2

Preliminary calculus

2.2 *Find from first principles the first derivative of* $(x + 3)^2$ *and compare your answer with that obtained using the chain rule.*

Using the definition of a derivative, we consider the difference between $(x+\Delta x+3)^2$ and $(x + 3)^2$, and determine the following limit (if it exists):

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x + \Delta x + 3)^2 - (x + 3)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[(x + 3)^2 + 2(x + 3)\Delta x + (\Delta x)^2] - (x + 3)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(2(x + 3)\Delta x + (\Delta x)^2}{\Delta x} \\ &= 2x + 6. \end{aligned}$$

The limit does exist, and so the derivative is $2x + 6$.
Rewriting the function as $f(x) = u^2$, where $u(x) = x + 3$, and using the chain rule:

$$f'(x) = 2u \times \frac{du}{dx} = 2u \times 1 = 2u = 2x + 6,$$

i.e. the same, as expected.

2.4 *Find the first derivatives of*

(a) $x/(a + x)^2$, (b) $x/(1 - x)^{1/2}$, (c) $\tan x$, *as* $\sin x/\cos x$,

(d) $(3x^2 + 2x + 1)/(8x^2 - 4x + 2)$.

In each case, using (2.13) for a quotient:

(a) $$f'(x) = \frac{[\,(a+x)^2 \times 1\,] - [\,x \times 2(a+x)\,]}{(a+x)^4} = \frac{a^2 - x^2}{(a+x)^4} = \frac{a-x}{(a+x)^3};$$

(b) $$f'(x) = \frac{[\,(1-x)^{1/2} \times 1\,] - [\,x \times -\frac{1}{2}(1-x)^{-1/2}\,]}{1-x} = \frac{1 - \frac{1}{2}x}{(1-x)^{3/2}};$$

(c) $$f'(x) = \frac{[\,\cos x \times \cos x\,] - [\,\sin x \times (-\sin x)\,]}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x;$$

(d) $$f'(x) = \frac{[(8x^2 - 4x + 2) \times (6x+2)] - [(3x^2 + 2x + 1) \times (16x - 4)]}{(8x^2 - 4x + 2)^2}$$

$$= \frac{x^3(48 - 48) + x^2(16 - 24 + 12 - 32) + \cdots}{(8x^2 - 4x + 2)^2}$$

$$\frac{\cdots + x(-8 + 12 + 8 - 16) + (4 + 4)}{(8x^2 - 4x + 2)^2}$$

$$= \frac{-28x^2 - 4x + 8}{(8x^2 - 4x + 2)^2} = \frac{-7x^2 - x + 2}{(4x^2 - 2x + 1)^2}.$$

2.6 *Show that the function* $y(x) = \exp(-|x|)$ *defined as*

$$\begin{aligned} \exp\ x \quad & \textit{for } x < 0, \\ 1 \quad & \textit{for } x = 0, \\ \exp(-x) \quad & \textit{for } x > 0, \end{aligned}$$

is not *differentiable at* $x = 0$. *Consider the limiting process for both* $\Delta x > 0$ *and* $\Delta x < 0$.

For $x > 0$, let $\Delta x = \eta$. Then,

$$y'(x > 0) = \lim_{\eta \to 0} \frac{e^{-0-\eta} - 1}{\eta}$$

$$= \lim_{\eta \to 0} \frac{1 - \eta + \frac{1}{2!}\eta^2 \cdots - 1}{\eta} = -1.$$

For $x < 0$, let $\Delta x = -\eta$. Then,

$$y'(x > 0) = \lim_{\eta \to 0} \frac{e^{0-\eta} - 1}{-\eta}$$

$$= \lim_{\eta \to 0} \frac{1 - \eta + \frac{1}{2!}\eta^2 \cdots - 1}{-\eta} = 1.$$

The two limits are not equal, and so $y(x)$ is not differentiable at $x = 0$.

2.8 *If* $2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$, *show that* $dy/dx = 16$ *when* $x = 1$.

For this equation neither x nor y can be made the subject of the equation, i.e neither can be written explicitly as a function of the other, and so we are forced to use implicit differentiation. Starting from

$$2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$$

implicit differentiation, and the use of the chain rule when differentiating $\sin y$ with respect to x, gives

$$2\frac{dy}{dx} + \cos y \frac{dy}{dx} = 4x^3 + 12x^2.$$

When $x = 1$ the original equation reduces to $2y + \sin y = 2\pi$ with the obvious (and unique, as can be verified from a simple sketch) solution $y = \pi$. Thus, with $x = 1$ and $y = \pi$,

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{4 + 12}{2 + \cos \pi} = 16.$$

2.10 *The function* $y(x)$ *is defined by* $y(x) = (1 + x^m)^n$.

(a) *Use the chain rule to show that the first derivative of* y *is* $nmx^{m-1}(1+x^m)^{n-1}$.

(b) *The binomial expansion (see section 1.5) of* $(1 + z)^n$ *is*

$$(1 + z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!} z^r + \cdots .$$

Keeping only the terms of zeroth and first order in dx, *apply this result twice to derive result* (a) *from first principles.*

(c) *Expand* y *in a series of powers of* x *before differentiating term by term. Show that the result is the series obtained by expanding the answer given for* dy/dx *in part* (a).

(a) Writing $1 + x^m$ as u, $y(x) = u^n$, and so $dy/du = nu^{n-1}$, whilst $du/dx = mx^{m-1}$. Thus, from the chain rule,

$$\frac{dy}{dx} = nu^{n-1} \times mx^{m-1} = nmx^{m-1}(1 + x^m)^{n-1}.$$

(b) From the defining process for a derivative,

$$
\begin{aligned}
y'(x) &= \lim_{\Delta x \to 0} \frac{[1 + (x + \Delta x)^m]^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[1 + x^m(1 + \frac{\Delta x}{x})^m]^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[1 + x^m(1 + \frac{m\Delta x}{x} + \cdots)]^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(1 + x^m + mx^{m-1}\Delta x + \cdots)^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left[(1 + x^m)\left(1 + \frac{mx^{m-1}\Delta x}{1+x^m} + \cdots\right)\right]^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(1 + x^m)^n\left(1 + \frac{mnx^{m-1}\Delta x}{1+x^m} + \cdots\right) - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{mn(1 + x^m)^{n-1}x^{m-1}\Delta x + \cdots}{\Delta x} \\
&= nmx^{m-1}(1 + x^m)^{n-1},
\end{aligned}
$$

i.e. the same as the result in part (a).

(c) Expanding in a power series before differentiating:

$$
\begin{aligned}
y(x) &= 1 + nx^m + \frac{n(n-1)}{2!}\,x^{2m} + \cdots \\
&\quad + \frac{n(n-1)\cdots(n-r+1)}{r!}\,x^{rm} + \cdots, \\
y'(x) &= m\,nx^{m-1} + \frac{2m\,n(n-1)}{2!}\,x^{2m-1} + \cdots \\
&\quad + \frac{rm\,n(n-1)\cdots(n-r+1)}{r!}\,x^{rm-1} + \cdots.
\end{aligned}
$$

Now, expanding the result given in part (a) gives

$$
\begin{aligned}
y'(x) &= nmx^{m-1}(1 + x^m)^{n-1} \\
&= nmx^{m-1}\left(1 + \cdots + \frac{(n-1)(n-2)\cdots(n-s)}{s!}\,x^{ms} + \cdots\right) \\
&= nmx^{m-1} + \cdots + \frac{mn(n-1)(n-2)\cdots(n-s)}{s!}\,x^{ms+m-1} + \cdots.
\end{aligned}
$$

This is the same as the previous expansion of $y'(x)$ if, in the general term, the index is moved by one, i.e. $s = r - 1$.

