INSTRUCTOR'S SOLUTIONS MANUAL

FUNDAMENTALS OF DIFFERENTIAL EQUATIONS
NINTH EDITION

AND

FUNDAMENTALS OF DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS
SEVENTH EDITION

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Contents

Notes to the Instructor

One goal in our writing has been to create flexible texts that afford the instructor a variety of topics and make available to the student an abundance of practice problems and projects. We recommend that the instructor read the discussion given in the preface in order to gain an overview of the prerequisites, topics of emphasis, and general philosophy of the text.

Supplements

Student's Solutions Manual: Contains complete, worked-out solutions to most odd-numbered exercises, providing students with an excellent study tool.

Companion Web site:

Instructor's MAPLE/MATLAB/MATHEMATICA manuals: By Thomas W. Polaski (Winthrop University), Bruno Welfert (Arizona State University), and Maurino Bautista (Rochester Institute of Technology). A collection of worksheets and projects to aid instructors in integrating computer algebra systems into their courses. Available in the Pearson Instructor Resource Center at `www.pearsonhighered.com/irc`.

MATLAB Manual ISBN: 0321977238/9780321977236
MAPLE Manual ISBN: 0321977149/9780321977144
MATHEMATICA Manual ISBN: 0321977750/9780321977755

Computer Labs

Projects

Although the projects that appear at the end of the chapters in the text can be worked out by the conscientious student working alone, making them *group* projects may add a social element that encourages discussion and interactions that simulate a professional work place atmosphere. Group sizes of 3 or 4 seem to be optimal. Moreover, requiring that each individual student separately write up the group's solution as a formal technical report for grading by the instructor also contributes to the professional flavor.

Typically, our students each work on 3 or 4 projects per semester. If class time permits, oral presentations can be scheduled and help to improve the communication skills of the students.

The role of the instructor is, of course, to help the students solve these elaborate problems on their own and to recommend additional reference material when appropriate.

Some additional Group Projects are presented in this guide (see page 10).

Technical Writing Exercises

The technical writing exercises at the end of most chapters invite students to make documented responses to questions dealing with the concepts in the chapter. This not only gives students an opportunity to improve their writing skills, but it helps them organize their thoughts and better understand the new concepts. Moreover, many questions deal with critical thinking skills that will be useful in their careers as engineers, scientists, or mathematicians.

Since most students have little experience with technical writing, it may be necessary to return *ungraded* the first few technical writing assignments with comments and have the students redo the the exercise. This has worked well in our classes and is much appreciated by the students. Handing out a "model" technical writing response is also helpful for the students.

Student Presentations

It is not uncommon for an instructor to have students go to the board and present a solution to a problem. Differential equations is so rich in theory and applications that it is an excellent course to allow (require) a student to give a presentation on a special application (e.g., almost any topic from Chapters 3 and 5), on a new technique not covered in class (e.g., material from Section 2.6, the Projects), or on additional theory (e.g., material from Chapter 6 which generalizes the results in Chapter 4). In addition to improving students' communication skills, these "special" topics are long remembered by the students. Working in groups of 3 or 4 and sharing the presentation responsibilities can add substantially to the interest and quality of the presentation. Students should also be encouraged to enliven their communication by building physical models, preparing part of their lectures with the aid of video technology, and utilizing appropriate internet web sites.

Homework Assignments

We would like to share with you an obvious, non-original, but effective method to encourage students to do homework problems.

An essential feature is that it requires little extra work on the part of the instructor or grader.

We assign homework problems (about 5 of them) after each lecture. At the end of the week (Fridays), students are asked to turn in their homework (typically, 3 sets) for that week. We then choose at random one problem from each assignment (typically, a total of 3) that will be graded. (The point is that the student does not know in advance which problems will be chosen.) Full credit is given for any of the chosen problems for which there is evidence that the student has made an honest attempt at solving. The homework problem sets are returned to the students at the next meeting (Mondays) with grades like 0/3, 1/3, 2/3, or 3/3 indicating the proportion of problems for which the student received credit. The homework grades are tallied at the end of the semester and count as one test grade. Certainly, there are variations on this theme. The point is that students are motivated to do their homework.

Syllabus Suggestions

To serve as a guide in constructing a syllabus for a one-semester or two-semester course, the prefaces to the texts list sample outlines that emphasize methods, applications, theory, partial differential equations, phase plane analysis, computation, or combinations of these. As a further guide in making a choice of subject matter, we provide (starting on the next page) a listing of text material dealing with some common areas of emphasis.

Numerical, Graphical, and Qualitative Methods

The sections and projects dealing with numerical, graphical, and qualitative techniques of solving differential equations include:

Section 1.3: *Direction Fields*

Section 1.4: *The Approximation Method of Euler*

Project A for Chapter 1: *Picard's Method*

Project B for Chapter 1: *The Phase Line*

Project D for Chapter 1: *Taylor Series Method*

Section 3.6: *Improved Euler's Method*, which includes step-by-step outlines of the improved Euler's method subroutine and improved Euler's method with tolerance. These outlines are easy for the student to translate into a computer program (pp. 127–128).

Section 3.7: *Higher-Order Numerical Methods*: *Taylor and Runge-Kutta*, which includes outlines for the Fourth Order Runge-Kutta subroutine and algorithm with tolerance (see pp. 135–136).

