Problem Solutions – Chapter 2

Equilibrium in stars

Problem 2.21. Critical mass

(a) Find critical mass in terms of density ρ and temperature T directly from (2) w/o defining Jeans length and speed of sound.

Equate potential and kinetic energies, from (2),

$$\frac{1}{2} m_H v^2 \approx \frac{G M_c m_H}{R_c}$$

where R_c is the radius of the critical cloud. Substitute $3kT/2$ for the left side to find

$$\frac{kT}{m_H} \approx \frac{G M_c}{R_c} \tag{2.21.s1}$$

Also, for a constant density cloud,

$$M_c \approx \frac{4\pi}{3} R_c^3 \, \rho \tag{2.21.s2}$$

Eliminate R_c from (s1) and (s2) and solve for M_c to find,

➡ $$M_c \approx \left(\frac{kT}{G\, m_{av}}\right)^{3/2} \frac{1}{\rho^{1/2}} \tag{2.21.s3}$$

as required.

(b) Evaluate M_c for ISM with T = 100 K and $n_H = 5 \times 10^5$ m^{-3}

Plug given values into (s3), where $\rho = n_H m_H = 8.37 \times 10^{-22}$ kg m^{-3}, to find

➡ $$M_c = 4.7 \times 10^{34} \text{ kg} = 23\,500\, M_\odot$$

This is about midway in the range of giant molecular cloud masses.

(c) Size of critical region

From (s2), $R \approx (M_c/\rho)^{1/3} = 3.8 \times 10^{18}$ m = 400 LY.

Or, from (6), the speed of sound is

$$v_s \approx \left(\frac{5}{3} \frac{kT}{m_{av}}\right)^{1/2} = 1170 \text{ m/s}$$

giving the Jeans length, from (5),

➡ $$\lambda_J = \frac{v_s}{(G\rho)^{1/2}} \approx 520 \text{ LY}$$

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Problem 2.22 – Shrinking cloud

A cloud of critical mass M_0 shrinks while maintaining the critical Jeans condition.

(a) Find $T(\rho, M_0, m)$ and evaluate change of T for decrease in radius by factor of 10.

Proceed as in Prob. 2.21a, but solve for T. I.e., solve (21.s3) for T to obtain

➡ $$T = \frac{Gm_{\text{av}}}{k} M_0^{2/3} \rho^{1/3} \qquad (2.22.\text{s}1)$$

As radius decreases by factor of 10, with mass held constant, ρ increases by a factor of 10^3. From (s1), therefore, temperature increases a factor of 10. This is directly in accord with (21.s1).

(b) Find and comment on potential and kinetic energies in initial and final states.

The Jeans condition is

$$\frac{1}{2} m v^2 \approx \frac{GM_0 m}{R} \qquad (2.22.\text{s}2)$$

Multiply both sides by the number N of particles and note that $M_0 = Nm$,

$$N \frac{1}{2} m v^2 \approx \frac{GM_0^2}{R}$$

so we find

$$E_{\text{k,tot}} \approx -E_{\text{p,tot}}$$
$$E_{\text{k,tot}} + E_{\text{p,tot}} \approx 0$$

Thus the total energy all during the collapse approximates zero because we specified the Jeans criterion to be obeyed during the process. The system remains on the verge of being unbound. In fact this is exactly what the Jeans criterion (s2) tells us. Note that the virial theorem would have the system more firmly bound, with $2E_{\text{k,tot}} + E_{\text{p,tot}} \approx 0$.

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Problem 2.31. Hydrostatic equilibrium

(a) Explain minus sign in equation of hydrostatic equilibrium.

From (12), the equation of hydrostatic equilibrium is:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{-G\,\mathcal{M}(r)\,\rho(r)}{r^2} = -\rho(r)\,g(r) \qquad (2.31.\text{s}1)$$

The minus sign simply states that as one moves out in the star (d$r > 0$), the pressure decreases (d$P < 0$).

(b) Derive the equation of hydrostatic equilibrium in spherical coordinates making use of the gradient of the pressure

$$\nabla P = (\partial P/\partial r)\hat{\mathbf{r}} \tag{2.31.s2}$$

which is strictly a radial vector. Since $\partial P/\partial r<0$, the gradient vector is directed inward.