2.12 *Find the positions and natures of the stationary points of the following functions:*

(a) $x^3 - 3x + 3$; (b) $x^3 - 3x^2 + 3x$; (c) $x^3 + 3x + 3$;
(d) $\sin ax$ *with* $a \neq 0$; (e) $x^5 + x^3$; (f) $x^5 - x^3$.

In each case, we need to determine the first and second derivatives of the function. The zeros of the 1st derivative give the positions of the stationary points, and the values of the 2nd derivatives at those points determine their natures.

(a) $$y = x^3 - 3x + 3; \quad y' = 3x^2 - 3; \quad y'' = 6x.$$

$y' = 0$ has roots at $x = \pm 1$, where the values of y'' are ± 6. Therefore, there is a minimum at $x = 1$ and a maximum at $x = -1$.

(b) $$y = x^3 - 3x^2 + 3x; \quad y' = 3x^2 - 6x + 3; \quad y'' = 6x - 6.$$

$y' = 0$ has a double root at $x = 1$, where the value of y'' is 0. Therefore, there is a point of inflection at $x = 1$, but no other stationary points. At the point of inflection, the tangent to the curve $y = y(x)$ is horizontal.

(c) $$y = x^3 + 3x + 3; \quad y' = 3x^2 + 3; \quad y'' = 6x.$$

$y' = 0$ has no real roots, and so there are no stationary points.

(d) $$y = \sin ax; \quad y' = a \cos ax; \quad y'' = -a^2 \sin ax.$$

$y' = 0$ has roots at $x = (n + \frac{1}{2})\pi/a$ for integer n. The corresponding values of y'' are $\mp a^2$, depending on whether n is even or odd. Therefore, there is a maximum for even n and a minimum where n is odd.

(e) $$y = x^5 + x^3; \quad y' = 5x^4 + 3x^2; \quad y'' = 20x^3 + 6x.$$

$y' = 0$ has, as its only real root, a double root at $x = 0$, where the value of y'' is 0. Thus, there is a (horizontal) point of inflection at $x = 0$, but no other stationary point.

(f) $$y = x^5 - x^3; \quad y' = 5x^4 - 3x^2; \quad y'' = 20x^3 - 6x.$$

$y' = 0$ has a double root at $x = 0$ and simple roots at $x = \pm(\frac{3}{5})^{1/2}$, where the respective values of y'' are 0 and $\pm 6(\frac{3}{5})^{1/2}$. Therefore, there is a point of inflection at $x = 0$, a maximum at $x = -(\frac{3}{5})^{1/2}$ and a minimum at $x = (\frac{3}{5})^{1/2}$.

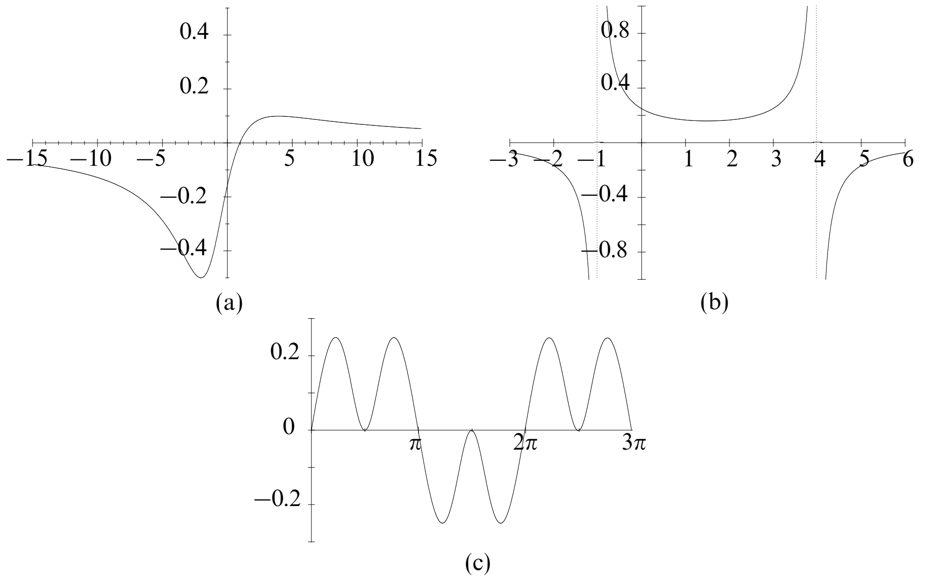

Figure 2.1 The solutions to exercise 2.14.

> ***2.14*** *By finding their stationary points and examining their general forms, determine the range of values that each of the following functions $y(x)$ can take. In each case make a sketch-graph incorporating the features you have identified.*
>
> (a) $y(x) = (x-1)/(x^2+2x+6)$.
> (b) $y(x) = 1/(4+3x-x^2)$.
> (c) $y(x) = (8\sin x)/(15+8\tan^2 x)$.

See figure 2.1 (a)–(c).

(a) Some simple points to calculate for

$$y = \frac{x-1}{x^2+2x+6}$$

are $y(0) = -\frac{1}{6}$, $y(1) = 0$ and $y(\pm\infty) = 0$, and, since the denominator has no real roots ($2^2 < 4 \times 1 \times 6$), there are no infinities. Its 1st derivative is

$$y' = \frac{-x^2+2x+8}{(x^2+2x+6)^2} = \frac{-(x+2)(x-4)}{(x^2+2x+6)^2}.$$

Thus there are turning points only at $x = -2$, with $y(-2) = -\frac{1}{2}$, and at $x = 4$, with $y(4) = \frac{1}{10}$. The former must be a minimum and the latter a maximum. The range in which $y(x)$ lies is $-\frac{1}{2} \le y \le \frac{1}{10}$.

(b) Some simple points to calculate for

$$y = \frac{1}{4 + 3x - x^2}.$$

are $y(0) = \frac{1}{4}$ and $y(\pm\infty) = 0$, approached from negative values. Since the denominator can be written as $(4-x)(1+x)$, the function has infinities at $x = -1$ and $x = 4$ and is positive in the range of x between them.

The 1st derivative is

$$y' = \frac{2x - 3}{(4 + 3x - x^2)^2}.$$

Thus there is only one turning point; this is at $x = \frac{3}{2}$, with corresponding $y(\frac{3}{2}) = \frac{4}{25}$. Since $\frac{3}{2}$ lies in the range $-1 < x < 4$, at the ends of which the function $\to +\infty$, the stationary point must be a minimum. This sets a lower limit on the positive values of $y(x)$ and so the ranges in which it lies are $y < 0$ and $y \geq \frac{4}{25}$.

(c) The function

$$y = \frac{8 \sin x}{15 + 8 \tan^2 x}$$

is clearly periodic with period 2π.

Since $\sin x$ and $\tan^2 x$ are both symmetric about $x = \frac{1}{2}\pi$, so is the function. Also, since $\sin x$ is antisymmetric about $x = \pi$ whilst $\tan^2 x$ is symmetric, the function is antisymmetric about $x = \pi$.

Some simple points to calculate are $y(n\pi) = 0$ for all integers n. Further, since $\tan(n + \frac{1}{2})\pi = \infty$, $y((n + \frac{1}{2})\pi) = 0$. As the denominator has no real roots there are no infinities.

Setting the derivative of $y(x) \equiv 8u(x)/v(x)$ equal to zero, i.e. writing $vu' = uv'$, and expressing all terms as powers of $\cos x$ gives (using $\tan^2 z = \sec^2 z - 1$ and $\sin^2 z = 1 - \cos^2 z$)

$$\begin{aligned}
(15 + 8\tan^2 x)\cos x &= 16 \sin x \tan x \sec^2 x, \\
15 + \frac{8}{\cos^2 x} - 8 &= \frac{16(1 - \cos^2 x)}{\cos^4 x}, \\
7\cos^4 x + 24\cos^2 x - 16 &= 0.
\end{aligned}$$

This quadratic equation for $\cos^2 x$ has roots of $\frac{4}{7}$ and -4. Only the first of these gives real values for $\cos x$ of $\pm\frac{2}{\sqrt{7}}$. The corresponding turning values of $y(x)$ are $\pm\frac{8}{7\sqrt{21}}$. The value of y always lies between these two limits.

