2 Derivatives

2.1 Derivatives and Rates of Change

Suggested Time and Emphasis

1–2 classes   Essential material

Points to Stress

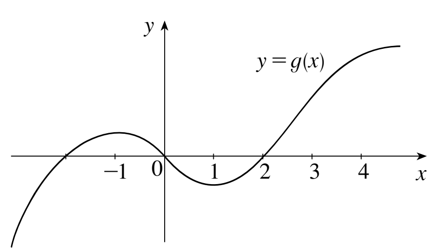
1. The slope of the tangent line as the limit of the slopes of secant lines (graphically, numerically, algebraically).
2. Physical examples of instantaneous rates of change (velocity, reaction rate, marginal cost, and so on) and their units.
3. The derivative notations $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.
4. Using f' to write an equation of the tangent line to a curve at a given point.
5. Using f' as an approximate rate of change when working with discrete data.

Quiz Questions

- **TEXT QUESTION** Why is it necessary to take a limit when computing the slope of the tangent line?
  ANSWER There are several possible answers here. Examples include the following:
  - By definition, the slope of the tangent line is the limit of the slopes of secant lines.
  - You don't know where to draw the tangent line unless you pick two points very close together.

  The idea is to get them thinking about this question.
- **DRILL QUESTION** For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

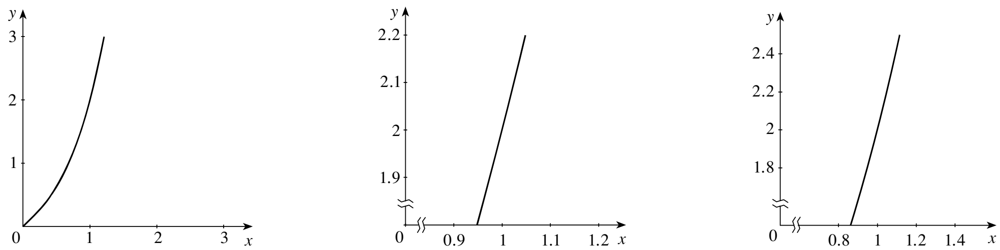
  0 $g'(-2)$ $g'(0)$ $g'(2)$ $g'(4)$



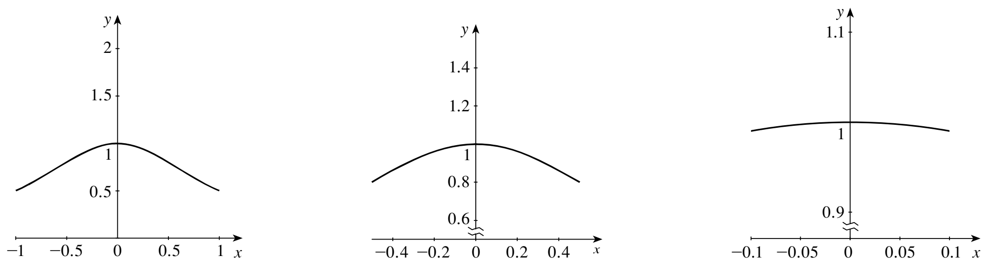
ANSWER $g'(0) < 0 < g'(4) < g'(-2) < g'(2)$

Materials for Lecture

- Review the geometry of the tangent line, and the concept of "locally linear". Estimate the slope of the line tangent to $y = x^3 + x$ at $(1, 2)$ by looking at the slopes of the lines between $x = 0.9$ and $x = 1.1$, $x = 0.99$ and $x = 1.01$, and so forth. Illustrate these secant lines on a graph of the function, redrawing the figure when necessary to illustrate the "zooming in" process.



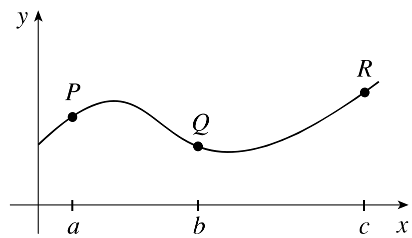
  Similarly examine $y = \frac{1}{x^2+1}$ at $(0, 1)$.



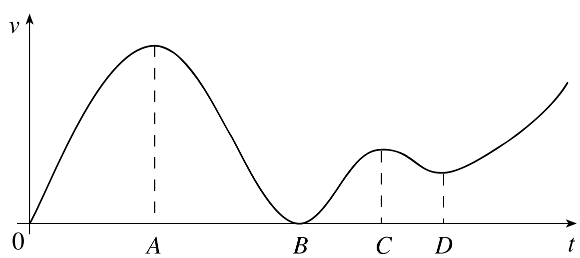
- If "A Jittery Function" was covered in Section 1.7, look at $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$ Poll the class: Is there a tangent line at $x = 0$? Then examine what happens if you look at the limits of the secant lines.
- Have students estimate the slope of the tangent line to $y = \sin x$ at various points. Foreshadow the concept of concavity by asking them some open-ended questions such as the following: What happens to the function when the slope of the tangent is increasing? Decreasing? Zero? Slowly changing?
- Discuss how physical situations can be translated into statements about derivatives. For example, the budget deficit can be viewed as the derivative of the national debt. Describe the units of derivatives in real world situations. The budget deficit, for example, is measured in billions of dollars per year. Another example: if $s(d)$ represents the sales figures for a magazine given d dollars of advertising, where s is the number of magazines sold, then $s'(d)$ is in magazines per dollar spent. Describe enough examples to make the pattern evident.
- Note that the text shows that if $f(x) = x^2 - 8x + 9$, then $f'(a) = 2a - 8$. Thus, $f'(55) = 102$ and $f'(100) = 192$. Demonstrate that these quantities cannot be easily estimated from a graph of the function. Foreshadow the treatment of a as a variable in Section 2.2.
- If a function models discrete data and the quantities involved are orders of magnitude larger than 1, we can use the approximation $f'(x) \approx f(x+1) - f(x)$. (That is, we can use $h = 1$ in the limit definition of the derivative.) For example, let $f(t)$ be the total population of the world, where t is measured in years since 1800. Then $f(211)$ is the world population in 2011, $f(212)$ is the total population in 2012, and $f'(211)$ is approximately the change in population from 2011 to 2012. In business, if $f(n)$ is the total cost of producing n objects, $f'(n)$ approximates the cost of producing the $(n+1)$th object.

Workshop/Discussion

- "Thumbnail" derivative estimates: graph a function on the board and have the class call out rough values of the derivative. Is it larger than 1? About 1? Between 0 and 1? About 0? Between -1 and 0? About -1? Smaller than -1? This is good preparation for Group Work 2 ("Oiling Up Your Calculators").
- Draw a function like the following, and first estimate slopes of secant lines between $x = a$ and $x = b$, and between $x = b$ and $x = c$. Then order the five quantities $f'(a)$, $f'(b)$, $f'(c)$, m_{PQ}, and m_{QR} in decreasing order. [Answer: $f'(b) < m_{PQ} < m_{QR} < f'(c) < f'(a)$.]



- Start the following problem with the students: A car is travelling down a highway away from its starting location with distance function $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$, where t is in hours, and d is in miles.
  1. How far has the car travelled after 1, 2, and 3 hours?
  2. What is the average velocity over the intervals [0, 1], [1, 2], and [2, 3]?
- Consider a car's velocity function described by the graph below.



  1. Ask the students to determine when the car was stopped.
  2. Ask the students when the car was accelerating (that is, when the velocity was increasing). When was the car decelerating?
  3. Ask the students to describe what is happening at times A, C, and D in terms of both velocity and acceleration. What is happening at time B?
- Estimate the slope of the tangent line to $y = \sin x$ at $x = 1$ by looking at the following table of values.

x	$\sin x$	$\dfrac{\sin x - \sin 1}{x - 1}$
0	0	0.841471
0.5	0.4794	0.724091
0.9	0.7833	0.581441
0.99	0.8360	0.544501
0.999	0.8409	0.540723
1.0001	0.8415	0.540260
1.001	0.8420	0.539881

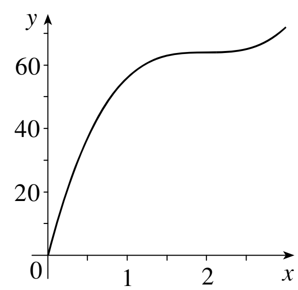
- Demonstrate some sample computations similar to Example 4, such as finding the derivative of $f(t) = \sqrt{1+t}$ at $t = 3$, or of $g(x) = x - x^2$ at $x = 1$.

Group Work 1: Follow that Car

Start this problem by giving the students the function $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$ and having them sketch its graph. Ask them how far the car has traveled after 1, 2, and 3 hours, and then show them how to compute the average velocity for [0, 1], [1, 2], and [2, 3].

ANSWERS

1.



2. It appears to stop at $t = 2$.

3. 8 mi/h, 2 mi/h, 0.08 mi/h

4. 0 mi/h. This is where the car stops.

Group Work 2: Oiling Up Your Calculators

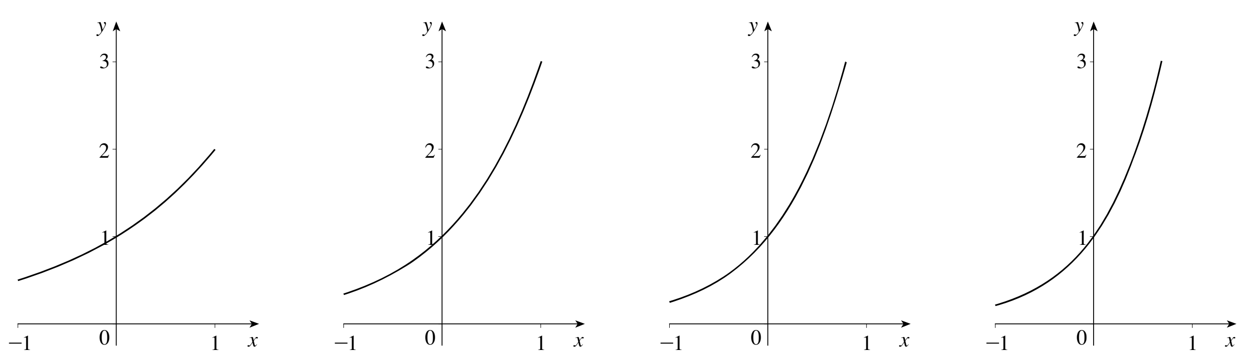
As long as the students have the ability to estimate the slope of a curve at a point, they don't need to have been exposed to the exponential function to do this activity. The exponential function and the number e will be covered in Chapter 6, and this exercise is a good initial introduction to the concept.

ANSWERS

1. If the students do this numerically, they should be able to get some pretty good estimates of $\ln 3 \approx 1.098612$. If they use graphs, they should be able to get 1.1 as an estimate.

2. 0.7 is a good estimate for a graph, and $\ln 2 \approx 0.693147$ is attainable numerically.

3. As a increases, the slope of the curve at $x = 0$ is increasing, as can be seen below.



4. The slope is less than 1 at $a = 2$ and greater than 1 at $a = 3$. Now apply the Intermediate Value Theorem.

5. The students are estimating e and should get 2.72 at a minimum level of accuracy.

Group Work 3: Connect the Dots

Closure is particularly important on this exercise. At this point in the course, many students will have the impression that all reasonable estimates are equally valid, so it is crucial that students discuss Problem 4. If there is student interest, this table can generate a rich discussion. Can A' ever be negative? What would that mean in real terms? What would $\left(A'\right)'$ mean in real terms in this instance?

ANSWERS

1. $A'(3500) \approx 0.06$ %/$ It is likely to be an overestimate, because the function is concave down at $p = 3500$.
2. After spending $3500, consumer approval is increasing at the rate of about 0.06 % for every additional dollar spent.
3. Percent per dollar
4. $A'(\$3550) \approx 0.06$ %/$. This is a better estimate because the same figures now give a two-sided approximation of the limit of the difference quotient.

Homework Problems

CORE EXERCISES 1, 3, 11, 13, 19, 27, 39, 45

SAMPLE ASSIGNMENT 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 27, 29, 35, 39, 43, 45

EXERCISE	D	A	N	G
1		×		
3		×		×
5		×		
7		×		
9		×		×
11				×
13		×		
15		×		
17	×			×

EXERCISE	D	A	N	G
19		×		
21				×
27		×		
29		×		
35	×			
39		×		
43		×	×	
45		×		

GROUP WORK 1, SECTION 2.1

Follow that Car

Here, we continue with the analysis of the distance $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$ of a car, where d is in miles and t is in hours.

1. Draw a graph of $d(t)$ from $t = 0$ to $t = 3$.

2. Does the car ever stop?

3. What is the average velocity over [1, 3]? over [1.5, 2.5]? over [1.9, 2.1]?

4. Estimate the instantaneous velocity at $t = 2$. Give a physical interpretation of your answer.

GROUP WORK 2, SECTION 2.1

Oiling Up Your Calculators

1. Estimate the slope of the line tangent to $y = 3^x$ at $x = 0$ using a method of your choosing.

2. Estimate the slope of the line tangent to $y = 2^x$ at $x = 0$ using a method of your choosing.

3. It is a fact that, as a increases, the slope of the line tangent to $y = a^x$ at $x = 0$ also increases in a continuous way. Geometrically, why should this be the case?

4. Prove that there is a special value of a for which the slope of the line tangent to $y = a^x$ at $x = 0$ is 1.

5. By trial and error, find an estimate of this special value of a, accurate to two decimal places.

GROUP WORK 3, SECTION 2.1

Connect the Dots

A company does a study on the effect of production value p of an advertisement on its consumer approval rating A. After interviewing eight focus groups, they come up with the following data:

Production Value	**Consumer Approval**
\$1000	32%
\$2000	33%
\$3000	46%
\$3500	55%
\$3600	61%
\$3800	65%
\$4000	69%
\$5000	70%

Assume that $A(p)$ gives the consumer approval percentage as a function of p.

1. Estimate $A'(\$3500)$. Is this likely to be an overestimate or an underestimate?

2. Interpret your answer to Problem 1 in real terms. What does your estimate of $A'(\$3500)$ tell you?

3. What are the units of $A'(p)$?

4. Estimate $A'(\$3550)$. Is your estimate better or worse than your estimate of $A'(\$3500)$? Why?

WRITING PROJECT **Early Methods for Finding Tangents**

The history of calculus is a fascinating and too-often neglected subject. Most people who study history never see calculus, and vice versa. We recommend assigning this section as extra credit to any motivated class, and possibly as a required group project, especially for a class consisting of students who are not science or math majors.

The students will need clear instructions detailing what their final result should look like. For example, recommend a page or two about Fermat's or Barrow's life and career, followed by two or three technical pages describing the alternate method of finding tangent lines as in the project's directions, and completed by a final half page of meaningful conclusion.

2.2 The Derivative as a Function

Suggested Time and Emphasis

1 class Essential material

Points to Stress

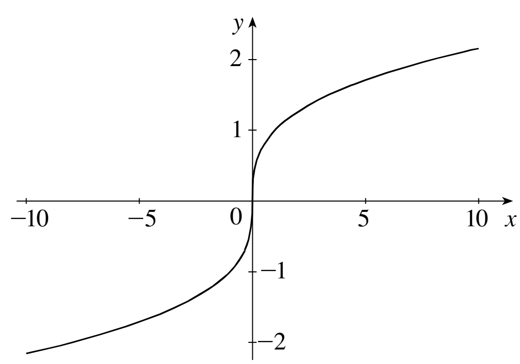
1. The concept of a differentiable function interpreted graphically, algebraically, and descriptively.
2. Obtaining the derivative function f' by first considering the derivative at a point x, and then treating x as a variable.
3. How a function can fail to be differentiable.
4. Sketching the derivative function given a graph of the original function.
5. Second and higher derivatives

Quiz Questions

TEXT QUESTION The previous section discussed the derivative $f'(a)$ for some function f. This section discusses the derivative $f'(x)$ for some function f. What is the difference, and why is it significant enough to merit separate sections?

ANSWER a is considered a constant, x is considered a variable. So $f'(a)$ is a number (the slope of the tangent line) and $f'(x)$ is a function.

DRILL QUESTION Consider the graph of $f(x) = \sqrt[3]{x}$. Is this function defined at $x = 0$? Continuous at $x = 0$? Differentiable at $x = 0$? Why?

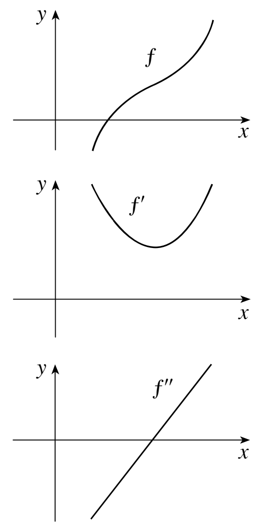


ANSWER It is defined and continuous, but not differentiable because it has a vertical tangent.

Materials for Lecture

- Ask the class this question: "If you were in a car, blindfolded, ears plugged, all five senses neutralized, what quantities would you still be able to perceive?" (Answers: acceleration and jerk.) Many students incorrectly add velocity to this list. Stress that acceleration is perceived as a force (hence $F = ma$) and that "jerk" causes the uncomfortable sensation when the car stops suddenly.
- Review definitions of differentiability, continuity, and the existence of a limit.
- Sketch f' from a graphical representation of $f(x) = \left|x^2 - 4\right|$, noting where f' does not exist. Then sketch $\left(f'\right)'$ from the graph of f'. Point out that differentiability implies continuity, and not vice versa.

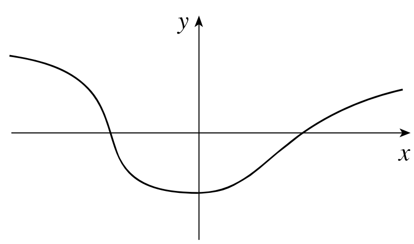
- Examine graphs of f and f' aligned vertically as shown. If you wish to foreshadow f'', add its graph below. Discuss what it means for f' to be positive, negative or zero. Then discuss what it means for f' to be increasing, decreasing or constant.



- If the group work "A Jittery Function" was covered in Section 1.7, then examine the differentiability of $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$ at $x = 0$ and elsewhere (if not already done in Section 3.1).
- Show that if $f(x) = x^4 - x^2 + x + 1$, then $f^{(5)}(x) \equiv 0$. Conclude that if $f(x)$ is a polynomial of degree m, then $f^{(m+1)}(x) \equiv 0$.

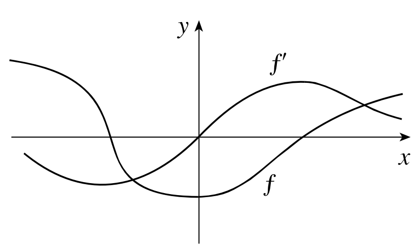
Workshop/Discussion

- Estimate derivatives from the graph of $f(x) = \sin x$. Do this at various points, and plot the results on the blackboard. See if the class can recognize the graph as a graph of the cosine curve.
- Given the graph of f below, have students determine where f has a horizontal tangent, where f' is positive, where f' is negative, where f' is increasing (this may require some additional discussion), and where f' is decreasing. Then have them sketch the graph of f'.



(For more exercises of this type using a wide variety of functions, use TEC Visual 2.2.)

ANSWER There is a horizontal tangent near $x = 0$. f' is positive to the right of 0, negative to the left. f' is increasing between the x-intercepts, and decreasing outside of them.



- Compute $f'(x)$ and $g'(x)$ if $f(x) = x^2 + x + 2$ and $g(x) = x^2 + x + 4$. Point out that $f'(x) = g'(x)$ and discuss why the constant term is not important. Next, compute $h'(x)$ if $h(x) = x^2 + 2x + 2$. Point out that the graph of $h'(x)$ is just the graph of $f'(x)$ shifted up one unit, so the linear term just shifts derivatives. For more explorations on how the coefficients in polynomials and other functions affect first and second derivatives, use TEC Module 2.2.

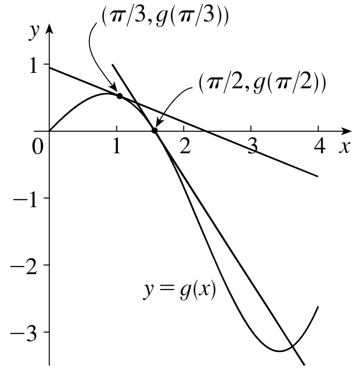
- Consider the function $f(x) = \sqrt{|x|}$. Show that it is not differentiable at 0 in two ways: by inspection (it has a cusp); and by computing the left- and right-hand limits of $f'(x)$ at $x = 0$ ($\lim_{x\to 0^+} f'(x) = \infty$, $\lim_{x\to 0^-} f'(x) = -\infty$).
- TEC Visual 2.2 can be used to develop students' ability to look at the graph of a function and visualize the graph of that function's derivative. The key feature of this module is that it allows the students to mark various features of the derivative *directly on the graph of the function* (for example, where the derivative is positive or negative). Then, after using this information and sketching a graph of the derivative, they can view the actual graph of the derivative and check their work.

Group Work 1: Tangent Lines and the Derivative Function

This simple exercise reinforces that although we are moving to thinking of the derivative as a function of x, it is still the slope of the line tangent to the graph of f.

ANSWERS

1, 3.



