Chapter 2

Systems of Linear Equations

2.1 Introduction to Systems of Linear Equations

1. We follow Example 2.1 and justify our assertion by applying the definition of ***linear***.
$x - \pi y + \left(\sqrt[3]{5}\right) z = 0$ ***is*** linear ***because*** power of z is 1 and π, $\sqrt[3]{5}$ are constants.

2. We follow Example 2.1 and justify our assertion by applying the definition of ***linear***.
$x^2 + y^2 + z^2 = 1$ is ***not*** linear ***because*** x, y, z occur to the power 2.

3. $x^{-1} + 7y + z = \sin\frac{\pi}{9}$ is ***not*** linear ***because*** x occurs to the power -1.

4. $2x - xy - 5z = 0$ is ***not*** linear ***because*** the product xy is of degree 2.

5. $3\cos x - 4y + z = \sqrt{3}$ is ***not*** linear ***because*** $\cos x$ is not linear.

6. $(\cos 3)\,x - 4y + z = \sqrt{3}$ ***is*** linear ***because*** $\cos 3$ and $\sqrt{3}$ are constants.

7. As in Section 1.3, we put the equation of this line into general form $ax + by = c$.
$2x + y = 7 - 3y$ is equivalent to $2x + 4y = 7$ after adding $3y$ to both sides.
Note: When the equation is ***linear*** there is no restriction on x and y. Why?

8. We begin by determining the restrictions on the variables x and y.
Typical sources are 1) division, 2) square roots, and 3) domains (like $\log x \Rightarrow x > 0$).

Step 1. Determine restriction ***type***. With $\dfrac{x^2 - y^2}{x - y} = 1$, it is division.

Step 2. Set the denominator equal to zero to determine the restriction.
We have $x - y = 0 \Rightarrow x = y$. So, the ***restriction*** is $x \neq y$.

Step 3. Simplify the given equation using algebra.

$$\frac{x^2 - y^2}{x - y} = 1 \overset{\text{factor}}{\Rightarrow} \frac{(x-y)(x+y)}{x-y} = 1 \overset{\text{cancel}}{\Rightarrow} x + y = 1.$$

Note: This tells us the given function is equivalent to the line $x + y = 1$ provided $x \neq y$.

9. We begin by determining the restrictions on the variables x and y.
Typical sources are 1) division, 2) square roots, and 3) domains (like $\log x \Rightarrow x > 0$).

Step 1. Determine restriction ***type***. With $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{4}{xy}$, it is division.

Step 2. Set the denominators equal to zero to determine the restriction.
We have $x = 0$, $y = 0$, and $xy = 0$. So, the ***restriction*** is $x, y \neq 0$.

Step 3. Simplify the given equation using algebra.

$$\frac{1}{x} + \frac{1}{y} = \frac{4}{xy} \overset{\substack{\text{common}\\ \text{denominator}}}{\Rightarrow} \frac{y}{xy} + \frac{x}{xy} = \frac{4}{xy} \overset{\substack{\text{multiply both sides}\\ \text{by xy}}}{\Rightarrow} x + y = 4.$$

Note: This tells us the given function is equivalent to the line $x + y = 4$ provided $x, y \neq 0$.

10. We begin by determining the restrictions on the variables x and y.
Typical sources are 1) division, 2) square roots, and 3) domains (like $\log x \Rightarrow x > 0$).

Step 1. Determine restriction ***type***. With $\log_{10} x - \log_{10} y = 2$, it is domains.

Step 2. Apply the domain restrictions to determine the overall restriction.
In this case, we have the overall restriction of $x > 0$ and $y > 0$.

Step 3. Simplify the given equation using algebra.

$$\log_{10} x - \log_{10} y = 2 \overset{\substack{\text{properties of}\\ \text{logarithms}}}{\Rightarrow} \log_{10} \frac{x}{y} = 2 \overset{\substack{\text{treat as}\\ \text{exponents}}}{\Rightarrow}$$

$$10^{\log_{10} \frac{x}{y}} = 10^2 \overset{\substack{\text{cancel and}\\ \text{simplify}}}{\Rightarrow} \frac{x}{y} = 100 \overset{\substack{\text{put in}\\ \text{general form}}}{\Rightarrow} x - 100y = 0.$$

Note: This tells us the given function is equivalent to the line $x - 100y = 0$ provided $x, y > 0$.

11. As in Example 2.2(a), we set $x = t$ and solve for y.
Setting $x = t$ in $3x - 6y = 0$ gives us $3t - 6y = 0$. Solving for y yields $6y = 3t \Rightarrow y = \frac{1}{2}t$.
So, we see the complete set of solutions can be written in the parametric form $[t, \frac{1}{2}t]$.

Note: We could have set $y = t$ to get $3x - 6t = 0$ and solved for x so $x = 2t$ and $[2t, t]$.

12. As in Example 2.2(a), we set $x_1 = t$ and solve for x_2.
Setting $x_1 = t$ yields $2t + 3x_2 = 5$. Solving for x_2 yields $3x_2 = 5 - 2t \Rightarrow x_2 = \frac{5}{3} - \frac{2}{3}t$.
So, a complete set of solutions written in parametric form is $[t, \frac{5}{3} - \frac{2}{3}t]$.

Note: We could have set $x_2 = t$ and solved for x_1 to get the parametric form $[\frac{5}{2} - \frac{3}{2}t, t]$.

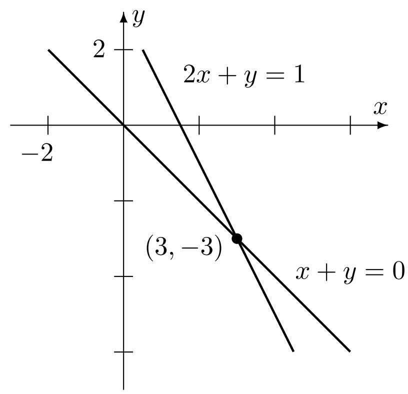
13. As in Example 2.2(b), we set $y = s$, $z = t$ and solve for x. (Why is this a good choice?)
This substitution yields $x + 2s + 3t = 4$. Solving for x yields $x = 4 - 2s - 3t$.
So, a complete set of solutions written in parametric form is $[4 - 2s - 3t, s, t]$.

14. As in Example 2.2(b), we set $x_1 = s$, $x_2 = t$ and solve for x_3.

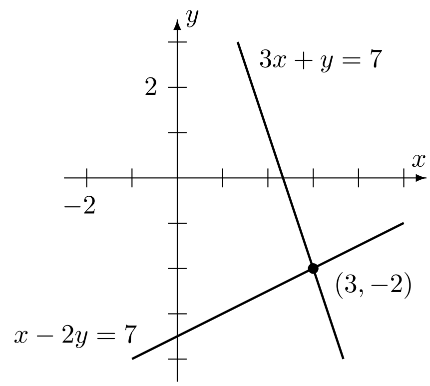
This substitution yields $4s + 3t + 2x_3 = 1$. Solving for x_3 yields $x_3 = \frac{1}{2} - 2s - \frac{3}{2}t$.

So, a complete set of solutions written in parametric form is $[s, t, \frac{1}{2} - 2s - \frac{3}{2}t]$.