Project F for Chapter 3: *Stability of Numerical Methods*

Project G for Chapter 3: *Period Doubling and Chaos*

Section 4.8: *Qualitative Considerations for Variable Coefficient and Non-linear Equations*, which discusses the energy integral lemma, as well as the Airy, Bessel, Duffing, and van der Pol equations.

Section 5.3: *Solving Systems and Higher-Order Equations Numerically*, which describes the vectorized forms of Euler's method and the Fourth Order Runge-Kutta method, and discusses an application to population dynamics.

Section 5.4: *Introduction to the Phase Plane*, which introduces the study of trajectories of autonomous systems, critical points, and stability.

Section 5.8: *Dynamical Systems, Poincaré Maps, and Chaos*, which discusses the use of numerical methods to approximate the Poincarè map and how to interpret the results.

Project A for Chapter 6: *Computer Algebra Systems and Exponential Shift*

Project D for Chapter 6: *Higher-Order Difference Equations*

Project A for Chapter 8: *Alphabetization Algorithms*

Project D for Chapter 10: *Numerical Method for* $\Delta u = f$ *on a Rectangle*

Project D for Chapter 11: *Shooting Method*

Project E for Chapter 11: *Finite-Difference Method for Boundary Value Problems*

Section 12.8: *Neurons and the FitzHugh-Nagumo Equations*, which uses direction fields to establish the onset of action potentials on axons.

Project C for Chapter 12: *Computing Phase Plane Diagrams*

Project D for Chapter 12: *Ecosystem of Planet GLIA-2*

Section 13.1: *Introduction: Successive Approximations*

Appendix B: *Newton's Method*

Appendix C: *Simpson's Rule*

Appendix E: *Method of Least Squares*

Appendix F: *Runge-Kutta Procedure for Equations*

Appendix G: *Software for Analyzing Differential Equations*

The instructor who wishes to emphasize numerical methods should also note that the text contains an extensive chapter on series solutions of differential equations (Chapter 8).

Engineering/Physics Applications

Since Laplace transforms is a subject vital to engineering, we have included a detailed chapter on this topic – see Chapter 7. Stability is also an important subject for engineers, so we have included an introduction to the subject in Section 5.4 along with an extensive discussion in Chapter 12. Further material dealing with engineering/physics applications include:

Project A for Chapter 2: Oil Spill in a Canal

Project C for Chapter 2: *Torricelli's Law of Fluid Flow*.

Project I for Chapter 2: *Designing a Solar Collector*.

Section 3.1: *Mathematical Modeling*.

Section 3.2: *Compartmental Analysis*, which contains a discussion of mixing problems and of population models.

Section 3.3: *Heating and Cooling off Buildings*, which discusses temperature variations in the presence of air conditioning or furnace heating.

Section 3.4: *Newtonian Mechanics*.

Section 3.5: *Electrical Circuits*.

Project C for Chapter 3: *Curve of Pursuit*.

Project D for Chapter 3: *Aircraft Guidance in a Crosswind.*

Section 4.1: *Introduction: The Mass-Spring Oscillator.*

Section 4.8: *Qualitative Considerations for Variable-Coefficient and Non-linear Equations.*

Section 4.9: *A Closer Look at Free Mechanical Vibrations.*

Section 4.10: *A Closer Look at Forced Mechanical Vibrations.*

Project B for Chapter 4: *Apollo Re-entry*

Project C for Chapter 4: *Simple Pendulum*

Project H for Chapter 4: *Gravity Train*

Section 5.1: *Interconnected Fluid Tanks.*

Section 5.4: *Introduction to the Phase PLane.*

Section 5.6: *Coupled Mass-Spring Systems.*

Section 5.7: *Electrical Systems.*

Section 5.8: *Dynamical Systems, Poincaré Maps, and Chaos* .

Project A for Chapter 5: *Designing a Landing System for Interplanetary Travel.*

Project C for Chapter 5: *Things that Bob.*

Project D for Chapter 5: *Hamiltonian Systems.*

Project G for Chapter 5: *Phase-Locked Loops*

Project C for Chapter 6: *Transverse Vibrations of a Beam.*

Chapter 7: *Laplace Transforms*, which in addition to basic material includes discussions of transfer functions, the Dirac delta function, and frequency response modelling.

Project B for Chapter 8, *Spherically Symmetric Solutions to Schrödinger's Equation for the Hydrogen Atom*

Project D for Chapter 8, *Buckling of a Tower*

Project E for Chapter 8, *Aging Spring and Bessel Functions*

Section 9.6: *Complex Eigenvalues*, includes discussion of normal (natural) frequencies.

Project B for Chapter 9: *Matrix Laplace Transform Method*.

Project C for Chapter 9: *Undamped Second-Order Systems*.

Chapter 10: *Partial Differential Equations*, which includes sections on Fourier series, the heat equation, wave equation, and Laplace's equation.

Project A for Chapter 10: *Steady-State Temperature Distribution in a Circular Cylinder*.

Project B for Chapter 10: *A Laplace Transform Solution of the Wave Equation*.

Project E for Chapter 10: *The Telegrapher's Equation and the Cable Equation*

Project A for Chapter 11: *Hermite Polynomials and the Harmonic Oscillator*.

Section 12.4: *Energy Methods*, which addresses both conservative and non-conservative autonomous mechanical systems.