2.16 *The curve* $4y^3 = a^2(x+3y)$ *can be parameterised as* $x = a\cos 3\theta$, $y = a\cos\theta$.

(a) *Obtain expressions for* dy/dx (i) *by implicit differentiation and* (ii) *in parameterised form. Verify that they are equivalent.*
(b) *Show that the only point of inflection occurs at the origin. Is it a stationary point of inflection?*
(c) *Use the information gained in* (a) *and* (b) *to sketch the curve, paying particular attention to its shape near the points* $(-a, a/2)$ *and* $(a, -a/2)$ *and to its slope at the 'end points'* (a, a) *and* $(-a, -a)$.

(a) (i) Differentiating the equation of the curve implicitly:

$$12y^2\frac{dy}{dx} = a^2 + 3a^2\frac{dy}{dx}, \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a^2}{12y^2 - 3a^2}.$$

(ii) In parameterised form:

$$\frac{dy}{d\theta} = -a\sin\theta, \quad \frac{dx}{d\theta} = -3a\sin 3\theta, \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-a\sin\theta}{-3a\sin 3\theta}.$$

But, using the results from section 1.2, we have that

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &= 2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta\\ &= \sin\theta(4\cos^2\theta - 1),\end{aligned}$$

thus giving dy/dx as

$$\frac{dy}{dx} = \frac{1}{12\cos^2\theta - 3} = \frac{a^2}{12a^2\cos^2\theta - 3a^2},$$

with $a\cos\theta = y$, i.e. as in (i).

(b) At a point of inflection $y'' = 0$. For the given function,

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right) \times \frac{dy}{dx} = -\frac{a^2}{(12y^2 - 3a^2)^2} \times 24y \times \frac{a^2}{12y^2 - 3a^2}.$$

This can only equal zero at $y = 0$, when $x = 0$ also. But, when $y = 0$ it follows from (a)(i) that $dy/dx = 1/(-3) = -\frac{1}{3}$. As this is non-zero the point of inflection is not a stationary point.

(c) See figure 2.2. Note in particular that the curve has vertical tangents when $y = \pm a/2$ and that $dy/dx = \frac{1}{9}$ at $y = \pm a$, i.e. the tangents at the end points of the 'S'-shaped curve are not horizontal.

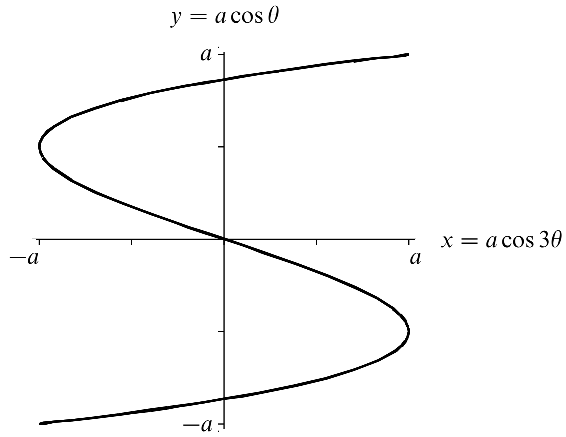

Figure 2.2 The parametric curve described in exercise 2.16.

2.18 *Show that the maximum curvature on the catenary $y(x) = a\cosh(x/a)$ is $1/a$. You will need some of the results about hyperbolic functions stated in subsection 3.7.6.*

The general expression for the curvature, ρ^{-1}, of the curve $y = y(x)$ is

$$\frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{3/2}},$$

and so we begin by calculating the first two derivatives of y. Starting from $y = a\cosh(x/a)$, we obtain

$$y' = a\frac{1}{a}\,\sinh\frac{x}{a},$$
$$y'' = \frac{1}{a}\,\cosh\frac{x}{a}.$$

Therefore the curvature of the catenary at the point (x, y) is given by

$$\frac{1}{\rho} = \frac{\dfrac{1}{a}\,\cosh\dfrac{x}{a}}{\left[1 + \sinh^2\dfrac{x}{a}\right]^{3/2}} = \frac{1}{a}\,\frac{\cosh\dfrac{x}{a}}{\cosh^3\dfrac{x}{a}} = \frac{a}{y^2}.$$

To obtain this result we have used the identity $\cosh^2 z = 1 + \sinh^2 z$. We see that the curvature is maximal when y is minimal; this occurs when $x = 0$ and $y = a$. The maximum curvature is therefore $1/a$.

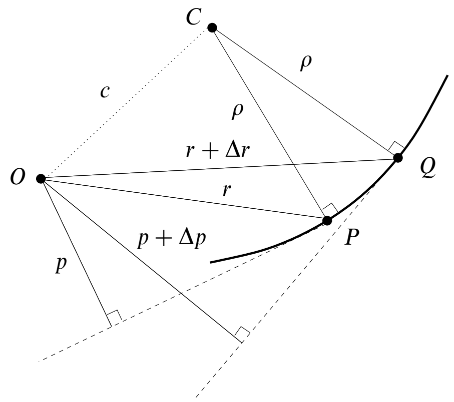

Figure 2.3 The coordinate system described in exercise 2.20.

2.20 *A two-dimensional coordinate system useful for orbit problems is the tangential-polar coordinate system (figure 2.3). In this system a curve is defined by r, the distance from a fixed point O to a general point P of the curve, and p, the perpendicular distance from O to the tangent to the curve at P. By proceeding as indicated below, show that the radius of curvature at P can be written in the form $\rho = r\,dr/dp$.*

Consider two neighbouring points P and Q on the curve. The normals to the curve through those points meet at C, with (in the limit $Q \to P$) $CP = CQ = \rho$. Apply the cosine rule to triangles OPC and OQC to obtain two expressions for c^2, one in terms of r and p and the other in terms of $r + \Delta r$ and $p + \Delta p$. By equating them and letting $Q \to P$ deduce the stated result.

We first note that cos OPC is equal to the sine of the angle between OP and the tangent at P, and that this in turn has the value p/r. Now, applying the cosine rule to the triangles OCP and OCQ, we have

$$\begin{aligned} c^2 &= r^2 + \rho^2 - 2r\rho\cos OPC = r^2 + \rho^2 - 2\rho p \\ c^2 &= (r + \Delta r)^2 + \rho^2 - 2(r + \Delta r)\rho\cos OQC \\ &= (r + \Delta r)^2 + \rho^2 - 2\rho(p + \Delta p). \end{aligned}$$

Subtracting and rearranging then yields

$$\rho = \frac{r\Delta r + \frac{1}{2}(\Delta r)^2}{\Delta p},$$

or, in the limit $Q \to P$, that $\rho = r(dr/dp)$.

2.22 *If $y = \exp(-x^2)$, show that $dy/dx = -2xy$ and hence, by applying Leibnitz' theorem, prove that for $n \geq 1$*

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0.$$

With $y(x) = \exp(-x^2)$,

$$\frac{dy}{dx} = -2x\exp(-x^2) = -2xy.$$

We now take the nth derivatives of both sides and use Leibnitz' theorem to find that of the product xy, noting that all derivatives of x beyond the first are zero:

$$y^{(n+1)} = -2[\,(y^{(n)})(x) + n(y^{(n-1)})(1) + 0\,].$$

i.e.

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0,$$

as stated in the question.

