leq m \leq 3$ and $1 \leq n \leq 4$. The following table, which shows the values of $m^3 + 2n^2$, shows that there is no solution to $m^3 + 2n^2 = 36$:

n \ m	1	2	3
1	3	10	29
2	9	16	35
3	19	26	45
4	33	40	59

34. Notice that $2m^2 + 4n^2 - 1$ is odd and $2(m + n)$ is even. Therefore $2m^2 + 4n^2 - 1 \neq 2(m + n)$ for all positive integers m and n.

36. We consider two cases: n is even, n is odd. First suppose that n is even. By Exercise 9, Section 2.1, the product of even integers is even. Therefore $n^2 = n \cdot n$ is even. Again by Exercise 9, Section 2.1, $n^3 = n^2 \cdot n$ is even. By Exercise 7, Section 2.1, the sum of even integers is even. Therefore $n^3 + n$ is even.

    Now suppose that n is odd. By Exercise 10, Section 2.1, the product of odd integers is odd. Therefore $n^2 = n \cdot n$ is odd. Again by Exercise 10, Section 2.1, $n^3 = n^2 \cdot n$ is odd. By Exercise 8, Section 2.1, the sum of odd integers is even. Therefore $n^3 + n$ is even. In either case, $n^3 + n$ is even.

38. First, note that from Exercise 37, for all x,
$$|-x| = |(-1)x| = |-1||x| = |x|.$$
Example 2.2.7 states that for all x, $x \leq |x|$. Using these results, we consider two cases: $x + y \geq 0$ and $x + y < 0$. If $x + y \geq 0$, we have
$$|x + y| = x + y \leq |x| + |y|.$$
If $x + y < 0$, we have
$$|x + y| = -(x + y) = -x + -y \leq |-x| + |-y| = |x| + |y|.$$

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40. Suppose that $xy > 0$. Then either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. If $x > 0$ and $y > 0$,

$$\operatorname{sgn}(xy) = 1 = 1 \cdot 1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

If $x < 0$ and $y < 0$,

$$\operatorname{sgn}(xy) = 1 = -1 \cdot -1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

Next, suppose that $xy = 0$. Then either $x = 0$ or $y = 0$. Thus either $\operatorname{sgn}(x) = 0$ or $\operatorname{sgn}(y) = 0$. In either case, $\operatorname{sgn}(x)\operatorname{sgn}(y) = 0$. Therefore

$$\operatorname{sgn}(xy) = 0 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

Finally, suppose that $xy < 0$. Then either $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$. If $x > 0$ and $y < 0$,

$$\operatorname{sgn}(xy) = -1 = 1 \cdot -1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

If $x < 0$ and $y > 0$,

$$\operatorname{sgn}(xy) = -1 = -1 \cdot 1 = \operatorname{sgn}(x)\operatorname{sgn}(y).$$

41. $|xy| = \operatorname{sgn}(xy)xy = \operatorname{sgn}(x)\operatorname{sgn}(y)xy = [\operatorname{sgn}(x)x][\operatorname{sgn}(y)y] = |x||y|$

43. Suppose that $x \geq y$. Then

$$\max\{x, y\} = x \qquad \text{and} \qquad |x - y| = x - y.$$

Thus

$$\max\{x, y\} = x = \frac{2x}{2} = \frac{x + y + x - y}{2} = \frac{x + y + |x - y|}{2}.$$

The other case is $x < y$. Then

$$\max\{x, y\} = y \qquad \text{and} \qquad |x - y| = y - x.$$

Thus

$$\max\{x, y\} = y = \frac{2y}{2} = \frac{x + y + y - x}{2} = \frac{x + y + |x - y|}{2}.$$

44. Suppose that $x \geq y$. Then

$$\min\{x, y\} = y \qquad \text{and} \qquad |x - y| = x - y.$$

Thus

$$\min\{x, y\} = y = \frac{2y}{2} = \frac{x + y - (x - y)}{2} = \frac{x + y - |x - y|}{2}.$$

The other case is $x < y$. Then

$$\min\{x, y\} = x \qquad \text{and} \qquad |x - y| = y - x.$$

Thus

$$\min\{x, y\} = x = \frac{2x}{2} = \frac{x + y - (y - x)}{2} = \frac{x + y - |x - y|}{2}.$$

45.
$$\begin{aligned}\max\{x, y\} + \min\{x, y\} &= \frac{x + y + |x - y|}{2} + \frac{x + y - |x - y|}{2} \\ &= \frac{x + y + |x - y| + x + y - |x - y|}{2} \\ &= \frac{2x + 2y}{2} = x + y.\end{aligned}$$

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47. Suppose that n is odd. Then $n = 2k+1$. Now $n+2 = (2k+1)+2 = 2(k+1)+1$ is odd.

