4. (a) Continuity of the first derivative.

(b) For $x_i \le x \le x_{i+1}$:

$$g_i'(x) = g'(x_i)\frac{x - x_{i+1}}{x_i - x_{i+1}} + g'(x_{i+1})\frac{x - x_i}{x_{i+1} - x_i}.$$

Integrating and substituting $g_i(x_i) = f(x_i)$ and $g_i(x_{i+1}) = f(x_{i+1})$, we obtain

$$g'(x_i) + g'(x_{i+1}) = 2\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \quad i = 0, \ldots, N-1$$

These are N equations for the $N+1$ unknowns $g'(x_0), \ldots, g'(x_N)$. One additional equation is required and it can be $g'(x_0) = g'(x_1)$, which means that the interpolant in the first interval is a straight line.

(c) For non-periodic equally-spaced data, the solution of (1.7) requires $O(2N)$ divisions and $O(3N)$ of each additions and multiplications, ignoring the effort in computing the right-hand side. Solving the system in (b) is only $O(N)$ additions.

5. Solve first for $g''(x_0), \ldots, g''(x_N)$ as explained in the text and then differentiate (1.6) to get the first derivative at the data points.

For $x_0 \le x_i \le x_{N-1}$:

$$g'(x_i) = g_i'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - g''(x_i)\frac{h}{3} - g''(x_{i+1})\frac{h}{6}.$$

For x_N:

$$g'(x_N) = g_{N-1}'(x_N) = \frac{f(x_N) - f(x_{N-1})}{h} + g''(x_{N-1})\frac{h}{6} + g''(x_N)\frac{h}{3}.$$

6. (a) For $\sigma = 0$, (1.3) is recovered. For $\sigma \to \infty$ we obtain

$$g_i(x) = f(x_i)\frac{x - x_{i+1}}{x_i - x_{i+1}} + f(x_{i+1})\frac{x - x_i}{x_{i+1} - x_i},$$

which is a straight line.

(b) The given differential equation for g_i is second order, linear, and non-homogeneous. Its solution is:

$$g_i(x) = C_1 e^{\sigma x} + C_2 e^{-\sigma x} - \frac{g''(x_i) - \sigma^2 f(x_i)}{\sigma^2}\frac{x - x_{i+1}}{x_i - x_{i+1}} - \frac{g''(x_{i+1}) - \sigma^2 f(x_{i+1})}{\sigma^2}\frac{x - x_i}{x_{i+1} - x_i}.$$

Differentiating:

$$g_i'(x) = C_1\sigma e^{\sigma x} - C_2\sigma e^{-\sigma x} + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{1}{\sigma^2}\frac{g''(x_{i+1}) - g''(x_i)}{x_{i+1} - x_i}.$$

C_1, C_2, and the second derivatives at the data points are determined as in Section 1.2 with (1.4) and (1.5) replaced by the two equations above.

7. (b,c) `polint`, `spline`, and `splint` are used to obtain the interpolations in Fig. 1.2. The predicted tuition in 2001 is \$10,836 using Lagrange polynomial and \$34,447 using cubic spline. The Lagrange polynomial does a pretty good job interpolating the data but behaves very poorly away from it; the predicted tuition is way too low. The cubic spline behaves well for both interpolation and extrapolation.

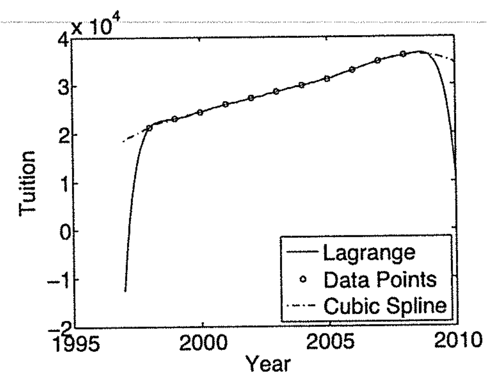

Figure 1.2: Exercise 7.

8. (a) Using `polint`, the interpolation is shown in Fig 1.3. The prediction in 2009 is -38.40 which is unrealistic.