2.24 *Determine what can be learned from applying Rolle's theorem to the following functions $f(x)$:* (a)e^x; (b)x^2+6x; (c)$2x^2+3x+1$; (d)$2x^2+3x+2$; (e)$2x^3-21x^2+60x+k$. (f)*If $k = -45$ in* (e), *show that $x = 3$ is one root of $f(x) = 0$, find the other roots, and verify that the conclusions from* (e) *are satisfied.*

(a) Since the derivative of $f(x) = e^x$ is $f'(x) = e^x$, Rolle's theorem states that between any two consecutive roots of $f(x) = e^x = 0$ there must be a root of $f'(x) = e^x = 0$, i.e. another root of the same equation. This is clearly a contradiction and it is wrong to suppose that there is more than one root of $e^x = 0$. In fact, there are no finite roots of the equation and the only zero of e^x lies formally at $x = -\infty$.

(b) Since $f(x) = x(x+6)$, it has zeros at $x = -6$ and $x = 0$. Therefore the (only) root of $f'(x) = 2x + 6 = 0$ must lie between these values; it clearly does, as $-6 < -3 < 0$.

(c) With $f(x) = 2x^2 + 3x + 1$ and hence $f'(x) = 4x + 3$, any roots of $f(x) = 0$ (actually -1 and $-\frac{1}{2}$) must lie on either side of the root of $f'(x) = 0$, i.e. $x = -\frac{3}{4}$. They clearly do.

(d) This is as in (c), but there are no real roots. However, it can be more generally stated that if there are two values of x that give $2x^2 + 3x + k$ equal values then they lie one on each side of $x = -\frac{3}{4}$.

(e) With $f(x) = 2x^3 - 21x^2 + 60x + k$,

$$f'(x) = 6x^2 - 42x + 60 = 6(x-5)(x-2)$$

and $f'(x) = 0$ has roots 2 and 5. Therefore, if $f(x) = 0$ has three real roots α_i with $\alpha_1 < \alpha_2 < \alpha_3$, then $\alpha_1 < 2 < \alpha_2 < 5 < \alpha_3$.

(f) When $k = -45$, $f(3) = 54 - 189 + 180 - 45 = 0$ and so $x = 3$ is a root of $f(x) = 0$ and $(x-3)$ is a factor of $f(x)$. Writing $f(x) = 2x^3 - 21x^2 + 60x - 45$ as $(x-3)(a_2x^2 + a_1x + a_0)$ and equating coefficients gives $a_2 = 2$, $a_1 = -15$ and $a_0 = 15$. The other two roots are therefore

$$\frac{15 \pm \sqrt{225 - 120}}{4} = \frac{1}{4}(15 \pm \sqrt{105}) = 1.19 \text{ or } 6.31.$$

Result (e) is verified in this case since $1.19 < 2 < 3 < 5 < 6.31$.

2.26 *Use the mean value theorem to establish bounds*

(a) *for* $-\ln(1-y)$, *by considering* $\ln x$ *in the range* $0 < 1 - y < x < 1$,
(b) *for* $e^y - 1$, *by considering* $e^x - 1$ *in the range* $0 < x < y$.

(a) The mean value theorem applied to $\ln x$ within limits $1 - y$ and 1 gives

$$\frac{\ln(1) - \ln(1-y)}{1 - (1-y)} = \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad (*)$$

for some x in the range $1 - y < x < 1$. Now, since $1 - y < x < 1$ it follows that

$$\frac{1}{1-y} > \frac{1}{x} > 1,$$

$$\Rightarrow \quad \frac{1}{1-y} > \frac{-\ln(1-y)}{y} > 1,$$

$$\Rightarrow \quad \frac{y}{1-y} > -\ln(1-y) > y.$$

The second line was obtained by substitution from (*).

(b) The mean value theorem applied to $e^x - 1$ within limits 0 and y gives

$$\frac{e^y - 1 - 0}{y - 0} = e^x \quad \text{for some } x \text{ in the range } 0 < x < y.$$

Now, since $0 < x < y$ it follows that

$$1 < e^x < e^y,$$

$$\Rightarrow \quad 1 < \frac{e^y - 1}{y} < e^y,$$

$$\Rightarrow \quad y < e^y - 1 < ye^y.$$

Again, the second line was obtained by substitution for x from the mean value theorem result.

> ***2.28*** *Use Rolle's theorem to deduce that if the equation $f(x) = 0$ has a repeated root x_1 then x_1 is also a root of the equation $f'(x) = 0$.*
>
> (a) *Apply this result to the 'standard' quadratic equation $ax^2 + bx + c = 0$, to show that a necessary condition for equal roots is $b^2 = 4ac$.*
> (b) *Find all the roots of $f(x) = x^3 + 4x^2 - 3x - 18 = 0$, given that one of them is a repeated root.*
> (c) *The equation $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2 = 0$ has a repeated integer root. How many real roots does it have altogether?*

If two roots of $f(x) = 0$ are x_1 and x_2, i.e. $f(x_1) = f(x_2) = 0$, then it follows from Rolle's theorem that there is some x_3 in the range $x_1 \le x_3 \le x_2$ for which $f'(x_3) = 0$. Now let $x_2 \to x_1$ to form the repeated root; x_3 must also tend to the limit x_1, i.e. x_1 is a root of $f'(x) = 0$ as well as of $f(x) = 0$.

(a) A quadratic equation $f(x) = ax^2 + bx + c = 0$ only has two roots and so if they are equal the common root α must also be a root of $f'(x) = 2ax + b = 0$, i.e. $\alpha = -b/2a$. Thus

$$a\frac{b^2}{4a^2} + b\frac{-b}{2a} + c = 0.$$

It then follows that $c - (b^2/4a) = 0$ and that $b^2 = 4ac$.

(b) With $f(x) = x^3 + 4x^2 - 3x - 18$, the repeated root must satisfy

$$f'(x) = 3x^2 + 8x - 3 = (3x - 1)(x + 3) = 0 \quad \text{i.e. } x = \frac{1}{3} \text{ or } x = -3.$$

Trying the two possibilities: $f(\frac{1}{3}) \neq 0$ but $f(-3) = -27 + 36 + 9 - 18 = 0$. Thus $f(x)$ must factorise as $(x + 3)^2(x - b)$, and comparing the constant terms in the two expressions for $f(x)$ immediately gives $b = 2$. Hence, $x = 2$ is the third root.

(c) Here $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2$. As previously, we examine $f'(x) = 0$, i.e. $f'(x) = 4x^3 + 12x^2 + 14x + 6 = 0$. This has to have an integer solution and, by inspection, this is $x = -1$. We can therefore factorise $f(x)$ as the product $(x + 1)^2(a_2x^2 + a_1x + a_0)$. Comparison of the coefficients gives immediately that $a_2 = 1$ and $a_0 = 2$. From the coefficients of x^3 we have $2a_2 + a_1 = 4$; hence $a_1 = 2$. Thus $f(x)$ can be written

$$f(x) = (x + 1)^2(x^2 + 2x + 2) = (x + 1)^2[(x + 1)^2 + 1].$$

The second factor, containing only positive terms, can have no real zeros and hence $f(x) = 0$ has only two real roots (coincident at $x = -1$).

2.30 *Find the following indefinite integrals:*

(a) $\int (4 + x^2)^{-1}\, dx$; (b) $\int (8 + 2x - x^2)^{-1/2}\, dx$ for $2 \le x \le 4$;
(c) $\int (1 + \sin\theta)^{-1}\, d\theta$; (d) $\int (x\sqrt{1 - x})^{-1}\, dx$ for $0 < x \le 1$.

We make reference to the 12 standard forms given in subsection 2.2.3 and, where relevant, select the appropriate model.