    Now suppose that $n+2$ is odd. Then $n+2 = 2k+1$. Now $n = (2k+1)-2 = 2(k-1)+1$ is odd.

    Therefore n is odd if and only if $n+2$ is odd.

49. Suppose that $A \subseteq C$ and $B \subseteq C$. Let $x \in A \cup B$. Then either $x \in A$ or $x \in B$. If $x \in A$, since $A \subseteq C$, $x \in C$. If $x \in B$, since $B \subseteq C$, $x \in C$. In either case, $x \in C$. Therefore $A \cup B \subseteq C$.

    Now suppose that $A \cup B \subseteq C$. Let $x \in A$. Then $x \in A \cup B$. Since $A \cup B \subseteq C$, $x \in C$. Therefore $A \subseteq C$. Let $x \in B$. Then $x \in A \cup B$. Since $A \cup B \subseteq C$, $x \in C$. Therefore $B \subseteq C$. We conclude that $A \subseteq C$ and $B \subseteq C$. It follows that $A \subseteq C$ and $B \subseteq C$ if and only if $A \cup B \subseteq C$.

50. Suppose that $C \subseteq A$ and $C \subseteq B$. Let $x \in C$. Since $C \subseteq A$, $x \in A$. Since $C \subseteq B$, $x \in B$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Therefore $C \subseteq A \cap B$.

    Now suppose that $C \subseteq A \cap B$. Let $x \in C$. Then $x \in A \cap B$. In particular, $x \in A$. Therefore $C \subseteq A$. Again let $x \in C$. Then $x \in A \cap B$. In particular, $x \in B$. Therefore $C \subseteq B$. Thus $C \subseteq A$ and $C \subseteq B$. It follows that $C \subseteq A$ and $C \subseteq B$ if and only if $C \subseteq A \cap B$.

53. [(a) → (b)] We assume that $A \cap B = \varnothing$ and prove that $B \subseteq \overline{A}$. Let $x \in B$. If $x \in A$, we obtain the contradiction $A \cap B \neq \varnothing$. Thus $x \notin A$. Hence $x \in \overline{A}$. Therefore $B \subseteq \overline{A}$.

    [(b) → (c)] We assume that $B \subseteq \overline{A}$ and prove that $A \triangle B = A \cup B$.

    Let $x \in A \triangle B$. By definition, $A \triangle B = (A \cup B) - (A \cap B)$, thus $x \in A \cup B$. Therefore $A \triangle B \subseteq A \cup B$.

    Let $x \in A \cup B$. We first prove that $x \notin A \cap B$. Suppose, by way of contradiction, that $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $B \subseteq \overline{A}$, $x \in \overline{A}$, which implies that $x \notin A$. We have the desired contradiction. Therefore $x \notin A \cap B$. Now $x \in (A \cup B) - (A \cap B) = A \triangle B$. Therefore $A \cup B \subseteq A \triangle B$. It follows that $A \triangle B = A \cup B$.

    [(c) → (a)] We assume that $A \triangle B = A \cup B$ and prove that $A \cap B = \varnothing$. Suppose, by way of contradiction, that $A \cap B$ is not empty. Then there exists $x \in A \cap B$. Then $x \in A \cup B$. This implies that $x \notin A \triangle B$. Since $x \in A \cup B$, $A \triangle B \neq A \cup B$, which is a contradiction. Therefore $A \cap B = \varnothing$.

54. [(a) → (b)] We assume that $A \cup B = U$ and prove that $\overline{A} \cap \overline{B} = \varnothing$. Taking the complement of both sides of the equation $A \cup B = U$ and using De Morgan's law and the 0/1 law (Theorem 1.1.22), we obtain

$$\overline{A} \cap \overline{B} = \overline{A \cup B} = \overline{U} = \varnothing.$$

    [(b) → (c)] We assume that $\overline{A} \cap \overline{B} = \varnothing$ and prove that $\overline{A} \subseteq B$. Replace A by $\overline{B}$ and B by $\overline{A}$ in Exercise 53(a) to obtain $\overline{B} \cap \overline{A} = \varnothing$. Since Exercise 53(a) is equivalent to Exercise 53(b), we obtain $\overline{A} \subseteq \overline{\overline{B}}$ or $\overline{A} \subseteq B$.