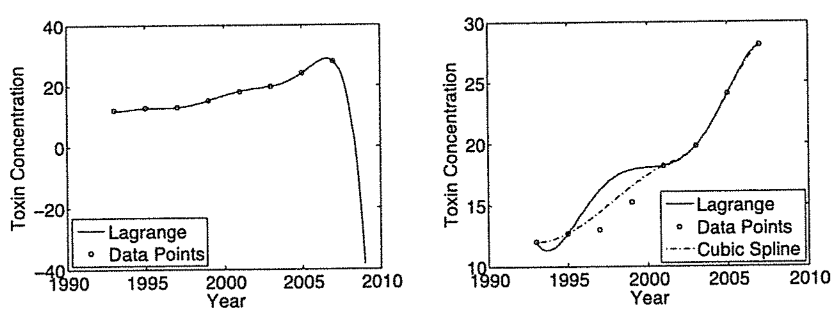

Figure 1.3: Exercise 8.

(b,c) Results are shown in Fig. 1.3. The predicted values are

	Lagrange	Spline
1997	16.23	14.44
1999	17.88	16.52

The predictions using the cubic spline are better.

9. The second order Lagrange polynomial passing through x_{i-1}, x_i, and x_{i+1} is

$$\begin{aligned} P(x) &= \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}y_{i-1} + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}y_i + \\ &\quad \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}y_{i+1}. \end{aligned}$$

Differentiating and evaluating at $x = x_i$, we obtain:

$$\begin{aligned} P'(x_i) &= \frac{(x_i-x_{i+1})y_{i-1}}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} + \frac{(x_i-x_{i-1})+(x_i-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}y_i + \\ &\quad \frac{(x_i-x_{i-1})y_{i+1}}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} \\ P''(x_i) &= \frac{2y_{i-1}}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} + \frac{2y_i}{(x_i-x_{i-1})(x_i-x_{i+1})} + \\ &\quad \frac{2y_{i+1}}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}. \end{aligned}$$

For uniformly spaced data, these reduce to:

$$P'(x_i) = \frac{y_{i+1}-y_{i-1}}{2\Delta} \quad \text{and} \quad P''(x_i) = \frac{y_{i+1}-2y_i+y_{i-1}}{\Delta^2}.$$

10. Let $\mathbf{v}$ be the vector whose points are the values of the polynomial $L_k(x)$ at the grid points $x_0, \ldots, x_N$, i.e. $v_i = L_k(x_i) = \delta_{ik}$. The derivative of $L_k(x)$ at x_j is $\left.\frac{d}{dx}L_k(x)\right|_{x=x_j} = L_k'(x_j)$ which is also given by

$$(D\mathbf{v})_j = \sum_{l=0}^{N} d_{jl}v_l = \sum_{l=0}^{N} d_{jl}\delta_{lk} = d_{jk}.$$

Thus $d_{jk} = L_k'(x_j)$. Now, taking the logarithm of $L_k(x) = \alpha_k \prod_{\substack{i=0\\i\neq k}}^{N}(x-x_i)$ and differentiating gives

$$\log L_k(x) = \log \alpha_k + \sum_{\substack{i=0\\i\neq k}}^{N} \log(x-x_i) \quad \text{and} \quad \frac{L_k'(x)}{L_k(x)} = \sum_{\substack{i=0\\i\neq k}}^{N} \frac{1}{x-x_i}.$$

Evaluating the last expression at $x = x_k$ gives (3):

$$L_k'(x_k) = d_{kk} = \sum_{\substack{i=0\\i\neq k}}^{N} \frac{1}{x_k-x_i}.$$

The same expression cannot be evaluated at $x \neq x_k$ since the denominator will be zero. We proceed further as follows:

$$L_k'(x) = L_k(x) \sum_{\substack{i=0 \\ i \neq k}}^{N} \frac{1}{x - x_i} = \alpha_k \prod_{\substack{l=0 \\ l \neq k}}^{N} (x - x_l) \sum_{\substack{i=0 \\ i \neq k}}^{N} \frac{1}{x - x_i} = \alpha_k \sum_{\substack{i=0 \\ i \neq k}}^{N} \prod_{\substack{l=0 \\ l \neq i,k}}^{N} (x - x_l).$$

This gives

$$L_k'(x_j) = \alpha_k \sum_{\substack{i=0 \\ i \neq k}}^{N} \prod_{\substack{l=0 \\ l \neq i,k}}^{N} (x_j - x_l).$$

The product is non zero only when $i = j$. Thus:

$$L_k'(x_j) = d_{jk} = \alpha_k \prod_{\substack{l=0 \\ l \neq j,k}}^{N} (x_j - x_l) = \frac{\alpha_k}{x_j - x_k} \prod_{\substack{l=0 \\ l \neq j}}^{N} (x_j - x_l) = \frac{\alpha_k}{\alpha_j (x_j - x_k)}.$$

11. (a) Looking at the contour plot (figure 1.4) we can estimate the value of $f(1.5, 1.5)$ to be 2.7.