(a) Using model 9,

$$\int \frac{1}{4 + x^2}\, dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + c.$$

(b) We rearrange the integrand in the form of model 12:

$$\int \frac{1}{\sqrt{8 + 2x - x^2}}\, dx = \int \frac{1}{\sqrt{8 + 1 - (x - 1)^2}}\, dx = \sin^{-1}\frac{x - 1}{3} + c.$$

(c) See equation (2.35) and the subsequent text.

$$\begin{aligned}\int \frac{1}{1 + \sin\theta}\, d\theta &= \int \frac{1}{1 + \dfrac{2t}{1 + t^2}}\,\frac{2}{1 + t^2}\, dt \\ &= \int \frac{2}{(1 + t)^2}\, dt \\ &= -\frac{2}{1 + t} + c \\ &= -\frac{2}{1 + \tan\dfrac{\theta}{2}} + c.\end{aligned}$$

(d) To remove the square root, set $u^2 = 1 - x$; then $2u\, du = -dx$ and

$$\begin{aligned}\int \frac{1}{x\sqrt{1 - x}}\, dx &= \int \frac{1}{(1 - u^2)u} \times -2u\, du \\ &= \int \frac{-2}{1 - u^2}\, du \\ &= \int \left(\frac{-1}{1 - u} + \frac{-1}{1 + u}\right) du \\ &= \ln(1 - u) - \ln(1 + u) + c \\ &= \ln\frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} + c.\end{aligned}$$

2.32 *Express $x^2(ax+b)^{-1}$ as the sum of powers of x and another integrable term, and hence evaluate*

$$\int_0^{b/a} \frac{x^2}{ax+b}\,dx.$$

We need to write the numerator in such a way that every term in it that involves x contains a factor $ax+b$. Therefore, write x^2 as

$$x^2 = \frac{x}{a}(ax+b) - \frac{b}{a^2}(ax+b) + \frac{b^2}{a^2}.$$

Then,

$$\begin{aligned}\int_0^{b/a} \frac{x^2}{ax+b}\,dx &= \int_0^{b/a} \left(\frac{x}{a} - \frac{b}{a^2} + \frac{b^2}{a^2(ax+b)}\right) dx \\ &= \left[\frac{x^2}{2a} - \frac{bx}{a^2} + \frac{b^2}{a^3}\ln(ax+b)\right]_0^{b/a} \\ &= \frac{b^2}{a^3}\left(\ln 2 - \frac{1}{2}\right).\end{aligned}$$

An alternative approach, consistent with the wording of the question, is to use the binomial theorem to write the integrand as

$$\frac{x^2}{ax+b} = \frac{x^2}{b}\left(1+\frac{ax}{b}\right)^{-1} = \frac{x^2}{b}\sum_{n=0}^{\infty}\left(-\frac{ax}{b}\right)^n.$$

Then the integral is

$$\begin{aligned}\int_0^{b/a} \frac{x^2}{ax+b}\,dx &= \frac{1}{b}\int_0^{b/a} \sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{b}\right)^n x^{n+2}\,dx \\ &= \frac{1}{b}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{b}\right)^n \frac{1}{n+3}\left(\frac{b}{a}\right)^{n+3} \\ &= \frac{b^2}{a^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+3}.\end{aligned}$$

That these two solutions are the same can be seen by writing $\ln 2 - \frac{1}{2}$ as

$$\begin{aligned}\ln 2 - \frac{1}{2} &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right) - \frac{1}{2} \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+3}.\end{aligned}$$

2.34 *Use logarithmic integration to find the indefinite integrals J of the following:*

(a) $\sin 2x/(1+4\sin^2 x)$;
(b) $e^x/(e^x - e^{-x})$;
(c) $(1 + x\ln x)/(x\ln x)$;
(d) $[x(x^n + a^n)]^{-1}$.

To use logarithmic integration each integrand needs to be arranged as a fraction that has the derivative of the denominator appearing in the numerator.

(a) Either by noting that $\sin 2x = 2\sin x\cos x$ and so is proportional to the derivative of $\sin^2 x$ or by recognising that $\sin^2 x$ can be written in terms of $\cos 2x$ and constants and that $\sin 2x$ is then its derivative, we have

$$J = \int \frac{\sin 2x}{1+4\sin^2 x}\,dx$$
$$= \int \frac{2\sin x\cos x}{1+4\sin^2 x}\,dx = \frac{1}{4}\ln(1+4\sin^2 x) + c,$$

or

$$J = \int \frac{\sin 2x}{1+2(1-\cos 2x)}\,dx = \frac{1}{4}\ln(3-2\cos 2x) + c.$$

These two answers are equivalent since $3 - 2\cos 2x = 3 - 2(1 - 2\sin^2 x) = 1 + 4\sin^2 x$.

(b) This is straightforward if it is noticed that multiplying both numerator and denominator by e^x produces the required form:

$$J = \int \frac{e^x}{e^x - e^{-x}}\,dx = \int \frac{e^{2x}}{e^{2x}-1}\,dx = \frac{1}{2}\ln(e^{2x}-1) + c.$$

An alternative, but longer, method is to write the numerator as $\cosh x + \sinh x$ and the denominator as $2\sinh x$. This leads to $J = \frac{1}{2}(x + \ln\sinh x)$, which can be re-written as

$$J = \tfrac{1}{2}(\ln e^x + \ln\sinh x) = \tfrac{1}{2}\ln(e^x\sinh x) = \tfrac{1}{2}\ln(e^{2x}-1) + \tfrac{1}{2}\ln\tfrac{1}{2}.$$

The $\frac{1}{2}\ln\frac{1}{2}$ forms part of c.

(c) Here we must first divide the numerator by the denominator to produce two separate terms, and then twice apply the result that $1/z$ is the derivative of $\ln z$:

$$J = \int \frac{1+x\ln x}{x\ln x}\,dx = \int \left(\frac{1}{x\ln x} + 1\right)dx = \ln(\ln x) + x + c.$$

(d) To put the integrand in a form suitable for logaritmic integration, we must first multiply both numerator and denominator by nx^{n-1} and then use partial fractions so that each denominator contains x only in the form x^m, of which mx^{m-1} is the derivative.

$$\begin{aligned} J &= \int \frac{dx}{x(x^n + a^n)} = \int \frac{nx^{n-1}}{nx^n(x^n + a^n)}\,dx \\ &= \frac{1}{na^n} \int \left(\frac{nx^{n-1}}{x^n} - \frac{nx^{n-1}}{x^n + a^n} \right) dx \\ &= \frac{1}{na^n} \left[\, n \ln x - \ln(x^n + a^n) \,\right] + c \\ &= \frac{1}{na^n} \ln \left(\frac{x^n}{x^n + a^n} \right) + c. \end{aligned}$$

2.36 *Find the indefinite integrals J of the following functions involving sinusoids:*

(a) $\cos^5 x - \cos^3 x$;
(b) $(1 - \cos x)/(1 + \cos x)$;
(c) $\cos x \sin x/(1 + \cos x)$;
(d) $\sec^2 x/(1 - \tan^2 x)$.

(a) As the integrand contains only odd powers of $\cos x$, take $\cos x$ out as a common factor and express the remainder in terms of $\sin x$, of which $\cos x$ is the derivative:

$$\begin{aligned} \cos^5 x - \cos^3 x &= [\,(1 - \sin^2 x)^2 - (1 - \sin^2 x)\,] \cos x \\ &= (\sin^4 x - \sin^2 x) \cos x. \end{aligned}$$

Hence,

$$J = \int (\sin^4 x - \sin^2 x) \cos x \, dx = \frac{1}{5} \sin^5 x - \frac{1}{3} \sin^3 x + c.$$

A more formal way of expressing this approach is to say ‘set $\sin x = u$ with $\cos x \, dx = du$.’