The radial outward force on a gas element due to the pressure gradient is $F_P = -\Delta P\, dA$, or in terms of the gradient and making use of $dA = r^2 d\Omega$

$$\boldsymbol{F}_P = -\nabla P\, dr\, r^2\, d\Omega \tag{2.31.s3}$$

The gravitational force (11) may be written in vector form as

$$\boldsymbol{F}_G = \frac{-G\,\mathcal{M}(r)}{r^2}\rho(r)\, r^2\, d\Omega\, dr\, \hat{\boldsymbol{r}} \tag{2.31.s4}$$

Require force balance, $\boldsymbol{F}_P + \boldsymbol{F}_G = 0$, to obtain from (s3) and (s4),

$$\nabla P = -\frac{G\,\mathcal{M}(r)}{r^2}\rho(r)\,\hat{\boldsymbol{r}}$$

➡ $$\nabla P = -g(r)\,\rho(r)\,\hat{\boldsymbol{r}}$$

From the definition of the gradient (s2), and noting the equivalence of the partial and total derivative in this case (one variable), we can obtain directly the scalar version (s1), as required.

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Problem 2.41 – Thermal time constant for ball of hot gas

Isothermal ball of gas of mass M, radius R and temperature T_v obeys the virial theorem . It radiates from surface with $T_{eff} = fT_v$ where f is a constant factor and shrinks as it radiates. Its potential energy may be approximated as $E_p = -GM^2/R$.

(a) Find dependence on M, R and T_v of following quantities:

(*i*) Thermal energy E_k:

From virial theorem,

➡ $$E_k = -E_p/2 \approx GM^2/2R \propto M^2/R \tag{2.41.s1}$$

(*ii*) Temperature T_v:

Write the virial theorem, (s1) in terms of temperature,

$$\frac{3}{2}kT_v\frac{M}{m_p/2} = \frac{GM^2}{2R}$$

to obtain

➡ $$T_\mathrm{v} = \frac{GMm_\mathrm{p}}{6kR} \propto \frac{M}{R} \qquad (2.41.s2)$$

The temperature increases as the radius decreases.

(*iii*) Luminosity

$$L = 4\pi R^2\, \sigma T_\mathrm{eff}^4$$

where $T_\mathrm{eff} = f\mathrm{T}_\mathrm{v}$ and T_v is given in (s2),

➡ $$L = 4\pi R^2\, \sigma \left(f \frac{GMm_\mathrm{p}}{6kR} \right)^4 \propto f^4\, \frac{M^4}{R^2} \equiv \frac{b}{R^2} \qquad (2.41.s3)$$

where $b = 2.334 \times 10^{-66} f^4 M^4$ is a constant during the shrinkage. The luminosity increases strongly, as R^{-2}, as the star shrinks – for our (artificial) constant f. (I do not know if there is any physical basis for a constant f.)

(b) Find *R(t)*.

The total energy is

$$E_\mathrm{tot} = E_\mathrm{k} + E_\mathrm{p} = -\frac{E_\mathrm{p}}{2} = -\frac{GM^2}{2R} \equiv -\frac{a}{R} \qquad (2.41.s4)$$

where $a = GM^2/2 = 3.34 \times 10^{-11}\, M^2$ is a constant during the cooling and shrinkage. This energy is expended by radiation from the surface at the rate L,

$$\mathrm{d}E_\mathrm{tot} = -L\, \mathrm{d}t$$

$$\mathrm{d}\left(-\frac{a}{R}\right) = -\left(\frac{b}{R^2}\right) \mathrm{d}t$$

$$\frac{a}{R^2}\, \mathrm{d}R = -\left(\frac{b}{R^2}\right) \mathrm{d}t$$

Integrate from the initial radius R_0 (at time $t = 0$) to arbitrary radius at time t,

$$\int_{R_0}^{R} \mathrm{d}R = -\int_0^t \frac{b}{a}\, \mathrm{d}t$$

giving,

➡ $$R = R_0 - \frac{b}{a}\, t \qquad (2.41.s5)$$

where $b/a = 7.0 \times 10^{-56} f M^2$ (SI units; M in kg).

The radius decreases linearly with time toward zero as long as it remains in thermal equilibrium with luminosity increasing as R^{-2} according to (s3). It gets more and more luminous until it suddenly winks out. In fact, a real star would run into degeneracy pressure, shrinkage would stop, and it would become a white dwarf.

(c) For the mass and radius of the sun, find the time to reach zero radius, according to (s5) with $f = 1$, and comment

The time to reach "zero" radius according to (s5) would be

$$t = \frac{a}{b} R_0 = 1.431 \times 10^{55} \frac{R_\odot}{f^4 M_\odot^2} \qquad (2.41.s6)$$
$$t = 2500 f^{-4} \text{ s}$$

where we invoked the numerical values of a and b given above. For $f = 1$ and solar values of $R_\odot$ and $M_\odot$, the time to zero radius is only 42 minutes. Why is it so short? At $f = 1$, the effective radiating temperature T_{eff} equals the virial temperature, which from (s2) is huge, $T_{v,\odot} = 3.84 \times 10^6$ K. In fact, the surface of the sun has a much lower value, $T_{eff,\odot} = 5800$ K, so the sun radiates energy at a much slower rate. Our ad hoc factor f simulates the role as opacity in that it controls the rate of energy reaching and radiating from the surface.