2. $y = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$

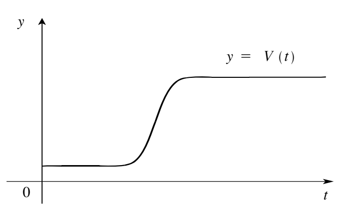
4. $y = \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)\left(x - \frac{\pi}{3}\right) + \frac{\pi}{6}$

Group Work 2: The Revenge of Orville Redenbacher

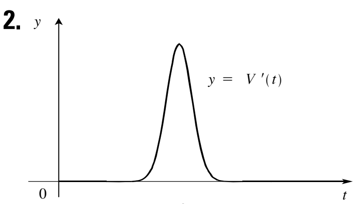
In an advanced class, or a class in which one group has finished far ahead of the others, ask the students to repeat the exercise substituting "$D(t)$, the density function" for $V(t)$.

ANSWERS

1.

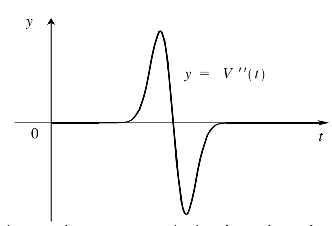


2.



Units are cm^3/s.

3.



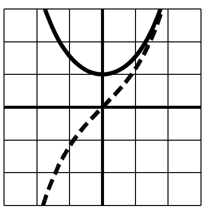
When the second derivative is zero, the first derivative has a maximum, meaning the popcorn is expanding the fastest.

Group Work 3: The Derivative Function

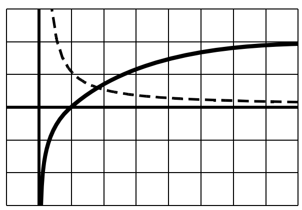
Give each group of between three and five students the picture of all eight graphs. They are to sketch the derivative functions by first estimating the slopes at points, and plotting the values of $f'(x)$. Each group should also be given a large copy of one of the graphs, perhaps on acetate. When they are ready, with this information they can draw the derivative graph on the same axes. For closure, project their solutions on the wall and point out salient features. Perhaps the students will notice that the derivatives turn out to be positive when their corresponding functions are increasing. Concavity can even be introduced at this time. Large copies of the answers are provided, in case the instructor wishes to overlay them on top of students' answers for reinforcement. Note that the derivative of graph 6 ($y = e^x$) is itself. Also note that the derivative of graph 1 ($y = \cosh x$) is *not* a straight line. Leave at least 15 minutes for closure. The whole exercise should take about 45–60 minutes, but it is really, truly worth the time.

If a group finishes early, have them discuss where f' is increasing and where it is decreasing. Also show that where f is increasing, f' is positive, and where f is decreasing, f' is negative.

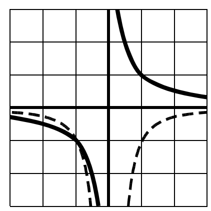
ANSWER



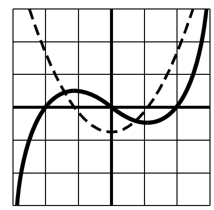
Graph 1



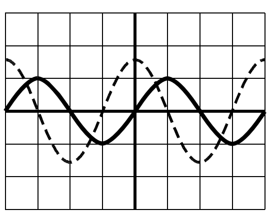
Graph 2



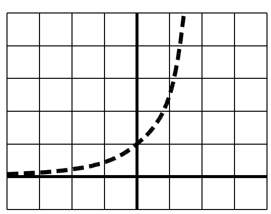
Graph 3



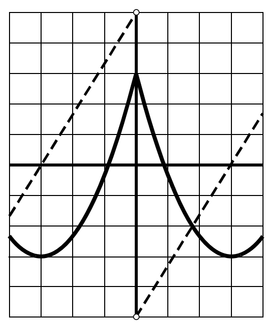
Graph 4



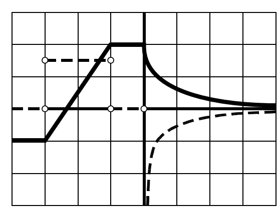
Graph 5



Graph 6



Graph 7



Graph 8

Homework Problems

CORE EXERCISES 1, 3, 13, 23, 35, 45

SAMPLE ASSIGNMENT 1, 3, 5, 13, 19, 23, 25, 27, 35, 37, 41, 45

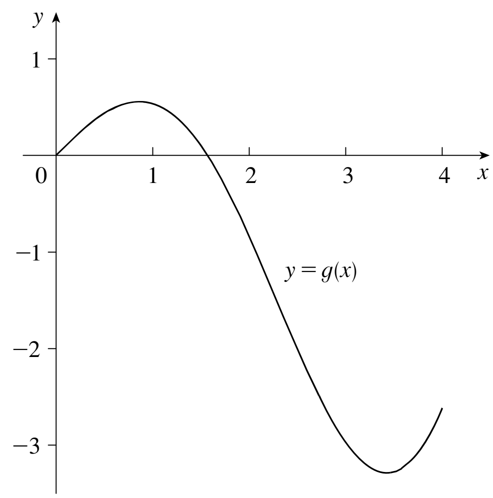
EXERCISE	D	A	N	G
1				×
3	×			×
5				×
13	×			×
19		×		
23		×		

EXERCISE	D	A	N	G
25		×		
27		×		
35				×
37				×
41	×			×
45		×		×

GROUP WORK 1, SECTION 2.2

Tangent Lines and the Derivative Function

The following is a graph of $g(x) = x \cos x$.



It is a fact that the derivative of this function is $g'(x) = \cos x - x \sin x$.

1. Sketch the line tangent to $g(x)$ at $x = \frac{\pi}{2} \approx 1.57$ on the graph above.

2. Find an equation of the tangent line at $x = \frac{\pi}{2}$.

3. Now sketch the line tangent to $g(x)$ at $x = \frac{\pi}{3} \approx 1.05$.

4. Find an equation of the tangent line at $x = \frac{\pi}{3}$.

GROUP WORK 2, SECTION 2.2

The Revenge of Orville Redenbacher

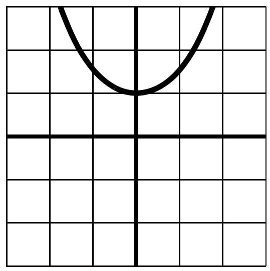
1. Consider a single kernel of popcorn in a popcorn popper. Let $V(t)$ be the volume in cm^3 of the kernel at time t seconds. Draw a graph of $V(t)$, including as much detail as you can, up to the time that the kernel is taken from the popper.

2. Now sketch a graph of the derivative function $V'(t)$. What are the units of $V'(t)$?

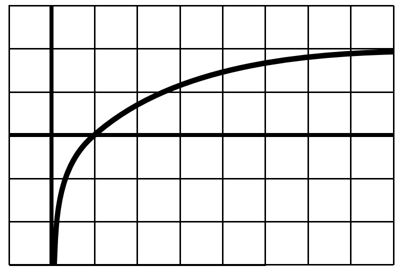
3. Finally, sketch a graph of $V''(t)$. What does it mean when $V''(t) = 0$?

The Derivative Function

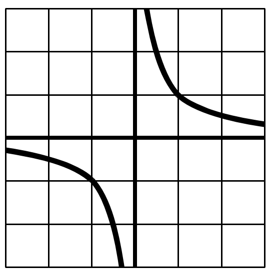
The graphs of several functions f are shown below. For each function, estimate the slope of the graph of f at various points. From your estimates, sketch graphs of f'.