Project A for Chapter 12: *Solitons and Korteweg-de Vries Equation*.

Project B for Chapter 12: *Burger's Equation*.

Students of engineering and physics would also find Chapter 8 on series solutions particularly useful, especially Section 8.8 on special functions.

Biology/Ecology Applications

Project C for Chapter 1: *The Phase Plane*, which discusses the logistic population model and bifurcation diagrams for population control.

Problem 40 in Exercises 2.3, which discusses the Hodgkins-Huxley model for axon activity

Project A for Chapter 2: *Oil Spill in a Canal*.

Project B for Chapter 2: *Differential Equations in Clinical Medicine.*

Section 3.1: *Mathematical Modelling.*

Section 3.2: *Compartmental Analysis*, which contains a discussion of mixing problems and population models.

Project A for Chapter 3: *Dynamics of HIV Infection.*

Project B for Chapter 3: *Aquaculture*, which deals with a model of raising and harvesting catfish.

Section 5.1: *Interconnected Fluid Tanks*, which introduces systems of equations.

Section 5.3: *Solving Systems and Higher-Order Equations Numerically*, which contains an application to population dynamics.

Section 5.5: *Applications to Biomathematics: Epidemic and Tumor Growth Models.*

Project B for Chapter 5: *Spread of Staph Infections in Hospitals – Part I.*

Project E for Chapter 5: *Cleaning Up the Great Lakes*

Project F for Chapter 5: *The 2014-2015 Ebola Epidemic.*

Problem 19 in Exercises 10.5 , which involves chemical diffusion through a thin layer.

Section 12.8: *Neurons and the Fitz-Nagumo Equations*

Project D for Chapter 12: *Ecosystem on Planet GLIA-2*

Project E for Chapter 12: *Spread of Staph Infections in Hospitals – Part II.*

Economics Applications

Project C for Chapter 1: *Applications to Economics*

Project H for Chapter 2: *Utility Functions and Risk Aversion*

Project E for Chapter 3: *Market Equilibrium: Stability and Time Paths*

The basic content of the remainder of this instructor's manual consists of supplemental projects, answers to the even-numbered problems, and detailed solutions to most of them. These answers are not available any place else since the text and the *Student's Solutions Manual* only provide answers and solutions to odd-numbered problems.

We would appreciate any comments you may have concerning the answers in this manual. These comments can be sent to the authors' email addresses below. We also would encourage sharing with us (the authors and users of the texts) any of your favorite projects.

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Projects for Chapter 3

Delay Differential Equations

In our discussion of mixing problems in Section 3.2, we encountered the initial value problem

$$x'(t) = 6 - \frac{3}{500}\, x\,(t - t_0)\,, \qquad (0.1)$$
$$x(t) = 0 \quad \text{for} \quad x \in [-t_0, 0]\,,$$

where t_0 is a positive constant. The equation in (0.1) is an example of a **delay differential equation**. These equations differ from the usual differential equations by the presence of the shift $(t - t_0)$ in the argument of the unknown function $x(t)$. In general, these equations are more difficult to work with than are regular differential equations, but quite a bit is known about them.[1]

(a) Show that the simple linear delay differential equation

$$x' = ax(t - b), \qquad (0.2)$$

where a, b are constants, has a solution of the form $x(t) = Ce^{st}$ for any constant C, provided s satisfies the transcendental equation $s = ae^{-bs}$.

(b) A solution to (0.2) for $t > 0$ can also be found using the **method of steps**. Assume that $x(t) = f(t)$ for $-b \le t \le 0$. For $0 \le t \le b$, equation (0.2) becomes

$$x'(t) = ax(t - b) = af(t - b),$$

and so

$$x(t) = \int_0^t af(\nu - b)d\nu + x(0).$$

Now that we know $x(t)$ on $[0, b]$, we can repeat this procedure to obtain

$$x(t) = \int_b^t ax(\nu - b)d\nu + x(b)$$

for $b \le x \le 2b$. This process can be continued indefinitely.

[1]See, for example, *Differential–Difference Equations*, by R. Bellman and K. L. Cooke, Academic Press, New York, 1963, or *Ordinary and Delay Differential Equations*, by R. D. Driver, Springer–Verlag, New York, 1977

Use the method of steps to show that the solution to the initial value problem

$$x'(t) = -x(t-1), \qquad x(t) = 1 \quad \text{on} \quad [-1, 0],$$

is given by

$$x(t) = \sum_{k=0}^{n} (-1)^k \frac{[t-(k-1)]^k}{k!}, \qquad \text{for} \quad n-1 \le t \le n,$$

where n is a nonnegative integer. (This problem can also be solved using the Laplace transform method of Chapter 7.)

(c) Use the method of steps to compute the solution to the initial value problem given in (0.1) on the interval $0 \le t \le 15$ for $t_0 = 3$.

Extrapolation

When precise information about the *form* of the error in an approximation is known, a technique called **extrapolation** can be used to improve the rate of convergence.

Suppose the approximation method converges with rate $O(h^p)$ as $h \to 0$ (cf. Section 3.6). From theoretical considerations, assume we know, more precisely, that

$$y(x;h) = \phi(x) + h^p a_p(x) + O\left(h^{p+1}\right), \tag{0.3}$$

where $y(x;h)$ is the approximation to $\phi(x)$ using step size h and $a_p(x)$ is some function that is independent of h (typically, we do not know a formula for $a_p(x)$, only that it exists). Our goal is to obtain approximations that converge at the faster rate than $O(h^{p+1})$.