    [(c) → (a)] We assume that $\overline{A} \subseteq B$ and prove that $A \cup B = U$. Since U is a universal set, we automatically have $A \cup B \subseteq U$.

    Let $x \in U$. If $x \in A$, then $x \in A \cup B$. If $x \notin A$, then $x \in \overline{A}$. Since $\overline{A} \subseteq B$, $x \in B$. Again $x \in A \cup B$. Therefore $U \subseteq A \cup B$. It follows that $A \cup B = U$.

Problem-Solving Corner: Proofs

1. The least upper bound of a nonempty finite set of real numbers is the maximum number in the set.

2. Call the given set X. We prove that the least upper bound of X is 1. Since

$$1 - \frac{1}{n} < 1$$

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for all positive integers n, 1 is an upper bound of X. Let a be an upper bound for X. Suppose, by way of contradiction, that $a < 1$. Since the integers are unbounded, there exists a positive integer k such that

$$\frac{1}{1-a} < k.$$

Multiplying by $(1-a)/k$ gives

$$\frac{1}{k} < 1 - a,$$

which, in turn, is equivalent to

$$a < 1 - \frac{1}{k}.$$

This contradicts the fact that a is an upper bound of X. Thus $1 \le a$ and 1 is the least upper bound of X.

3. Let b be the least upper bound of Y. If $x \in X$, then $x \in Y$ and $x \le b$. Thus b is an upper bound of X. If a is the least upper bound of X, $a \le b$.

4. 0

5. Let $Z = \{x + y \mid x \in X \text{ and } y \in Y\}$ and let $z \in Z$. Then $z = x + y$ for some $x \in X, y \in Y$. Now $z = x + y \le a + b$. Therefore Z is bounded above by $a + b$.

   Let c be an upper bound of Z. Suppose, by way of contradiction, that $c < a + b$. Let $\varepsilon = a + b - c$. Now $a - \varepsilon/2$ is not an upper bound of X so there exists $x \in X$ such that

$$a - \frac{\varepsilon}{2} < x.$$

   Similarly, there exists $y \in Y$ such that

$$b - \frac{\varepsilon}{2} < y.$$

   Adding the previous inequalities gives

$$c = a + b - \varepsilon < x + y,$$

   which contradicts the fact that c is an upper bound of Z. Therefore $c \ge a + b$ and $a + b$ is the least upper bound of Z.

6. Since a is a greatest lower bound for X and b is a lower bound for X, $b \le a$. Since b is a greatest lower bound for X and a is a lower bound for X, $a \le b$. Therefore $a = b$.

7. Let X be a nonempty set of real numbers bounded below. Let Y be the set of lower bounds of X. The set Y is nonempty since X is bounded below. Let x be an element of X. For every $y \in Y$, we have $y \le x$ since y is a lower bound of X. Therefore Y is bounded above by x. Thus Y is has a least upper bound, say a.

   Next we show that a is a lower bound of X. Suppose, by way of contradiction, that a is not a lower bound of X. Then there exists $x \in X$ such that $x < a$. Then x is not an upper bound of Y. Therefore there exists $y \in Y$ such that $x < y$. But this contradicts the fact that y is a lower bound of X. Therefore a is a lower bound of X.

   Finally, we show that a is the greatest lower bound of X. Let b be a lower bound of X. Then $b \in Y$. Since a is an upper bound of Y, $b \le a$. Therefore a is the greatest lower bound of X.

8. Since $a + \varepsilon > a$, $a + \varepsilon$ is not a lower bound of X. Therefore there exists $x \in X$ such that $a + \varepsilon > x$. Since a is a lower bound of X, $x \ge a$.

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9. Let tX denote the set

$$\{tx \mid x \in X\}.$$

We must prove that

(a) $z \geq ta$ for every $z \in tX$ (i.e., ta is an lower bound for tX),

(b) if b is an lower bound for tX, then $b \leq ta$ (i.e., ta is the greatest lower bound for tX).