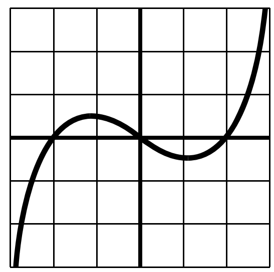
Graph 1



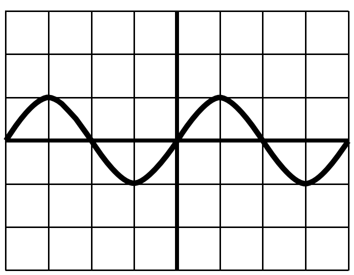
Graph 2



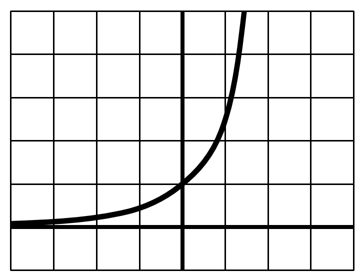
Graph 3



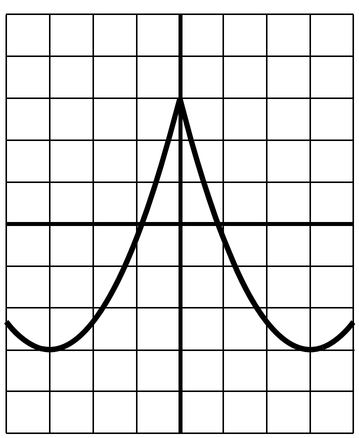
Graph 4



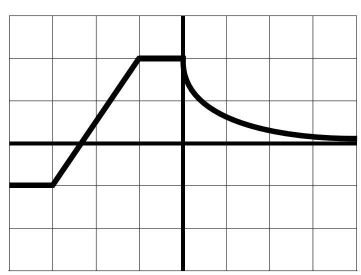
Graph 5



Graph 6

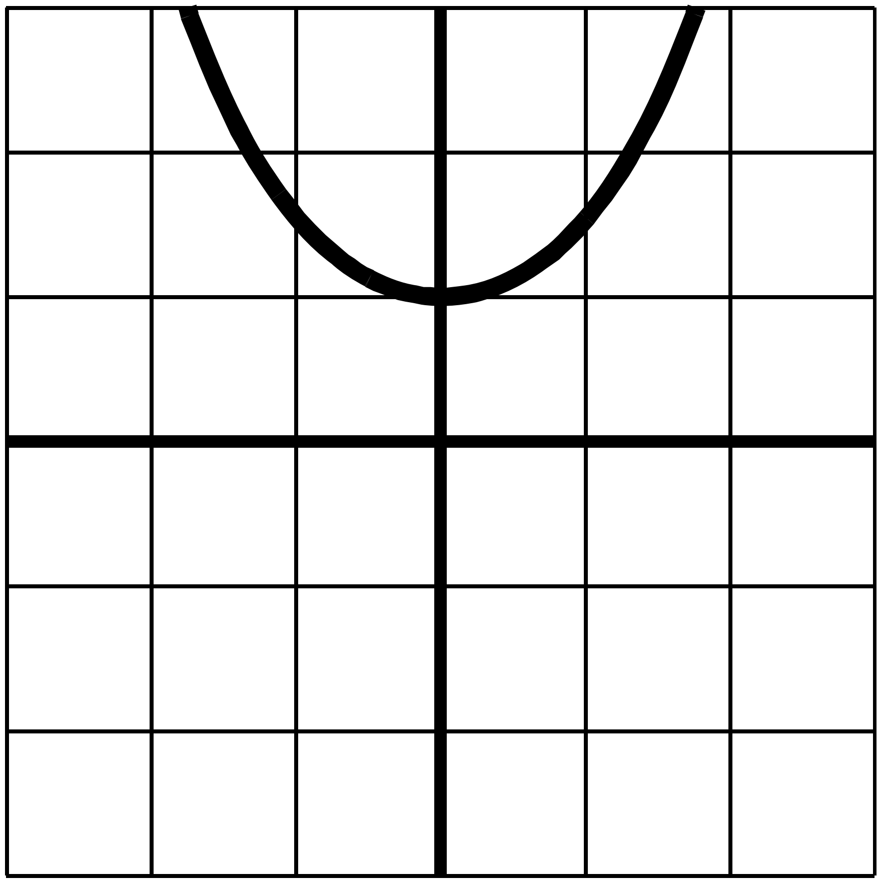


Graph 7

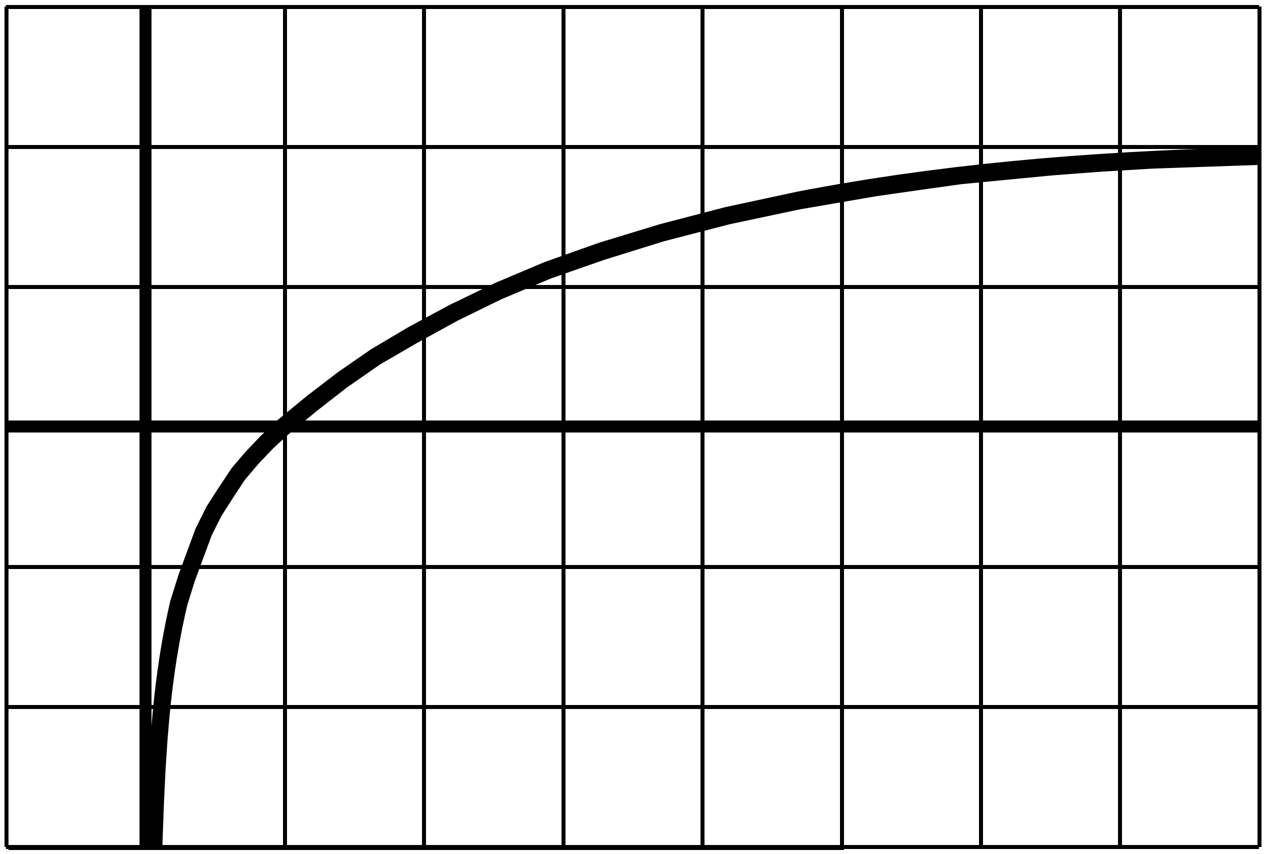


Graph 8

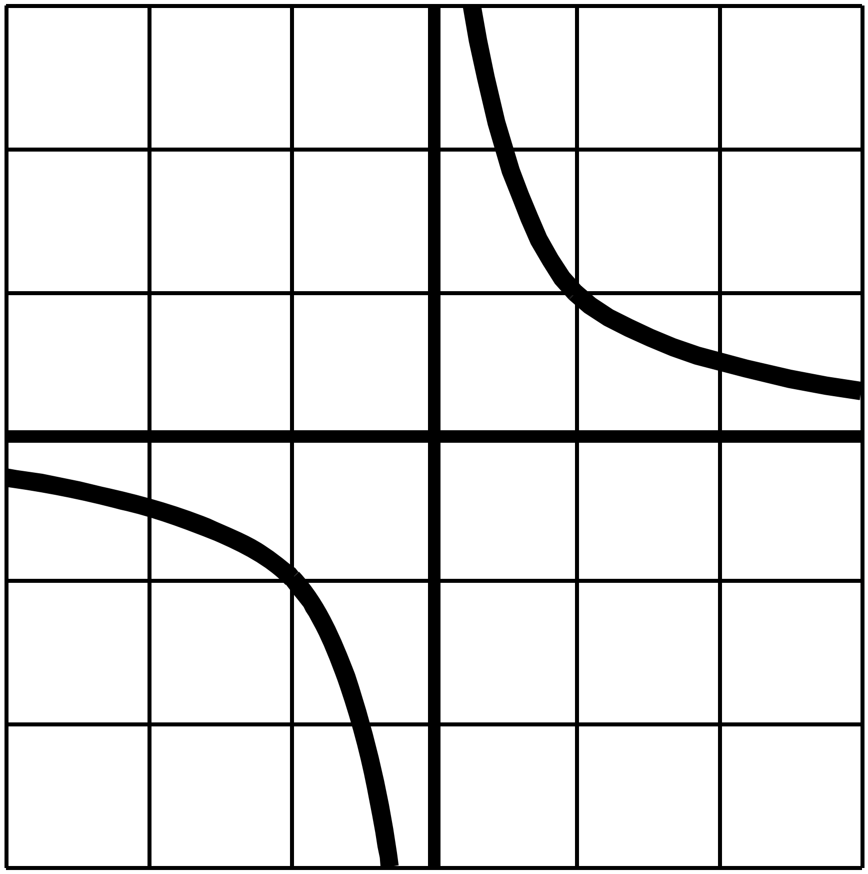
Graph 1



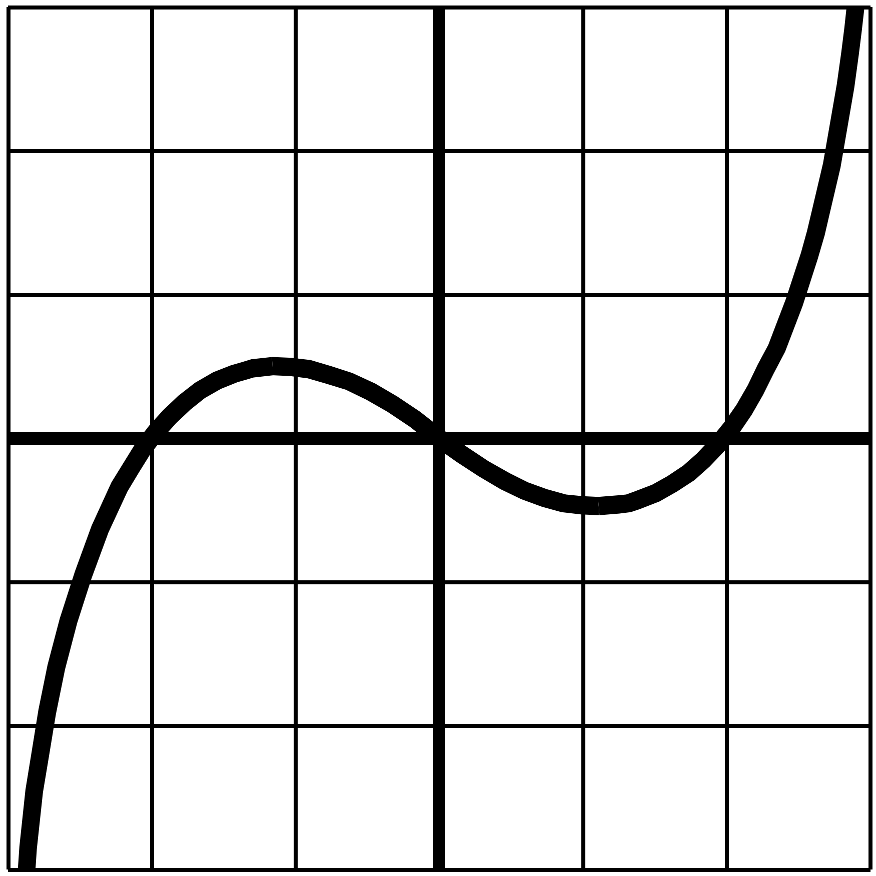
Graph 2



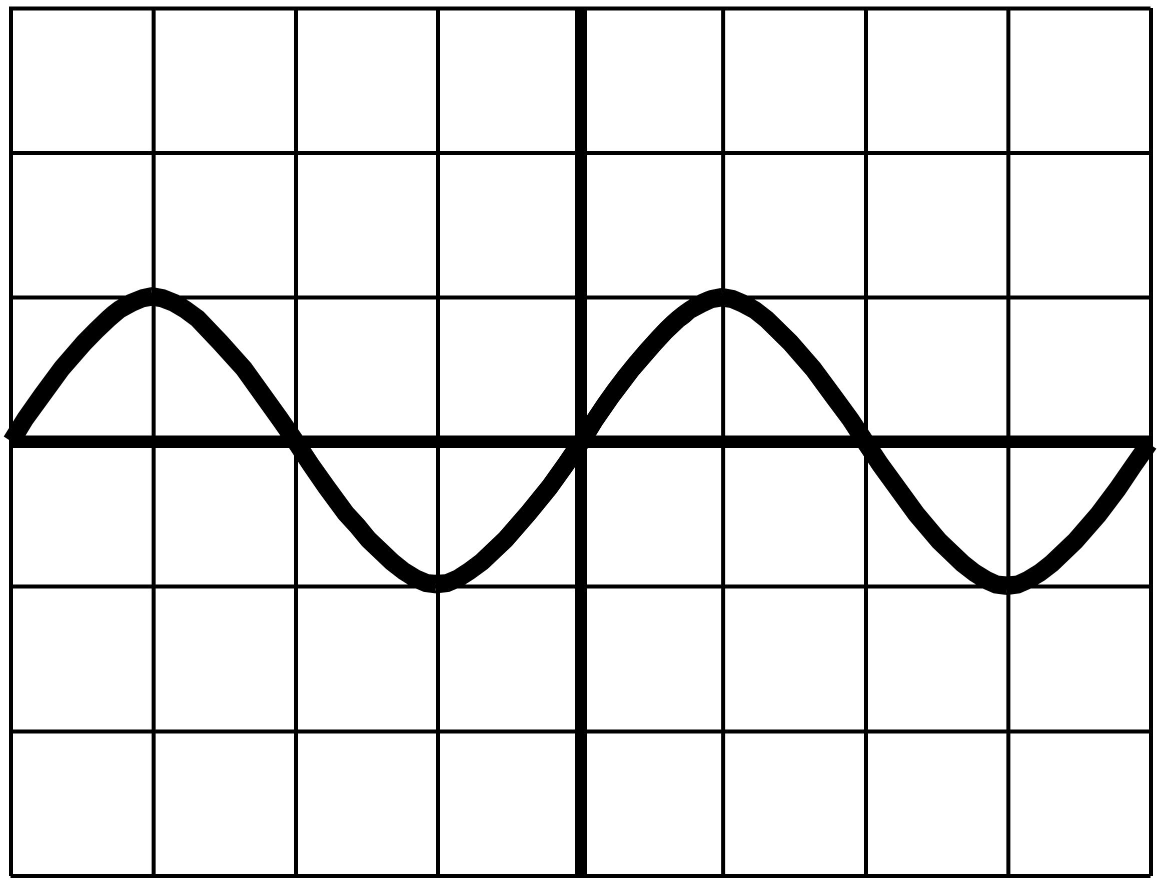
Graph 3



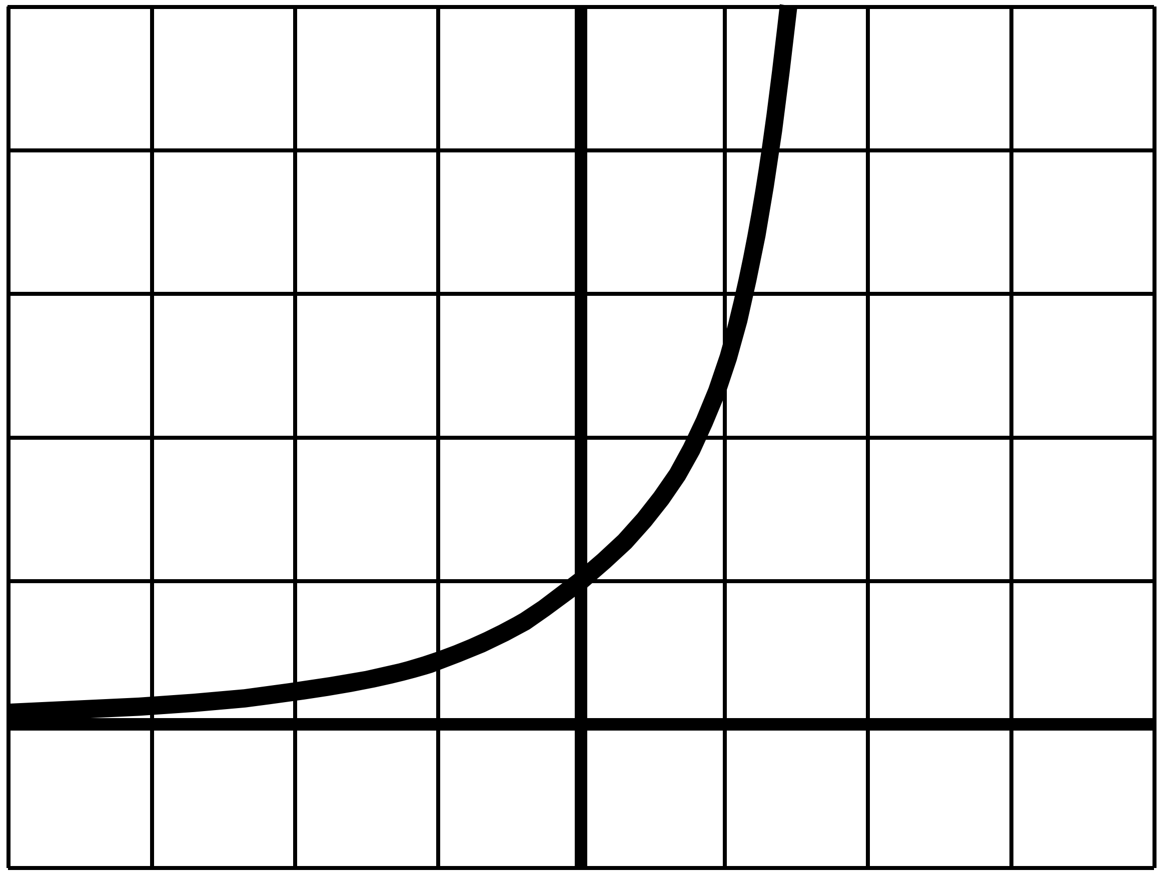
Graph 4



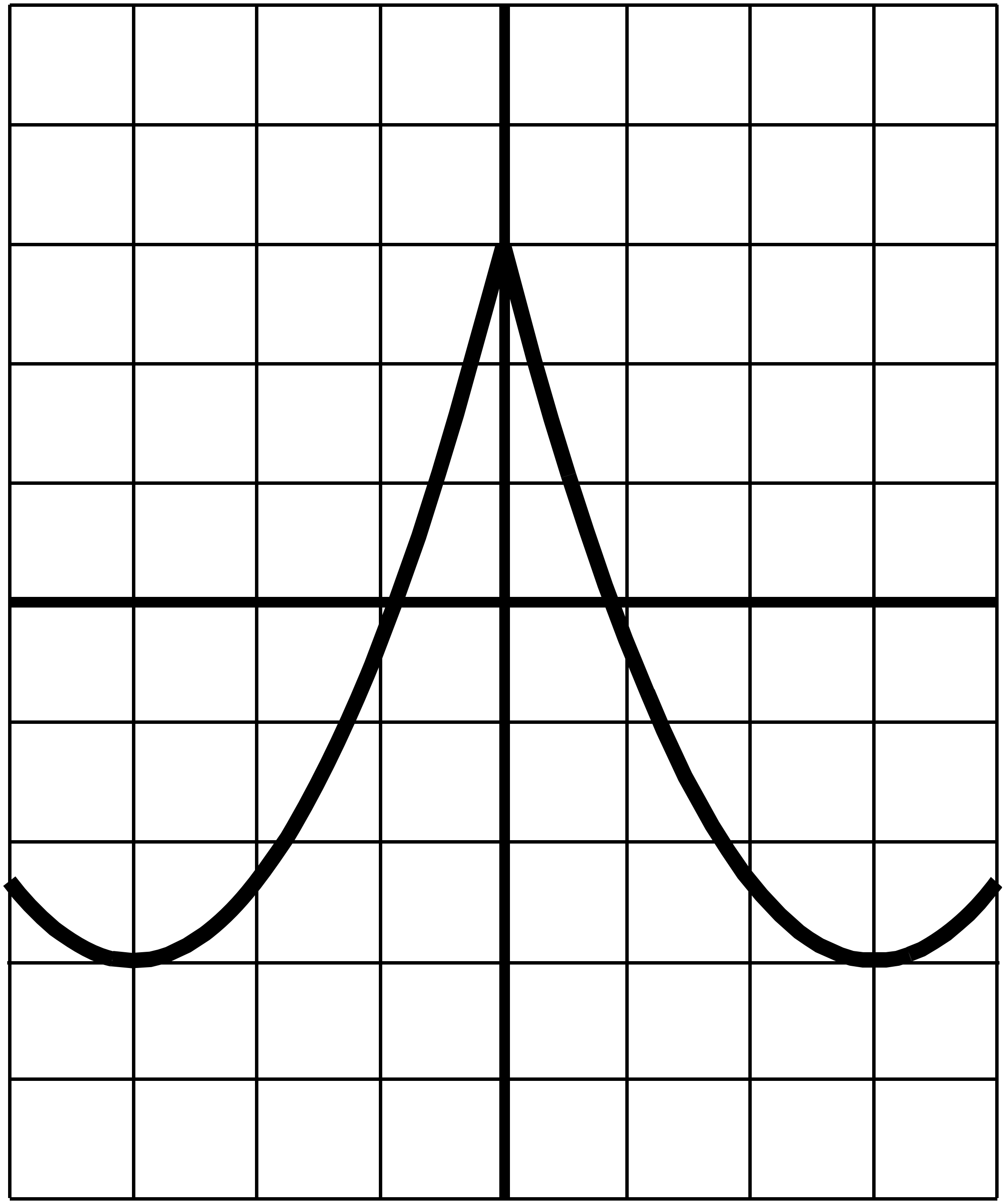
Graph 5



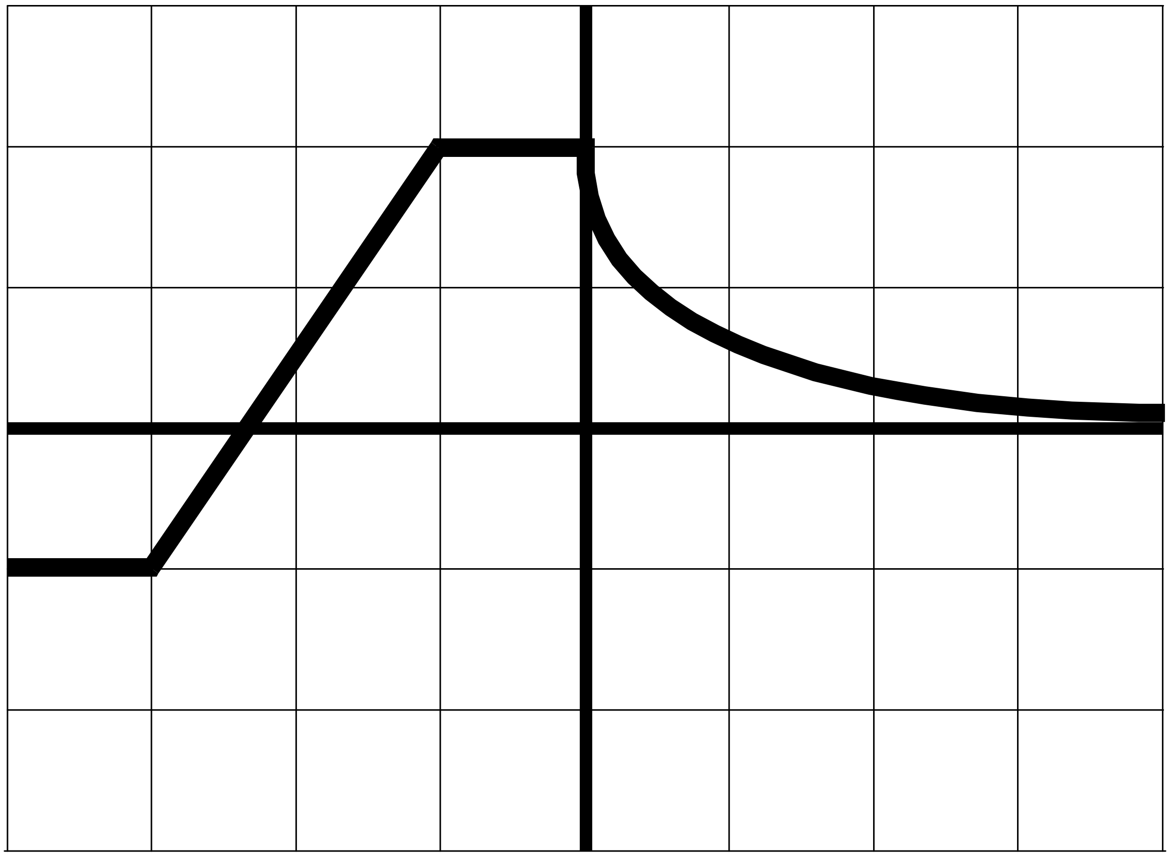
Graph 6



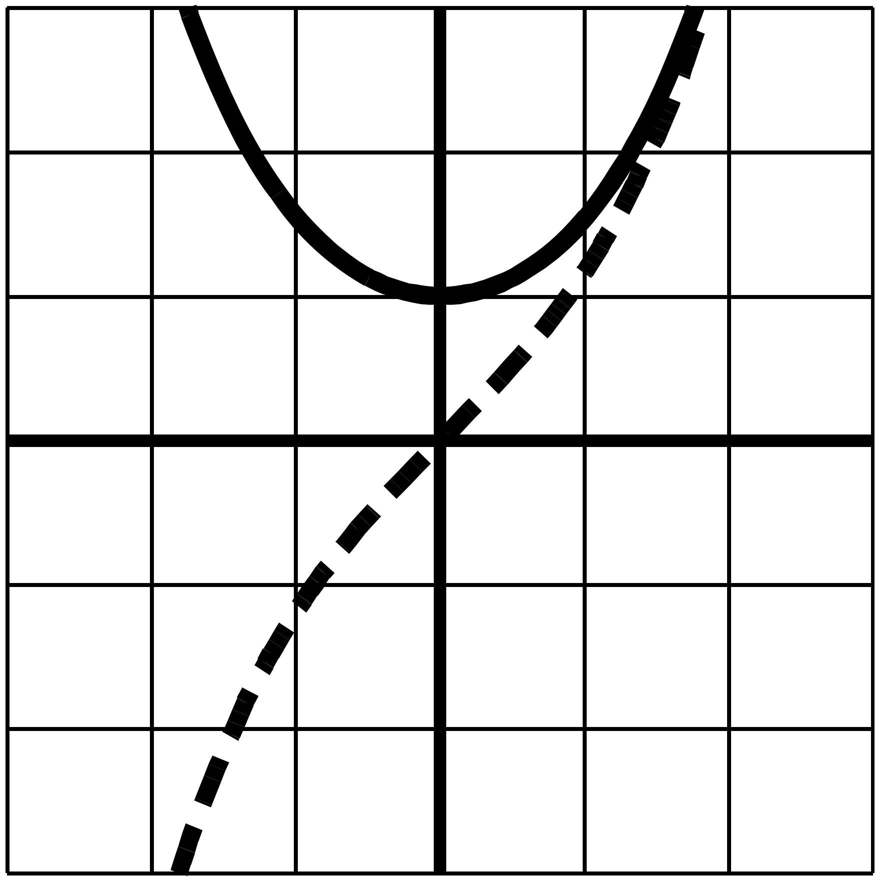
Graph 7



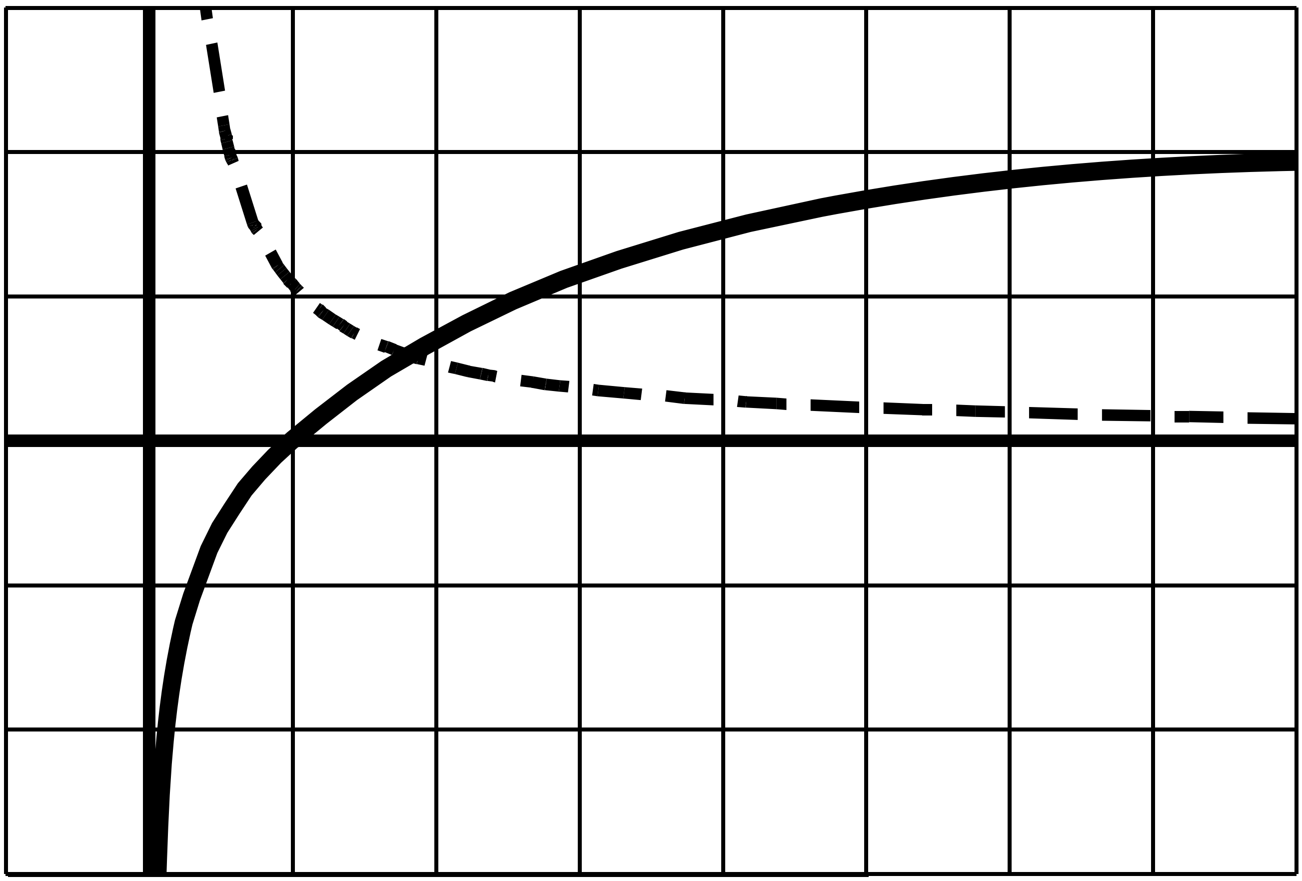
Graph 8



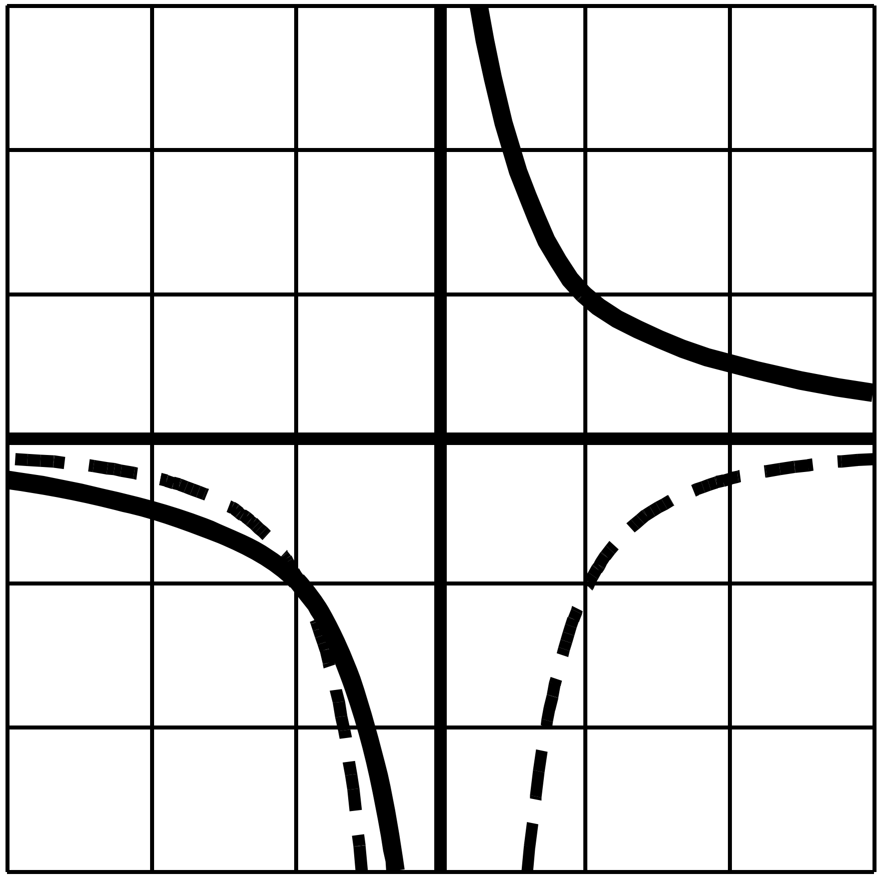
Answer 1



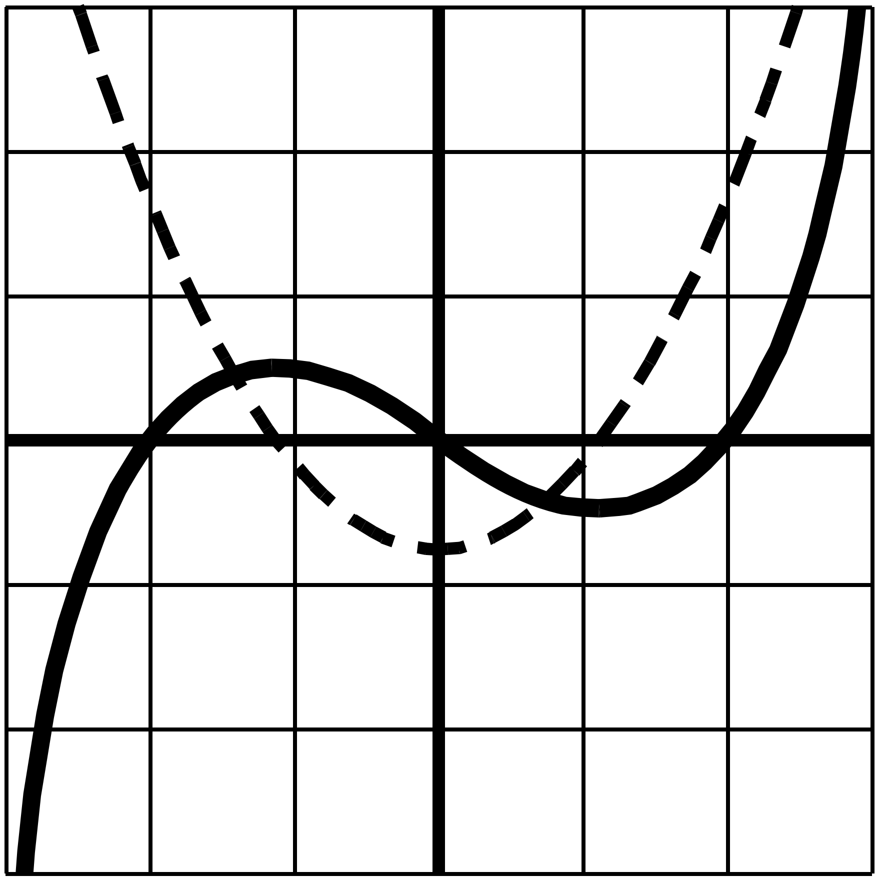
Answer 2



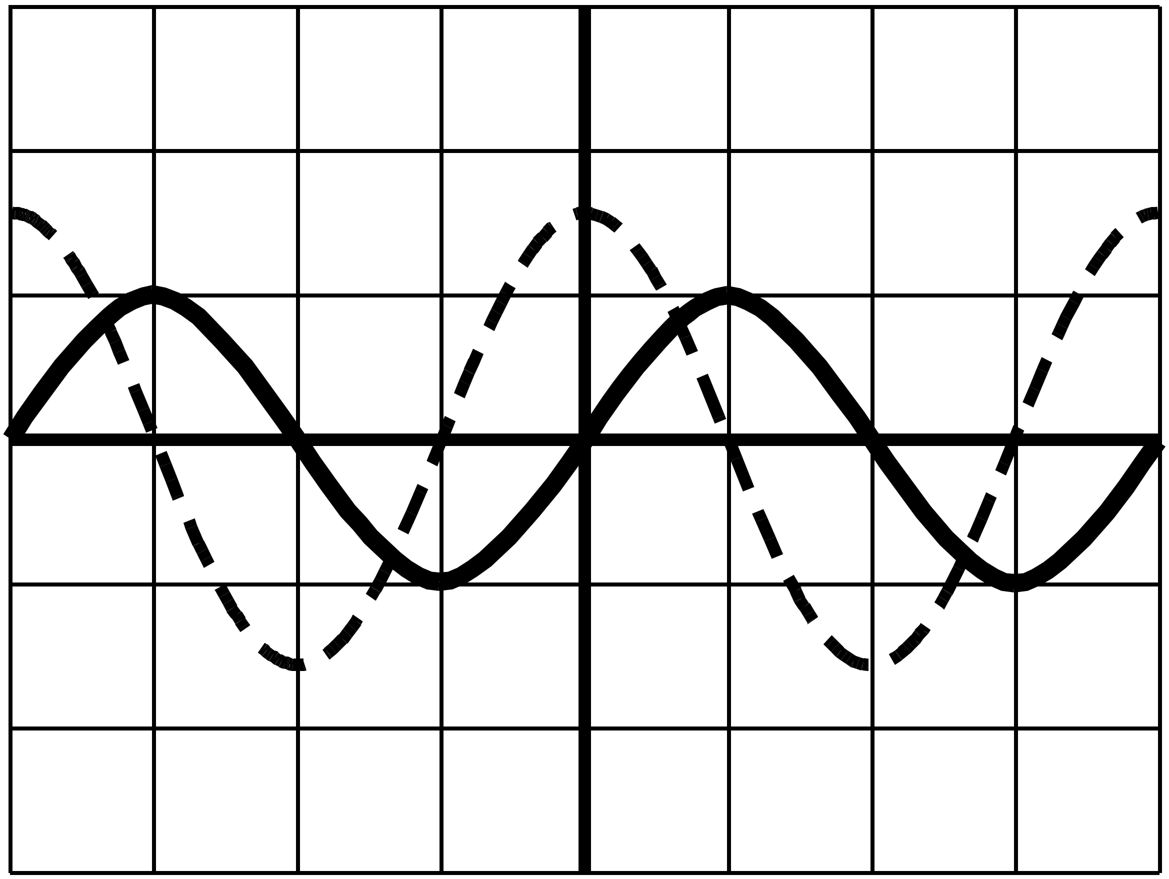
Answer 3



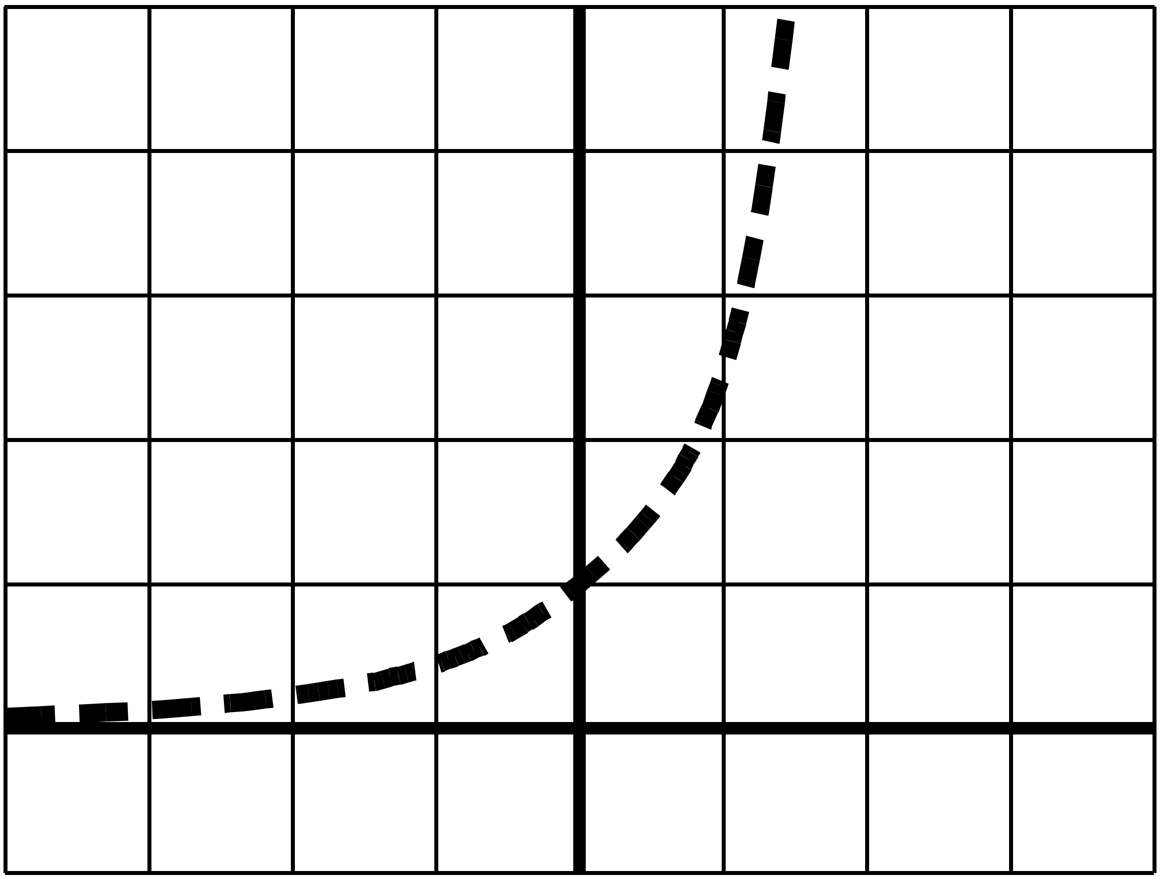
Answer 4



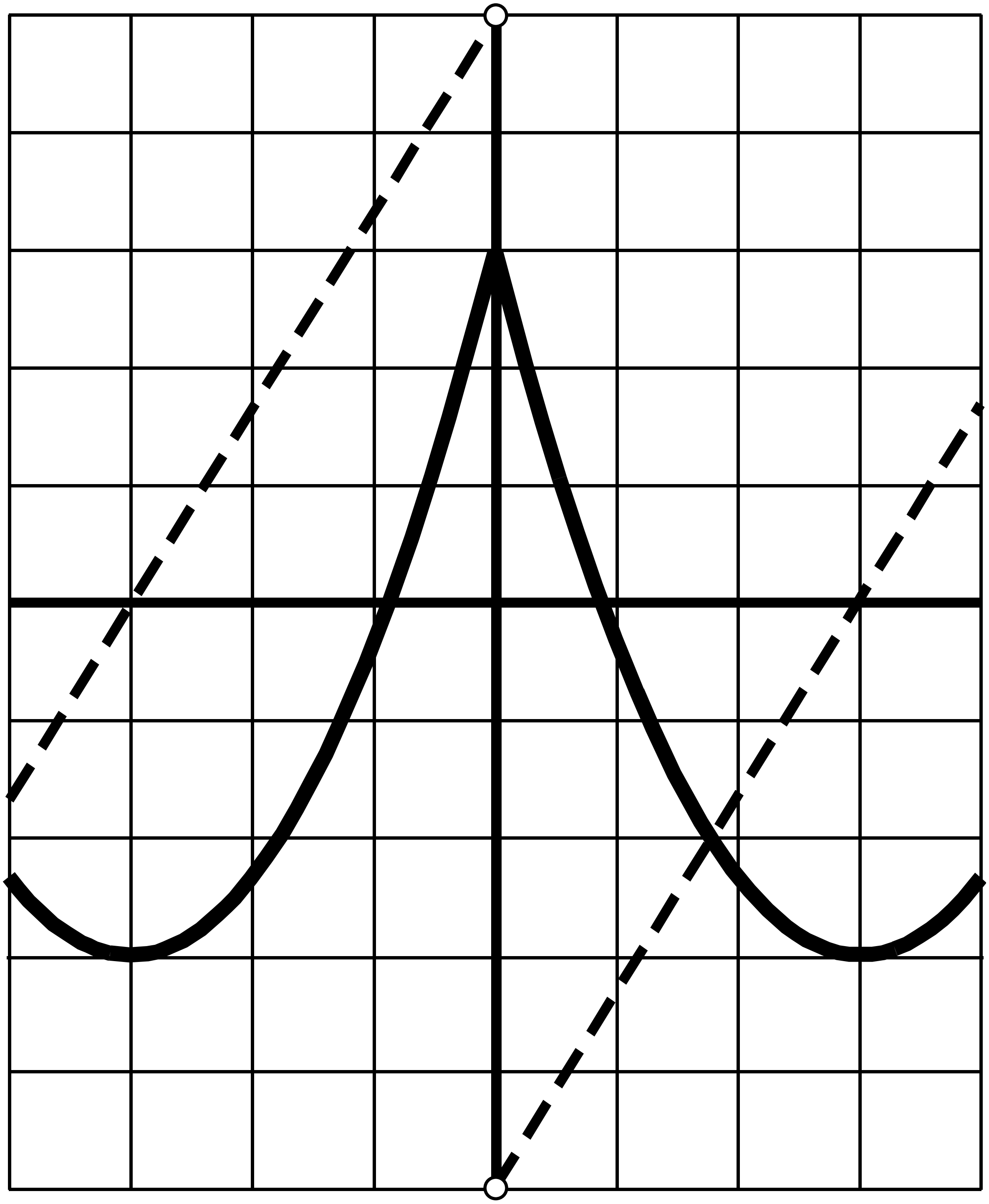
Answer 5



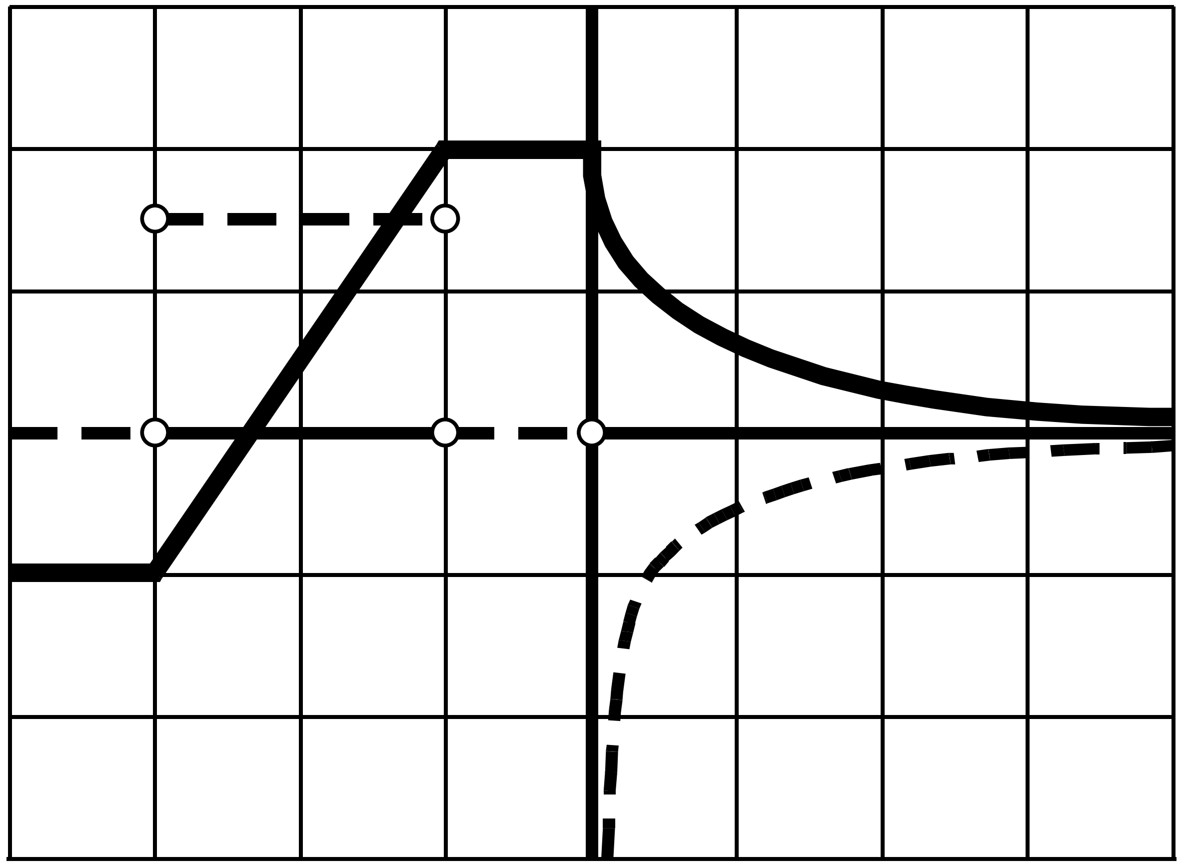
Answer 6



Answer 7



Answer 8



2.3 Differentiation Formulas

Suggested Time and Emphasis

2–3 classes Essential material

Points to Stress

1. The Power, Constant Multiple, Sum and Difference Rules, and how they are developed from the limit definition of the derivative.
2. Justification of the Product and Quotient Rules.
3. The computation of derivatives using the above rules.
4. The definition of the normal line to a curve at a point.

Quiz Questions

- **TEXT QUESTION** Why don't we use the Quotient Rule every time we encounter a quotient?
  ANSWER Sometimes algebraic simplification can make the problem much easier.
- **DRILL QUESTION** Compute the derivative of $\left(x^6 - \frac{1}{8}x^4\right)\left(\sqrt{x} + \pi\right)$.
  ANSWER $\dfrac{x^6 - \frac{1}{8}x^4}{2\sqrt{x}} + \left(6x^5 - \frac{1}{2}x^3\right)\left(\sqrt{x} + \pi\right)$

Materials for Lecture

- As an introductory exercise, draw the function $f(x) = \dfrac{x^3}{3}$. Ask the students to estimate slopes at several points, perhaps using secant lines. Create a table of x versus $f'(x)$ and try to get them to see the pattern. Then review the idea of the derivative function. Similarly, examine the derivatives of $f(x) = 5x + 2$ and $f(x) = 3$.
- Let $f(x) = x^3 + 2x^2 + 3x + 4$. Find a point a, both graphically and algebraically, where $f'(a) = 2$. Then ask them to find where the tangent line to the function $f(x) = x^3 - x + 1$ is parallel to the line $y = x$.
- Derive the Product Rule, and show its relationship to the Constant Multiple Rule (For example, one can find $\left[3e^x\right]'$ using *either* rule, but $\left[xe^x\right]'$ requires the Product Rule.)
- State and demonstrate a proof of the Quotient Rule via the Reciprocal Rule:

  Let $fg = 1$. Then by the Product Rule, $f'g + g'f = 0 \quad \Rightarrow \quad f'g = -g'f \quad \Rightarrow \quad f' = -\dfrac{g'f}{g} = -\dfrac{g'}{g^2}$ since $f = \dfrac{1}{g}$. This is the Reciprocal Rule: If $f = \dfrac{1}{g}$, then $f' = -\dfrac{g'}{g^2}$.

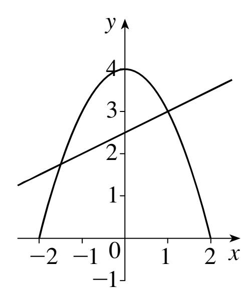
  This result allows us to prove the Quotient Rule:

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \left(f \cdot \frac{1}{g}\right)' = f'\left(\frac{1}{g}\right) + f\left(\frac{1}{g}\right)' && \text{(by the Product Rule)}\\ &= \frac{f'}{g} + f\left(-\frac{g'}{g^2}\right) && \text{(by the Reciprocal Rule)}\\ &= \frac{f'g - fg'}{g^2}\end{aligned}$$

- Show that, if $f(x) = x^4 - x^2 + x + 1$, then $f^{(5)}(x) \equiv 0$. Conclude that if $f(x)$ is a polynomial of degree m, then $f^{(m+1)}(x) \equiv 0$.

Workshop/Discussion

- Do a complex-looking differentiation that requires algebraic simplification, such as $f(x) = \dfrac{x^2\sqrt[3]{x} + x\sqrt{x^3} - \left(\sqrt{2}x\right)^2}{\left(x^{2/3}\right)^2}$
- After the students have mastered the basics of the Power Rule, have them differentiate some notationally tricky functions such as x^{π}, $\sqrt[\pi]{x}$, and $\pi^{\sqrt{2}}$.
- Give some examples in which the automatic use of the Quotient Rule is not the best strategy to follow, for example, $f(x) = \dfrac{x^2 + \sqrt{x} - \sqrt[3]{x}}{x}$, $g(x) = \dfrac{x^3 - 2x}{17}$, or $h(x) = \dfrac{3}{x}$. The idea is to get the students to think and simplify first (if they can) before using any of the rules.
- Do an example like Exercise 53. If you actually use the Witch of Agnesi, the students may be interested to hear the history of the curve: Italian mathematician Maria Agnesi (1718–1799) was a scholar whose first paper was published when she was nine years old. She called a particular curve *versiera*, or "turning curve". John Colson from Cambridge confused the word with *avversiera*, or "wife of the devil," and translated it "witch".
- Graph $f(x) = 4 - x^2$ and compute the equations of the tangent line and the normal line at $x = 1$. Draw those lines and point out that, as predicted, they are perpendicular.



Group Work 1: Doing a Lot with a Little

This exercise starts out by showing what can be done with the Power Rule, and ends by foreshadowing the Chain Rule. The first page should be handed out separately, and then the second sheet handed out to groups who finish early. Emphasize that the solution to Problem 5 should resemble that of Problem 4 in form. If a group finishes both sheets far ahead of the others, ask them to figure out a formula for the derivative of $f(x) = (g(x))^n$, and to come up with a few examples to check their formula. (Notice that when we state the Power Rule, we allow n to be any real number.)

ANSWERS (Notation may vary)

1. $f'(x) = 10x^9 + 7x^8 + 4x^7 - 35x^6 - 1.98x^5 + 5\pi x^4 - 4\sqrt{2}x^3$

2. $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$, $g'(x) = -\dfrac{3}{x^4} + \dfrac{3}{4\sqrt[4]{x^7}}$, $h'(x) = \frac{9}{2}x^{7/2} - x^{-3/2}$

3. $f'(x) = 64x^3$, $g'(x) = 15x^{14}$

4. This follows immediately when the given functions are expanded.

5. $f'(x) = n(kx)^{n-1}k$, $g'(x) = n\left(x^k\right)^{n-1} \cdot kx^{k-1}$

Group Work 2: Find the Error

This is the first of several exercises where students will try to find mistakes in somebody else's reasoning. When first faced with a task like this, some students will pick a line towards the end, show it is false, and then consider the task completed. It is important to stress you want them to find the reasoning error; what the person who did the work did incorrectly to get that false line.