We start by replacing h by $h/2$ in (0.3) to get

$$y\left(x;\frac{h}{2}\right) = \phi(x) + \frac{h^p}{2^p} a_p(x) + O\left(h^{p+1}\right).$$

If we multiply both sides by 2^p and subtract equation (0.3), we find

$$2^p y\left(x;\frac{h}{2}\right) - y(x;h) = (2^p - 1)\,\phi(x) + O\left(h^{p+1}\right).$$

Solving for $\phi(x)$ yields

$$\phi(x) = \frac{2^p y\,(x;h/2) - y(x;h)}{2^p - 1} + O\left(h^{p+1}\right).$$

Hence,

$$y^*\left(x;\frac{h}{2}\right):=\frac{2^p y\left(x;h/2\right)-y(x;h)}{2^p-1}$$

has a rate of convergence of $\mathrm{O}\left(h^{p+1}\right)$.

(a) Assuming

$$y^*\left(x;\frac{h}{2}\right)=\phi(x)+h^{p+1}a_{p+1}(x)+\mathrm{O}\left(h^{p+2}\right),$$

show that

$$y^{**}\left(x;\frac{h}{4}\right):=\frac{2^{p+1}y^*\left(x;h/4\right)-y^*(x;h/2)}{2^{p+1}-1}$$

has a rate of convergence of $\mathrm{O}\left(h^{p+2}\right)$.

(b) Assuming

$$y^{**}\left(x;\frac{h}{4}\right)=\phi(x)+h^{p+2}a_{p+2}(x)+\mathrm{O}\left(h^{p+3}\right),$$

show that

$$y^{***}\left(x;\frac{h}{8}\right):=\frac{2^{p+2}y^{**}\left(x;h/8\right)-y^{**}(x;h/4)}{2^{p+2}-1}$$

has a rate of convergence of $\mathrm{O}\left(h^{p+3}\right)$.

(c) The results of using Euler's method (with $h=1$, $1/2$, $1/4$, $1/8$) to approximate the solution to the initial value problem

$$y'=y,\quad y(0)=1$$

at $x=1$ are given in Table 1.2, page 26. For Euler's method, the extrapolation procedure applies with $p=1$. Use the results in Table 1.2 to find an approximation to $e=y(1)$ by computing $y^{***}(1;1/8)$. [Hint: Compute $y^*\left(1;1/2\right)$, $y^*\left(1;1/4\right)$, and $y^*\left(1;1/8\right)$; then compute $y^{**}\left(1;1/4\right)$ and $y^{**}\left(1;1/8\right)$.]

(d) Table 1.2 also contains Euler's approximation for $y(1)$ when $h=1/16$. Use this additional information to compute the next step in the extrapolation procedure; that is, compute $y^{****}(1;1/16)$.

Feedback and the Op Amp

The operational amplier (op amp) depicted in Figure 0.1(a) is a nonlinear device. Thanks to internal power sources, concatenated transistors, etc., it delivers a huge negative voltage at the output terminal O whenever the voltage at its inverting terminal $(-)$ exceeds that at its noninverting terminal $(+)$, and a huge positive voltage when the situation is reversed. One could express $E_{\text{out}} \approx G(E_{\text{in}}^{+} - E_{\text{in}}^{-})$ with a large *gain* G (sometimes 1000 or more), but the approximation would be too unreliable for many applications. Engineers have come up with a way to tame this unruly device by employing *negative feedback*, as illustrated in Figure 0.1(b). By connecting the output to the *inverting* input terminal, the op amp acts like a policeman, preventing any unbalance between the inverting and noninverting input voltages. With such a connection, then, the inverting and noninverting voltages are maintained at the same value: 0 V (electrical *ground*), for the situation depicted.

Furthermore, the input terminals of the op amp do not draw any current; whatever currentis fed to the inverting terminal is immediately redirected to the feedback path. As a result the current drawn from the indicated source $E(t)$ is governed by the equivalent circuit shown in Figure 0.1(c):