We first prove part (a). Let $z \in tX$. Then $z = tx$ for some $x \in X$. Since a is an upper bound for X, $x \leq a$. Multiplying by t and noting that $t < 0$, we have $z = tx \geq ta$. Therefore, $z \geq ta$ for every $z \in tX$ and the proof of part (a) is complete.

Next we prove part (b). Let b be a lower bound for tX. Then $tx \geq b$ for every $x \in X$. Dividing by t and noting that $t < 0$, we have $x \leq b/t$ for every $x \in X$. Therefore b/t is an upper bound for X. Since a is the least upper bound for X, $b/t \geq a$. Multiplying by t and noting again that $t < 0$, we have $b \leq ta$. Therefore ta is the greatest lower bound for tX. The proof is complete.

Section 2.3

3.

1.	$\neg p \vee r$	
2.	$\neg r \vee q$	
3.	p	
4.	$\neg p \vee q$	from 1,2
5.	q	from 3,4

4.

1.	$\neg p \vee t$	
2.	$\neg q \vee s$	
3.	$\neg r \vee s$	
4.	$\neg r \vee t$	
5.	$p \vee q \vee r \vee u$	
6.	$t \vee q \vee r \vee u$	from 1,5
7.	$s \vee t \vee r \vee u$	from 2,6
8.	$s \vee t \vee u$	from 3,7

6. $(p \leftrightarrow r) \equiv (p \rightarrow r)(r \rightarrow p) \equiv (\neg p \vee r)(\neg r \vee p)$

1.	$\neg p \vee r$	
2.	$\neg r \vee p$	
3.	r	
4.	p	from 2,3

8.

1.	$a \vee \neg b$	
2.	$a \vee c$	
3.	$\neg a$	
4.	$\neg d$	
5.	b	negated conclusion
6.	$\neg b$	from 1,3

Now 5 and 6 combine to give a contradiction.

Section 2.4

In some of these solutions, the Basis Steps are omitted.

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2. $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2)$
$$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$$

3. $1(1!) + 2(2!) + \cdots + n(n!) + (n+1)(n+1)!$
$$= (n+1)! - 1 + (n+1)(n+1)! = (n+2)! - 1$$

5. $1^2 - 2^2 + \cdots + (-1)^{n+1}n^2 + (-1)^{n+2}(n+1)^2$
$$= \frac{(-1)^{n+1}n(n+1)}{2} + (-1)^{n+2}(n+1)^2 = \frac{(-1)^{n+2}(n+1)(n+2)}{2}$$

6. $1^3 + 2^3 + \cdots + n^3 + (n+1)^3$
$$= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2$$

8. $\frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} + \cdots + \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n+2)} + \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n+2)(2n+4)}$
$$= \frac{1}{2} - \frac{1 \cdot 3 \cdots (2n+1)}{2 \cdot 4 \cdots (2n+2)} + \frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n+2)(2n+4)}$$
$$= \frac{1}{2} - \frac{1 \cdot 3 \cdots (2n+3)}{2 \cdot 4 \cdots (2n+4)}$$

9. $\frac{1}{2^2-1} + \frac{1}{3^2-1} + \cdots + \frac{1}{(n+1)^2-1} + \frac{1}{(n+2)^2-1}$
$$= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} + \frac{1}{(n+2)^2-1}$$
$$= \frac{3}{4} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

10. $1 \cdot 2^2 + 2 \cdot 3^2 + \cdots + (n+1)(n+2)^2 = \frac{n(n+1)(n+2)(3n+5)}{12} + (n+1)(n+2)^2$
$$= \frac{n(n+1)(n+2)(3n+5) + 12(n+1)(n+2)^2}{12}$$
$$= \frac{(n+1)(n+2)[3n^2 + 5n + 12(n+2)]}{12}$$
$$= \frac{(n+1)(n+2)(3n^2 + 17n + 24)}{12}$$
$$= \frac{(n+1)(n+2)(n+3)(3n+8)}{12}$$

12. The solution is similar to that for Exercise 11, which is given in the book.

14. First note that
$$\frac{1 \cdot 3 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdots (2n)(2n+2)} \leq \frac{1}{\sqrt{n+1}} \frac{2n+1}{2n+2}.$$

The proof will be complete if we can show that
$$\frac{2n+1}{(2n+2)\sqrt{n+1}} \leq \frac{1}{\sqrt{n+2}}.$$

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