If a student still doesn't understand the idea, put it this way: "The person who wrote this listens to what you just said, and says, 'What did I do wrong?' Can you give an answer that will help that person avoid making similar mistakes in the future?"

ANSWER The function "$\underbrace{x + x + \cdots + x}_{x \text{ times}}$" is defined only for integer values of x and is thus not a differentiable function.

Group Work 3: Back and Forth

This exercise foreshadows antiderivatives and gives students an opportunity to practice using the derivative rules they've learned so far.

The students pair up, and decide who is A and who is B. Seat the A's on one side of the room and the B's on the other side. All the A's get one sheet, and all the B's get the other sheet. The students compute five derivatives, without simplifying, and write their answers in the space provided. Emphasize that they should write only their unsimplified answers, not the work leading up to them, in the blanks. Then they trade papers with their partner and try to undo what their partner has done, that is, find the antiderivative.

If a pair finishes early, have them repeat the exercise, making up their own functions, and simplifying at will.

When closing this exercise, have the class notice that there was no way to recover the constant terms in Problems 1 and 5. Ask what this implies about the general problem of finding a function whose derivative is equal to a given function.

ANSWERS

FORM A $f'(x) = 20x^3 + 3x$ (the 4 is unrecoverable), $g'(x) = x^{-1/2} - x^{-3/4}$, $h'(x) = \left(x^2 + 2x + 4\right)\left(3x^2 - 1\right) + (2x + 2)\left(x^3 - x - 3\right)$, $j'(x) = \dfrac{\left(\sqrt{x} + 1\right)\left(4x^3 - 4\right) - \left[\left(x^4 - 4x + 3\right) / \left(2\sqrt{x}\right)\right]}{\left(\sqrt{x} + 1\right)^2}$, $k'(x) = -x^{-4/3}$ (the 42 is unrecoverable)

FORM B $f'(x) = -6x^2 + 8\sqrt{x}$ (the 8 is unrecoverable), $g'(x) = \frac{1}{5}\left[\left(3x^2\right)\left(x^3 + x\right) + \left(x^3 + 1\right)\left(3x^2 + 1\right) + 12x\right]$,
$h'(x) = \left(x^3 + x^2 + 2x\right)\left(10x - 8x^3 + 8\right) + \left(5x^2 - 2x^4 + 8x\right)\left(3x^2 + 2x + 2\right)$,
$j'(x) = 1 + 2x$ ($x \neq 0$), $k'(x) = -\frac{22}{3}x^{-2/3}$

Group Work 4: Sparse Data

This exercise allows the students to practice the rules they have learned, with a minimum of algebraic manipulation. The students should work on these problems in groups of three or four, perhaps choosing groups of students with similar algebraic proficiency. The first four problems are more routine than the last two, and Problem 5 uses the General Power Rule, which was illustrated in Group Work 1.

ANSWERS **1.** 0 **2.** -48 **3.** $\frac{43}{25}$ **4.** -18 **5.** $\frac{1}{3}$

Homework Problems

CORE EXERCISES 5, 7, 17, 23, 25, 29, 61, 63

SAMPLE ASSIGNMENT 5, 7, 9, 11, 15, 17, 23, 25, 27, 29, 35, 51, 55, 59, 61, 63, 75

EXERCISE	D	A	N	G
5		×		
7		×		
9		×		
11		×		
15		×		
17		×		
23		×		
25		×		
27		×		

EXERCISE	D	A	N	G
29		×		
35		×		
51		×		
55		×		
59		×		
61		×		
63		×		
75		×		

GROUP WORK 1, SECTION 2.3

Doing a Lot with a Little

Section 2.3 introduces the Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$, where n is any real number. The good news is that this rule, combined with the Constant Multiple and Sum Rules, allows us to take the derivative of even the most formidable polynomial with ease! To demonstrate this power, try Problem 1:

1. *A formidable polynomial:*

$$f(x) = x^{10} + \tfrac{7}{9}x^9 + \tfrac{1}{2}x^8 - 5x^7 - 0.33x^6 + \pi x^5 - \sqrt{2}x^4 - 42$$

Its derivative:

$$f'(x) =$$

The ability to differentiate polynomials is only one of the things we've gained by establishing the Power Rule. Using some basic definitions, and a touch of algebra, there are all kinds of functions that can be differentiated using the Power Rule.

2. *All kinds of functions:*

$f(x) = \sqrt[3]{x} + \sqrt[5]{2}$ $\qquad g(x) = \dfrac{1}{x^3} - \dfrac{1}{\sqrt[4]{x^3}}$ $\qquad h(x) = \dfrac{x^5 - 3\sqrt{x} + 2}{\sqrt{x}}$

Their derivatives:

$f'(x) =$ $\qquad g'(x) =$ $\qquad h'(x) =$

Unfortunately, there are some deceptive functions that look like they should be straightforward applications of the Power and Constant Multiple Rules, but actually require a little thought.

3. *Some deceptive functions:*

$f(x) = (2x)^4$ $\qquad g(x) = \left(x^3\right)^5$

Their derivatives:

$f'(x) =$ $\qquad g'(x) =$

The process you used to take the derivative of the functions in Problem 3 can be generalized. In the first case, $f(x) = (2x)^4$, we had a function that was of the form $(kx)^n$, where k and n were constants ($k = 2$ and $n = 4$). In the second case, $g(x) = \left(x^3\right)^5$, we had a function of the form $\left(x^k\right)^n$. Now we are going to find a pattern, similar to the Power Rule, that will allow us to find the derivatives of these functions as well.

4. Show that your answers to Problem 3 can also be written in this form:

$$f'(x) = 4(2x)^3 \cdot 2 \qquad\qquad g'(x) = 5\left(x^3\right)^4 \cdot 3x^2$$

And now it is time to generalize the Power Rule. Consider the two general functions, and try to find expressions for the derivatives similar in form to those given in Problem 4. You may assume that n is an integer.

5. *Two general functions*:

$$f(x) = (kx)^n \qquad\qquad g(x) = \left(x^k\right)^n$$

Their derivatives:

$$f'(x) =$$
$$g'(x) =$$

Find the Error

It is a bright Spring morning. You have just finished your Chemistry lab, and have a Physics class starting in a half hour, so you have a little bit of time to sit on a park bench and relax by leafing through your *Calculus* book. Suddenly, you notice a wild-eyed, hungry-looking stranger looking over your shoulder.

"Lies! Lies!" he yells. "That book there is filled with nothing but lies!"

"Why, you are mistaken," you explain. "My *Calculus* book is chock-a-block with knowledge and useful wisdom."

"Oh yeah? Well what would your calculus book say about THIS?" he demands, and hands you a piece of paper with the following written on it:

For $x > 0$:

$$\begin{aligned}
x &= \underbrace{1 + 1 + \cdots + 1}_{x \text{ times}} \\
x^2 &= \underbrace{x + x + \cdots + x}_{x \text{ times}} \\
D\left(x^2\right) &= D\left(\underbrace{x + x + \cdots + x}_{x \text{ times}}\right) \\
D\left(x^2\right) &= \underbrace{D(x) + D(x) + \cdots + D(x)}_{x \text{ times}} \\
2x &= \underbrace{1 + 1 + \cdots + 1}_{x \text{ times}} \\
2x &= x \\
2 &= 1
\end{aligned}$$

"Put THAT in your pipe and smoke it!" At that, the gentleman runs off, screaming, "I'll be back!" into the wind.

Is all of mathematics wrong? Is two really equal to one? Are "two for one" specials really no bargain at all? Is "six of one" really not "half a dozen of the other"? Or is there a mistake in your new friend's reasoning? If so, what is it?

GROUP WORK 3, SECTION 2.3
Back and Forth (Form A)

Compute the following derivatives. Write your answers at the bottom of this sheet, where indicated. When finished, fold the top of the page backward along the dotted line and hand to your partner.

Do not simplify.

1. $f(x) = 5x^4 + \frac{3}{2}x^2 - 4$

2. $g(x) = 2\sqrt{x} - 4\sqrt[4]{x}$

3. $h(x) = \left(x^2 + 2x + 4\right)\left(x^3 - x - 3\right)$

4. $j(x) = \dfrac{x^4 - 4x + 3}{\sqrt{x} + 1}$

5. $k(x) = \dfrac{3}{\sqrt[3]{x}} + 42$

- -

ANSWERS

$f'(x) =$

$g'(x) =$

$h'(x) =$

$j'(x) =$

$k'(x) =$

GROUP WORK 3, SECTION 2.3

Back and Forth (Form B)

Compute the following derivatives. Write your answers at the bottom of this sheet, where indicated. When finished, fold the top of the page backward along the dotted line and hand to your partner.

Do not simplify.

1. $f(x) = -2x^3 + \frac{\sqrt{8}}{2}x^2 - 8$

2. $g(x) = \dfrac{(x^3+1)(x^3+x) + 6x^2}{5}$

3. $h(x) = (x^3 + x^2 + 2x)(5x^2 - 2x^4 + 8x)$

4. $j(x) = \dfrac{x^2 + x^3}{x}$

5. $k(x) = \sqrt{11} - 22\sqrt[3]{x}$

ANSWERS

$f'(x) =$

$g'(x) =$

$h'(x) =$

$j'(x) =$

$k'(x) =$

GROUP WORK 4, SECTION 2.3

Sparse Data

Assume that $f(x)$ and $g(x)$ are differentiable functions about which we know very little. In fact, assume that all we know about these functions is the following table of data:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	3	1	-5	8
-1	-9	7	4	1
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	?

This isn't a lot of information. For example, we can't compute $f'(3)$ with any degree of accuracy. But we are still able to figure some things out, using the rules of differentiation.

1. Let $h(x) = \left(\sqrt[3]{x}\right)^4 f(x)$. What is $h'(0)$?

2. Let $j(x) = -4f(x)g(x)$. What is $j'(1)$?

3. Let $k(x) = \dfrac{xf(x)}{g(x)}$. What is $k'(-2)$?

4. Let $l(x) = x^3 g(x)$. If $l'(2) = -48$, what is $g'(2)$?

5. Let $m(x) = \dfrac{1}{f(x)}$. What is $m'(1)$?

APPLIED PROJECT **Building a Better Roller Coaster**

This project models a typical hill in a roller coaster ride using two lines as the sides and a parabola for the peak area. It also discusses how to smooth this model to have a continuous second derivative by using cubic connecting functions between the parabola and the two lines. A computer algebra system is needed to solve the resulting equations. In their report, students should address the question, "*Why* do we want the second derivative to be continuous?"

2.4 Derivatives of Trigonometric Functions

Suggested Time and Emphasis

1 class Essential material

Points to Stress

Formulas for the derivatives of the standard trigonometric functions.

Quiz Questions

- **TEXT QUESTION** Why does the text bother going through all the fuss of computing $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$ and $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta}$?

  ANSWER When deriving the formulas for the derivatives for $\sin \theta$ and $\cos \theta$, these limits arise when taking the limits of the difference quotients. These computations are necessary to finish the derivations.

- **DRILL QUESTION** What is $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - \tan\left(\frac{\pi}{4}\right)}{h}$?

  (A) 2 (B) $-\frac{\sqrt{2}}{2}$ (C) 0 (D) 1 (E) Does not exist

  ANSWER (A)

Materials for Lecture

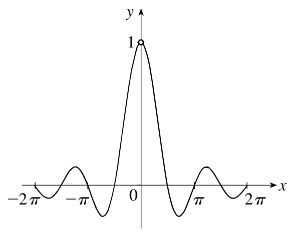
- Many students may need a review of notation: $\sin^2 x = (\sin x)^2$, $\sin x^2 = \sin\left(x^2\right)$, $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$, $\sin x^{-1} = \sin \frac{1}{x}$, but $\sin^{-1} x$ represents the inverse sine of x, $\arcsin x$, and not any of the previous functions.
- Demonstrate simple harmonic motion in different ways such as observing the end of a vertical spring, marking the edge of a spinning disk, or swinging an object on a chain.
- Have the students set their calculators to degrees and approximate the derivative of $\cos x$ at $x = \frac{\pi}{2}$ by zooming in on the graph of $\cos x$. Repeat the exercise with the calculators set to radians. Discuss the reason why the answers are different, and why only one is considered correct. Show how the slope of the tangent to the graph of $\sin x$ at $x = 0$ is *not* 1 if the x-axis is calibrated in degrees instead of radians.

  ANSWER The derivation of $(\sin \theta)' = \cos \theta$ involved using the fundamental trigonometric limit, which assumed θ was in radians.

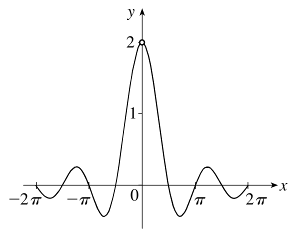
Workshop/Discussion

- If Group Work 2 in Section 1.4 ("Just Two Solutions") was assigned, this is a good time to revisit it. The students are now in a position to get farther, but will wind up having to solve $x = -\cot x$. They can approximate this number on their calculators, or you can revisit the activity a third time when covering Newton's Method.
- Demonstrate that $\lim_{x \to 0} \frac{\sin ax}{ax} = 1$ for any positive a. Then ask students to find $\lim_{x \to 0} \frac{\sin ax}{x}$. Show how this argument can be extended to derive the formulas $\frac{d}{dx} \sin ax = a \cos ax$ and $\frac{d}{dx} \cos ax = -a \sin ax$.

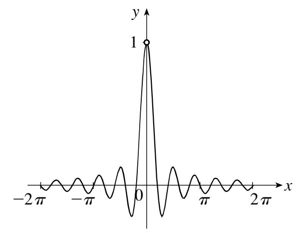
Finally, demonstrate that your results make sense by drawing graphs of $\frac{\sin ax}{ax}$ and $\frac{\sin ax}{x}$ for various values of a.



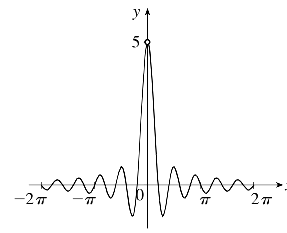
$y = \frac{\sin 2x}{2x}$



$y = \frac{\sin 2x}{x}$



$y = \frac{\sin 5x}{5x}$



$y = \frac{\sin 5x}{x}$

- Consider $f(x) = \frac{1}{2}x + \cos x$, $0 \le x \le 2\pi$. Discuss local maxima and minima of $f(x)$. Repeat for $g(x) = \frac{9}{10}x + \cos x$ and $h(x) = x + \cos x$. Discuss why h is qualitatively different from f and g.

Group Work 1: The Magnificent Six

After showing the students that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, it is possible to use the Quotient Rule to derive the trigonometric derivatives on their own, and the process of deriving these formulas is good practice at using the rules learned so far.

ANSWERS

1. $\cos x$ **2.** $-\sin x$ **3.** $\sec^2 x$

4. $\tan x \sec x$ **5.** $-\csc^2 x$ **6.** $-\cot x \csc x$

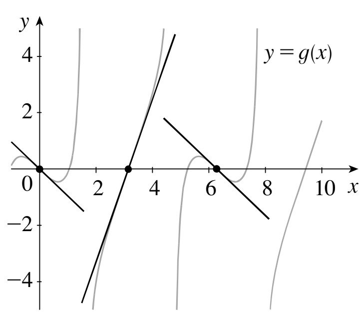
Group Work 2: Using Our New Knowledge

ANSWERS

1. $-1, 3, -1$

2. $y = -x$, $y = 3x - 3\pi$, $y = -x + 2\pi$

3.



4. There is no tangent line at $y = \frac{\pi}{2}$ because the function has a vertical asymptote there.

Group Work 3: When the Lights Go Down in the City

This activity will help the students understand the relationship between a trigonometric function in the abstract, and a trigonometric function as a model for real situations.

Creative use of technology can be encouraged here. It is important to stress to the students that Problem 2 assumes that they are looking at only a one-month window. Problem 6 foreshadows the technique of linear approximation covered in Section 2.9.

ANSWERS

1. Maximum: 1, minimum: 20 **2.** It is part of a cosine curve. **3.** December **4.** May
5. 3.095 minutes per day (0.5158 hours per day) **6.** $3.095 \cdot 31 = 95.94$, accurate to within about 1%.

Homework Problems

CORE EXERCISES 5, 7, 15, 21, 39

SAMPLE ASSIGNMENT 3, 5, 7, 9, 13, 15, 17, 21, 23, 25, 29, 35, 39, 41, 45

EXERCISE	D	A	N	G
3		×		
5		×		
7		×		
9		×		
13		×		
15		×		
17		×		
21		×		

EXERCISE	D	A	N	G
23		×		
25		×		×
29		×		
35		×		
39		×		
41		×		
45		×		

GROUP WORK 1, SECTION 2.4

The Magnificent Six

The derivative of $f(x) = \sin x$ was derived for you in class. From this one piece of information, it is possible to figure out formulas for the derivatives of the other five trigonometric functions. Using the trigonometric identities you know, compute the following derivatives. Simplify your answers as much as possible.

1. $(\sin x)' =$

2. $(\cos x)' =$

3. $(\tan x)' =$

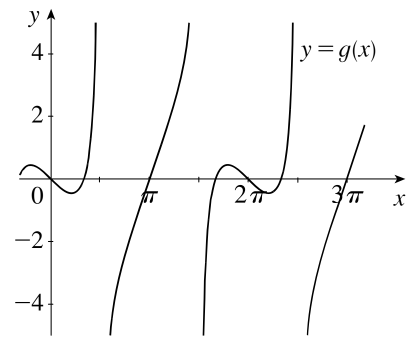
4. $(\sec x)' =$

5. $(\cot x)' =$

6. $(\csc x)' =$

GROUP WORK 2, SECTION 2.4
Using Our New Knowledge

The following is a graph of $g(x) = \tan x - 2\sin x$.



There are some things we can say about the graph just by looking at the picture, although our intuition may sometimes mislead us.

1. Compute $g'(0)$, $g'(\pi)$, and $g'(2\pi)$.

2. Find equations of the lines tangent to this curve at $x = 0$, $x = \pi$, and $x = 2\pi$.

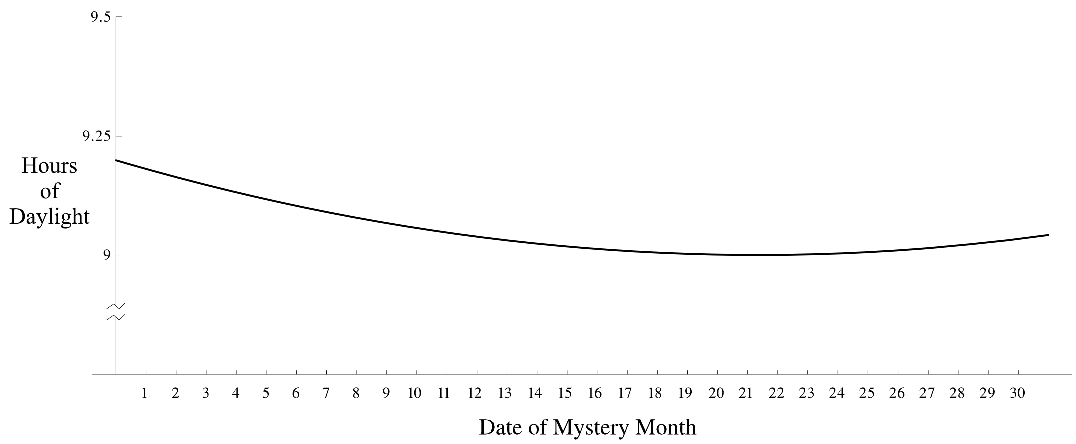
3. Graph the equations you found in Problem 2, and make sure they look as they should.

4. What happens when you try to find the equation of the line tangent to this curve at $x = \frac{\pi}{2}$? Why?

GROUP WORK 3, SECTION 2.4

When the Lights Go Down in the City

The number of hours of daylight in Summitville, Canada varies between 9 hours and 15 hours per day. A model for the number of daylight hours on day t is $D(t) = 12 - 3\cos(0.0172(t + 11))$, $0 < t \leq 365$. ($t = 1$ corresponds to January 1.) The graph for a particular month looks like this:



1. On approximately what day of the month does this graph achieve its minimum? Its maximum?

2. Why does this graph have the shape that it does?

3. What month is this graph likely to represent?

4. For which month would you expect to see a graph shaped like this one, only upside-down?

5. How rapidly are we gaining daylight 90 days after the minimum occurs?

6. A newspaper in Summitville states that during the period of 31 days starting from day 68 after the minimum, we gain 1 hour and 35 minutes of sunlight. Use the rate of change computed in Problem 5 to estimate the change in hours of sunlight over this period. How close is your estimate to the figure reported in the newspaper?