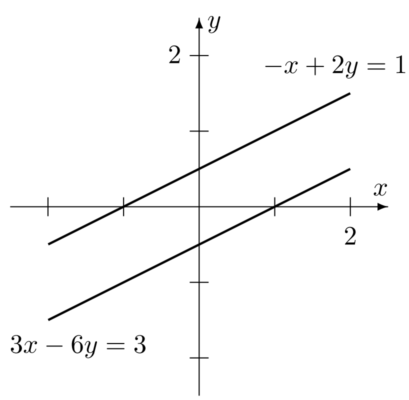
15. The lines intersect at $(3, -3)$,
so the unique solution is $[3, -3]$.
To solve, subtract 2^{nd} from $1^{\text{st}} \Rightarrow$
$-x = -3 \Leftrightarrow x = 3$,
so substitution $\Rightarrow y = -3$.



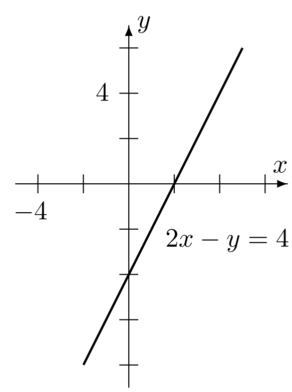
16. The lines intersect at $(3, -2)$,
so the unique solution is $[3, -2]$.
To solve, subtract $3 \times 1^{\text{st}}$ from $2^{\text{nd}} \Rightarrow$
$7y = -14 \Leftrightarrow y = -2$,
so substitution $\Rightarrow x = 3$.



17. The lines are parallel $\Rightarrow$ no solution.
This system is inconsistent.
Add $3 \times 2^{\text{nd}}$ to $1^{\text{st}} \Rightarrow 0 = 6$.



18. The graphs intersect in the line $2x - y = 4 \Rightarrow$
There are infinitely many solutions, $[t, 2t - 4]$.
The 2^{nd} is $-\frac{3}{5} \times 1^{\text{st}}$.



19. As in Example 2.5, we start from the last equation and work backward.
We find successively $y = 3$ and $x = 1 + 2(3) = 7$. So, the unique solution is $[x, y] = [7, 3]$.

20. As in Example 2.5, we start from the last equation and work backward.
We find successively $v = 3$ and $u = \frac{5}{2} + \frac{3}{2}(3) = 7$. So, the unique solution is $[u, v] = [7, 3]$.

21. We find the solution $[x, y, z] = \left[\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right]$ using back substitution. Details below.

$$\begin{aligned} 3z = -1 &\Rightarrow & z &= -\tfrac{1}{3} \\ 2y - z = 1 &\Rightarrow 2y = 1 + z \Rightarrow y = \tfrac{1}{2} + \tfrac{1}{2}z \Rightarrow & y &= \tfrac{1}{2} + \tfrac{1}{2}\left(-\tfrac{1}{3}\right) = \tfrac{1}{3} \\ x - y + z = 0 &\Rightarrow x = y - z \Rightarrow & x &= \left(\tfrac{1}{3}\right) - \left(-\tfrac{1}{3}\right) = \tfrac{2}{3} \end{aligned}$$

22. We find the solution $[x_1, x_2, x_3] = [0, 0, 0]$ using back substitution.
Note: This follows immediately from the fact that all three equations are equal to zero.

23. We find the solution $[x_1, x_2, x_3, x_4] = [5, -2, 1, 1]$ using back substitution. Details below.
We find $x_3 = x_4 = 1$, $x_2 = -1 - 1 = -2$, and $x_1 = 1 - (-2) + 1 + 1 = 5$.

24. We combine the techniques of Examples 2 and 5 to find $[x, y, z] = [2 + 5t, -1 + 2t, t]$.
Details: We let $z = t$ to get $y = -1 + 2t$, so $x = 5 + 3(-1 + 2t) - t = 2 + 5t$.

25. Working forward, we find $x = 2$, $y = -3 - 2(2) = -7$, and $z = -10 + 4(-7) + 3(2) = -32$.
So the unique solution to the system is $[x, y, z] = [2, -7, -32]$.

26. Working forward, we find $x_1 = -1$, $x_2 = 5 + \frac{1}{2}(-1) = \frac{9}{2}$, and $x_3 = 7 - 2\left(\frac{9}{2}\right) - \frac{3}{2}(-1) = -\frac{1}{2}$.
So the unique solution is $[x_1, x_2, x_3] = \left[-1, \frac{9}{2}, -\frac{1}{2}\right]$.

27. As in the solution to Example 2.6, we create the augmented matrix from the coefficients.

The system $\begin{aligned} x - y &= 0 \\ 2x + y &= 3 \end{aligned}$ has $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right]$ as its augmented matrix.

28. As in the solution to Example 2.6, we create the augmented matrix from the coefficients.

The system $\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + x_3 &= 0 \\ -x_1 + 2x_2 - 2x_3 &= 0 \end{aligned}$ has $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 \end{array}\right]$ as its augmented matrix.

29. The system $\begin{aligned} x + 5y &= -1 \\ -x + y &= -5 \\ 2x + 4y &= 4 \end{aligned}$ has $\left[\begin{array}{cc|c} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right]$ as its augmented matrix.

30. The system $\begin{aligned} a - 2b + d &= 2 \\ -a + b - c - 3d &= 1 \end{aligned}$ has $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ -1 & 1 & -1 & -3 & 1 \end{array}\right]$ as its augmented matrix.

31. The augmented matrix $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right]$ becomes $\begin{aligned} y + z &= 1 \\ x - y &= 1 \\ 2x - y + z &= 1 \end{aligned}$ as a system.

32. The augmented matrix $\left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right]$ becomes $\begin{array}{r} a-b+\quad\;\; 3d+\;\; e=2 \\ a+b+2c+\;\; d-\;\; e=4 \\ b+\quad\;\; 2d+3e=0 \end{array}$.

33. As in Example 2.4(a), we add $(x-y)+(2x+y)=0+3$ to get $3x=3 \Rightarrow x=1$ and $y=1$. A quick check confirms that $[1,1]$ is indeed the unique solution of the system.

34. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 28.

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 1 \\ -1 & 2 & -2 & 0 \end{array}\right] \xrightarrow[R_2+R_1]{R_2-2R_1} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 3 & -3 & 1 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1/3 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1/3 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \xrightarrow[R_2+R_3]{R_1-R_3} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \Rightarrow$$

$x_1=\frac{2}{3}$, $x_2=-\frac{1}{3}$, and $x_3=-\frac{2}{3}$. So the solution is $[x_1,x_2,x_3]=\left[\frac{2}{3},-\frac{1}{3},-\frac{2}{3}\right]$.

35. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 29.

$$\left[\begin{array}{rr|r} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{rr|r} 1 & 5 & -1 \\ 0 & 6 & -6 \\ 0 & -6 & 6 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{rr|r} 1 & 5 & -1 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$y=-1$ and $x=-1-5(-1)=4$, so the solution is $[x,y]=[4,-1]$.

36. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 30.

$$\left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ -1 & 1 & -1 & -3 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ 0 & -1 & -1 & -2 & 3 \end{array}\right] \Rightarrow$$

$d=t$, $c=s$, $b=-3-s-2t$, and $a=2+2(-3-s-2t)-t=-4-2s-5t$, so the solution is $[a,b,c,d]=[-4-2s-5t,-3-s-2t,s,t]$.

37. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 31.

$$\left[\begin{array}{rrr|r} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow[2R_2]{R_1 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 2 & -2 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$\Rightarrow 0=2 \Rightarrow$ No solution.