(d) Find the value of f that gives the solar effective temperature.

From the definition of f in our statement of the problem above,

$$f = \frac{T_{eff,\odot}}{T_{v,\odot}} = \frac{5800}{3.84 \times 10^6} = 1.51 \times 10^{-3} \qquad (2.41.s7)$$

(e) With this value of f, find time to zero radius, and also the time it would take for the solar angular diameter to shrink by 0.01″.

(*i*) Time to zero radius.

From (s6),

➡ $$t = \frac{2500}{f^4} = 4.81 \times 10^{14} \text{ s} = 1.52 \times 10^7 \text{ yr} \qquad (2.41.s8)$$

This is essentially the thermal (Kelvin) time scale, (34).

(*ii*) Time for today's sun to decrease *diameter* $\Delta\theta = 0.01''$.

Express the angle in terms of known defined parameters and invoke (s5),

$$\Delta\theta = \frac{2(R - R_0)}{D} = -2 \frac{b}{a} \frac{t}{D}$$

where D is the solar distance, 1 AU. Solve for t and substitute values ($\Delta\theta = -0.01'' = -4.85 \times 10^{-8}$ rad; $D = 1.5 \times 10^{11}$ m; a/b is found in (s6) and f in (s7))

➡ $$t = -\frac{a}{b}\frac{D}{2}\Delta\theta = -\frac{1.431\times10^{55}}{f^4 M_\odot^2}\frac{D}{2}\Delta\theta = 2.5\times10^{9}\text{ s} = 79.2\text{ yr}$$

If a small 0.01″ secular decrease could be disentangled from larger variations of order 0.1″, possibly associated with the solar cycle, one might actually have been able to detect it in historical times – if in fact the sun were shrinking. Luminosity or temperature measures would also be possibilities. For a discussion of historical records of solar diameter, see Thuillier, G., Sofia, S., and Haberreiter, M.: 2005, *Adv. Space Res.* **35**, 329 (and later erratum and comments in the same journal).

(f) Find when the sun would have been 3 times its current size. How would luminosity and effective temperature compare to today's values?

Set $R = 3R_0$ in (s5) and solve for t to find, referencing (s6) and (s8),

➡ $$t = -2\frac{a}{b}R_0 = -2\times 1.52\times10^{7}\text{ yr} = -3.0\times10^{7}\text{ yr} \qquad (2.41.\text{s8})$$

Since $T_v \propto R^{-1}$, $T_{eff} \propto T_v$ and $L \propto R^{-2}$, we have, for this earlier time,

➡ $$T_{3R_0} = \frac{5800}{3} = 1600\text{ K};$$
$$L_{3R_0} = \frac{4\times10^{26}}{9} = 4.5\times10^{25}\text{ W}$$

Thus the temperature and luminosity of the sun would be quite different. It would be interesting to find out what evidence there may be against this (planetary data?).

Keep in mind that our linear relation between T_{eff} and T_v is quite arbitrary. One should properly obtain the temperature profile of the star and its atmosphere at each stage of its shrinkage to properly obtain the rates of energy release.

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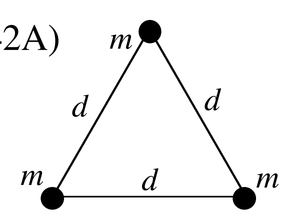
Problem 2.42 – Virial theorem and cluster of 3 galaxies in triangular configuration.

(2.42A)



Given: galaxies at corners of equilateral triangle in plane with separations d = 500 000 LY. Each has visible mass $m = 2\times10^{41}$ kg. Measured line of sight velocities are $v_{i,los}$ = +150, –200, +240 km/s where i = 1,2,3 corresponds to the three galaxies. True velocities are v_i.

(a) Find twice total kinetic and potential energies and their ratio.

The kinetic energy term is

$$2\Sigma E_\mathrm{k} = 2\sum_1^3 \left(\frac{1}{2} m v_i^2\right) = N\, m\, \frac{\Sigma\, v_i^2}{N} \tag{2.42.s1}$$

$$= 3\, m \left\langle v_i^2 \right\rangle_\mathrm{av} = 9\, m \left\langle v_{i,\,\mathrm{los}}^2 \right\rangle_\mathrm{av}$$

where N=3 is the number of galaxies, where we invoked the definition of an average, and where, for random distributions of directions, $\langle v_i^2 \rangle_\mathrm{av} = 3 \langle v_{i,\mathrm{los}}^2 \rangle_\mathrm{av}$. Use the three given values to find $\langle v_{i,\mathrm{los}}^2 \rangle_\mathrm{av} = 4 \times 10^{10}$ m^2/s^2 and substitute into (s1),

$$2\Sigma E_\mathrm{k} = 9\, m \left\langle v_{i,\,\mathrm{los}}^2 \right\rangle_\mathrm{av} = 7.2 \times 10^{52}\ \mathrm{J} \tag{2.42.s2}$$