2.5 The Chain Rule

Suggested Time and Emphasis

$1\frac{1}{2}$–2 classes Essential material

Points to Stress

1. A justification of the Chain Rule by interpreting derivatives as rates of change.

2. The use of the Chain Rule to compute derivatives.

Quiz Questions

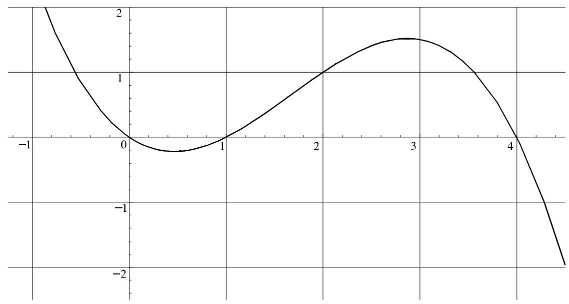
- **TEXT QUESTION** The text presents the two forms of the Chain Rule: $(f(g(x)))' = f'(g(x))\,g'(x)$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. Do these two equations say the same thing? Explain your answer.
  ANSWER They do. Let $y = f(u)$ and $u = g(x)$. Then the statement $f(g(x))' = f'(g(x))\,g'(x)$ becomes $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.
- **DRILL QUESTION** Compute $\frac{d}{dx}\sin x^2$ and $\frac{d}{dx}\sin^2 x$.
  ANSWER $2x\cos x^2$, $2\sin x\cos x$

Materials for Lecture

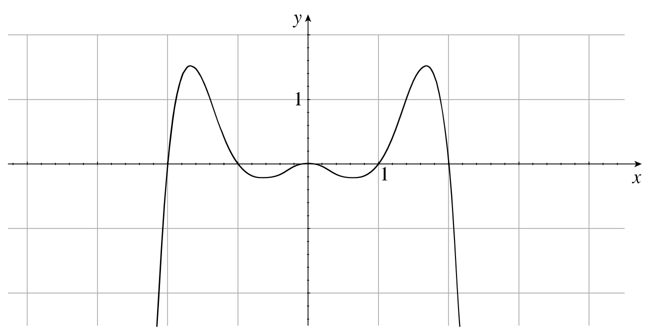
- The following is one way to introduce the Chain Rule:
  Before formally discussing the Chain Rule, do two examples of differentiating multi-nested functions. Explain to the students that you aren't going to justify anything yet, but that you just want them to see the pattern before getting into the material. After every step, say something like, "The derivative of $\sin x$ is $\cos x$, so the derivative of the sine of this stuff is the cosine of this stuff, times the derivative of what's left." After the students have seen the pattern with functions like $\left(\sin\left(\cos\left(x^2+4x+5\right)\right)\right)^{33}$, you should justify the Chain Rule and discuss the details.
- Show how to compute derivatives, using the Chain Rule, in one line. Take the derivative of $\sin\left(x^4+1\right)$, first by using the Chain Rule explicitly [$f(u) = \sin u$, $u(x) = x^4+1$], and then by inspection [the derivative of $\sin\left(x^4+1\right)$, which is $\cos\left(x^4+1\right)$ times the derivative of x^4+1, which is $4x^3$.]
- Address the question: "Where do you stop when using the Chain Rule?" For example, why is it false that
  $\frac{d}{dx}\sin\left(x^5+4x^2\right) \stackrel{?}{=} \left[\cos\left(x^5+4x^2\right)\right]\left(5x^4+8x\right)\left(20x^3+8\right)\left(60x^2\right)(120x)(120)$?
  One way to help students decide "when to stop" is to draw their attention to the text's Reference Page 5 (Differentiation Rules). One stops when the derivative is one of the primitive rules such as the ones on that page.
- Justify the Chain Rule using rate of change arguments, such as the following: One factory converts sugar to chocolate ($c = 8s$) and another converts chocolate to candy bars ($b = 16c$). Finding the rate at which sugar is converted to candy bars can be used to help justify the Chain Rule, particularly if the units of the relevant quantities are emphasized.

Workshop/Discussion

- Compute some derivatives, such as those of $\sec\left(x^2+x\right)$, $\left(\dfrac{\cos x}{x^2+1}\right)^3$, $\sqrt[3]{x+\cos x^2}$, and $\cos\left(\sin\left(x^2\right)\right)$.
- Compute the equation of the line tangent to $y=\cos\left(x+\dfrac{\pi}{\sqrt{x+1}}\right)$ at $(0,-1)$.
- Draw the following graph of $f(x)$ (or copy it onto a transparency).



Tell the students that $f'(0)=-1$, $f'(1)=\frac{3}{4}$, $f'(2)=1$, and $f'(4)=-3$. Define $g(x)=f\left(x^2\right)$. First compute $g(-1)$, $g(0)$, $g(1)$, $g\left(\sqrt{2}\right)$, and $g(2)$. Then compute $g'(0)$, $g'(-2)$, and $g'(2)$. Finally, sketch $g(x)$ as below. Use the graph to verify the values for $g(x)$ and $g'(x)$ computed above.



- Foreshadow Section 9.4 by showing how we can use the Chain Rule to demonstrate that e^{kx} solves the differential equation $\dfrac{dy}{dx}=ky$ and is the only solution.

Group Work 1: Unbroken Chain

This is meant to be a gentle introduction to the mechanics of taking derivatives using the Chain Rule. You may be surprised at the difficulty some groups have with Problem 4 of the exercise, but by the end they all should be ready to go home and practice.

Start by "warming the class up" as a large group by having them take the derivatives of functions like $x^{3.24}$, $\sin x$, $\sqrt{x}$, $\tan x$, and so on. This quick review is important, because the exercise works best if their mental focus is on the Chain Rule, as opposed to formulas they should already know.

While helping the individual groups, don't volunteer that the answers to most of the questions are supersets of the previous questions. They are supposed to discover this pattern for themselves.

If a group finishes early, give them a function like $\cos\left(x^2+\sqrt{x}\right)\sin(1/x)$ to try.

When they are finished, write the solutions to Problems 4, 6, and 7 on the board. Ask the students if they need you to write the solutions to the earlier ones. After they say "no", try to get them to explain why it isn't necessary. (If they say "yes", refuse and ask them why you are refusing.)

ANSWERS

1. $\cos 3x \cdot 3$ **2.** $3(\sin 3x)^2 \cos 3x \cdot 3$ **3.** $3(\sin 3x)^2 \cos 3x \cdot 3 + 5$

4. $5\left[(\sin 3x)^3 + 5x\right]^4 \left[3(\sin 3x)^2 (\cos 3x)\, 3 + 5\right]$ (check their parentheses carefully)

5. $1 - x^{-2}$ **6.** $\frac{1}{2}\left[x + (1/x)\right]^{-1/2}\left(1 - x^{-2}\right)$

7. $\left[(\sin 3x)^3 + 5x\right]^5 \frac{1}{2}\left(x+\frac{1}{x}\right)^{-1/2}\left(1-x^{-2}\right) + 5\left[(\sin 3x)^3 + 5x\right]^4\left[3(\sin 3x)^2(\cos 3x)\,3+5\right]\left(\sqrt{x+\frac{1}{x}}\right)$

(If the students don't write out the answer to Exercise 7, instead referring to the answers to previous parts, don't penalize them; they have gotten the point.)

Group Work 2: Chain Rule Without Formulas

This exercise works best with pairs or groups of three. Before handing it out, write both forms of the Chain Rule on the board. If a group finishes early, ask them where $h' = 0$ and over which intervals h' is constant. (This turns out to be a tricky problem.)

ANSWERS **1.** $f'(3)\,g'(1) \approx 3$ **2.** $f'(0)\,g'(0) \approx -\frac{3}{2}$ **3.** $g'(2)$ does not exist, so $h'(2)$ does not exist.

Group Work 3: Examining a Strange Graph

Have the students first answer the questions just by looking at the graph, and then go back and verify their intuition using calculus. If the students find this curve interesting, you can point out another interesting property. Consider the line segment going from $(0,-1)$ to $(0,1)$. The curve gets arbitrarily close to *every* point on this segment, although it never actually touches the segment. If we consider the combined segment and curve we get a mathematical object that is "connected" but not "path connected".

If a group finishes early, perhaps ask them to figure out what the graph of $\tan(1/x)$ will look like, and to verify their guess using their calculators.

ANSWERS

1. $y' = -\dfrac{\cos(1/x)}{x^2}$. As $x \to \infty$, $y' \to 0$. Therefore the function has a horizontal asymptote. Or, one can argue that as $x \to \infty$, $1/x \to 0$, so $\sin(1/x) \to 0$.

2. The function does not approach a specific y-value as $x \to 0$. (One can look at either the function or its derivative as $x \to 0$.)

3. The slope of the curve approaches 0.

4. The slope oscillates, but its peaks and valleys get larger and larger without bound as $x \to 0$.

Homework Problems

CORE EXERCISES 7, 23, 25, 47, 51, 55

SAMPLE ASSIGNMENT 7, 9, 13, 15, 19, 23, 25, 47, 49, 51, 53, 55

EXERCISE	D	A	N	G
7		×		
9		×		
13		×		
15		×		
19		×		
23		×		

EXERCISE	D	A	N	G
25		×		
47		×		
49		×		
51		×		
53		×		
55		×		×

GROUP WORK 1, SECTION 2.5
Unbroken Chain

For each of the following functions of x, write the equation for the derivative function. This will go a lot more smoothly if you remember the Sum, Product, Quotient, and Chain Rules... especially the Chain Rule! Please do us both a favor and don't simplify the answers.

1. $f(x) = \sin 3x$ $\qquad f'(x) =$

2. $g(x) = (\sin 3x)^3$ $\qquad g'(x) =$

3. $h(x) = (\sin 3x)^3 + 5x$ $\qquad h'(x) =$

4. $j(x) = \left[(\sin 3x)^3 + 5x\right]^5$ $\qquad j'(x) =$

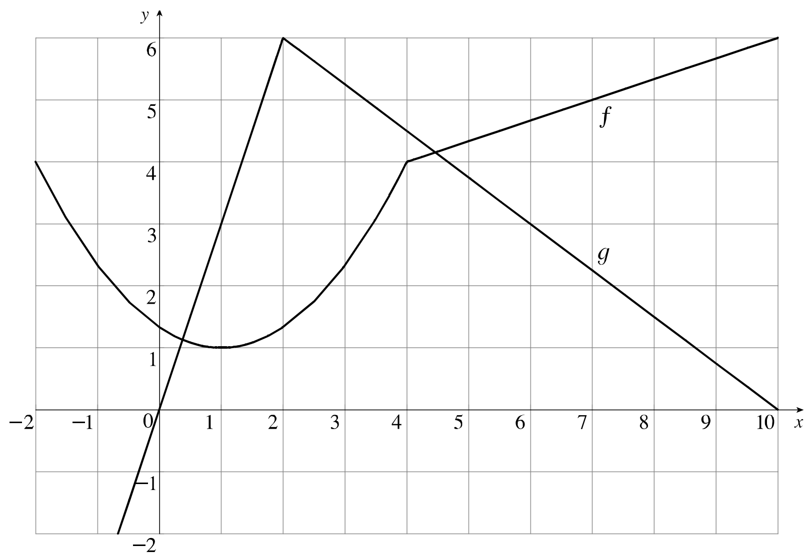
5. $k(x) = x + \dfrac{1}{x}$ $\qquad k'(x) =$

6. $l(x) = \sqrt{x + \dfrac{1}{x}}$ $\qquad l'(x) =$

7. $m(x) = \left(\sqrt{x + \dfrac{1}{x}}\right)\left[(\sin 3x)^3 + 5x\right]^5$ $\qquad m'(x) =$

GROUP WORK 2, SECTION 2.5
Chain Rule Without Formulas

Consider the functions f and g given by the following graph:



Define $h = f \circ g$.

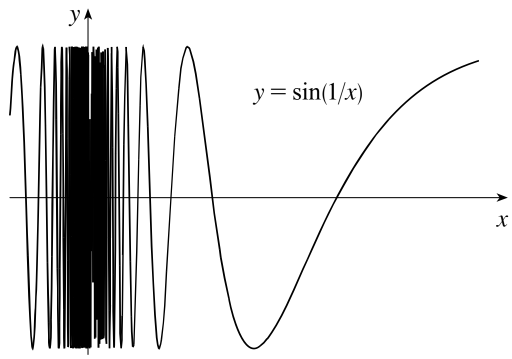
1. Compute $h'(1)$.

2. Compute $h'(0)$.

3. Does $h'(2)$ exist?

GROUP WORK 3, SECTION 2.5
Examining a Strange Graph

Several times in this course, we have looked at the graph of $y = \sin(1/x)$.



There are some things we can say about the graph just by looking at the picture, although our intuition may deceive us.

1. As we move farther and farther to the right, does the graph oscillate forever, or does it approach some y-value?

2. As we move closer and closer to zero, does the graph oscillate forever, or approach some y-value?

3. What happens to the slope of the curve as we go farther and farther to the right?

4. What happens to the slope of the curve as we approach zero?

Since intuition could fail us, please consider the function $y = \sin(1/x)$ directly, and prove that your answers to the above questions are correct. If it turns out that you were wrong above, then correct your answer and note why your intuition led you astray.

SPECIAL SECTION **Derivative Hangman**

I recommend using this exercise just after covering the Chain Rule, for a class of students who need more practice computing derivatives. It is designed to keep all the students involved and practicing both computing derivatives, and checking their work. Divide the class into teams of 4–6 students each. Put blanks representing the letters of a mystery word or phrase on the board. The game then proceeds as follows:

One representative from each team goes to the blackboard. The teacher then puts up a function either on the blackboard, or using the overhead projector. Everyone in the room tries to compute the derivative. The people at the board cannot speak, but their teammates can work together, speaking quietly.

The first person at the board to compute the derivative slaps the board, blows a whistle, or claps their hands. The teacher calls on him or her to state the solution. Then each other team gets a chance to accept the answer, or challenge.

The team that wins (first to have their representative get it right, or first to challenge successfully) gets to guess a letter of the puzzle. If they guess A, for example, all instances of A in the mystery phrase are filled in:

__ __ A __ __ A __ __ __   __ __ __ __ __ __ A

Whether or not their letter was in the phrase, they then get a chance to guess at the puzzle ("QUADRATIC FORMULA", in this case). If they get it right, the round is over and they win. If not, each team sends up a new representative and the game continues.

If this game is officiated with care and enthusiasm, all the students will be involved and working every time a new problem is put on the board.

APPLIED PROJECT **Where Should a Pilot Start Descent?**

This project can be used as an out-of-class assignment, or as an extended in-class exercise. At this point in the course, some students may be asking about opportunities for extra credit, and an oral report based on this project would be a worthwhile extra-credit activity.

The project includes a computation of the minimum distance from the airport at which an airplane should begin its descent. A nice addition to this project would be the actual figure (or range of figures) used by a local airport, obtained by a few well-placed telephone calls.

2.6 Implicit Differentiation

Suggested Time and Emphasis

1 class   Essential material

Points to Stress

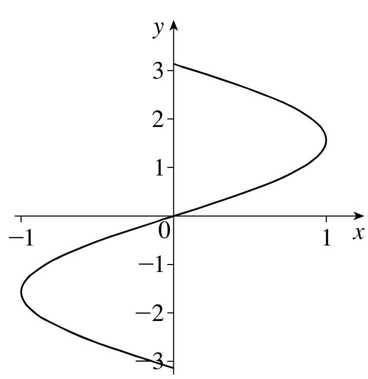
1. The concepts of implicit functions and implicit curves.
2. The technique of implicit differentiation.
3. The derivatives of the arcsine and arctangent functions.

Quiz Questions

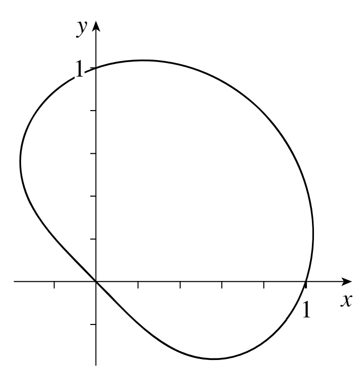
- **TEXT QUESTION** Describe what is being illustrated by Figure 3. Make sure your answer is as complete as possible.
  ANSWER The implicit curve $x^3 + y^3 = 6xy$ does not define a function. Figure 3 illustrates several functions, each of which is implicitly defined by $x^3 + y^3 = 6xy$.
- **DRILL QUESTION** If $x^2 + xy = 10$, find $\frac{dy}{dx}$ when $x = 2$.
  ANSWER $-\frac{7}{2}$

Materials for Lecture

- Go over the definition of implicit curves, and the method of implicit differentiation. A good starting example is the curve defined by $x = \sin y$ (which can be easily graphed and visualized). Another example is the curve $x + y = \left(x^2 + y^2\right)^2$, which can be graphed using polar coordinates.
  ANSWER



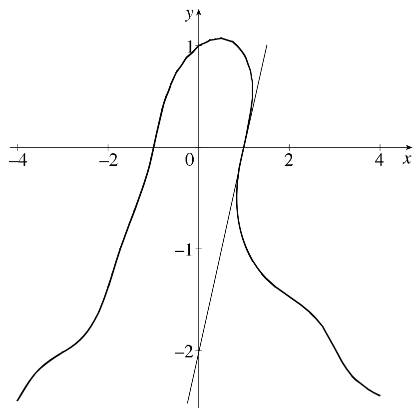
$x = \sin y$



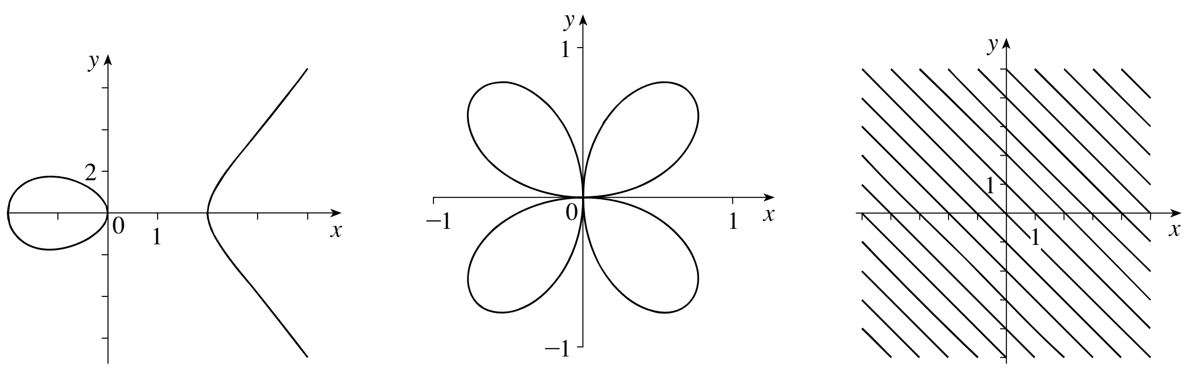
$x + y = \left(x^2 + y^2\right)^2$,
$r = \sqrt[3]{\cos\theta + \sin\theta}$

- Derive the equation of the line tangent to the curve $x^2 - \sin(xy) + y^3 = 1$ at the point $(1, 0)$. Sketch the curve as below and draw the tangent line.

  ANSWER The tangent line is $y = 2x - 2$.



- Discuss $f(x) = \sin\left(\sin^{-1} x\right)$. Remind the students that the domain is $[-1, 1]$ and that $f(x) = x$ on that domain. Also show that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for $0 \le x \le 1$.
- Display some interesting looking implicit curves such as the following:



$y^2 = x^3 - 4x$
an elliptic curve

$x^6 + y^6 = 4x^2y^2 - 3y^2x^4 - 3x^2y^4$
a four-leaved rose

$\sin(\pi(x + y)) = 0$

Have the students figure out a test to see if a given point is on the implicit curve. For example, is $(2, 0)$ on the first graph? Is $(0.6, 0.2)$ on the second? Is $(1.2, 2.8)$ on the third? Have the students determine the slopes of the lines in the third graph, and show that they are parallel.

ANSWER Substituting the coordinates into the equations shows that $(2, 0)$ is on the first graph, $(0.6, 0.2)$ is not on the second, and $(1.2, 2.8)$ is on the third. The lines on the third graph all have slope -1, and are therefore parallel.

Workshop/Discussion

- If the students have access to appropriate graphing technology, have them try to come up with interesting-looking implicit curves. Perhaps have an award for the most aesthetically pleasing one.
- Consider $r^2 + 2s\sqrt{t} = rt$. Show the students how to compute dr/dt when s is held constant, dr/ds and ds/dr when t is held constant, and dt/ds when r is held constant.
- Have the students differentiate $y^2 = x^7 - 6x$ implicitly, and then differentiate $y = \sqrt{x^7 - 6x}$ using the Chain Rule.

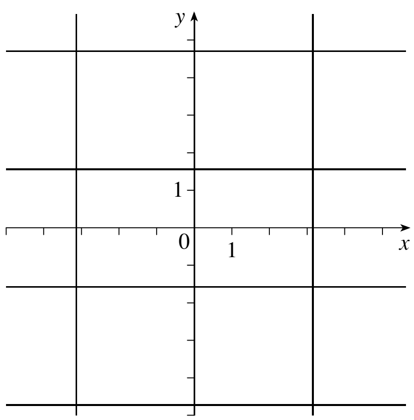
- If $f(x)^4 = (x + f(x))^3$ and $f(1) = 2$, find $f'(1)$.

Group Work 1: Implicit Curves

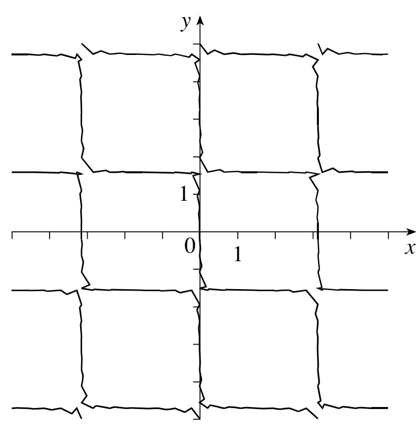
Computer algebra systems are notoriously bad at graphing implicit functions. Even simple functions such as the ones described above in Materials for Lecture point 5 are often poorly graphed by implicit function plotters. This activity describes an implicit curve which many calculators graph inaccurately, but which can be analyzed using a little bit of algebra.