$$E(t) = \frac{1}{C}\int I(t)dt \quad \text{or} \quad I(t) = C\frac{dE}{dt},$$

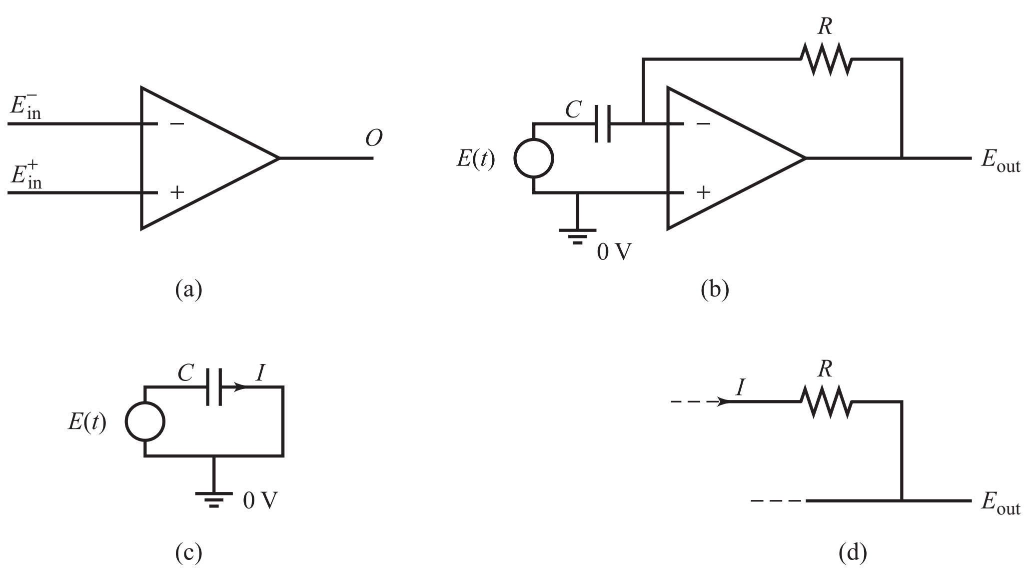

Figure 0.1: Op amp differentiator

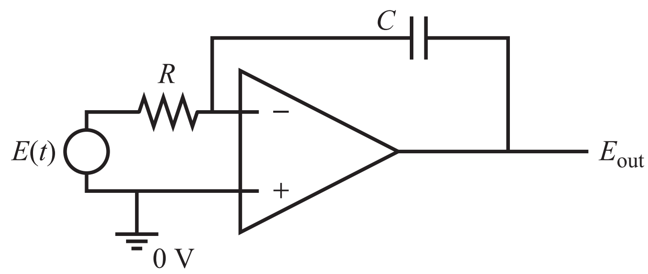

Figure 0.2: Op amp integrator

and this current I flows through the resistor R in Figure 0.1(d), causing a voltage drop from 0 to $-RI$. In other words, the output voltage $E_{\text{out}} = -RI = -RC(dE/dt)$ is a scaled and inverted replica of the derivative of the source voltage. The circuit is an **op amp differentiator.**

(a) Mimic this analysis to show that the circuit in Figure 0.2 is an **op amp integrator** with

$$E_{\text{out}} = -\frac{1}{RC}\int E(t)dt,$$

up to a constant that depends on the initial charge on the capacitor.

(b) Design op amp integrators and differentiators using negative feedback but with inductors instead of capacitors. (In most situations, capacitors are less expensive than inductors, so the previous designs are preferred.)

Bang-Bang Controls

In Example 3 of Section 3.3 (page 93), it was assumed that the amount of heating or cooling supplied by a furnace or air conditioner is proportional to the difference between the actual temperature and the desired temperature; recall the equation

$$U(t) = K_U[T_D - T(t)].$$

In many homes the heating/cooling mechanisms deliver a *constant* rate of heat flow, say,

$$U(t) = \begin{cases} K_1, & \text{if} \quad T(t) > T_D, \\ K_2, & \text{if} \quad T(t) < T_D \end{cases}$$

(with $K_1 < 0$).

(a) Modify the differential equation (9) in Example 3 on page 93 so that it describes the temperature of a home employing this "bang-bang" control law.

(b) Suppose the initial temperature $T(0)$ is greater than T_D. Modify the constants in the solution (12), page 94, so that the formula is valid as long as $T(t) > T_D$.

(c) If the initial temperature $T(0)$ is less than T_D, what values should the constants in (12) take to make the formula valid for $T(t) < T_D$?

(d) How does one piece the solutions in (b) and (c) to obtain a complete time description of the temperature $T_{(t)}$?

Projects for Chapter 5

Effects of Hunting on Predator–Prey Systems

As discussed in Section 5.3 (p. 191), cyclic variations in the population of predators and their prey have been studied using the Volterra-Lotka predator–prey model

$$\frac{dx}{dt} = Ax - Bxy\,, \tag{0.4}$$

$$\frac{dy}{dt} = -Cy + Dxy\,, \tag{0.5}$$

where A, B, C, and D are positive constants, $x(t)$ is the population of prey at time t, and $y(t)$ is the population of predators. It can be shown that such a system has a periodic solution. That is, there exists some constant T such that $x(t) = x(t+T)$ and $y(t) = y(t+T)$ for all t. The periodic or cyclic variation in the population has been observed in various systems such as sharks–food fish, lynx–rabbits, and ladybird beetles–cottony cushion scale. Because of this periodic behavior, it is useful to consider the average population $\overline{x}$ and $\overline{y}$ defined by

$$\overline{x} := \frac{1}{T}\int_0^t x(t)dt\,, \qquad \overline{y} := \frac{1}{T}\int_0^t y(t)dt\,.$$

(a) Show that $\overline{x} = C/D$ and $\overline{y} = A/B$. [Hint: Use equation (0.4) and the fact that $x(0) = x(T)$ to show that

$$\int_0^T [A - By(t)]\,dt = \int_0^T \frac{x'(t)}{x(t)}\,dt = 0\,.$$

(b) To determine the effect of indiscriminate hunting on the population, assume hunting reduces the rate of change in a population by a constant times the population. Then the predator–prey system satisfies the new set of equations

$$\frac{dx}{dt} = Ax - Bxy - \varepsilon x = (A-\varepsilon)x - Bxy\,, \tag{0.6}$$

$$\frac{dy}{dt} = -Cy + Dxy - \delta y = -(C+\delta)y + Dxy\,, \tag{0.7}$$

where ε and δ are positive constants with $\varepsilon < A$. What effect does this have on the average population of prey? On the average population of predators?