(b) This integral can be found either by writing the numerator and denominator in terms of sinusoidal functions of $x/2$ or by making the substitution $t = \tan(x/2)$. Using first the half-angle identities, we have

$$\begin{aligned} J &= \int \frac{1 - \cos x}{1 + \cos x}\,dx = \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} \\ &= \int \tan^2 \frac{x}{2}\,dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx = 2 \tan \frac{x}{2} - x + c. \end{aligned}$$

The second approach (see subsection 2.2.7) is

$$\begin{aligned}
J &= \int \frac{1 - \dfrac{1-t^2}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}} \, \frac{2\,dt}{1+t^2} \\
&= \int \frac{2t^2}{1+t^2}\, dt \\
&= \int 2\, dt - \int \frac{2}{1+t^2}\, dt \\
&= 2t - 2\tan^{-1} t + c = 2\tan\frac{x}{2} - x + c.
\end{aligned}$$

(c) This integrand, containing only sinusoidal functions, can be converted to an algebraic one by writing $t = \tan(x/2)$ and expressing the functions appearing in the integrand in terms of it,

$$\begin{aligned}
\frac{\cos x \sin x}{1 + \cos x}\, dx &= \frac{\dfrac{1-t^2}{1+t^2}\dfrac{2t}{1+t^2}\dfrac{2}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}}\, dt \\
&= \frac{2t(1-t^2)}{(1+t^2)^2}\, dt \\
&= 2t\left[\frac{A}{(1+t^2)^2} + \frac{B}{1+t^2}\right] dt,
\end{aligned}$$

with $A + B(1+t^2) = 1 - t^2$, implying that $B = -1$ and $A = 2$.
And so, recalling that $1 + t^2 = \sec^2(x/2) = 1/[\cos^2(x/2)]$,

$$\begin{aligned}
J &= \int \left(\frac{4t}{(1+t^2)^2} - \frac{2t}{1+t^2}\right) dt \\
&= -\frac{2}{1+t^2} - \ln(1+t^2) + c \\
&= -2\cos^2\frac{x}{2} + \ln(\cos^2\frac{x}{2}) + c.
\end{aligned}$$

(d) We can either set $\tan x = u$ or show that the integrand is $\sec 2x$ and then use the result of exercise 2.35; here we use the latter method.

$$\int \frac{\sec^2 x}{1 - \tan^2 x}\, dx = \int \frac{1}{\cos^2 x - \sin^2 x}\, dx = \int \sec 2x\, dx.$$

It then follows from the earlier result that $J = \frac{1}{2}\ln(\sec 2x + \tan 2x) + c$. This can also be written as $\frac{1}{2}\ln[(1+\tan x)/(1-\tan x)] + c$.

***2.38** Determine whether the following integrals exist and, where they do, evaluate them:*

(a) $\int_0^\infty \exp(-\lambda x)\,dx$; (b) $\int_{-\infty}^\infty \frac{x}{(x^2+a^2)^2}\,dx$;

(c) $\int_1^\infty \frac{1}{x+1}\,dx$; (d) $\int_0^1 \frac{1}{x^2}\,dx$;

(e) $\int_0^{\pi/2} \cot\theta\,d\theta$; (f) $\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx$.

(a) This is an infinite integral and so we must examine the result of letting the range of a finite integral go to infinity:

$$\int_0^\infty e^{-\lambda x}\,dx = \lim_{R\to\infty}\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^R = \lim_{R\to\infty}\left[\frac{1}{\lambda} - \frac{e^{-\lambda R}}{\lambda}\right].$$

The limit as $R \to \infty$ does exist if $\lambda > 0$ and is then equal to λ^{-1}.

(b) This is also an infinite integral. However, because of the antisymmetry of the integrand, the integral is zero for all finite values of R. It therefore has a limit as $R \to \infty$ of zero, which is consequently the value of the integral.

$$\int_{-\infty}^\infty \frac{x}{(x^2+a^2)^2}\,dx = \lim_{R\to\infty}\left[\frac{-1}{2(x^2+a^2)}\right]_{-R}^R = \lim_{R\to\infty}[0] = 0.$$

(c) The integral is elementary over any finite range $(1, R)$ and so we must examine its behaviour as $R \to \infty$:

$$\int_1^\infty \frac{1}{x+1}\,dx = \lim_{R\to\infty}\,[\ln(1+x)]_1^R = \lim_{R\to\infty}\ln\frac{1+R}{2} = \infty.$$

The limit is not finite and so the integral does not exist.

(d) The integrand, $1/x^2$ is undefined at $x = 0$ and so we must examine the behaviour of the integral with lower limit ϵ as $\epsilon \to 0$.

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\epsilon\to 0}\left[-\frac{1}{x}\right]_\epsilon^1 = \lim_{\epsilon\to 0}\left(-1+\frac{1}{\epsilon}\right) = \infty.$$

As the limit is not finite the integral does not exist.

(e) Again, a infinite quantity ($\cot 0$) appears in the integrand and the limit test has to be applied.

$$\int_0^{\pi/2}\cot\theta\,d\theta = \int_0^{\pi/2}\frac{\cos\theta}{\sin\theta}\,d\theta$$
$$= \lim_{\epsilon\to 0}\,[\,\ln(\sin\theta)\,]_\epsilon^{\pi/2} = \lim_{\epsilon\to 0}[\,0 - \ln(\sin\epsilon)\,] = -(-\infty).$$

The limit is not finite and so the integral does not exist.

(f) Yet again, the integrand has an infinity (at $x = 1$) and the limit test has to be applied

$$\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx = \lim_{z\to 1}\Big[-(1-x^2)^{1/2}\Big]_0^z = 0 + 1 = 1.$$

This time the limit does exist; the integral is defined and has value 1.

2.40 *Show, using the following methods, that the indefinite integral of* $x^3/(x+1)^{1/2}$ *is*

$$J = \tfrac{2}{35}(5x^3 - 6x^2 + 8x - 16)(x+1)^{1/2} + c.$$

(a) *Repeated integration by parts.*
(b) *Setting* $x + 1 = u^2$ *and determining* dJ/du *as* $(dJ/dx)(dx/du)$.

(a) Evaluating the successive integrals produced by the repeated integration by parts:

$$\int \frac{x^3}{(x+1)^{1/2}}\,dx = 2x^3\sqrt{x+1} - \int 3x^2\, 2\sqrt{x+1}\,dx,$$
$$\int x^2\sqrt{x+1}\,dx = \frac{2}{3}x^2(x+1)^{3/2} - \int 2x\,\frac{2}{3}(x+1)^{3/2}\,dx,$$
$$\int x(x+1)^{3/2}\,dx = \frac{2}{5}x(x+1)^{5/2} - \int \frac{2}{5}(x+1)^{5/2}\,dx,$$
$$\int (x+1)^{5/2}\,dx = \frac{2}{7}(x+1)^{7/2}.$$

And so, remembering to carry forward the multiplicative factors generated at each stage, we have

$$\begin{aligned} J &= \sqrt{x+1}\left[2x^3 - 4x^2(x+1) + \frac{16}{5}x(x+1)^2 - \frac{32}{35}(x+1)^3\right] + c \\ &= \frac{2\sqrt{x+1}}{35}\left[5x^3 - 6x^2 + 8x - 16\right] + c. \end{aligned}$$

(b) Set $x + 1 = u^2$, giving $dx = 2u\,du$, to obtain

$$\begin{aligned} J &= \int \frac{(u^2-1)^3}{u}\,2u\,du \\ &= 2\int (u^6 - 3u^4 + 3u^2 - 1)\,du. \end{aligned}$$

This integral is now easily evaluated to give

$$\begin{aligned}
J &= 2\left(\frac{1}{7}u^7 - \frac{3}{5}u^5 + u^3 - u\right) + c \\
&= \frac{2u}{35}(5u^6 - 21u^4 + 35u^2 - 35) + c \\
&= \frac{2u}{35}\,[\,5(x^3 + 3x^2 + 3x + 1) - 21(x^2 + 2x + 1) + 35(x + 1) - 35\,] + c \\
&= \frac{2\sqrt{x+1}}{35}\,[\,5x^3 - 6x^2 + 8x - 16\,] + c.
\end{aligned}$$

i.e. the same final result as for method (a).