ANSWERS

1. All lines of the form $x = \pi k$, $y = \frac{\pi}{2} + \pi k$, k an integer.



2. Maple gives the graph below.



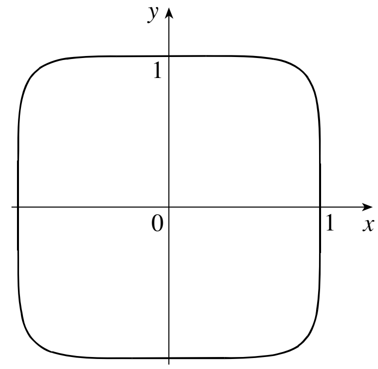
3. $dy/dx = 0$ or is undefined when $x = \pi k$. The derivative must be taken carefully to obtain this result.

Group Work 2: Circles and Astroids

The basic idea of this exercise is for students to visualize flat circles and astroids, and to compute slopes by implicit differentiation. The question about where the slope is 1 or -1 can be addressed first visually and then analytically. As a follow-up question, students can be asked to show that the answers are always the points of intersection with the lines $y = x$ and $y = -x$.

ANSWERS

1. $\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^5$. The slope of the tangent is 1 at $\left(\pm 2^{-1/6}, \mp 2^{-1/6}\right)$ and -1 at $\left(\pm 2^{-1/6}, \pm 2^{-1/6}\right)$.



2. If $p/q = \frac{4}{3}$, the slope is 1 at $\left(\pm 2^{-3/4}, \mp 2^{-3/4}\right)$ and -1 at $\left(\pm 2^{-3/4}, \pm 2^{-3/4}\right)$. If $p/q = \frac{2}{5}$, the slope is 1 at $\left(\pm 2^{-5/2}, \mp 2^{-5/2}\right)$ and -1 at $\left(\pm 2^{-5/2}, \pm 2^{-5/2}\right)$.

Group Work 3: Looking for the Minimum

This exercise gives practice with the derivative of arcsine, reviews earlier material by bringing it back to tangent lines, and foreshadows local extrema.

ANSWERS

1. $g'(x) = \dfrac{2x + \cos x}{\sqrt{-x^4 - 2x^2 \sin x + \cos^2 x}}$

2. $y + 0.2346 = 0.0004597\,(x - 0.45)$

3. It occurs to the left, because the function is increasing at $x = -0.45$.

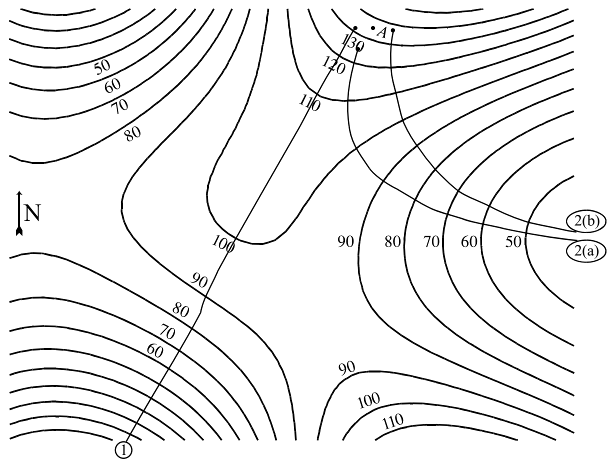
4. $x \approx -0.45018$

Group Work 4: A Walk in the Park

Before beginning this exercise, discuss the concepts of orthogonal trajectories (discussed in Exercises 49–52) and path of steepest descent. Perhaps do a quick example on the blackboard, and then hand out the activity. Problem 4 requires some deep reasoning.

ANSWERS

1, 2.



3. The steepest descent lines are always perpendicular to the contour lines.

4. Yes, there are. There are precarious balance points between the paths that go to one valley or the other. These are *points of unstable equilibrium*.

Homework Problems

CORE EXERCISES 7, 23, 25, 35, 49

SAMPLE ASSIGNMENT 1, 7, 9, 13, 17, 23, 25, 27, 35, 37, 49, 59

EXERCISE	D	A	N	G
1		×		
7		×		
9		×		
13		×		
17		×		
23		×		

EXERCISE	D	A	N	G
25		×		
27		×		
35		×		
37		×		
49		×		×
59		×		

GROUP WORK 1, SECTION 2.6
Implicit Curves

Consider the implicit function $\sin x \cos y = 0$.

1. Without using technology, graph this function. You have to think carefully, but you can get it.

2. If you have access to technology that can graph implicit functions, have it graph this function. Do you get a good graph?

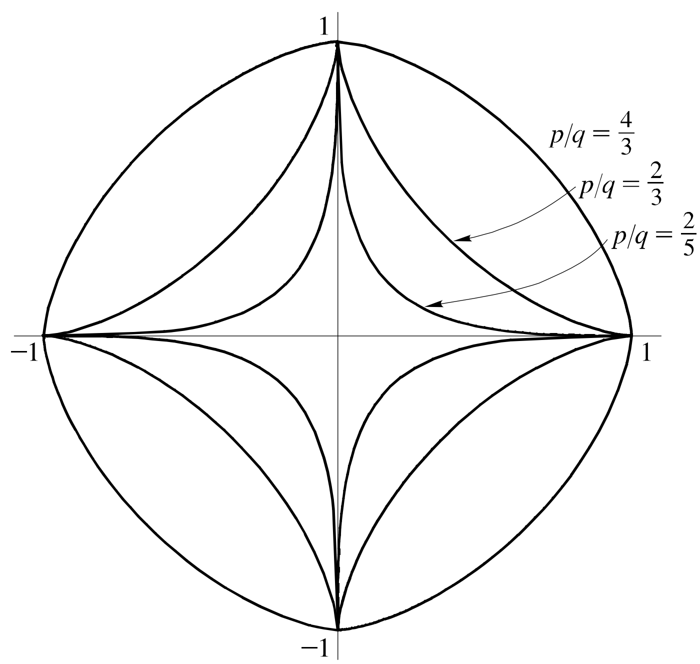
3. Use implicit differentiation to compute $\frac{dy}{dx}$. Does your graph confirm or contradict your answer?

GROUP WORK 2, SECTION 2.6

Circles and Astroids

1. Consider the "flat" circle $x^6 + y^6 = 1$. At what point(s) is the slope of the tangent line equal to 1? Where is it equal to -1?

2. Below are some curves $x^{p/q} + y^{p/q} = 1$, where p is even and q is odd. These curves are sometimes called *astroids* when $p/q < 1$.

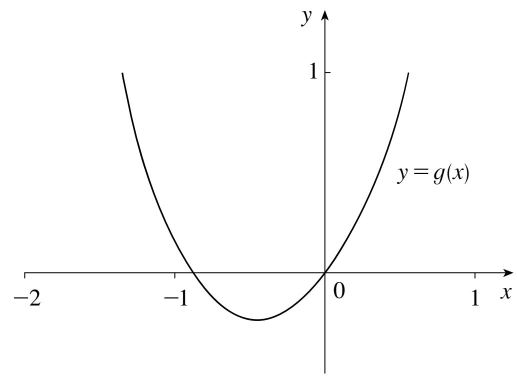


At what point(s) is the slope of the tangent line equal to 1 or -1 if $p/q = \frac{4}{3}$? How about if $p/q = \frac{2}{5}$?

GROUP WORK 3, SECTION 2.6

Looking for the Minimum

The graph of $g(x) = \arcsin\left(x^2 + \sin x\right)$ is shown below. Clearly there is a minimum value somewhere between $x = -1$ and $x = 0$.



1. Find a formula for $g'(x)$.

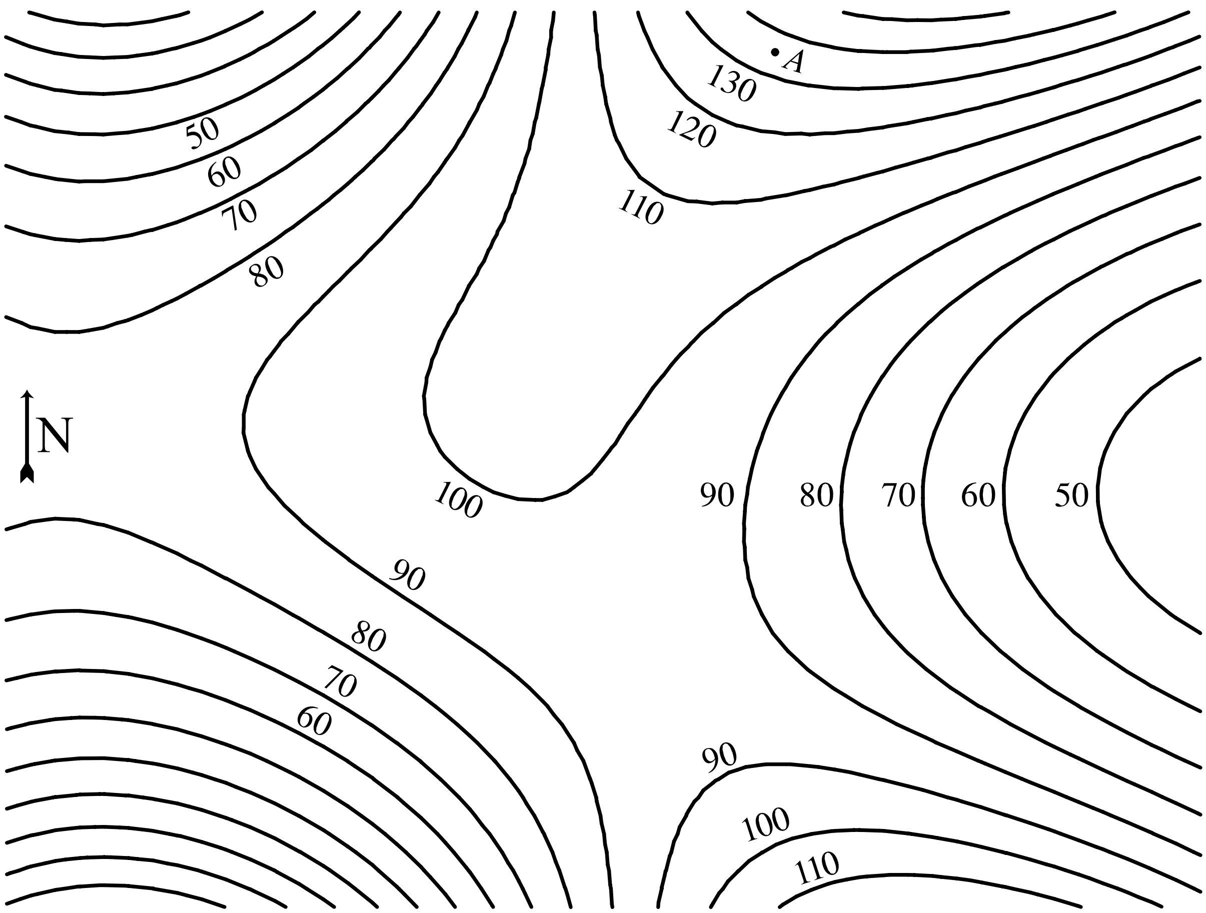
2. Find an equation of the line tangent to this curve at $x = -0.45$. (Round all numbers to four significant figures.)

3. Does the minimum value of $g(x)$ occur to the left or to the right of $x = -0.45$? How do you know?

4. Estimate the location of the minimum value of $g(x)$. Then use technology to see how close your estimate is to the actual location.

A Walk in the Park

The following is a contour map of a region in Orange Rock National Park.



1. Suppose you start a little to the west of point A. Draw the path of steepest descent from this point to the edge of the map.

2. (a) Now start a little bit southwest of point A, and trace the path of steepest descent.

(b) Repeat this starting at a point a little east of point A.

3. What assumptions are you making in drawing your paths?

4. Are there any paths starting near point A that do *not* fall into one of the three valleys that are in the park? Explain your reasoning.

LABORATORY PROJECT **Families of Implicit Curves**

This exciting project puts the abilities of a CAS to use quite nicely. Students should be encouraged to take the last part of Problem 1(b) seriously by exploring many values of c, not just the ones explicitly mentioned. With a CAS, this takes only a few keystrokes. In Problem 2, students should be encouraged to play with the equation by putting constants in front of other terms and noting what effect this has on the graph.

2.7 Rates of Change in the Natural and Social Sciences

Suggested Time and Emphasis

1 class Essential material

Points to Stress

1. The concepts of average and instantaneous rate of change.
2. Some uses of derivatives in physics and in other disciplines.

Quiz Questions

- **TEXT QUESTION** This section discusses many different kinds of examples. What is the main idea underlying them all?

  ANSWER All of them involve expressing quantities as an average rate of change, and then using the idea of the derivative to compute an instantaneous rate of change.
- **DRILL QUESTION** The magnitude F of the force exerted by the Earth on an object is inversely proportional to the square of the distance r from that body to the center of the Earth.

  (a) Write an equation expressing F as a function of r.

  (b) Write an equation expressing dF/dr as a function of r.

  (c) What is the physical meaning of dF/dr?

  ANSWER

  (a) $F = \dfrac{k}{r^2}$ (b) $\dfrac{dF}{dr} = -\dfrac{2k}{r^3}$

  (c) dF/dr tells how fast the force changes as a result of a slight change in the object's distance from the center of the Earth.

Materials for Lecture

- Bring in a taut string, rubber band, violin, or guitar. Illustrate that when the string is plucked, the pitch depends on the length. Discuss Exercise 28, solving it as a class.
- Go over Examples 6 and 7 in detail (or different examples, based on the makeup of the student population).
- Foreshadow Exercise 34 by defining "stable population" and discussing some of the underlying concepts.

Workshop/Discussion

- Discuss some of the issues involved in using a continuous function to model discrete data. For example, ask if taking the derivative of a step function like "cost" is a valid thing to do.
- Do a velocity/distance linear motion problem, such as the one below:

  Let $s(t) = t^4 - 8t^3 + 18t^2$ be the distance function for a particle.

  1. Find the position at $t = 1$, $t = 2$, $t = 3$, and $t = 6$.

  2. Find the velocity at $t = 2$ and $t = 4$.

  3. Determine when the particle is at rest. When is the acceleration zero?

  4. Find the total distance traveled on the intervals $[0, 1]$, $[0, 2]$, $[0, 3]$, and $[0, 6]$.

5. When is the particle speeding up? Slowing down? This motion can be visualized and analyzed graphically.

Group Work 1: Follow That Particle!

Students are asked to analyze the motion of a typical particle.

ANSWERS

1. $0, 3, 22, \approx 1.1$

2. $v(t) = -4t^3 + 15t^2 - 1, -1, 10, 27, \approx -118.6$

3. At rest: at $t \approx 3.7$. Moving forward: $0 \leq t \lesssim 3.7$

4. $\int_1^2 |f(x)|\,dx$ is larger

5. $a(t) = -12t^2 + 30t$

6. Speeding up: $0 < t < 2.5$. Slowing down: $2.5 < t < 5$.

Group Work 2

To help with the homework assignment, put the students into groups, ideally grouping similar majors together, and have each group work on a different problem from the upcoming assignment. After finishing their work, each group should present their solution to the class. Each student will then have a start on several of the problems from the assignment.

Homework Problems

CORE EXERCISES 1, 7, 15, 29

SAMPLE ASSIGNMENT 1, 5, 7, 9, 15, 17, 19, 29

EXERCISE	D	A	N	G
1		×		×
5	×			×
7		×		
9		×		
15		×		
17		×		
19		×		
29	×	×		

GROUP WORK 1, SECTION 2.7

Follow That Particle!

For 4.95 seconds, a particle moves in a straight line according to the position function

$$f(t) = \left(t^3 + 1\right)(5 - t) - 5$$

where t is measured in seconds and f in feet.

Answer the following questions. You can visualize this motion and verify many of your answers using a graph. First attempt all the problems by hand, and then graph the position function to verify your answers.

1. What is the position of the particle at $t = 0$, $t = 1$, $t = 2$, $t = 4.95$?

2. Find the velocity of the particle at time t. What is the velocity of the particle at $t = 0$, $t = 1$, $t = 2$, $t = 4.95$?

3. When is the particle at rest? When is the particle moving forward?

4. Find the total distance traveled by the particle on the intervals [0, 1] and [1, 2]. Which is larger and why?

5. Find the acceleration of the particle at time t.

6. When was the particle speeding up? Slowing down?

2.8 Related Rates

Suggested Time and Emphasis

1 class Essential material

Points to Stress

1. The concept of related rates (first two paragraphs of the text).
2. The classic procedure for handling related rates, including the warning to the side of the procedure in the text.
3. The value of careful diagrams and good notation.

Quiz Questions

- **TEXT QUESTION** In Example 2 in the text, what is the physical meaning of the negative sign in the expression $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$?

  ANSWER The value of y is getting smaller, because the ladder is moving downward.
- **DRILL QUESTION** If one side of a rectangle, a, is increasing at a rate of 3 inches per minute while the other side, b, is decreasing at a rate of 3 inches per minute, which of the following must be true about the area A of the rectangle?

  (A) A is always increasing
  (B) A is always decreasing
  (C) A is decreasing only when $a < b$
  (D) A is decreasing only when $a > b$
  (E) A is constant.

  ANSWER (D)

Materials for Lecture

- Begin with a quick review of implicit differentiation, particularly when an implicit function in x and y is differentiated with respect to time or some other third variable. Have the students read the first two paragraphs of the section, and try to see why implicit differentiation is going to be useful in solving related rates problems. Then present a sample problem such as Exercise 9, using the strategy outlined in the text. Deliberately start to make the error referred to, to see if the students catch it.
- Bring balloons into class, and show the students (or have them discover for themselves) how the radius naturally grows more slowly as time goes on, assuming air comes in at a constant rate (for example, one breath every 30 seconds).
- Revisit Example 2 in the text. Compute the velocity of the ladder when it is $\frac{1}{1000}$ inch off the ground ($y = 0.001$). Show how that at some point, the tip of the ladder will exceed the speed of light. Have the students discuss what they think the problem is. (This can be done even with a large class; give them a few minutes.) Since the conclusion that "the tip really does exceed the speed of light" is impossible, the only possible conclusion to draw is that the model is faulty. Take a yardstick and actually do the experiment. (The tip of the yardstick does not stay in contact with the wall.) If the room is such that the students cannot

all see the result of the experiment, have a few volunteers come up to watch and describe what happens, and encourage the students to try the experiment at home with a ruler or other similar object.

Workshop/Discussion

- Work this problem with the class: You are blowing a bubble with bubble gum and can blow air into the bubble at a rate of 3 in^3/s.
  (a) At what rate is the volume V increasing with respect to the radius when the radius r is 1 inch? When the radius is 3 inches?
  (b) How fast is the radius increasing with respect to time when $r = 1$ inch? When $r = 3$ inches?
  (c) Suppose you increase your effort when $r = 3$ inches and begin to blow in air at a rate of 4 in^3/s. How fast is the radius increasing now?
- Do some challenging related rates problems, such as Exercises 38 and 45.
- Many children notice that when they eat a spherical lollipop (as opposed to the disk-shaped kind) it seems like at first they can lick and lick and lick without it seeming to get smaller, and then toward the end it disappears quickly. If they tell an adult, it is usually attributed to imagination or the subjectivity of passing time. Have the students try to come up with a mathematical explanation.
  ANSWER If a student is licking at a constant rate, dV/dt is constant. However, the perceived change in size of the lollipop is based on the *diameter* of the sphere, which decreases more quickly near the end.

Group Work 1: Find the Error

This exercise illustrates a common error that many students make. You may want to project the problem on an overhead, and give the class a few minutes to discuss it. The exercise can stand alone, or be handed out as a warm-up.

Group Work 2: Nobody Escapes the Cube

This is a good introduction to related rates problems, requiring the students to express the volume of a cube in terms of its surface area.

ANSWERS **1.** 2 in^2/s **2.** $\frac{1}{2}$ in^3/s

Group Work 3: The Swimming Pool

The students shouldn't work on this activity until they've had a chance to see or try some basic related rates problems. Be prepared to give plenty of guidance to the students.

ANSWERS

1. $V = \begin{cases} 500h + \frac{125}{8}h^2 & \text{if } 0 < h < 16 \\ 1500h - 12{,}000 & \text{if } 16 \le h < 20 \end{cases} \quad \Rightarrow \quad \frac{dV}{dt} = \begin{cases} 500 + \frac{125}{4}h & \text{if } 0 < h < 16 \\ 1500\frac{dh}{dt} & \text{if } 16 \le h < 20 \end{cases}$

2. You would need dV/dt, the rate at which the pool is being filled. Note that you would not need h; if you knew dV/dt and the pool was empty at $t = 0$, you could calculate V and then compute h.