The potential energy term is

$$-\Sigma E_\mathrm{p} = \sum_\mathrm{pairs} \frac{Gm^2}{d} = 3\,\frac{Gm^2}{d} = 1.7 \times 10^{51}\ \mathrm{J} \tag{2.42.s3}$$

The ratio is

$$\mathrm{Ratio} = \frac{2\Sigma E_\mathrm{k}}{-\Sigma E_\mathrm{p}} = \frac{7.2 \times 10^{52}\ \mathrm{J}}{1.7 \times 10^{51}\ \mathrm{J}} = 42 \tag{2.42.s4}$$

For galaxies obeying the virial theorem, the ratio should be unity. We conclude that, even though the cluster appears to be in stable equilibrium (sort of), the speeds are too high. Thus it is actually flying apart, or there is more mass that is not visible.

(b) Find an expression for the "virial mass" M_v required to satisfy the virial theorem in terms of the observed velocities and G and d. Assume all the mass is in the three galaxies so your expressions heretofore still apply.

Express the two terms of the virial theorem, (s2) and (s3) in terms of the total mass $M = 3m$ and plug into the virial theorem, $2E_\mathrm{k} + E_\mathrm{p} = 0$.

$$3\, M_\mathrm{v} \left\langle v_{i,\,\mathrm{los}}^2 \right\rangle_\mathrm{av} + \frac{GM_\mathrm{v}^2}{3d} = 0 \tag{2.42.s5}$$

This defines a new mass, the *virial mass,* $M_\mathrm{v} > M$. Solve for M_v,

➡ $$M_\mathrm{v} = \frac{9d}{G} \left\langle v_{i,\,\mathrm{los}}^2 \right\rangle_\mathrm{av} \tag{2.42.s6}$$

(c) Apply the observed values and find the ratio of M_v to the visible mass, $M = 3m$.

Substitute the given values into (s6) to find $M_\mathrm{v} = 2.6 \times 10^{43}$ kg. The desired ratio is

➡ $$\frac{M_\mathrm{v}}{M} = \frac{M_\mathrm{v}}{3m} = \frac{2.6 \times 10^{43}}{6 \times 10^{41}} = 42$$

This is the same ratio obtained in (s4). In retrospect, this is expected because the ratio of the kinetic to potential energy scales as M^{-1}. The ratio (s4) can thus be lowered to unity by increasing M a factor of 42.

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Problem 2.51 – Thermal time constant for white dwarf star

White dwarf of $M = 1.0\ M_\odot$, $R = 10^{-2}\ R_\odot$, $T_{\rm eff} = 33\ 000$ K, isothermal core of carbon nuclei at $T_{\rm c} = 3 \times 10^7$ K.

The luminosity is

$$L = 4\pi R^2 \sigma T_{\rm eff}^4 = 4.1 \times 10^{25}\ \text{W}$$

and its thermal energy content,

$$U = \frac{3}{2} kT_{\rm c} \frac{M}{12\ m_{\rm amu}} = 6.24 \times 10^{40}\ \text{J}$$

where $m_{\rm amu} = 1.66 \times 10^{-27}$ kg is the atomic mass unit.

The thermal time constant is then

$$\tau_{\rm K} = \frac{U}{L} = \frac{6.24 \times 10^{40}}{4.1 \times 10^{25}} = 1.5 \times 10^{15}\ \text{s} = 4.8 \times 10^7\ \text{yr.}$$

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Problem 2.52 – Dynamical time scales

(a) Find expression for orbital period of satellite just above surface of celestial body of mass M and radius R. Compare to $\tau_{\rm dyn} = (G\rho)^{-1/2}$ (37).

Apply Newton's second and gravitational laws

$$\frac{GMm}{R^2} = m\omega^2 R$$

Solve for ω and use $\rho_{\rm av} = (3/4\pi)\ (M/R^3)$ to eliminate M,

$$\omega = \left(\frac{GM}{R^3}\right)^{1/2} = \left(\frac{4\pi}{3} G\rho_{\rm av}\right)^{1/2}$$

leading to a period,

$$P = \frac{2\pi}{\omega} = 2\pi\left(\frac{4\pi}{3} G\rho_{\rm av}\right)^{-1/2} = \frac{\sqrt{3\pi}}{(G\rho)^{1/2}} = 3.1\ \tau_{\rm dyn}$$

The orbital period is the same order of magnitude as the dynamical time.