38. As shown after Example 2.6, we row reduce the augmented matrix from Exercise 32.

$$\left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right] \xrightarrow[2R_3]{R_2-R_1} \left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 2 & 2 & -2 & -2 & 2 \\ 0 & 2 & 0 & 4 & 6 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{rrrrr|r} 1 & -1 & 0 & 3 & 1 & 2 \\ 0 & 2 & 2 & -2 & -2 & 2 \\ 0 & 0 & -2 & 6 & 8 & -2 \end{array}\right]$$

Using back substitution, we get: $e=t$, $d=s$, $c=\left(-\frac{1}{2}\right)(-2-6s-8t)=1+3s+4t$.

$b=\left(\frac{1}{2}\right)(2-2(1+3s+4t)+2s+2t)=-2s-3t$, $a=2+(-2s-3t)-3s-t=2-5s-4t$.

So, the solution is $[a,b,c,d,e]=[2-5s-4t,-2s-3t,1+3s+4t,s,t]$.

39. The key to this problem is simple substitution.

(a) The fact that $x = t$ tells us that x is a free variable. What does that tell us?
The linear equations we are looking for must be multiples of each other. Why?
Substituting $t = x$ into $y = 3 - 2t$ yields $y = 3 - 2x \Rightarrow 2x + y = 3$.
Any multiple of this equation will create the system we are looking for.
For example, $2x + y = 3$ and $4x + 2y = 6$ (which is just $2\times$ the equation $2x + y = 3$).

(b) Substituting $s = y$ into $y = 3 - 2x$ yields $s = 3 - 2x \Rightarrow x = \frac{3}{2} - \frac{1}{2}s$.
The parametric solution then becomes $x = \frac{3}{2} - \frac{1}{2}s$ and $y = s$.

40. The key to this problem is simple substitution.

(a) Substituting $t = x_1$ into $x_2 = 1 + t$, $x_3 = 2 - t$ yields $x_2 = 1 + x_1$, $x_3 = 2 - x_1$.
These equations lead immediately to the system: $-x_1 + x_2 = 1$, $x_1 + x_3 = 2$.

(b) Substituting $s = x_3$ into $x_3 = 2 - x_1$ yields $s = 2 - x_1 \Rightarrow x_1 = 2 - s$.
Then substituting $2 - s = x_1$ into $x_2 = 1 + x_1$ yields $x_2 = 1 + (2 - s) \Rightarrow x_2 = 3 - s$.
The parametric solution then becomes $x_1 = 2 - s$, $x_2 = 3 - s$, and $x_3 = s$.

41. Let $u = \frac{1}{x}$, and $v = \frac{1}{y}$. Then the system of equations becomes $2u + 3v = 0$, $3u + 4v = 1$.
Solving the second equation for v gives $v = \frac{1}{4} - \frac{3}{4}u$. So, substitution $\Rightarrow 2u + 3\left(\frac{1}{4} - \frac{3}{4}u\right) = 0$.
Thus $u = 3$ and $v = \frac{1}{4} - \frac{3}{4}(3) = -2$. So, the solution is $[x, y] = \left[\frac{1}{3}, -\frac{1}{2}\right]$.

42. Let $u = x^2$, and $v = y^2$. So, the system becomes $u + 2v = 6$, $u - v = 3$.
Subtracting the second equation from the first gives $3v = 3 \Rightarrow v = 1$.
Substituting this into the second equation gives $u = 3 + 1 = 4$. Thus $u = 4$ and $v = 1 \Rightarrow$
The solution set is $[x, y] = \left[\pm\sqrt{4}, \pm\sqrt{1}\right]$. That is, $\{[2, 1], [2, -1], [-2, 1], [-2, -1]\}$.

43. Let $u = \tan x$, $v = \sin y$, $w = \cos z \Rightarrow u - 2v = 2$, $u - v + w = 2$, $v - w = -1$.
We form the augmented matrix and row reduce it to find the solution of the system:

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 \end{array}\right].$$

Using back substitution $w = \frac{1}{2}$, $v = -\frac{1}{2}$, $u = 2 + 2\left(-\frac{1}{2}\right) = 1 \Rightarrow [u, v, w] = \left[1, -\frac{1}{2}, \frac{1}{2}\right]$.
Since $x = \tan^{-1} u$, $y = \sin^{-1} v$, $z = \cos^{-1} w$, the solution is $[x, y, z] = \left[\frac{\pi}{4}, -\frac{\pi}{6}, \frac{\pi}{3}\right]$.

44. Let $r = 2^a$, and $s = 3^b$. Then the system becomes $-r + 2s = 1$, $3r - 4s = 1$.
Adding three times the first equation to the second gives $2s = 4 \Rightarrow s = 2$.
Substituting $s = 2$ into $-r + 2s = 1$ yields $-r + 2(2) = 1 \Rightarrow r = 3 \Rightarrow [r, s] = [3, 2]$.
Since $a = \log_2 r$ and $b = \log_3 s$, the solution is $[a, b] = [\log_2 3, \log_3 2]$.

Exploration: Lies My Computer Told Me

1. $\begin{aligned} x+y &= 0 \\ x+\tfrac{801}{800}y &= 1 \end{aligned} \Rightarrow \begin{aligned} -800x-800y &= 0 \\ 800x+801y &= 800 \end{aligned} \Rightarrow \begin{aligned} x &= -800 \\ y &= 800 \end{aligned}$

2. $\begin{aligned} x+y &= 0 \\ x+1.0012y &= 1 \end{aligned} \Rightarrow \begin{aligned} -x-y &= 0 \\ x+1.0012y &= 1 \end{aligned} \Rightarrow \begin{aligned} x &= -833.33 \\ y &= 833.33 \end{aligned}$

3. $\begin{aligned} x+y &= 0 \\ x+1.00y &= 1 \end{aligned} \Rightarrow \begin{aligned} -x-y &= 0 \\ x+1.00y &= 1 \end{aligned} \Rightarrow 0=1 \Rightarrow$ No solution.

4. Even small changes in slope can cause lines to be widely divergent for large values of x.

To eight significant digits, we have:

$$\begin{aligned} 4.552x+7.083y &= 1.931 \\ 1.731x+2.693y &= 2.001 \end{aligned} \Rightarrow \begin{aligned} 7.879512x+12.260673y &= 3.342561 \\ -7.879512x-12.258536y &= -9.108552 \end{aligned} \Rightarrow$$

$$\begin{aligned} x &= 4198.8301 \\ y &= -2698.1708 \end{aligned}$$

To four significant digits, we have:

$$\begin{aligned} 4.552x+7.083y &= 1.931 \\ 1.731x+2.693y &= 2.001 \end{aligned} \Rightarrow \begin{aligned} 7.879x+12.26y &= 3.342 \\ -7.879x-12.25y &= -9.108 \end{aligned} \Rightarrow \begin{aligned} x &= 896.8 \\ y &= -576.4 \end{aligned}$$

To two significant digits, we have:

$$\begin{aligned} 4.5x+7.0y &= 1.9 \\ 1.7x+2.6y &= 2.0 \end{aligned} \Rightarrow \begin{aligned} 7.6x+11y &= 3.2 \\ -7.6x-11y &= -9 \end{aligned} \Rightarrow 0=-9 \Rightarrow \text{No solution.}$$

2.2 Direct Methods for Solving Linear Systems

1. No, this matrix is not in row echelon form. Why not? Give at least one reason.
The leading entry in row 3 appears to the left of the leading entry in row 2.