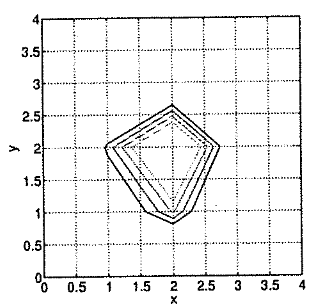

Figure 1.4: Contour plot on course data; from dark to light: $f = 2.4, 2.6, 2.8, 3.0$.

(b) Using equation (1.7) in the text, the following linear system should be solved for the second derivative.

$$\begin{pmatrix} 2/3 & 1/6 & 0 & 1/6 \\ 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \\ 1/6 & 0 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} g_{xx}(0,i) \\ g_{xx}(1,i) \\ g_{xx}(2,i) \\ g_{xx}(3,i) \end{pmatrix} = \begin{pmatrix} f(3,i) - 2f(0,i) + f(1,i) \\ f(2,i) - 2f(1,i) + f(0,i) \\ f(3,i) - 2f(2,i) + f(1,i) \\ f(0,i) - 2f(3,i) + f(2,i) \end{pmatrix}$$

For example, for $i = 0$ the solution to this system is

$g_{xx}(0,0) = 0.8466, \quad g_{xx}(1,0) = -0.0233, \quad g_{xx}(2,0) = -0.8460, \quad g_{xx}(3,0) = 0.0226,$

and from equation (1.6) in the text, $g(x, 0)$ for $1 \leq x \leq 2$ will be:

$$g(x,0)|_{1 \leq x \leq 2} = \frac{g_{xx}(1,0)}{6} \left[(2-x)^3 - (2-x) \right] + \frac{g_{xx}(2,0)}{6} \left[(x-1)^3 - (x-1) \right] + g(1,0)\,(2-x) + g(2,0)\,(x-1)\,.$$

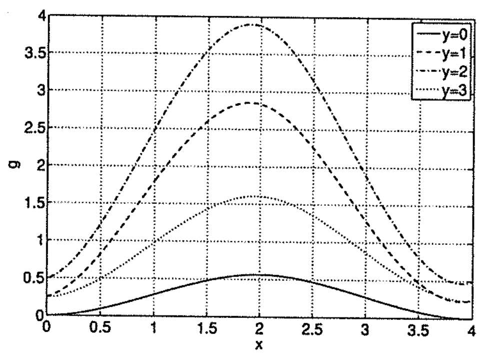

Figure 1.5: $g(x,i)$ for $i = 1, 2, 3, 4$.

The same procedure can be repeated for other intervals.

(c) From solution of part (b) we obtain:

$g(1.5,0) = 0.4819, \quad g(1.5,1) = 2.6082, \quad g(1.5,2) = 3.5588, \quad g(1.5,3) = 1.4326.$

The following system has to be solved for g_{yy} values.

$$\begin{pmatrix} 2/3 & 1/6 & 0 & 1/6 \\ 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \\ 1/6 & 0 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} g_{yy}(1.5,0) \\ g_{yy}(1.5,1) \\ g_{yy}(1.5,2) \\ g_{yy}(1.5,3) \end{pmatrix} = \begin{pmatrix} g(1.5,3) - 2g(1.5,0) + g(1.5,1) \\ g(1.5,0) - 2g(1.5,1) + g(1.5,2) \\ g(1.5,1) - 2g(1.5,2) + g(1.5,3) \\ g(1.5,2) - 2g(1.5,3) + g(1.5,0) \end{pmatrix}. \tag{1.1}$$

After solving this system we obtain

$$g_{yy}(1.5,1) = -1.7637, \quad g_{yy}(1.5,2) = -4.6150.$$

Therefore, $g(1.5,y)$ for $1 \leq y \leq 2$ will be:

$$g(1.5,y)|_{1\leq y\leq 2} = \frac{-1.7637}{6}\left[(2-y)^3 - (2-y)\right] + \frac{-4.6150}{6}\left[(y-1)^3 - (y-1)\right] + 2.6082\,(2-y) + 3.5588\,(y-1).$$

Substituting $y = 1.5$ results in $g(1.5,1.5) = 3.4821$.