(b) Find the dynamical time for the following objects

	Object	ρ(kg/m^3)	$\tau_{dyn} = (G\rho)^{-1/2}$ (s)
(*i*)	Core of sun	$100 \times 10^3 = 10^5$	386 s ≈ 6 min
(*ii*)	Earth	5.5×10^3	1656 s ≈ 28 min
(*iii*)	White dwarf	1.8×10^9	2.9 s
(*iv*)	Neutron star	6.7×10^{17}	0.15 ms
(*v*)	Proton	1.2×10^{17}	0.35 ms

The neutron star has an average density just a few times that of the proton, so the dynamic times are comparable.

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Problem 2.53 – Diffusion time in 2-D space

Find the time for particles of speed v to diffuse a distance R in the presence of n scatterers/m^2, each with cross section σ (m^{-1}).

For the 1-D problem, we have, from (44) for N steps of size ℓ,

$$x_{rms} = N^{1/2}\ \ell \qquad (2.53.s1)$$

For the 2-D problem, only 1/2 the steps are along the x axis, so the number of steps along the x axis is

$$N_x = \frac{N}{2}$$

giving, analogously to (s1),

$$x_{rms} = \sqrt{\frac{N}{2}}\ \ell$$

Similarly, along the y axis,

$$y_{rms} = \sqrt{\frac{N}{2}}\ \ell$$

In the 2-D space,

$$r_{rms}^2 = x_{rms}^2 + y_{rms}^2 = 2\left(\frac{N}{2}\ell^2\right) = N\ell^2$$

The total number of steps required to reach $R = r_{rms}$ is thus

$$N = \frac{R^2}{\ell^2}$$

the same expression (47) obtained in the text for the 3-D case. At speed v, the time to go N steps is

➡ $$\tau = N\frac{\ell}{v} = \frac{R^2}{\ell^2}\frac{\ell}{v} = \frac{R^2\sigma n}{v}$$

where we invoked $\ell = (\sigma n)^{-1}$, where σ is the cross section.

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Problem 2.61 – Coulomb barrier for two protons

Two protons separated 1.4×10^{-15} m from one another, center to center.

(a) Magnitude of electrostatic force

➡ $$F = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = 9 \times 10^9 \left(\frac{1.6 \times 10^{-19}}{1.4 \times 10^{-15}}\right)^2$$

$$F = 117 \text{ N}$$

(b) Temperature required so protons come to within this distance

Set average kinetic energy, $3kT/2$, equal to the potential energy at this distance,

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = \frac{3}{2}kT$$

and solve for T.

$$T = \frac{2}{3}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{rk}$$

➡ $$= 7.9 \times 10^9 \text{ K}$$

This is more than two orders of magnitude greater than the temperature at the center of the sun, 1.6 $\times$ 10^7 K.

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Problem 2.62 – Energy release in pp chain

(a) Mass of hydrogen required to power a 1-kW heater for one year.

Energy required is E = 1000 W $\times$ 3.16 $\times$ 10^7 s = 3.16 $\times$ 10^{10} J, and the energy yield per kg is $\Delta E/\Delta m$ = 6.4 $\times$ 10^{14} J/kg (62). Thus, the mass required is

➡ $$M = \frac{E}{\Delta E/\Delta m} = \frac{3.16 \times 10^{10}}{6.4 \times 10^{14}} = 49 \times 10^{-6}\text{ kg} = 49 \text{ mg} \qquad (2.62.\text{s1})$$

(b) Mass required to power sun for one microsecond

Energy required is $E = 10^{-6} L_\odot = 4 \times 10^{20}$ J. Thus, following (s1),

➡ $$M = \frac{E}{\Delta E/\Delta m} = \frac{4 \times 10^{20}}{6.4 \times 10^{14}} = 625 \times 10^3 \text{ kg} = 625 \text{ metric tons}$$

(c) Show electric charge is conserved for each of the nuclear interactions in Fig. 6. Note that the interactions given are nuclear, not atomic

The electric charge in units of the electron charge for the involved particles are

–1 for e^-

0 for γ ray and ν

+1 for ^1H, ^2H, and e^+

+2 for ^3He and ^4He

+3 for ^7Li

+4 for ^7Be and ^8Be

+5 for ^8B

Apply these to the interactions to show, by inspection, that charge is conserved.

(d) Energy yield per nucleon for oil burning, and for wood burning.

Oil yield is 140 000 BTU/gal where 1 gal H_2O weighs 3.78 kg, oil density about equal to water and 1 BTU = 1055 J). Thus oil yield becomes ~3.91×10^7 J/kg = 0.42 eV/nucleon where we applied the conversion from J to eV and the number of nucleons in a kg ~$1/m_p$.