2. This matrix is in row echelon form, but not reduced row echelon form. Why not?.
There are many reasons. For example, the leading entry in row 1 is 7 not 1.

3. This matrix is in row echelon form, and also reduced row echelon form. Why is the 3 okay?
The 3 occurs in a column that does not contain a leading 1.

4. This matrix is in row echelon form, and also reduced row echelon form. Why are the 0s okay?
All three rows are zero, so no leading 1s are required.

5. No, this matrix is not in row echelon form. Why not? Give a reason.
The row of all zeroes is not at the bottom.

6. No, this matrix is not in row echelon form. Why not? Give a reason.
The leading entry in row 3 appears to the left of the leading entry in row 2.

7. No, this matrix is not in row echelon form. Why not? Give a reason.
The leading entry in row 2 appears underneath the leading entry in row 1.

8. This matrix is in row echelon form, but not reduced row echelon form. Why not?
The leading entry in row 4 is not a 1. Could we have given another reason?

9. (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. (b) ... $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

10. (a) $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{R_2 - \frac{1}{3}R_1} \begin{bmatrix} 3 & 2 \\ 0 & \frac{10}{3} \end{bmatrix}$ ... (b) ... $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

11. (a) $\begin{bmatrix} 3 & 5 \\ 5 & -2 \\ 2 & 4 \end{bmatrix} \xrightarrow[R_3 - \frac{2}{3}R_1]{R_2 - \frac{5}{3}R_1} \begin{bmatrix} 3 & 5 \\ 0 & -\frac{31}{3} \\ 0 & \frac{2}{3} \end{bmatrix} \xrightarrow[\frac{3}{2}R_3]{-\frac{3}{31}R_2} \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) Continuing from (a): $\begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 5R_2} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

12. (a) $\begin{bmatrix} 2 & -4 & -2 & 6 \\ 3 & -6 & 2 & 6 \end{bmatrix} \xrightarrow{R_2 - \frac{3}{2}R_1} \begin{bmatrix} 2 & -4 & -2 & 6 \\ 0 & 0 & 5 & -3 \end{bmatrix}$ ...

(b) ... from (a): $\begin{bmatrix} 2 & -4 & -2 & 6 \\ 0 & 0 & 5 & -3 \end{bmatrix} \xrightarrow[\frac{1}{5}R_2]{\frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -\frac{3}{5} \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & -2 & 0 & \frac{12}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \end{bmatrix}$.

13. (a) $\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix} \xrightarrow[-3R_3]{-3R_2} \begin{bmatrix} 3 & -2 & -1 \\ -6 & 3 & 3 \\ -12 & 9 & 3 \end{bmatrix} \xrightarrow[R_3+4R_1]{R_2+2R_1} \begin{bmatrix} 3 & -2 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$

(b) Continuing from (a): $\begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$

14. (a) $\begin{bmatrix} -2 & 6 & -7 \\ 3 & -9 & 10 \\ 1 & -3 & 3 \end{bmatrix} \xrightarrow{R_2+\frac{3}{2}R_1} \begin{bmatrix} -2 & 6 & -7 \\ 0 & 0 & -\frac{1}{2} \\ 1 & -3 & 3 \end{bmatrix} \xrightarrow{R_3+\frac{1}{2}R_1} \begin{bmatrix} -2 & 6 & -7 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \ldots$

(b) $\ldots \xrightarrow{R_3+R_2} \begin{bmatrix} -2 & 6 & -7 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2} \begin{bmatrix} -2 & 6 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+7R_2} \begin{bmatrix} -2 & 6 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \xrightarrow{R_4+29R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix} \xrightarrow{8R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$

$\xrightarrow{R_4-3R_2} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix} \xrightarrow{\substack{R_2+2R_1 \\ R_3+2R_1 \\ R_4-R_1}} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}.$

16. $R_j \leftrightarrow R_i$ undoes $R_i \leftrightarrow R_j$, $\frac{1}{k}R_i$ undoes kR_i, $R_i - kR_j$ undoes $R_i + kR_j$.

17. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1+\frac{7}{2}R_2} \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = B.$

So A and B are row equivalent. Convert A into B by $R_2 - 2R_1$, $-\frac{1}{2}R_1$, $R_1 + \frac{7}{2}R_2$.

18. $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2+2R_3} \begin{bmatrix} 3 & 1 & -1 \\ -1 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

$\xrightarrow{R_3+R_1} \begin{bmatrix} 3 & 1 & -1 \\ -1 & 3 & 2 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{R_2+\frac{1}{2}R_3} \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{bmatrix} = B.$

Therefore, the matrices A and B are row equivalent.

19. Performing $R_2 + R_1$ and $R_1 + R_2$ does *not* leave rows 1 and 2 identical.
After performing $R_2 + R_1$ the second row is now $R_2' = R_2 + R_1$.
So $R_1 + R_2$ is actually $R_1 + R_2' = R_1 + (R_2 + R_1) = 2R_1 + R_2$.
Performing $R_2 + R_1$ and $R_1 + R_2$ simultaneously annuls their linearity.

20. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} x_1 \\ x_2 + x_1 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} -x_2 \\ x_2 + x_1 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$

The net effect is to interchange the first and second rows.

21. Our first task is to show that $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{3R_2-2R_1} \begin{bmatrix} 3 & 1 \\ 0 & 10 \end{bmatrix}$ is *not* an elementary row operation.

Compare $3R_2 - 2R_1$ to the elementary row operations $R_i \leftrightarrow R_j$, kR_i, $R_i + kR_j$.
Clearly, $3R_2 - 2R_1$ is a combination of kR_i and $R_i + kR_j$ done at the same time.
Performing row operations simultaneously annuls their linearity.

One way to achieve the result is: $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2-\frac{2}{3}R_1} \begin{bmatrix} 3 & 1 \\ 0 & \frac{10}{3} \end{bmatrix} \xrightarrow{3R_2} \begin{bmatrix} 3 & 1 \\ 0 & 10 \end{bmatrix}.$

22. We must show that we can create a 1 in row 1, column 1 using $R_i \leftrightarrow R_j$, kR_i, or $R_i + kR_j$.

$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$ $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{2}{3} \\ 1 & 4 \end{bmatrix}.$ $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & -6 \\ 1 & 4 \end{bmatrix}.$

$R_i \leftrightarrow R_j$ is the most direct. That is, it requires the fewest operations.
kR_i requires fewer operations than $R_i + kR_j$, but $R_i + kR_j$ gives integer results.

23. Since **rank** = the number of nonzero rows in the row echelon form of a matrix, before we answer we should put each of the matrices into row echelon form.

(1) Since this matrix A is not in its row echelon form B, we must row reduce A first.
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. So, rank A = the number of nonzero rows in $B = 3$.

(2) A is in row echelon form, so we need only count the number of its nonzero rows.

Since $\begin{bmatrix} 7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has two nonzero rows, rank $A = 2$.

(3) A is in row echelon form, so we need only count the number of its nonzero rows.

Since $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has two nonzero rows, rank $A = 2$.

(4) A is in row echelon form, so we need only count the number of its nonzero rows.

Since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has no nonzero rows, rank $A = 0$.

(5) Since this matrix A is not in its row echelon form B, we must row reduce A first.
$\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. So, rank $A = 2$.

(6) Since this matrix A is not in its row echelon form B, we must row reduce A first.
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So, rank $A = 3$.