Homework Problems

CORE EXERCISES 3, 9, 13, 15, 23

SAMPLE ASSIGNMENT 3, 5, 7, 9, 11, 13, 15, 23, 27, 31, 35

EXERCISE	D	A	N	G
3		×		
5		×		
7		×		
9		×		
11	×	×		×
13	×	×		×

EXERCISE	D	A	N	G
15		×		
23		×		
27		×		
31		×		
35		×		

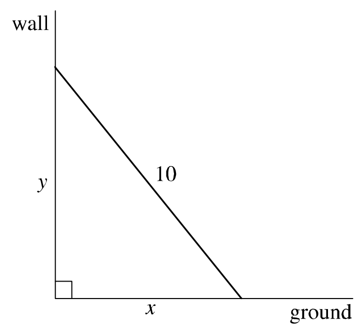
GROUP WORK 1, SECTION 2.8

Find the Error

It is a beautiful Spring evening. You and your wild-eyed, hungry-looking friends are sitting around, reading your Calculus books. You arrive at the following:

> **EXAMPLE 2** A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Your enthused roommates don't read the rest of the example, preferring to do the problem on their own. This is how they proceed:



"We want to find dy/dt. So we set up

$$x^2 + y^2 = 100$$

Now, we want dy/dt when $dx/dt = 1$ and $x = 6$. Substituting $x = 6$ gives us

$$36 + y^2 = 100 \text{ or } y^2 = 64$$

Now we take derivatives:

$$2y\frac{dy}{dt} = 0$$

giving $dy/dt = 0$."

The problem is, of course, that this answer doesn't make any sense.

1. Why does their answer not make any sense?

2. What error did they make? How could they correct it?

GROUP WORK 2, SECTION 2.8

Nobody Escapes the Cube

We are designing a computer graphic in which we zoom in on a cube. The volume V, surface area S, and side length x of the cube are all varying with respect to time. With this information, compute the following quantities, using the steps described in the text:

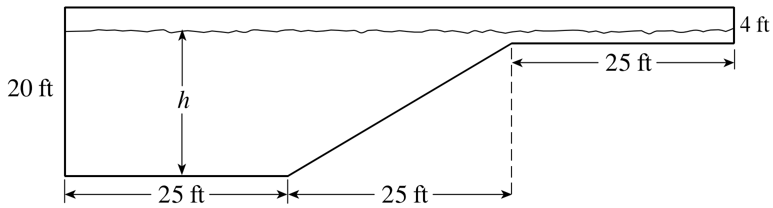
1. dS/dt when $x = 2$ inches and $dV/dt = 1 \text{ in}^3/\text{s}$.

2. dV/dt when $x = 2$ inches and $dS/dt = 1 \text{ in}^2/\text{s}$.

GROUP WORK 3, SECTION 2.8

The Swimming Pool

We wish to find the change in volume of a 20-foot-wide pool as it fills up with water. A cross-section of the pool is shown below.



1. Express dV/dt in terms of h, V, and dh/dt.

2. What additional information would you need to find dh/dt at $t = 10$ minutes?

2.9 Linear Approximations and Differentials

Suggested Time and Emphasis

1 class   Essential material (linear approximation) and optional material (differentials)

Points to Stress

1. The general equation of a line tangent to the graph of a function, and its use in approximating that function near a point.
2. The differential as the difference between the linearization of a function and the function itself.

Quiz Questions

- **TEXT QUESTION** What is the difference between the function $L(x)$ defined in the text and the equation of the tangent line $y = f(a) + f'(a)(x - a)$?
  ANSWER None
- **DRILL QUESTION** Write the equation of the straight line that best approximates the graph of $y = x + \cos x$ at the point (0, 1).
  ANSWER $y = x + 1$

Materials for Lecture

- Discuss the motivation for studying linear approximations. Ask, "In this age of computers, why not just plug numbers into the actual function?"
  ANSWERS
  1. A common modeling technique is to assume a function is locally linear, and then use the linear equation in calculations, since it is easier to manipulate.
  2. It is often easier physically to measure the derivative of a function than the function itself. Then the derivative measurements can be used to obtain an approximation of the function.
  3. When measuring a real phenomenon, there is often no easy-to-understand function that can be written in a line or two, and the best that can be obtained is a set of sample data points. The "underlying" function must be approximated.
  4. In the real world, the input to functions can be noisy or wiggly. It is easier to handle small input fluctuations if we assume that the output varies linearly.
  5. When a function is called thousands of times by a computer program, as occurs in computer graphics applications, the small time savings from using a linear function can result in savings of hours or even days.
- Discuss the meaning of the phrase "approximating along the tangent line" and its connections to linear approximation. Then present examples of linear approximation, such as $\sin x \approx x$ for x near 1 and $x + \cos x \approx x + 1$ for x near 0.
- Raise the question, "What if we want a more accurate model of a function?" Foreshadow the quadratic approximation (Taylor polynomial of order two) as an extension of the linear approximation. (The linear

approximation matches the function in the first derivative, so how can you make a function match the second derivative as well?)

- Graph $y = \sin x$ with its approximations at $x = 0$ and $x = \frac{\pi}{4}$. Discuss which is "better".
- To illustrate how controversial differentials once were, cite the quotation from Bishop Berkeley (1734) on differentials: "And what are these evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?"
- Bring in a carpenter's level. Show how, when the level is held perfectly straight, it can be used to measure acceleration. (The bubble moves when the level is accelerated, and returns to center at constant velocity). This can be done on an overhead projector, if the floor is flat. This is the principle used to make a simple accelerometer. Then discuss how, given acceleration measurements, it is possible to approximate velocity using the technique of linear approximation.

Workshop/Discussion

- Let $f(x) = x^{1.857}$. Find the linear approximation of $f(x)$ at $a = 1$ and use it to approximate f at $x = 1.1$, $x = 1.01$, and $x = 1.001$. Compare the approximations to the actual values the calculator gives for f at these points.
- In Example 2, discuss why we base our linear approximation at $x = 1$ rather than at $x = 0.99$ or 1.01.
- Practice using linear approximations with $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$, and use differentials to approximate Δy for $\Delta x = -1$ and $\Delta x = 1$.
- Have the students try to find a linear approximation for $|x|$ near $x = 0$, and explain why it is impossible.

Group Work 1: Four Variations on a Theme

This exercise explores four very different functions that have identical linear approximations near $x = 0$.

ANSWERS

1. $y = x$ in all cases.

2.

Function	Function Value at $x = 0.1$	Approximation at $x = 0.1$
f	0.09545	0.1
g	0.101	0.1
h	0.11007	0.1
j	0.09983	0.1

3. If the students need to, they can check the approximations for $x = 0.2$ or $x = 0.3$. The best approximation is the one to $j(x)$, and the worst is the one to $h(x)$. This is immediate from looking at the graphs. Notice that j and g have inflection points at $x = 0$.

Group Work 2: Linear Approximation

Some students may try to find approximations of the derivative functions. They should be reminded that we are approximating f, using the graph of f' as an aid.

ANSWERS

1. $f(x) \approx 1.75(x-2)+4$, so $f(1.98) \approx 3.965$ and $f(2.02) \approx 4.035$.
2. f is concave down, so the approximations are overestimates.
3. The estimates are both 7, because the function is horizontal when $x = 3$.

Homework Problems

CORE EXERCISES 1, 7, 11, 15, 23, 31

SAMPLE ASSIGNMENT 1, 3, 7, 9, 11, 13, 15, 17, 23, 25, 31, 33

EXERCISE	D	A	N	G
1		×		
3		×		
7		×		
9		×		
11		×		
13		×		

EXERCISE	D	A	N	G
15		×		
17		×		
23		×		
25		×		
31		×		
33		×		

Four Variations on a Theme

Consider the following four functions:

$$f(x) = -1 + \sqrt{2x+1} \qquad g(x) = x^3 + x \qquad h(x) = \tan^2 x + x \qquad j(x) = \sin x$$

1. Find the linearizations of f, g, h, and j at $a = 0$.

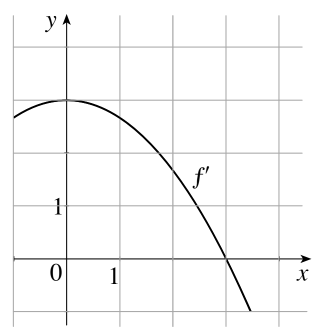
2. Compute the values of each of these functions at $x = 0.1$ and the values of their linearizations.

3. For which function is the approximation best? For which is it worst? Why?

GROUP WORK 2, SECTION 2.9

Linear Approximation

Consider this graph of $f'(x)$, the *derivative* of $f(x)$.



1. Suppose that $f(2) = 4$. Approximate $f(1.98)$ and $f(2.02)$ as best you can. Don't just guess. Show your work.

2. Determine whether your approximations were overestimates or underestimates.

3. Suppose you also know that $f(3) = 7$. Can you approximate $f(2.98)$ and $f(3.02)$? Explain your answer.

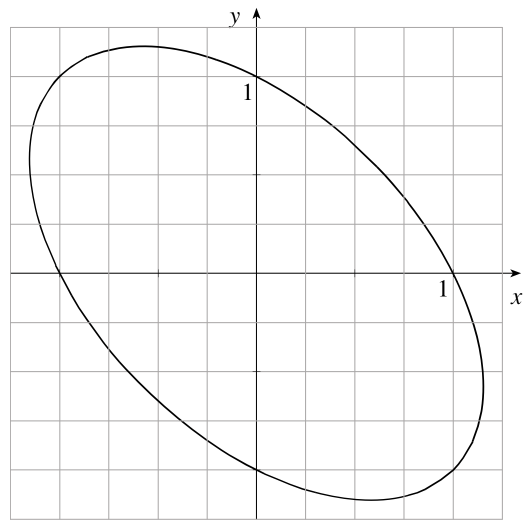
LABORATORY PROJECT **Taylor Polynomials**

This project provides a solid early introduction to Taylor polynomials as extensions of the tangent line approximation concept. A few examples involving $\cos x$ and $\sqrt{x+3}$ are explored in more detail. Students may be asked to explore their own function, and see what happens. Have them go beyond just working through the six questions, and try to demonstrate that they understand the pretty concept introduced in this project.

2 SAMPLE EXAM

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Consider the graph of $x^2 + xy + y^2 = 1$.



(a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

(b) Find all points where the tangent line is horizontal.

(c) Find all points where the tangent line is parallel to the line $y = -x$.

2. Let $f(x) = 7\sin(x + \pi) + \cos 2x$.

(a) Compute $f'(x)$, $f''(x)$, $f^3(x)$, and $f^4(x)$.

(b) Compute $f^{13}(0)$.

3. Assume that $f(x)$ and $g(x)$ are differential functions that we know very little about. In fact, assume that all we know of these function is the following table of data:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-2	3	1	-5	8
-1	-9	7	4	1
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	0

(a) Let $h(x) = g(x)\sin x$. What is $h'(0)$?

(b) Let $j(x) = \left[f(x) + x^2\right]^3$. What is $j'(1)$?

(c) Let $l(x) = \dfrac{\tan \pi x}{g(x)}$. What is $l'(-1)$?

4. Let $u(x)$ be an always positive function such that $u'(x) < 0$ for all real numbers.

(a) Let $f(x) = [u(x)]^2$. For what values of x will $f(x)$ be increasing?

(b) Let $g(x) = u(u(x))$. For what values of x will $g(x)$ be increasing?

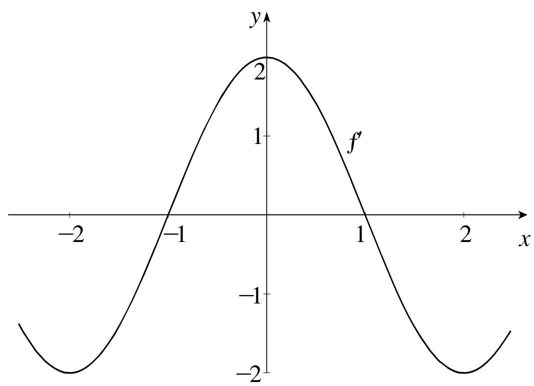
5. Let $f(x) = -x^3 - 2x^2 + x + 1$ and $g(x) = \sin x + 1$.

(a) Find the equation of the line tangent to $f(x)$ at $x = 0$.

(b) Show that $g(x)$ has the same tangent line as $f(x)$ at $x = 0$.

(c) Does this tangent line give a better approximation of $f(x)$ or $g(x)$ at $x = 1$? Give reasons for your answer.

6. The following is a graph of f', the derivative of some function f.



(a) Where is f increasing?

(b) Where does f have a local minimum? Where does f have a local maximum?

(c) Where is f concave up?

(d) Assuming that $f(0) = -1$, sketch a possible graph of f.

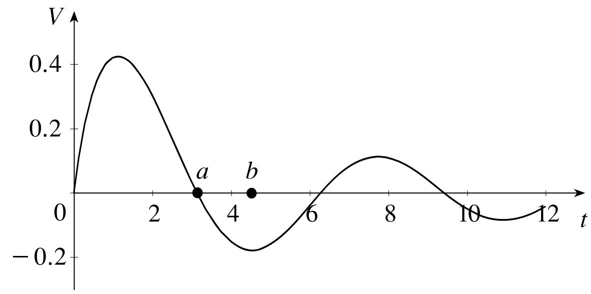
7. As a spherical raindrop evaporates, its volume changes at a rate proportional to its surface area A.

(a) If the constant of proportionality is K, find the rate of change of the radius r when $r = 4$.

(b) Show that the rate of change of the radius is always constant.

(c) Does part (b) mean that the rate of change of the volume is always constant? Why or why not?

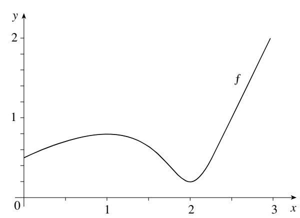
8. The voltage across a resistor R is given by $V(t) = \dfrac{1}{1+t}\sin t$. A graph of $V(t)$ is shown below.



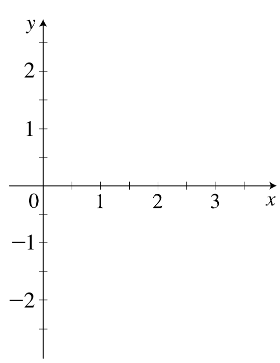
(a) How fast is the voltage changing after 2 seconds?

(b) Would you be better off using the linear approximation at $x = a$ to estimate $V(b)$, or using the linear approximation at $x = b$ to estimate $V(a)$? Justify your answer.

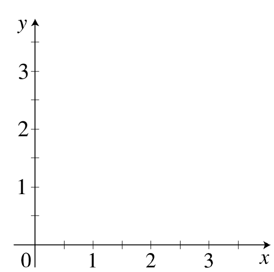
9. Let f be the function whose graph is given below.



(a) Sketch a plausible graph of f'.



(b) Sketch a plausible graph of a function F such that $F' = f$ and $F(0) = 1$.



10. Suppose that the line tangent to the graph of $y = f(x)$ at $x = 3$ passes through the points $(-2, 3)$ and $(4, -1)$.

(a) Find $f'(3)$.

(b) Find $f(3)$.

(c) What is the equation of the line tangent to f at 3?

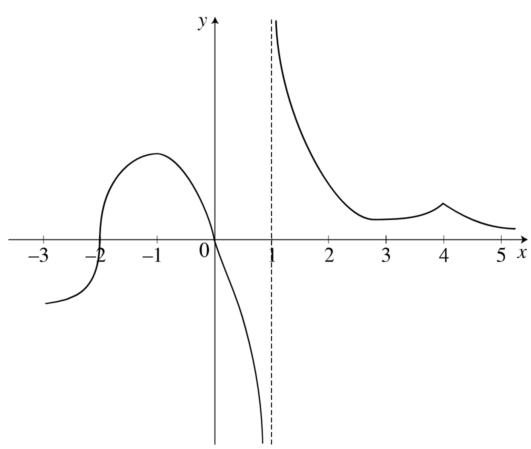
11. Each of the following limits represent the derivative of a function f at some point a. State a formula for f and the value of the point a.

(a) $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

(b) $\lim_{x\to 3} \frac{(x+1)^{3/2}-8}{x-3}$

(c) $\lim_{h\to 0} \frac{\sin(\pi(2+h))-0}{h}$

12. The graph of $f(x)$ is given below. For which value(s) of x is $f(x)$ not differentiable? Justify your answer(s).



2 SAMPLE EXAM SOLUTIONS

1. (a) $2x + x\dfrac{dy}{dx} + y + 2y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$

(b) Set $y + 2x = 0$ and $y = -2x$. Then $x^2 - 2x^2 + 4x^2 = 1 \Leftrightarrow 3x^2 = 1 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}} \Leftrightarrow y = \mp\frac{2}{\sqrt{3}}$, so the points are $\left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

(c) Set $-\dfrac{2x+y}{x+2y} = -1$ to get $y = x$. Then $x^2 + x^2 + x^2 = 3x^2 = 1 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}} \Leftrightarrow y = \pm\frac{1}{\sqrt{3}}$, so the points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

2. (a) $f'(x) = 7\cos(x+\pi) - 2\sin 2x$; $f''(x) = -7\sin(x+\pi) - 4\cos 2x$; $f^{(3)}(x) = -7\cos(x+\pi) + 8\sin 2x$; $f^{(4)}(x) = 7\sin(x+\pi) + 16\cos 2x$

(b) $f^{(13)}(x) = 7\cos(x+\pi) - 2^{13}\sin 2x$; $f^{(13)}(0) = 7\cos\pi = -7$

3. (a) $h'(x) = g'(x)\sin x + g(x)\cos x$; $h'(0) = g(0) = 9$

(b) $j'(x) = 3\left(f(x) + x^2\right)^2\left(f'(x) + 2x\right)$; $j'(1) = 3\left(f(1)+1\right)^2\left(f'(1)+2\right) = 3 \cdot 4^2 \cdot 4 = 192$

(c) $l'(x) = \dfrac{g(x)\cdot\pi\sec^2\pi x - g'(x)\tan\pi x}{g(x)^2}$; $l'(-1) = \dfrac{7\pi - 0}{7^2} = \dfrac{\pi}{7}$

4. (a) $f'(x) = 2u(x)u'(x) < 0$ for all x, since $u(x) > 0$ and $u'(x) < 0$. Never increasing.

(b) $g'(x) = u'(u(x)) \cdot u'(x) > 0$, since $u'(u(x))$ and $u'(x) < 0$. Always increasing.

5. (a) $f'(x) = -3x^2 - 4x + 1$; $f'(0) = 1$, $f(0) = 1$. Tangent line is $y = 1 + 1 \cdot x = 1 + x$

(b) $g'(x) = \cos x$; $g'(0) = 1$, $g(0) = 1$. Tangent line is $y = 1 + x$

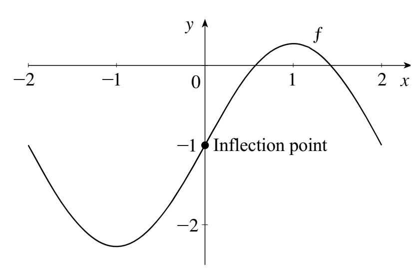
(c) At $x = 1$, $f(1) = -1$, $g(1) = \sin 1 + 1 \approx 1.841$. The tangent line approximation is $y = 1 + 1 = 2$. This is better for $g(x)$ at $x = 1$.

6. (a) f is increasing on $(-1, 1)$.

(b) Local minimum at $x = -1$; local maximum at $x = 1$

(c) f is concave up where $f'(x)$ is increasing, that is, on $(-2, 0)$.

(d)



7. (a) $\dfrac{dV}{dt} = KA$. $V = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. Since $A = 4\pi r^2$, we have $K4\pi r^2 = 4\pi r^2\dfrac{dr}{dt}$. Thus, $\dfrac{dr}{dt} = K$.

(b) By part (a), $\dfrac{dr}{dt} = K$ is constant. $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = 4K\pi r^2$. So $\dfrac{dV}{dt}$ depends on r^2 and is not constant.

8. (a) $V'(t) = \dfrac{1}{1+t}\cos t - \dfrac{1}{(1+t^2)}\sin t$; $V'(2) = \frac{1}{3}\cos t - \frac{1}{9}\sin 2 \approx -0.240$

(b) The tangent line at $x = b$ is horizontal. So the estimate for $V(a)$ using this linear approximation is $V(b)$, which is not very good. Thus, it is better to use the linear approximation at $x = a$ to estimate $V(b)$.

9. (a) Answers will vary. Look for:

(i) zeros at 1 and 2

(ii) f' positive for $x \in (0, 1)$ and $(2, 4)$

(iii) f' negative for $x \in (1, 2)$

(iv) f' flattens out for $x > 2.5$

(b) Answers will vary. Look for

(i) $F(0) = 1$

(ii) F always increasing

(ii) F is never perfectly flat

(iv) F is closest to being flat at $x = 2$

(v) F is concave up for $x \in (0, 1)$ and $x \in (2, 4)$

(vi) F is concave down for $x \in (1, 2)$

10. (a) $\frac{3-(-1)}{-2-4} = -\frac{2}{3}$

(b) The equation of the tangent line is $y - 3 = -\frac{2}{3}(x + 2)$, so $f(3) = -\frac{2}{3}(3 + 2) + 3 = -\frac{1}{3}$.

(c) The equation of the tangent line is $y - 3 = -\frac{2}{3}(x + 2)$.

11. (a) $f(x) = x^2, a = 3$ (b) $f(x) = (x + 1)^{3/2}, a = 3$ (c) $f(x) = \sin(\pi x), a = 2$

12. f isn't differentiable at $x = 1$, because it is not continuous there; at $x = -2$, because it has a vertical tangent there; and at $x = 4$, because it has a cusp there.