Compare to to pp chain. From (61) we had a yield of 26.75 MeV from 4 protons, which is a yield of $26.75 \times 1/4$ = 6.69 MeV per nucleon, ten million times greater.

Wood burning yield ~ 8000 BTU/lb = 1.86×10^7 J/kg = 0.20 eV/nucleon, about half that of oil.

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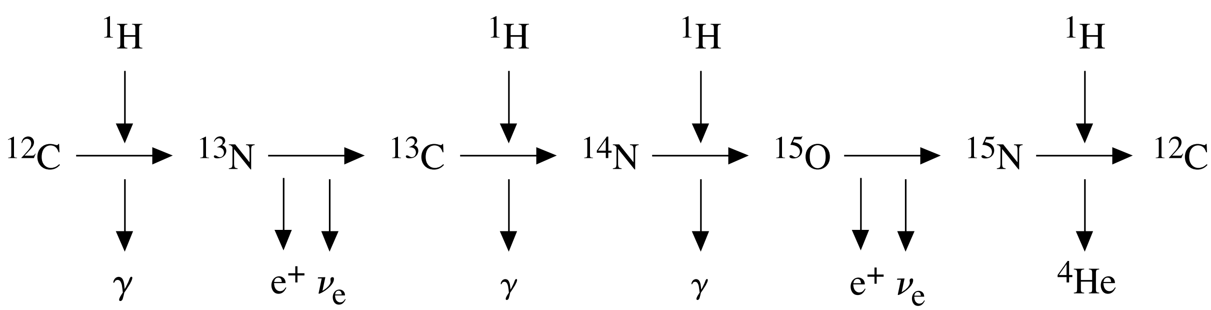
Problem 2.63 – Diagram for CNO chain

(2.63A)



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Problem 2.64 – Energy generation rate in sun.

(a) Find the coefficient ε_0 in the expression (66), from solar luminosity

Assume solar luminosity originates in the central $r_i = 0.1R_\odot$ of the sun where $\rho = 1.5 \times 10^5$ kg/m^3 and $T = 1.6 \times 10^7$ K. Use current hydrogen mass fraction for the sun center, $X = 0.36$. For an energy generation rate ϵ_{pp} (W/kg), the luminosity may be written

$$L_\odot = \frac{4}{3}\pi r_i^3 \rho \epsilon_{pp}$$

which we solve for ϵ_{pp}, and evaluate for the sun,

➡ $$\epsilon_{pp} = \frac{L_\odot}{\frac{4}{3}\pi r_i^3 \rho} = 1.8 \times 10^{-3} \text{ W/kg} \qquad (2.64.s1)$$

From (66),

$$\epsilon_{pp} = \epsilon_0 X^2 \left(\frac{\rho}{10^5 \text{ kg/m}^3}\right)\left(\frac{T}{10^7 \text{ K}}\right)^4$$

Solve for ϵ_0 and substitute in given values to find

➡ $$\epsilon_0 = \frac{\epsilon_{pp}}{X^2 (\rho/10^5)(T/10^7)^4} = 1.4 \times 10^{-3} \text{ W/kg}$$

(b) Power from (i) 1 m^3 of water at the solar rate given (s1) and (*ii*) a large swimming pool 25 m × 15 m × 2 m.

(i) For 1 m^3, the mass of the water is M_1 = 1000 kg. (Water density is 1000 kg/m^3.) The "luminosity" of the cubic meter is thus

$$L_{\text{cubic meter}} = \epsilon_{pp}\left(\frac{\text{W}}{\text{kg}}\right) M_1 \text{ (kg)} = 1.8 \times 10^{-3} \times 1000$$
$$= 1.8 \text{ W}$$

which is minuscule.

(*ii*) The swimming pool has volume 750 m^3 and mass 750 × 10^3 kg, and hence a "luminosity" of

$$L_{\text{pool}} = \epsilon_{pp} M_{\text{pool}} = 1.78 \times 10^{-3} \times 7.5 \times 10^5$$
$$= 1.34 \times 10^3 \text{ W} = 1.34 \text{ kW}$$

which is about the output of one stove burner, still not much.

Moral: the solar furnace is a very dilute energy source.

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Problem 2.71 – Massive black hole and Eddington luminosity

(a) Eddington luminosity of MBH at center of Galaxy (GC); comparison to observed flare

Mass of MBH at GC is M = 3 × 10^6 $M_\odot$. Observed flare L_{flare} = 4 × 10^{28} W. Let μ_e = 1 (hydrogen completely ionized).