(c) Assume the hunting was done selectively, as in shooting only rabbits (or shooting only lynx). Then we have $\varepsilon > 0$ and $\delta = 0$ (or $\varepsilon = 0$ and $\delta > 0$) in (0.6)–(0.7). What effect does this have on the average populations of predator and prey?

(d) In a rural county, foxes prey mainly on rabbits but occasionally include a chicken in their diet. The farmers decide to put a stop to the chicken killing by hunting the foxes. What do you predict will happen? What will happen to the farmers' gardens?

Limit Cycles

In the study of triode vacuum tubes, one encounters the van der Pol equation[2]

$$y'' - \mu\left(1 - y^2\right)y' + y = 0\,,$$

where the constant μ is regarded as a parameter. In Section 4.8, we used the mass-spring oscillator analogy to argue that the non-zero solutions to the van der Pol equation with $\mu = 1$ should approach a periodic limit cycle. The same argument applies for any positive value of μ.

(a) Recast the van der Pol equation as a system in normal form and use software to plot some typical trajectories for $\mu = 0.1$, 1, and 10. Re-scale the plots if necessary until you can discern the limit cycle trajectory; find trajectories that spiral in, and ones that spiral out, to the limit cycle.

(b) Now let $\mu = -0.1$, -1, and -10. Try to predict the nature of the solutions using the mass-spring analogy. Then use the software to check your predictions. Are there limit cycles? Do the neighboring trajectories spiral into, or spiral out from, the limit cycles?

(c) Repeat parts (a) and (b) for the Rayleigh equation

$$y'' - \mu\left[1 - (y')^2\right]y' + y = 0\,.$$

[2] *Historical Footnote*: Experimental research by **E. V. Appleton** and **B. van der Pol** in 1921 on the oscillation of an electrical circuit containing a triode generator (vacuum tube) led to the non-linear equation now called 0 **van der Pol's equation**. Methods of solution were developed by van der Pol in 1926–1927. **Mary L. Cartwright** continued research into non-linear oscillation theory and together with **J. E. Littlewood** obtained existence results for forced oscillations in non-linear systems in 1945.

A Growth Model for Phytoplankton—Part I

Courtesy of Dr. Olivier Bernard and Dr. Jean-Luc Gouzé, INRIA

A chemostat is a stirred tank in which phytoplankton grow by consuming a nutrient (e.g., nitrate). The nutrient is supplied to the tank at a given rate, and a solution containing the phytoplankton and remaining nutrient is removed at an equal rate (cf. Figure 0.3). The chemostat reproduces *in vitro* the conditions of the growth of phytoplankton in the ocean; the phytoplankton is the first element of the marine food chain.

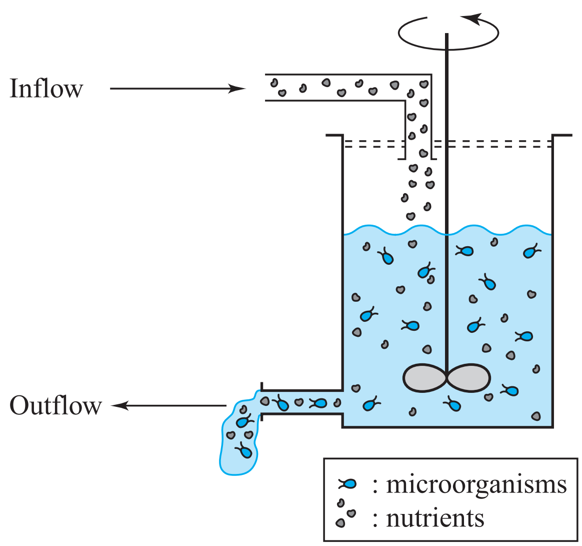

Figure 0.3: Chemostat

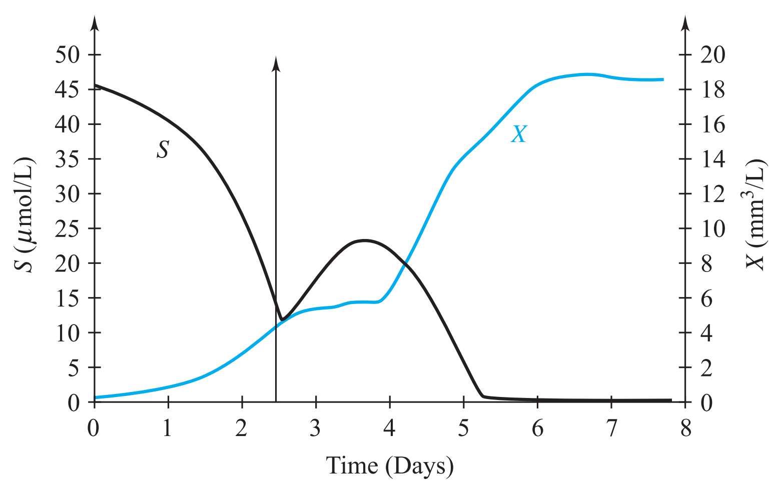

Figure 0.4: Nutrient/biovolume data

Let S denote the concentration (in μmol/liter) of the nutrient and X the biovolume (which is to be taken as an estimation of the biomass) of phytoplankton (in mm^3 of cells per liter of solution). A classical model (J. Monod, *La technique de culture continue: théorie et applications*. Annales de I'Institut Pasteur, 79, 1950) of the behavior of the chemostat is the following:

$$\begin{cases} \dfrac{DX}{dt} = \rho \dfrac{SX}{S+k} - \alpha X \\ \dfrac{dS}{dt} = \alpha(s_i - S) - \dfrac{\rho}{y} \dfrac{SX}{S+K}. \end{cases} \tag{0.8}$$

The unit for time is the day; the dilution rate α and the growth rate ρ are in day^{-1}; and the input concentration s_i and the constant k have the same units as S. Experimental (smoothed) data, obtained from the Station Zoologique of Villefrance-sur-Mer in France, are displayed in Figure 0.4.