(7) Since this matrix A is not in its row echelon form B, we must row reduce A first.
$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 - R_1 + R_2 + 2R_3]{R_2 - R_1 \quad R_3 - \frac{1}{2}R_1 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$. So, rank $A = 3$.

(8) A is in row echelon form, so we need only count the number of its nonzero rows.

Since $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has 3 nonzero rows, rank $A = 3$.

24. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

25. We have the following system of equations: $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 4 \end{bmatrix}$.

We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+3R_3} \left[\begin{array}{ccc|c} 13 & -1 & 0 & 21 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{array}\right] \xrightarrow[-R_3]{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 13 & -1 & 0 & 21 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$$\xrightarrow[-R_2]{R_1-R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ -13 & 1 & 0 & -21 \\ 0 & -1 & 1 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -13 & 1 & 0 & -21 \\ 0 & -1 & 1 & -4 \end{array}\right] \xrightarrow{R_2+13R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & -1 & 1 & -4 \end{array}\right]$$

$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$. So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

26. We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 2 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & -3 & 1 & 3 \\ 0 & 3 & -1 & -2 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & -3 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The third row is equivalent to the equation $0 = 1$ which clearly has no solution. Therefore, the system is inconsistent.

Does $R_3 = R_1 - R_2$ (excluding constants) cause the system to be inconsistent?

27. We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow{R_3+8R_1+10R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 = -8R_1 - 10R_2$ (excluding constants) does not cause a problem here? Why?
Since the system is homogeneous (all constants = 0), the system has at least one solution.

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The third row of $0 = 0$ tells us that $x_3 = t$ is a free variable.
Back substituting, we have $x_2 + t = 0 \Rightarrow x_2 = -t$ and $x_1 + t = 0 \Rightarrow x_1 = -t$.

So, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ or equivalently $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

28. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 3 & 3 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & -1 & 2 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & -1 & 2 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-2R_1} \\ \xrightarrow{R_3-2R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & -3 & 1 & 3 & 1 \\ 0 & -1 & 1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & -3 & 1 & 3 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{-R_2} \\ \xrightarrow{R_3-3R_2} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & -2 & 0 & -5 \end{array}\right]$$

The system has been reduced to

$$\begin{aligned} w + 2x - z &= 0 \\ x - y - z &= -2 \\ -2y &= -5 \end{aligned}$$

z is a free variable; setting $z = t$ and back-substituting gives

$$\begin{aligned} y &= \frac{5}{2} \\ x &= y + t - 2 = t + \frac{1}{2} \\ w &= t - 2x = -t - 1 \end{aligned}$$

29. Note that there are 3 equations but only 2 variables to satisfy them.
It is helpful, therefore, to begin by noting $R_3 = 9R_1 - 4R_2$.

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}\right] \xrightarrow{R_3-9R_1+4R_2} \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \ldots \longrightarrow \ldots \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

So, the solution is $\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

30. First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} -1 & 3 & -2 & 4 & 2 \\ 2 & -6 & 1 & -2 & -1 \\ 1 & -3 & 4 & -8 & -4 \end{array}\right] \xrightarrow[\substack{R_2+2R_1 \\ R_3+R_1}]{-R_1} \left[\begin{array}{rrrr|r} 1 & -3 & 2 & -4 & -2 \\ 0 & 0 & -3 & 6 & 3 \\ 0 & 0 & 2 & -4 & -2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{rrrr|r} 1 & -3 & 2 & -4 & -2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 2 & -4 & -2 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{rrrr|r} 1 & -3 & 2 & -4 & -2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{rrrr|r} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has been reduced to

$$\begin{aligned} x_1 - 3x_2 &= 0 \\ x_3 - 2x_4 &= -1 \end{aligned}$$

x_2 and x_4 are free variables; setting $x_2 = s$ and $x_4 = t$ and back-substituting gives

$$\begin{aligned} x_1 &= 3s \\ x_3 &= 2t - 1 \end{aligned}$$

31. From the beginning, we know this system has infinitely many solutions. Why?
Because this system has 5 variables and only 3 equations they have to satisfy.

We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{rrrrr|r} \frac{1}{2} & 1 & -1 & -6 & 0 & 2 \\ \frac{1}{6} & \frac{1}{2} & 0 & -3 & 1 & -1 \\ \frac{1}{3} & 0 & -2 & 0 & -4 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{rrrrr|r} 1 & 0 & -6 & 0 & -12 & 24 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since rank $A = 2$ and $5 - 2 = 3$, we get 3 free variables: $x_3 = r$, $x_4 = s$, and $x_5 = t$.
Back substituting, we get $x_2 = -10 - 2r + 6s - 6t$, $x_1 = 24 + 6r + 12t$.

$$\text{So, the solution is } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 24 \\ -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

32. We should be able to see that rank $A = 3$. What does this tell us?
There are 3 equations and 3 variables. So if there is a solution, it is unique. Why?
Because the Rank Theorem (2.2) tells us there are $3 - 3 = 0$ free variables.

$$\left[\begin{array}{rrr|r} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \text{The solution is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 0 \end{bmatrix}.$$

33. We form the augmented matrix and row reduce it as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow$$

The fourth row is equivalent to the equation $0 = 1$ which clearly has no solution. Therefore, the system is inconsistent.

Q: Rank $A = 3$. How does that relate to the fact that there is no solution?
A: The system has no solution when A has a zero row with corresponding constant $\neq 0$.

34. When there are 4 equations and 4 variables, if the solution exists it is unique.
Why? Because the Rank Theorem (2.2) tells us there are $4 - 4 = 0$ free variables.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 4 & 10 \\ 1 & 3 & 6 & 10 & 20 \\ 1 & 4 & 10 & 20 & 35 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \text{The solution is } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

35. Begin by thinking of this system as $[A|\mathbf{x}]$, then determine rank A by inspection.
Mentally performing $R_1 \leftrightarrow R_3$ to put matrix A into row echelon form,
makes it obvious that rank $A = 3$ (because A has 3 nonzero rows).

Since rank $A = 3$, this is a system of 3 equations and 3 variables.
Therefore, the system has a unique solution because there are $3 - 3 = 0$ free variables.

36. Begin by thinking of this system as $[A|\mathbf{x}]$, then determine rank A by inspection.
Mentally performing $R_3 - 2R_2$ implies the equation $0 = 2$.
This equation makes it obvious that this system has no solution.

Note: $R_3 = 2R_2$ implies rank $A = 2$. How does that relate to our answer?

37. Since this system has 4 variables and at most 3 equations,
it has infinitely many solutions. Why? There is at least one free variable.

38. Since this system has 5 variables and at most 3 equations,
it has infinitely many solutions. Why? There are at least two free variables.

39. We need only show that the condition $ad - bc \neq 0$ implies that rank $A = 2$. Why?
If rank $A = 2$, there are $2 - 2 = 0$ free variables so the system has a unique solution.

Case 1: $a = 0$, which implies both $b \neq 0$ and $c \neq 0$. Why? Because $0d - bc = -bc \neq 0$.

Row reduce A: $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$.

A is now in row echelon form with 2 nonzero rows. Therefore, rank $A = 2$.

Case 2: $c = 0$, which implies both $a \neq 0$ and $d \neq 0$. Why? Because $ad - b0 = ad \neq 0$.

Row reduce A: $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

A is now in row echelon form with 2 nonzero rows. Therefore, rank $A = 2$.

Case 3: $a \neq 0$ and $c \neq 0$.