2.42 *Define $J(m,n)$, for non-negative integers m and n, by the integral*

$$J(m,n) = \int_0^{\pi/2} \cos^m\theta \sin^n\theta \, d\theta.$$

(a) *Evaluate $J(0,0)$, $J(0,1)$, $J(1,0)$, $J(1,1)$, $J(m,1)$, $J(1,n)$.*
(b) *Using integration by parts prove that, for m and n both > 1,*

$$J(m,n) = \frac{m-1}{m+n}J(m-2,n) \quad \text{and} \quad J(m,n) = \frac{n-1}{m+n}J(m,n-2).$$

(c) *Evaluate (i) $J(5,3)$, (ii) $J(6,5)$, (iii) $J(4,8)$.*

(a) For these special values of m and/or n the integrals are all elementary, as follows.

$$\begin{aligned}
J(0,0) &= \int_0^{\pi/2} 1 \, d\theta = \frac{\pi}{2}, \\
J(0,1) &= \int_0^{\pi/2} \sin\theta \, d\theta = 1, \\
J(1,0) &= \int_0^{\pi/2} \cos\theta \, d\theta = 1, \\
J(1,1) &= \int_0^{\pi/2} \cos\theta\sin\theta \, d\theta = \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{2}, \\
J(m,1) &= \int_0^{\pi/2} \cos^m\theta\sin\theta \, d\theta = \frac{1}{m+1}, \\
J(1,n) &= \int_0^{\pi/2} \cos\theta\sin^n\theta \, d\theta = \frac{1}{n+1}.
\end{aligned}$$

(b) In order to obtain a reduction formula, we ‘sacrifice’ one of the cosine factors so that it can act as the derivative of a sine function, so allowing $\sin^n\theta$ to be integrated. The two extra powers of $\sin\theta$ generated by the integration by parts are then removed by writing them as $1-\cos^2\theta$.

$$\begin{aligned}
J(m,n) &= \int_0^{\pi/2} \cos^{m-1}\theta \sin^n\theta \cos\theta \, d\theta \\
&= \left[\frac{\cos^{m-1}\theta \sin^{n+1}\theta}{n+1}\right]_0^{\pi/2} \\
&\qquad - \int_0^{\pi/2} \frac{(m-1)\cos^{m-2}\theta(-\sin\theta)\sin^{n+1}\theta}{n+1}\, d\theta \\
&= 0 + \frac{m-1}{n+1}\int_0^{\pi/2} \cos^{m-2}\theta(1-\cos^2\theta)\sin^n\theta \, d\theta \\
&= \frac{m-1}{n+1}J(m-2,n) - \frac{m-1}{n+1}J(m,n). \\
J(m,n) &= \frac{m-1}{m+n}J(m-2,n).
\end{aligned}$$

Similarly, by 'sacrificing' a sine term to act as the derivative of a cosine term,

$$J(m,n) = \frac{n-1}{m+n}J(m,n-2).$$

(c) For these specific cases we apply the reduction formulae in (b) to reduce them to one of the forms evaluated in (a).

$$\text{(i)}\;\; J(5,3) = \frac{2}{8}J(5,1) = \frac{2}{8}\frac{1}{6} = \frac{1}{24},$$

$$\text{(ii)}\;\; J(6,5) = \frac{4}{11}\frac{2}{9}J(6,1) = \frac{4}{11}\frac{2}{9}\frac{1}{7} = \frac{8}{693},$$

$$\text{(iii)}\;\; J(4,8) = \frac{3}{12}\frac{1}{10}J(0,8) = \frac{3}{12}\frac{1}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{7\pi}{2048}.$$

2.44 *Evaluate the following definite integrals:*

(a) $\int_0^\infty xe^{-x}\,dx$; (b) $\int_0^1 \left[(x^3+1)/(x^4+4x+1)\right]\,dx$;
(c) $\int_0^{\pi/2}[a+(a-1)\cos\theta]^{-1}\,d\theta$ *with* $a>\frac{1}{2}$; (d) $\int_{-\infty}^{\infty}(x^2+6x+18)^{-1}\,dx$.

(a) Integrating by parts:

$$\int_0^\infty xe^{-x}\,dx = \left[-xe^{-x}\right]_0^\infty - \int_0^\infty -e^{-x}\,dx = 0 + \left[-e^{-x}\right]_0^\infty = 1.$$

(b) This is a logarithmic integration:

$$\int_0^1 \frac{x^3+1}{x^4+4x+1}\,dx = \frac{1}{4}\int_0^1 \frac{4x^3+4}{x^4+4x+1} = \frac{1}{4}\left[\,\ln(x^4+4x+1)\,\right]_0^1 = \frac{1}{4}\ln 6.$$

(c) Writing $t = \tan(\theta/2)$:

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{a+(a-1)\cos\theta}\,d\theta &= \int_0^1 \frac{1}{a+(a-1)\left(\dfrac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2}\\ &= \int_0^1 \frac{2}{2a-1+t^2}\,dt\\ &= \frac{2}{\sqrt{2a-1}}\left[\tan^{-1}\frac{t}{\sqrt{2a-1}}\right]_0^1\\ &= \frac{2}{\sqrt{2a-1}}\tan^{-1}\frac{1}{\sqrt{2a-1}}.\end{aligned}$$

(d) The denominator has no real zeros ($6^2 < 4\times 1\times 18$) and so, completing the square, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+6x+18}\,dx &= \int_{-\infty}^{\infty} \frac{1}{(x+3)^2+9}\,dx\\ &= \frac{1}{3}\left[\tan^{-1}\left(\frac{x+3}{3}\right)\right]_{-\infty}^{\infty}\\ &= \frac{1}{3}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{3}.\end{aligned}$$

2.46 *Find positive constants a, b such that $ax \le \sin x \le bx$ for $0 \le x \le \pi/2$. Use this inequality to find (to two significant figures) upper and lower bounds for the integral*

$$I = \int_0^{\pi/2} (1+\sin x)^{1/2}\,dx.$$

Use the substitution $t = \tan(x/2)$ to evaluate I exactly.

Consider $f(x) = (\sin x)/x$. Its derivative is

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{x - \tan x}{x^2}\,\cos x,$$

which is everwhere negative (or zero) in the given range. This shows that $f(x)$ is a monotonically decreasing function in that range and reaches its lowest value at the end of the range. This value must therefore be $\sin(\pi/2)/(\pi/2)$, i.e. $2/\pi$.