(i) When $t < 2.5$, the dilution rate α is zero ("batch culture"). It is known that k is in the range $0.1 \leq k \leq 1$.

(a) What are the units for the yield factor y?

(b) Write a linear approximation of the system (0.8) for $S >> k$ (i.e., S is much larger than k).

(c) Solve the approximated system in part (b) and use the solution for X and the experimental data in Figure 0.4 to obtain a numerical value for ρ. [*Hint*: Plot the logarithm of X against the time, and estimate the slope.] Use the equation for S and the data to obtain an estimation of y.

(d) Take $k = 0.5$ and the values for ρ and y obtained in (c). Using a computer software package and the initial conditions $X(0) = 0.15, S(0) = 45.84$, draw the numerical solutions $X(t), S(t)$ for the system (0.8) and for the approximated system of part (b). Is the approximation of part (b) a reasonable one?

(ii) For $t > 2.5$, the dilution rate is $\alpha = 1.06 \text{ day}^{-1}$. After a delay, the growth rate ρ of the phytoplankton changes because the cells adapt themselves to their new environment.

(e) Estimate the time T when the growth rate changes and obtain the new value for ρ. (As above, take $k = 0.5$.)

Project for Chapter 12

A Growth Model for Phytoplankton—Part II

Courtesy of Dr. Olivier Bernard and Dr. Jean-Luc Gouzé, INRIA, Sophia-Antipolis

In the first part of this project (Chapter 5), we used experimental data to identify the parameters of the following model for the chemostat:

$$\begin{cases} \dfrac{dX}{dt} = \rho \dfrac{SX}{S+k} - \alpha X, \\ \dfrac{dS}{dt} = \alpha(s_i - S) - \dfrac{\rho}{y}\dfrac{SX}{S+K}. \end{cases} \tag{0.9}$$

Suppose $\rho > \alpha$, as in the experiments.

(a) Compute the two equilibria (critical points) of the system.

(b) Determine the corresponding linear systems for the two equilibria. Study their stability and show that there is one saddle point and one stable node.

(c) Show that $S + X/y$ obeys a simple equation. Write the analytical solution and show that

$$\lim_{t\to\infty}(S + X/y) = S_i. \tag{0.10}$$

What does it mean on the phase diagram?

(d) For $t > 4$, re-plot the experimental data (Figure 0.4) in the (X, S)-plane.

(e) Using (c) and the fact that at the experimental equilibrium S is very small (cf. Figure 0.4), estimate s_i from the data and equation (0.10) (take $y = 0.13$).

(f) In the phase plane, draw the isoclines $dS/DX = 0$ and $dX/dS = 0$ corresponding to the estimated parameters $\rho = 2, k = 0.5, y = 0.13, \alpha = 1.06$, and the value of S_i obtained in (e). Compare with (d).

(g) Sketch a phase diagram for the system (0.9).

(h) Do you think that (0.9) is a reasonable model?

Project for Chapter 13

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Satellite Altitude Stability

In this problem, we determine the orientation at which a satellite in a circular orbit of radius r can maintain a relatively constant facing with respect to a spherical primary (e.g., a planet) of mass M. The torque of gravity on the asymmetric satellite maintains the orientation.

Suppose (x, y, z) and $(\overline{x}, \overline{y}, \overline{z})$ refer to coordinates in two systems that have a common origin at the satellite's center of mass. Fix the xyz-axes in the satellite as principal axes; then let the $\overline{z}$-axis point toward the primary and let the $\overline{x}$-axis point in the direction of the satellite's velocity. The xyz-axes may be rotated to coincide with the $\overline{xyz}$-axes by a rotation ϕ about the x-axis (roll), followed by a rotation θ about the resulting y-axis (pitch), and a rotation ψ about the final z-axis (yaw). Euler's equations from physics (with high terms omitted[3] to obtain approximate solutions valid near $(\phi, \theta, \psi) = (0, 0, 0)$) show that the equations for the rotational motion due to gravity acting on the satellite are

$$
\begin{aligned}
I_x\phi'' &= -4\omega_0^2\left(I_z - I_y\right)\phi - \omega_0\left(I_y - I_z - I_x\right)\psi' \\
I_y\theta'' &= -3\omega_0^2\left(I_x - I_z\right)\theta \\
I_z\psi'' &= -4\omega_0^2\left(I_y - I_x\right)\psi + \omega_0\left(I_y - I_z - I_x\right)\phi',
\end{aligned}
$$

where $\omega_0 = \sqrt{(GM)/r^3}$ is the angular frequency of the orbit and the positive constants I_x, I_y, I_z are the moments of inertia of the satellite about the x, y, and z-axes.