From (75)

$$L_{\mathrm{Edd,GC}} = 1.26 \times 10^{31}\ \mu_e\ M/M_\odot$$

➡ $$= 3.8 \times 10^{37}\ \ \mathrm{W}$$

This is about 9 orders of magnitude greater than the observed flare.

(b) Accretion rate at Eddington and flare luminosities for GC

From (80),

$$L \approx \frac{G M \frac{\mathrm{d}m}{\mathrm{d}t}}{R}$$

Solve for dm/dt and substitute $R = R_S = 2GM/c^2$, to find

$$\dot{m} = L \frac{2}{c^2}$$

giving for L_{flare} and L_{Edd}

➡ $$\dot{m}_{\mathrm{flare,GC}} = 8.9 \times 10^{11}\ \mathrm{kg/s} = 1.4 \times 10^{-11}\ M_\odot/\mathrm{yr} \quad (L = 4 \times 10^{28}\ \mathrm{W})$$
$$\dot{m}_{\mathrm{Edd,GC}} = 8.5 \times 10^{20}\ \mathrm{kg/s} = 1.34 \times 10^{-2}\ M_\odot/\mathrm{yr} \quad (L = 3.8 \times 10^{37}\ \mathrm{W})$$

(c) Flux from GC at the earth; compare to Sco X-1 and sun

Sco X-1 flux is 4×10^{-10} W/m^2. Solar flux in x-rays during a bright flare is ~10^{-6} W/m^2.· The solar constant is 1365 W/m^2. Distance to GC is 25 000 LY = 2.4×10^{20} m.

The flux W/m^2 as a function of luminosity is

$$\mathscr{F} = \frac{L}{4\pi D^2}$$

giving, for L_{flare} and L_{Edd} of the GC:

➡ $$\mathscr{F}_{\mathrm{flare,GC}} = 5.7 \times 10^{-14}\ \ \mathrm{W/m^2} \qquad (L = 4 \times 10^{28}\ \mathrm{W})$$
$$\mathscr{F}_{\mathrm{Edd,GC}} = 5.4 \times 10^{-5}\ \ \mathrm{W/m^2} \qquad (L = 3.8 \times 10^{37}\ \mathrm{W})$$

Comparisons of fluxes:

Sco X-1 is much brighter than the GC flare flux, but much less than the hypothetical Eddington flux from the GC. The sun is extremely bright in x-rays relative to the GC flare largely because of its much closer distance, but interestingly an Eddington flux from the GC would exceed the flux of the flaring sun (~10^{-6} W/m^2) in x rays by a factor of ~50 for the values given here. The bolometric flux from the sun (1365 W/m^2) dwarfs everything else.

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Problem 2.72 – Eddington luminosity of 1 kg of water

(a) What is Eddington limit for 1 kg of mass if $\mu_e = 1$?

Eddington limit for 1 kg is, for $\mu_e = 1$, from (75),

$$L_{Edd} = 1.26 \times 10^{31}\ \mu_e\, M/M_\odot\ \text{ W}$$
$$= 6.3\ \text{W}$$

(b) Would liquid start being ejected by radiation forces.

The answer is No for several reasons: (i) Attractive molecular forces would constrain liquid. (*ii*) Molecules with very few ions would not have many free electrons to absorb the radiation. In other words, μ_e (nucleons per electron) would be huge and the Eddington luminosity would much exceed 6 W. (*iii*) Earth gravity would strongly constrain the liquid to remain in the pan.

But note that thermal motion of water molecules would lead to slow evaporation and any heating would increase this.

(c) Liquid in space

In space, only molecular forces and self gravity would tend to keep the liquid together. At the low pressure it would instantly become gaseous. One can apply the Jeans criterion to the gas to find the temperature below which the potential energy of self gravity might conceivably dominate the kinetic energies of thermal motion. As a limit, assume the gas initially has the density of water.

The critical mass is given in (7) as

$$M_c \approx \left(\frac{kT}{G\, m_{av}}\right)^{3/2} \frac{1}{\rho^{1/2}}$$

Solve for T

$$T = \frac{G\, m_{av}}{k} M_c^{2/3} \rho^{1/3}$$

Substitute $\rho = 1000$ kg/m^3, $M_c = 1$ kg, $m_{av} = 18\ m_{amu} = 3.0 \times 10^{-26}$ kg to find $T = 1.5 \times 10^{-12}$ K. This is much less than any expected temperatures in space. For example, the cosmic microwave background (CMB) at 3 K or the hydrogen clouds in the interstellar medium at ~100 K (Table 10.2). Thus there would be no holding back the rapid dispersion of the liquid due to thermal motions. The ultraviolet light from stars would likely dissociate the water molecules before long.

The radiation forces should have negligible effect at these temperatures.