From the standard Maclaurin series for $\sin x$ (subsection 4.6.3)

$$f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots ,$$

and the limit of $f(x)$ as $x \to 0$ is 1. In summary,

$$\frac{2}{\pi} \le \frac{\sin x}{x} \le 1 \quad \text{for} \quad 0 \le x \le \frac{\pi}{2}.$$

It then follows that

$$\int_0^{\pi/2} (1 + \frac{2}{\pi}x)^{1/2}\, dx \le \int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx \le \int_0^{\pi/2} (1 + x)^{1/2}\, dx,$$

$$\left[\frac{\pi}{2}\,\frac{2}{3}(1 + \frac{2}{\pi}x)^{3/2}\right]_0^{\pi/2} \le \quad I \quad \le \left[\frac{2}{3}(1 + x)^{3/2}\right]_0^{\pi/2},$$

$$\frac{\pi}{3}\left[(2)^{3/2} - 1\right] \le \quad I \quad \le \frac{2}{3}\left[(1 + \frac{\pi}{2})^{3/2} - 1\right],$$

$$1.91 \le \quad I \quad \le \quad 2.08.$$

For an exact evaluation we use the standard half-angle formulae:

$$t = \tan\frac{x}{2}, \qquad \sin x = \frac{2t}{1 + t^2}, \qquad dx = \frac{2}{1 + t^2}\, dt.$$

Substitution of these gives

$$\begin{aligned}
\int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx &= \int_0^1 \left(1 + \frac{2t}{1 + t^2}\right)^{1/2} \frac{2}{1 + t^2}\, dt \\
&= \int_0^1 \frac{2 + 2t}{(1 + t^2)^{3/2}}\, dt \\
&= \int_0^1 \frac{2}{(1 + t^2)^{3/2}}\, dt + 2\left[-\frac{1}{(1 + t^2)^{1/2}}\right]_0^1.
\end{aligned}$$

To evaluate the first integral we turn it back into one involving sinusoidal functions and write $t = \tan\theta$ with $dt = \sec^2\theta\, d\theta$. Then the original integral becomes

$$\begin{aligned}
\int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx &= \int_0^{\pi/4} \frac{2\sec^2\theta}{\sec^3\theta}\, d\theta + 2\left[1 - \frac{1}{\sqrt{2}}\right] \\
&= \int_0^{\pi/4} 2\cos\theta\, d\theta + 2 - \sqrt{2} \\
&= 2[\sin\theta]_0^{\pi/4} + 2 - \sqrt{2} \\
&= \sqrt{2} - 0 + 2 - \sqrt{2} = 2.
\end{aligned}$$

An alternative evaluation can be made by setting $x = (\pi/2) - y$ and then writing $1 + \cos y$ in the form $2\cos^2(y/2)$. This gives the final value of 2 more directly.

Whichever method is used in (b), we note that, as it must (or at least should!) the exact value of the integral lies between our calculated bounds.

2.48 *Show that the total length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, which can be parameterised as $x = a\cos^3\theta$, $y = a\sin^3\theta$, is $6a$.*

We first check that $x^{2/3} + y^{2/3} = a^{2/3}$ can be parameterised as $x = a\cos^3\theta$ and $y = a\sin^3\theta$. This is so, since $a^{2/3}\cos^2\theta + a^{2/3}\sin^2\theta = a^{2/3}$ is an identity.

Now the element of length of the curve ds is given by $ds^2 = dx^2 + dy^2$ or, using the parameterisation,

$$\begin{aligned} ds &= \left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{1/2} d\theta \\ &= \left[\left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2\right]^{1/2} d\theta \\ &= 3a\cos\theta\sin\theta\, d\theta. \end{aligned}$$

The total length of the asteroid curve is four times its length in the first quadrant and therefore given by

$$s = 4 \times 3a \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta = 12a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 6a.$$

2.50 *The equation of a cardioid in plane polar coordinates is*

$$\rho = a(1 - \sin\phi).$$

Sketch the curve and find (i) *its area,* (ii) *its total length,* (iii) *the surface area of the solid formed by rotating the cardioid about its axis of symmetry and* (iv) *the volume of the same solid.*

For a sketch of the 'heart-shaped' (actually more apple-shaped) curve see figure 2.4.

To avoid any possible double counting, integrals will be taken from $\phi = \pi/2$ to $\phi = 3\pi/2$ and symmetry used for scaling up.

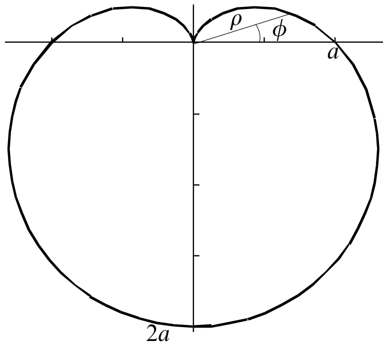

Figure 2.4 The cardioid discussed in exercise 2.50.

(i) Area. In plane polar coordinates this is straightforward.

$$\begin{aligned}\int \frac{1}{2}\rho^2\,d\phi &= 2\int_{\pi/2}^{3\pi/2} \frac{1}{2}a^2(1-\sin\phi)^2\,d\phi\\ &= a^2\int_{\pi/2}^{3\pi/2}(1-2\sin\phi+\sin^2\phi)\,d\phi\\ &= a^2(\pi - 0 + \tfrac{1}{2}\pi)\\ &= \frac{3\pi a^2}{2}.\end{aligned}$$

The third term in the integral was evaluated using the standard result that the average value of the square of a sinusoid over a whole number of quarter cycles is $\frac{1}{2}$.

(ii) Length. Since $ds^2 = d\rho^2 + \rho^2 d\phi^2$, the total length is

$$\begin{aligned}L &= 2\int_{\pi/2}^{3\pi/2}\left[\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2\right]^{1/2} d\phi\\ &= 2\int_{\pi/2}^{3\pi/2}(a^2\cos^2\phi + a^2 - 2a^2\sin\phi + a^2\sin^2\phi)^{1/2}\,d\phi\\ &= 2a\sqrt{2}\int_{\pi/2}^{3\pi/2}(1-\sin\phi)^{1/2}\,d\phi\\ &= 2a\sqrt{2}\int_0^{-\pi}(1-\cos\phi')^{1/2}(-d\phi') \quad \text{where} \quad \phi = \frac{1}{2}\pi - \phi'.\end{aligned}$$

Using the trigonometric half-angle formula $1 - \cos\theta = 2\sin^2(\theta/2)$, this integral is easily evaluated to give

$$L = 2a\sqrt{2}\int_{-\pi}^{0} \sqrt{2}\sin\frac{\phi'}{2}\,d\phi'$$
$$= 4a\left[-2\cos\frac{\phi'}{2}\right]_{-\pi}^{0} = -8a.$$

The negative sign is irrelevant and merely reflects the (inappropriate) choice of taking the positive square root of $\sin^2(\phi'/2)$. The total length of the curve is thus $8a$.

(iii) Surface area of the solid of rotation.
The elemental circular strip at any given value of ρ and ϕ has a total length of $2\pi\rho\cos\phi$ and a width ds (on the surface) given by $(ds)^2 = (d\rho)^2 + (\rho d\phi)^2$. This strip contributes an elemental surface area $2\pi\rho\cos\phi\, ds$ and so the total surface area S of the solid is given by

$$S = \int_{\pi/2}^{3\pi/2} 2\pi\rho\cos\phi\left[\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2\right]^{1/2} d\phi$$
$$= 2\sqrt{2}\pi a^2 \int_{\pi/2}^{3\pi/2} (1-\sin\phi)^{3/2}\cos\phi\, d\phi \quad \text{[using the result from (ii)]}$$
$$= 2\sqrt{2}\pi a^2 \left[-\frac{2}{5}(1-\sin\phi)^{5/2}\right]_{\pi/2}^{3\pi/2}$$
$$= -\frac{32\pi a^2}{5}.$$

Again, the minus sign is irrelevant and arises because, in the range of ϕ used, the elemental strip radius is actually $-\rho\cos\phi$.

(iv) Volume of the solid of rotation.
The height above the origin of any point is $\rho\sin\phi$ and so, for $\pi/2 \le \phi \le 3\pi/2$, the thickness of any elemental disc is $-d(\rho\sin\phi)$ whilst its area is $\pi\rho^2\cos^2\phi$.

It should be noted that this formulation allows correctly for the 'missing' part of the body of revolution – as it were, for the air that surrounds the 'stalk of the apple'. Whilst ϕ is in the range $\pi/2 \le \phi \le 5\pi/6$ (the upper limit being found by maximising $y = \rho\sin\phi = a(1-\sin\phi)\sin\phi$), negative volume is being added to the solid, representing 'the air'. For $5\pi/6 \le \phi \le \pi$ the solid acquires volume as if there were no air core. For the rest of the range, $\pi \le \phi \le 3\pi/2$, such considerations do not arise.