(a) Find constants $c_1, \ldots, c_5$ such that these equations can be written as two systems

$$
\frac{d}{dt}\begin{bmatrix} \phi \\ \psi \\ \phi' \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1 & 0 & 0 & c_2 \\ 0 & c_3 & c_4 & 0 \end{bmatrix}\begin{bmatrix} \phi \\ \psi \\ \phi' \\ \psi' \end{bmatrix}
$$

and

$$
\frac{d}{dt}\begin{bmatrix} \theta \\ \theta' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c_5 & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \theta' \end{bmatrix}.
$$

[3] The derivation of these equations is found in *Attitude Stabilization and Control of Earth Satellites*, by O. H. Gerlach, Space Science Reviews, #4 (1965), 541–566.

(b) Show that the origin is asymptotically stable for the first system in (a) if

$$\begin{aligned}&(c_2c_4 + c_3 + c_1)^2 - 4c_1c_3 > 0\,,\\&c_1c_3 > 0\,,\\&c_2c_4 + c_3 + c_1 > 0\end{aligned}$$

and hence deduce that $I_y > I_x > I_z$ yields an asymptotically stable origin. Are there other conditions on the moments of inertia by which the origin is stable?

(c) Show that, for the asymptotically stable configuration in (b), the second system in (a) becomes a harmonic oscillator problem, and find the frequency of oscillation in terms of I_x, I_y, I_z, and ω_0. Phobos maintains $I_y > I_x > I_z$ in its orientation with respect to Mars, and has angular frequency of orbit $\omega_0 = 0.82$ rad/hr. If $(I_x - I_z)/I_y = 0.23$, show that the period of the libration for Phobos (the period with which the side of Phobos facing Mars shakes back and forth) is about 9 hours.

CHAPTER 1: Introduction

EXERCISES 1.1: Background

2. This equation is an ODE because it contains no partial derivatives. Since the highest order derivative is d^2y/dx^2, the equation is a second order equation. This same term also shows us that the independent variable is x and the dependent variable is y. This equation is linear.

4. This equation is a PDE of the second order because it contains second partial derivatives. x and y are independent variables, and u is the dependent variable.

6. This equation is an ODE of the first order with the independent variable t and the dependent variable x. It is nonlinear.

8. ODE of the second order with the independent variable x and the dependent variable y, nonlinear.

10. ODE of the fourth order with the independent variable x and the dependent variable y, linear.

12. ODE of the second order with the independent variable x and the dependent variable y, nonlinear.

14. The velocity at time t is the rate of change of the position function $x(t)$, i.e., x'. Thus,

$$\frac{dx}{dt} = kx^4,$$

where k is the proportionality constant.

16. The equation is

$$\frac{dA}{dt} = kA^2,$$

where k is the proportionality constant.

EXERCISES 1.2: Solutions and Initial Value Problems

2. **(a)** Writing the given equation in the form $y^2 = 3 - x$, we see that it defines two functions of x on $x \le 3$, $y = \pm\sqrt{3-x}$. Differentiation yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\pm\sqrt{3-x}\right) = \pm\frac{d}{dx}\left[(3-x)^{1/2}\right] \\ &= \pm\frac{1}{2}(3-x)^{-1/2}(-1) = -\frac{1}{\pm 2\sqrt{3-x}} = -\frac{1}{2y}\,. \end{aligned}$$

(b) Solving for y yields

$$\begin{aligned} & y^3(x - x\sin x) = 1 \qquad \Rightarrow \qquad y^3 = \frac{1}{x(1-\sin x)} \\ \Rightarrow \qquad & y = \frac{1}{\sqrt[3]{x(1-\sin x)}} = [x(1-\sin x)]^{-1/3}\,. \end{aligned}$$

The domain of this function is $x \ne 0$ and

$$\sin x \ne 1 \qquad \Rightarrow \qquad x \ne \frac{\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots\,.$$

For $0 < x < \pi/2$, one has

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left\{[x(1-\sin x)]^{-1/3}\right\} = -\frac{1}{3}[x(1-\sin x)]^{-1/3-1}\frac{d}{dx}[x(1-\sin x)] \\ &= -\frac{1}{3}[x(1-\sin x)]^{-1}[x(1-\sin x)]^{-1/3}[(1-\sin x) + x(-\cos x)] \\ &= \frac{(x\cos x + \sin x - 1)y}{3x(1-\sin x)}. \end{aligned}$$

We also remark that the given relation is an implicit solution on any interval not containing points $x = 0, \pi/2 + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

4. Differentiating the function $x = 2\cos t - 3\sin t$ twice, we obtain

$$x' = -2\sin t - 3\cos t, \quad x'' = -2\cos t + 3\sin t.$$

Thus,

$$x'' + x = (-2\cos t + 3\sin t) + (2\cos t - 3\sin t) = 0$$

for any t on $(-\infty, \infty)$. Thus, the answer is "Yes".

6. Substituting $x = \cos 2t$ and $x' = -2\sin 2t$ into the given equation yields

$$(-2\sin 2t) + t\cos 2t = \sin 2t \qquad \Leftrightarrow \qquad t\cos 2t = 3\sin 2t\,.$$

Clearly, this is not an identity and, therefore, the function $x = \cos 2t$ is not a solution.