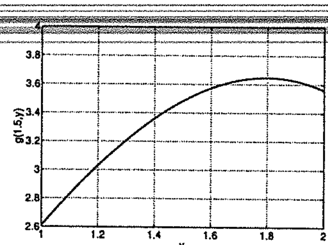

Figure 1.6: $g(1.5,y)$ for $1 \leq y \leq 2$

(d) After solving corresponding systems which are similar to (1.1) we obtain

$g_{yy}(1,1) = -1.2279, \quad g_{yy}(1,2) = -3.2994, \quad g_{yy}(2,1) = -1.8428, \quad g_{yy}(2,2) = -4.9700.$

Therefore, the polynomial expressions for $1 \le y \le 2$ will be

$$g(1,y)|_{1\le y\le 2} = \frac{-1.2279}{6}\left[(2-y)^3-(2-y)\right] + \frac{-3.2994}{6}\left[(y-1)^3-(y-1)\right] + 1.7995\,(2-y) + 2.4900\,(y-1)\,, \tag{1.2}$$

$$g(2,y)|_{1\le y\le 2} = \frac{-1.8428}{6}\left[(2-y)^3-(2-y)\right] + \frac{-4.9700}{6}\left[(y-1)^3-(y-1)\right] + 2.8357\,(2-y) + 3.8781\,(y-1)\,. \tag{1.3}$$

(e) In part (b) the g_{xx} values at the grid points are computed. We can use spline to interpolate these values in the y direction. We first solve the following system.

$$\begin{pmatrix} 2/3 & 1/6 & 0 & 1/6 \\ 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \\ 1/6 & 0 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} g_{xxyy}(1,0) \\ g_{xxyy}(1,1) \\ g_{xxyy}(1,2) \\ g_{xxyy}(1,3) \end{pmatrix} = \begin{pmatrix} g_{xx}(1,3) - 2g_{xx}(1,0) + g_{xx}(1,1) \\ g_{xx}(1,2) - 2g_{xx}(1,1) + g_{xx}(1,0) \\ g_{xx}(1,3) - 2g_{xx}(1,2) + g_{xx}(1,1) \\ g_{xx}(1,0) - 2g_{xx}(1,3) + g_{xx}(1,2) \end{pmatrix}$$

A similar system should be solved for $g_{xx}(2,y)$. The resulting numerical values are

$g_{xxyy}(1,1) = 0.9025, \quad g_{xxyy}(1,2) = 1.3381, \quad g_{xxyy}(2,1) = 2.7470, \quad g_{xxyy}(2,2) = 6.3511.$

The polynomial expressions for $1 \le y \le 2$ will be

$$g_{xx}(1,y)|_{1\le y\le 2} = \frac{0.9025}{6}\left[(2-y)^3-(2-y)\right] + \frac{1.3381}{6}\left[(y-1)^3-(y-1)\right] + -0.7701\,(2-y) + -0.9153\,(y-1)\,, \tag{1.4}$$

$$g_{xx}(2,y)|_{1\le y\le 2} = \frac{2.7470}{6}\left[(2-y)^3-(2-y)\right] + \frac{6.3511}{6}\left[(y-1)^3-(y-1)\right] + -3.8787\,(2-y) + -5.0800\,(y-1)\,. \tag{1.5}$$

(f) We can now use the information of (d) and (e) to do a cubic spline in the x direction. For $1 \le x \le 2$ and $1 \le y_0 \le 2$ we have

$$g(x,y_0)|_{1\le x\le 2} = \frac{g_{xx}(1,y_0)}{6}[(2-x)^3-(2-x)] + \frac{g_{xx}(2,y_0)}{6}[(x-1)^3-(x-1)] + g(1,y_0)(2-x) + g(2,y_0)(x-1), \tag{1.6}$$

where $g(1,y_0)$, $g(2,y_0)$, $g_{xx}(1,y_0)$, and $g_{xx}(2,y_0)$ should be substituted from equations (1.2), (1.3), (1.4), and (1.5) respectively. The resulting polynomial will be of the form

$$P(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{32}x^3y^2 + a_{23}x^2y^3 + a_{33}x^3y^3.$$