Row reduce A: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{aR_2}{\overset{cR_1}{\longrightarrow}} \begin{bmatrix} ac & bc \\ ac & ad \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} ac & bc \\ 0 & ad - bc \end{bmatrix}$.

A is now in row echelon form with 2 nonzero rows. Therefore, rank $A = 2$.

40. First row-reduce the system $\left[A \mid x\right]$ and then answer parts (a), (b), and (c):

$$\left[\begin{array}{cc|c} k & 1 & -2 \\ 2 & -2 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 2 & -2 & 4 \\ k & 1 & -2 \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}kR_1} \left[\begin{array}{cc|c} 2 & -2 & 4 \\ 0 & k+1 & -2-2k \end{array}\right]$$

(a) There are no values of k for which this system has no solution. Why?
The system has no solution when A has a zero row with corresponding constant $\neq 0$.
$2 + 2k = 0 \Rightarrow k = -1$ is the only value of k that creates a row of all zeros.
But $k = -1 \Rightarrow$ the constant $3 + 3k = 3 - 3 = 0$. What does this imply?

(b) The system has a unique solution for $k \neq -1$. Why?
From (a), we see when $k \neq -1$ then rank $A = 2$. So, there are $2 - 2 = 0$ free variables.

(c) The only value of k for which this system has infinitely many solutions is $k = -1$.
The system has infinitely many solutions when A has a zero row with constant $= 0$.
From (a), we see this is exactly the case when $k = -1$.

41. First row reduce the system $[A|\mathbf{x}]$ and then answers parts (a), (b), and (c).

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right] \xrightarrow{R_2-kR_1} \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right]$$

(a) When $k=-1$, this system has no solution. Why?
The system has no solution when A has a zero row with corresponding constant $\neq 0$.
$1-k^2=0 \Rightarrow k=\pm 1$ create a zero row in A.
And $k=-1 \Rightarrow$ the constant $1-k=1+1\neq 0$.

(b) When $k \neq \pm 1$, this system has a unique solution. Why?
From (a), we see when $k\neq \pm 1$ then rank $A=2$. So, there are $2-2=0$ free variables.

(c) When $k=1$, this system has infinitely many solutions.
The system has infinitely many solutions when A has a zero row with constant $=0$.
From (a), we see this is exactly the case when $k=1$.

42. First row-reduce the system $\left[A\,\middle|\,x\right]$ and then answer parts (a), (b), and (c):

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 4 & -1 & k \\ 2 & -1 & 4 & k^2 \end{array}\right] \begin{array}{c}\xrightarrow{R_2-R_1} \\ \xrightarrow{R_3-2R_2}\end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & -3 & 2 & k^2-4 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 0 & 0 & k^2+k-6 \end{array}\right]$$

(a) The system has no solution when A has a zero row with a corresponding nonzero constant. This occurs for $k^2+k-6\neq 0$, i.e. for $k\neq 2,-3$.

(b) The system has a unique solution when A has rank 3. Since A has a zero row, the system does not have a unique solution for any value of k.

(c) The system has an infinite number of solutions when A has a zero row with a zero constant. This occurs for $k^2+k-6=0$, i.e. for $k=2,-3$.

43. First row reduce the system $[A|\mathbf{x}]$ and then answers parts (a), (b), and (c).

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & k^2+k-2 & k+2 \end{array}\right]$$

(a) When $k=1$, this system has no solution. Why?
The system has no solution when A has a zero row with corresponding constant $\neq 0$.
$k^2+k-2=0 \Rightarrow k=1$ makes the bottom row 0, but the constant $k+2=1+2=3\neq 0$.

(b) When $k\neq 1,-2$, this system has a unique solution. Why?
When $k\neq 1,-2$, rank $A=3$. So, there are $3-3=0$ free variables.

(c) When $k=-2$, this system has infinitely many solutions.
The system has infinitely many solutions when A has a zero row with constant $=0$.
$k^2+k-2=0 \Rightarrow k=-2$ makes the bottom row 0 and the constant $k+2=-2+2=0$.

44. (a) The following system of n equations has infinitely many solutions:

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0 \\ 2x_1 + 2x_2 + \cdots + 2x_n &= 0 \\ &\vdots \\ nx_1 + nx_2 + \cdots + nx_n &= 0 \end{aligned}$$

Likewise, the following system of $n+1$ equations has infinitely many solutions:

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0 \\ 2x_1 + 2x_2 + \cdots + 2x_n &= 0 \\ &\vdots \\ nx_1 + nx_2 + \cdots + nx_n &= 0 \\ (n+1)\,x_1 + (n+2)\,x_2 + \cdots + (n+1)\,x_n &= 0 \end{aligned}$$

(b) The system of n equations $x_1 = 0$, $x_2 = 0$, …, $x_n = 0$ has the unique solution $x_i = 0$. as does the system of $2n$ equations $x_1 = 0$, $2x_1 = 0$ …, $x_n = 0$, $2x_n = 0$.

45. As in Example 2.14, find the line of intersection of $3x + 2y + z = -1$, $2x - y + 4z = 5$.

First, observe that there will be a line of intersection. Why?
The normal vectors of the two planes, $[3, 2, 1]$ and $[2, -1, 4]$ are not parallel.

The points that lie in the intersection of the two planes correspond to the points in the solution the system:

$$\begin{aligned} 3x + 2y + z &= -1 \\ 2x - y + 4z &= 5 \end{aligned}$$

Gauss-Jordan elimination yields: $\left[\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 2 & -1 & 4 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{9}{7} & \frac{9}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{17}{7} \end{array}\right]$

Replacing variables, we have: $\begin{aligned} x + \tfrac{9}{7}z &= \tfrac{9}{7} \\ y - \tfrac{10}{7}z &= -\tfrac{17}{7} \end{aligned} \Rightarrow \begin{aligned} z &= 1 - \tfrac{7}{9}x \\ y &= -\tfrac{17}{7} + \tfrac{10}{7}z \end{aligned}$

To eliminate fractions we set $x = 9t$, so $z = 1 - \frac{7}{9}(9t) = 1 - 7t$.
Substituting $z = 1 - 7t$ into $y = -\frac{17}{7} + \frac{10}{7}z$ yields: $y = -\frac{17}{7} + \frac{10}{7}(1 - 7t) = -1 - 10t$.
Summarizing, we now have $x = 9t$, $y = -1 - 10t$, and $z = 1 - 7t$.

Therefore, the line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -10 \\ -9 \end{bmatrix}$.

46. As in Example 2.14, find the line of intersection of $4x + y - z = 0$, $2x - y + 3z = 4$.

First, observe that there will be a line of intersection. Why?
The normal vectors of the two planes, $[4, 1, -1]$ and $[2, -1, 3]$ are not parallel.

The points that lie in the intersection of the two planes correspond to the points in the solution the system:
$$\begin{array}{rcrcrcr} 4x & + & y & - & z & = & 0 \\ 2x & - & y & + & 3z & = & 4 \end{array}$$

Gauss-Jordan elimination yields:
$$\left[\begin{array}{rrr|r} 4 & 1 & -1 & 0 \\ 2 & -1 & 3 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{7}{3} & -\frac{8}{3} \end{array}\right]$$

Replacing variables, we have:
$$\begin{array}{rcrcr} x & + & \frac{1}{3}z & = & \frac{2}{3} \\ y & - & \frac{7}{3}z & = & -\frac{8}{3} \end{array} \Rightarrow \begin{array}{rcrcr} z & = & 2 & - & 3x \\ y & = & -\frac{8}{3} & + & \frac{7}{3}z \end{array}$$

To begin we set $x = t$, so $z = 2 - 3(t) = 2 - 3t$.
Substituting $z = 2 - 3t$ into $y = -\frac{8}{3} + \frac{7}{3}z$ yields: $y = -\frac{8}{3} + \frac{7}{3}(2 - 3t) = 2 - 7t$.
Summarizing, we now have $x = t$, $y = 2 - 7t$, and $z = 2 - 3t$.