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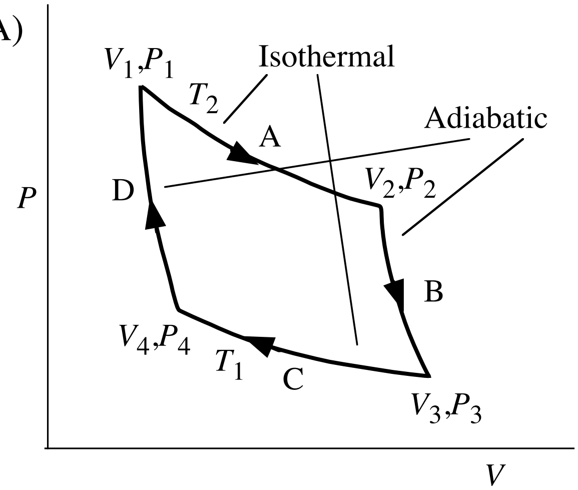
Problem 2.81 – Entropy change around Carnot cycle

(a) Demonstrate that d$S \equiv \delta Q/T$ integrated around the Carnot cycle is zero.

(2.81A)



Consider legs B and D.

These legs are adiabatic, so $\delta Q = 0$ for each segment of such a leg. From the definition of entropy change (just above), we have d$S = 0$. Integration along an entire adiabatic leg, thus gives

$$\Delta S_B = 0;\ \Delta S_D = 0 \tag{2.81.s1}$$

Now consider leg A.

This leg is isothermal so d$T = 0$ and hence the change of internal energyis $dU = C_V dT = 0$. Now apply the first law,

$$\delta Q = dU + \delta W \tag{2.81.s2}$$

$$\delta Q = C_V dT + P dV \tag{2.81.s3}$$
$$\delta Q = 0 + P dV$$

giving entropy change

$$dS \equiv \frac{\delta Q}{T} = \frac{P\,dV}{T} = R\frac{dV}{V} \tag{2.81.s4}$$

where we applied the ideal gas law for one mole, $PV = RT$. Integrate along leg A from position 1 to 2,

$$\int_1^2 dS = R\int_{V_1}^{V_2} \frac{dV}{V}$$

to obtain the entropy change for leg A.

$$\Delta S_A = R \ln \frac{V_2}{V_1} \tag{2.81.s5}$$

Similarly, for leg C,

$$\Delta S_C = R \ln \frac{V_4}{V_3} \tag{2.81.s6}$$

Return to legs B and D to find relations between the several volumes in the latter two results. We will find that the two expressions add to zero.
Apply the first law to leg B, from (s3) where $\delta Q = 0$ because it is adiabatic

$$0 = C_V \, dT + \frac{RT}{V} \, dV$$

where we again applied $PV = RT$. Integrate along leg B to obtain

$$C_V \ln \frac{T_1}{T_2} = -R \ln \frac{V_3}{V_2}$$

Rearrange,

$$\frac{V_3}{V_2} = \left(\frac{T_2}{T_1}\right)^{C_V/R} \tag{2.81.s7}$$

Similarly, for leg D, but inverting the expression,

$$\frac{V_4}{V_1} = \left(\frac{T_2}{T_1}\right)^{C_V/R} \tag{2.81.s8}$$

Hence, we find the expected result,

$$\frac{V_4}{V_1} = \frac{V_3}{V_2} \quad \text{or} \quad \frac{V_4}{V_3} = \frac{V_1}{V_2}$$

which, when applied to (s5) and (s6), tell us that

$$\Delta S_A + \Delta S_C = 0. \tag{2.81.s9}$$

Thus, the entropy change along all four legs, from (s1) and (s9) is zero as required,

➡ $$\Delta S = \Delta S_A + \Delta S_B + \Delta S_C + \Delta S_D = 0 \tag{2.81.s10}$$

(b) Show that PV^γ = constant for gas sample moving along adiabatic leg.

Given, $\gamma \equiv C_P/C_R$ where $C_P = C_V + R$ is the heat capacity at constant pressure.

Consider leg B. The ideal gas law at the starting point (Sketch A) is $P_2V_2 = RT_2$ and at the ending point, $P_3V_3 = RT_1$. We thus can rewrite (s7) as

$$\left(\frac{V_3}{V_2}\right)^{R/C_V} = \frac{P_2 V_2}{P_3 V_3} \qquad (2.81.s11)$$

Make use of the definitions in the problem statement to note that

$$R/C_V = (C_P - C_V)/C_V = \gamma - 1$$

which we use in (s11), which, after rearrangement, becomes

➡ $$P_3 V_3^{\gamma} = P_2 V_2^{\gamma}$$

This is the desired result.

END