Let's look at the terms in (1.6) that contribute to a_{33} (the terms that contain x^3y^3). Both

$$\frac{g_{xx}(1,y_0)}{6}(2-x)^3 \text{ and } \frac{g_{xx}(2,y_0)}{6}(x-1)^3$$

will contribute. Substituting for $g_{xx}(1,y_0)$ and $g_{xx}(2,y_0)$ from equations (1.4) and (1.5) and keeping only the terms with x^3y^3 results in

$$a_{33} = \frac{1}{36}(0.9025 - 1.3381 - 2.7470 + 6.3511) = 0.0880.$$

(g) From Equation (1.6) in part (f) we have

$$\begin{aligned} g(1.5,1.5) = \frac{g_{xx}(1,1.5)}{6}[(0.5)^3 - 0.5] &+ \frac{g_{xx}(2,1.5)}{6}[(0.5)^3 - (0.5)] + \\ +g(1,1.5)(0.5) &+ g(2,1.5)(0.5). \end{aligned} \quad (1.7)$$

From equations (1.2), (1.3), (1.4), and (1.5) we obtain

$$g_{xx}(1,1.5) = -0.9827, \quad g_{xx}(2,1.5) = -5.048, \quad g(1,1.5) = 2.4277, \quad g(2,1.5) = 3.7827$$

Substituting these values into (1.7) results in $g(1.5,1.5) = 3.4821$ which is the same as the result of part (c).

By interpolating the data to a fine mesh using splines, one can obtain a much smoother contour plot compared to the one shown in part (a).

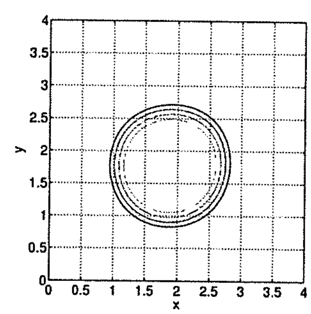

Figure 1.7: Contour plot after spline interpolation; from dark to light $f = 2.4, 2.6, 2.8, 3.0$.

Chapter 2

NUMERICAL DIFFERENTIATION - FINITE DIFFERENCES

1. (a)

$$\begin{aligned}
\frac{\delta(u_n v_n)}{\delta x} &= \frac{u_{n+1}v_{n+1} - u_{n-1}v_{n-1}}{2h} \\
u_n\frac{\delta v_n}{\delta x} + v_n\frac{\delta u_n}{\delta x} &= u_n\frac{v_{n+1} - v_{n-1}}{2h} + v_n\frac{u_{n+1} - u_{n-1}}{2h}
\end{aligned}$$

The two expressions are not equal in general.

(b) Both equal $\frac{u_{n+1}v_{n+1} - u_{n-1}v_{n-1}}{2h}$.

(c)

$$\begin{aligned}
\frac{\delta(\overline{\phi}_n\psi_n)}{\delta x} &= \frac{\phi_{n+2}\psi_{n+1} + \phi_n\psi_{n+1} - \phi_n\psi_{n-1} - \phi_{n-2}\psi_{n-1}}{4h} \\
\overline{\psi_n\frac{\delta\phi_n}{\delta x}} &= \frac{\phi_{n+2}\psi_{n+1} + \phi_n\psi_{n-1} - \phi_n\psi_{n+1} - \phi_{n-2}\psi_{n-1}}{4h}
\end{aligned}$$

Subtracting gives:

$$\frac{\delta(\overline{\phi}_n\psi_n)}{\delta x} - \overline{\psi_n\frac{\delta\phi_n}{\delta x}} = \phi_n\frac{\psi_{n+1} - \psi_{n-1}}{2h} = \phi_n\frac{\delta\psi_n}{\delta x}$$

(d)

$$\frac{\delta}{\delta x}\left(\frac{\delta u_n}{\delta x}\right) = \frac{\delta}{\delta x}\left(\frac{u_{n+1} - u_{n-1}}{2h}\right) = \frac{u_{n+2} - 2u_n + u_{n-2}}{4h^2}$$

The Taylor series of u_{n+2} and u_{n-2} about u_n are:

$$\begin{aligned}
u_{n+2} &= u_n + 2hu_n' + 2h^2u_n'' + \frac{4h^3}{3}u_n''' + \frac{2h^4}{3}u_n^{(iv)} + \cdots \\
u_{n-2} &= u_n - 2hu_n' + 2h^2u_n'' - \frac{4h^3}{3}u_n''' + \frac{2h^4}{3}u_n^{(iv)} + \cdots
\end{aligned}$$