Therefore, the line is
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -7 \\ -3 \end{bmatrix}.$$

47. When looking for examples, begin with familiar planes like $x = 0$, $y = 0$, and $z = 0$.

(a) Let's start with $x = 0$ and $y = 0$. These planes obviously intersect in the z-axis. Why?

As in Exercise 45: $\begin{array}{rcrcrcl} x &+& 0y &+& 0z &=& 0 \\ 0x &+& y &+& 0z &=& 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow x = 0, y = 0, z = t \Rightarrow$

The line of intersection of $x = 0$ and $y = 0$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, the z-axis.

All we need is one other plane that passes through the z-axis to complete our example.
It may help to sketch $\mathbb{R}^2$ and look for a line that passes through the origin.
One such line is $x = y$ which corresponds to the plane $x - y = 0$ in $\mathbb{R}^3$.

Sketch these three planes in $\mathbb{R}^3$ to confirm they intersect in the z-axis.

Q: How do we confirm these three planes intersect in the z-axis algebraically?
A: Check the intersection between $x = 0$ and $x - y = 0$. Why is that enough?

As in Exercise 45: $\begin{array}{rcrcrcl} x &+& 0y &+& 0z &=& 0 \\ x &-& y &+& 0z &=& 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow x = y = 0, \text{ and } z = t \Rightarrow$

The line of intersection of $x = 0$ and $x - y = 0$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Q: Start with $y = 0$ and $z = 0$ and then $x = 1$ and $y = 1$. Is there a pattern?

(b) Begin with $x = 0$ and $y = 0$. We need one plane that crosses across both of these.
It may help to visualize $\mathbb{R}^2$ and look for a line that cuts across the first quadrant.
It is obvious that the plane $x + y = 1$ will complete the example?

Sketch these three planes in $\mathbb{R}^3$ to confirm they intersect in pairs.
For example, $x = 0$ and $x + y = 1$ intersect in the line $[x, y, z] = [0, 1, 0] + t[0, 0, 1]$.

As in Exercise 45: $\begin{array}{rcrcrcl} x &+& 0y &+& 0z &=& 0 \\ x &+& y &+& 0z &=& 1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow x = 0,\ y = 1, \text{ and } z = t \Rightarrow$

The line of intersection of $x = 0$ and $x + y = 1$ is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(c) An obvious example is $x = 0$, $x = 1$, and $y = 0$. Why?
The normal vector for $x = 0$ and $x = 1$ is $[1, 0, 0]$
while the normal vector for $y = 0$ is $[0, 1, 0]$.

(d) The most obvious example is $x = 0$, $y = 0$, and $z = 0$.
Note that any example of $x = a$, $y = a$, and $z = a$ will work.
Are there any other obvious pattern examples that will work?

48. As in Example 2.15, if these lines intersect we need to determine the point of intersection. As pointed out in that example, we need to change the parameter for the first line to s. We want to find an $\mathbf{x} = [x, y, z]$ that satisfies both equations simultaneously. That is, we want $\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$ or $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Substituting the given $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$, and $\mathbf{v}$ into $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$, we obtain the equations:

$$s\begin{bmatrix}1\\2\\-1\end{bmatrix} - t\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\2\\0\end{bmatrix} - \begin{bmatrix}-1\\2\\1\end{bmatrix} \Rightarrow \begin{array}{rcrcr} s & + & t & = & 3\\ 2s & - & t & = & 0\\ -s & & & = & -1\end{array}$$

From this, the solution is easily found to be $s = 1, t = 2$.

$$\text{Therefore, the point of intersection is: } \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1\\2\\1\end{bmatrix} + 1\begin{bmatrix}1\\2\\-1\end{bmatrix} = \begin{bmatrix}0\\4\\0\end{bmatrix}.$$

Check that substituting $t = 2$ into the other equation gives the same point.

49. As in Example 2.15, if these lines intersect we need to determine the point of intersection. As pointed out in that example, we need to change the parameter for the first line to s. We want to find an $\mathbf{x} = [x, y, z]$ that satisfies both equations simultaneously. That is, we want $\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$ or $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Substituting the given $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$, and $\mathbf{v}$ into $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$, we obtain the equations:

$$s\begin{bmatrix}1\\0\\1\end{bmatrix} - t\begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\end{bmatrix} - \begin{bmatrix}3\\1\\0\end{bmatrix} \Rightarrow \begin{array}{rcrcr} s & - & 2t & = & -4\\ & - & 3t & = & 0\\ s & - & t & = & -1\end{array}$$

From this, there is clearly no solution since $t = 0$ implies $s = -4$ and -1 at the same time. Therefore, we conclude that these lines do not intersect.

50. Similar to Example 2.15, we need to find conditions on a, b, c to create a point of intersection. As pointed out in that example, we need to change the parameter for the first line to s. We want to find an $\mathbf{x} = [x, y, z]$ that satisfies both equations simultaneously. That is, we want $\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$ or $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Substituting the given $\mathbf{p}$, $\mathbf{q} = [a, b, c]$, $\mathbf{u}$, and $\mathbf{v}$ into $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$, we get:

$$s\begin{bmatrix}1\\1\\-1\end{bmatrix} - t\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} \Rightarrow \begin{array}{rcrcl} s & - & 2t & = & a - 1\\ s & - & t & = & b - 2\\ -s & & & = & c - 3\end{array}$$

From this, the conditions on a, b, c are easily found to be $Q = (s - 2t + 1, s - t + 2, -s + 3)$.

51. Recall the definition of the dot product: $\mathbf{u} \times \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$.
So, vectors that satisfy $\mathbf{u} \cdot \mathbf{x} = \mathbf{0}$ and $\mathbf{v} \cdot \mathbf{x} = \mathbf{0}$ correspond to the system:

$$\begin{array}{rcrcrcl} u_1x_1 & + & u_2x_2 & + & u_3x_3 & = & 0 \\ v_1x_1 & + & v_2x_2 & + & v_3x_3 & = & 0 \end{array}$$ which leads to the augmented matrix $\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right]$.

Gauss-Jordan elimination yields:

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right] \overset{u_1R_2}{\longrightarrow} \left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ u_1v_1 & u_1v_2 & u_1v_3 & 0 \end{array}\right] \overset{R_2 - v_1R_1}{\longrightarrow} \left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ 0 & u_1v_2 - u_2v_1 & u_1v_3 - u_3v_1 & 0 \end{array}\right]$$

The second row implies: $(u_1v_2 - u_2v_1)x_2 = (u_3v_1 - u_1v_3)x_3 \Rightarrow x_2 = \dfrac{u_3v_1 - u_1v_3}{u_1v_2 - u_2v_1}x_3$.

To clear the fraction, we let $x_3 = (u_1v_2 - u_2v_1)t \Rightarrow x_2 = (u_3v_1 - u_1v_3)t$

The first row implies: $u_1x_1 = -u_2x_2 - u_3x_3$.

So, $u_1x_1 = -u_2(u_3v_1 - u_1v_3)t - u_3(u_1v_2 - u_2v_1)t = (u_1u_2v_3 - u_1u_3v_2)t$.

Dividing both sides by u_1 yields: $x_1 = (u_2v_3 - u_3v_2)t$.

Therefore, as was to be shown: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} t$.

52. This problem has two distinct parts:
1) Prove that the given nonparallel lines do not intersect, that is that they are skew.
2) Find two parallel planes, one containing each line. We will use the result of Exercise 51.

1: Similar to Example 2.15, we will show these lines, ℓ_P and ℓ_Q do not intersect.
As pointed out in that example, we need to change the parameter for the first line to s.
We want to show there is no $\mathbf{x} = [x, y, z]$ that satisfies both equations simultaneously.
So, we consider $\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$ or $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Substituting the given $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$, and $\mathbf{v}$ into $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$, we obtain the equations:

$$s\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} - t\begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{rcrcr} 2s & & & = & -1 \\ -3s & - & 6t & = & 0 \\ s & + & t & = & -1 \end{array}$$

From this, there is clearly no solution since $s = -\frac{1}{2}$ implies $t = \frac{1}{4}$ and $-\frac{1}{2}$ at the same time.
Therefore, we conclude that these lines do not intersect.

2: From Exploration: The Cross Product in Chapter 1, $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
How does this help us find the parallel planes we are seeking? $\mathbf{u} \times \mathbf{v}$ can give us a normal.

From Exercise 51, taking $t = 1$ we have:

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} (-3)(-1) & - & (1)(6) \\ (1)(0) & - & (2)(-1) \\ (2)(6) & - & (-3)(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 12 \end{bmatrix}.$$

It is easy to confirm that $\mathbf{n} \cdot \mathbf{u} = 0$ and $\mathbf{n} \cdot \mathbf{v} = 0$.

So we are looking for planes of the form $-3x + 2y + 12z = d$.
One passing through $P = (1, 1, 0)$ and the other passing through $Q = (0, 1, -1)$.

Substituting $P = (1, 1, 0)$, we get $-3(1) + 2(1) + 12(0) = -1 = d$.
Substituting $Q = (0, 1, -1)$, we get $-3(0) + 2(1) + 12(-1) = -10 = d$.

Therefore, $\mathscr{P}$, $-3x + 2y + 12z = -1$, and $\mathscr{Q}$, $-3x + 2y + 12z = -10$,
are parallel planes containing ℓ_P and ℓ_Q respectively.

53. Following Example 2.16, we note that we need only use addition and multiplication.
We form augmented matrix and perform Gauss-Jordan elimination in $\mathbb{Z}_3$:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2 & 1 \end{array}\right] \xrightarrow{2R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$$

So the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

54. Following Example 2.17, we note that we need only use addition and multiplication. We form the augmented matrix and perform Gauss-Jordan elimination in $\mathbb{Z}_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, we have: $\begin{array}{rcrcrcl} x & + & y & & & = & 1 \\ & & y & + & z & = & 0 \end{array}$

Setting the free variable $z = t$ yields: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1+t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

Since $t = 0$ or 1, there are exactly two solutions: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

55. Following Example 2.17, we note that we need only use addition and multiplication. We form the augmented matrix and perform Gauss-Jordan elimination in $\mathbb{Z}_3$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3+2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So the solution is: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$

56. Following Example 2.17, we note that we need only use addition and multiplication. We form the augmented matrix and perform Gauss-Jordan elimination in $\mathbb{Z}_5$:

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 4 & 1 \end{array}\right] \xrightarrow{R_2+3R_1} \left[\begin{array}{cc|c} 3 & 2 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

The second row is equivalent to the equation $0 = 4$ which clearly has no solution. Therefore, the system is inconsistent.

57. Following Example 2.17, we note that we need only use addition and multiplication. We form the augmented matrix and perform Gauss-Jordan elimination in $\mathbb{Z}_7$:

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 4 & 1 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 3 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1+5R_2} \left[\begin{array}{cc|c} 3 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{5R_1} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 3 \end{array}\right]$$

So the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$

58. Following Example 2.17, we note that we need only use addition and multiplication. We form the augmented matrix and perform Gauss-Jordan elimination in $\mathbb{Z}_5$:

$$\left[\begin{array}{cccc|c} 1&0&0&4&1\\ 1&2&4&0&3\\ 2&2&0&1&1\\ 1&0&3&0&2 \end{array}\right] \xrightarrow[R_4+4R_1]{\substack{R_2+4R_1\\ R_3+3R_1}} \left[\begin{array}{cccc|c} 1&0&0&4&1\\ 0&2&4&1&2\\ 0&2&0&3&4\\ 0&0&3&1&1 \end{array}\right] \xrightarrow{R_3+4R_2} \left[\begin{array}{cccc|c} 1&0&0&4&1\\ 0&2&4&1&2\\ 0&0&1&2&2\\ 0&0&3&1&1 \end{array}\right] \xrightarrow[R_4+2R_3]{3R_2} \left[\begin{array}{cccc|c} 1&0&0&4&1\\ 0&1&2&3&1\\ 0&0&1&2&2\\ 0&0&0&0&0 \end{array}\right]$$

Therefore, we have:
$$\begin{array}{rcrcrcl} x_1 & & & + & 4x_4 & = & 1\\ & x_2 & + \; 2x_3 & + & 3x_4 & = & 1\\ & & x_3 & + & 2x_4 & = & 2 \end{array}$$

Setting the free variable $x_4 = t$ yields: $\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} = \begin{bmatrix} 1+\ t\\ 2+\ t\\ 2+3t\\ 0+\ t \end{bmatrix} = \begin{bmatrix} 1\\ 2\\ 2\\ 0 \end{bmatrix} + t\begin{bmatrix} 1\\ 1\\ 3\\ 1 \end{bmatrix}.$

Since $t = 0, 1, 2, 3,$ or 4, there are 5 solutions: $\begin{bmatrix} 1\\ 2\\ 2\\ 0 \end{bmatrix}, \begin{bmatrix} 2\\ 3\\ 0\\ 1 \end{bmatrix}, \begin{bmatrix} 3\\ 4\\ 3\\ 2 \end{bmatrix}, \begin{bmatrix} 4\\ 0\\ 1\\ 3 \end{bmatrix}$, and $\begin{bmatrix} 0\\ 1\\ 4\\ 4 \end{bmatrix}$.

59. Recall the Rank Theorem (which applies to systems over $\mathbb{Z}_p$ not just $\mathbb{R}^n$) says:
If the system is consistent, then: number of free variables $= n-$ rank (A).

In $\mathbb{Z}_p$, however, each free variable can only take on p different values.
If there is 1 free variable, there are $p^1 = p$ solutions as in Exercise 58.
If there are 2 free variables, there are $p \times p = p^2$ solutions.
In general, the total number of solutions is $p^{\text{number of free variables}} = p^{n-\text{rank}(A)}$.

60. Following Example 2.17, we note that we need only use addition and multiplication. Unlike all previous examples, however, 6 is not prime. Will that matter? Let's see. We form the augmented matrix and perform Gauss-Jordan elimination in $\mathbb{Z}_6$:

$$\left[\begin{array}{cc|c} 2&3&4\\ 4&3&2 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 2&3&4\\ 0&0&0 \end{array}\right]$$ So, we have $2x + 3y = 4$ in $\mathbb{Z}_6$.

Let's look at the multiplication table.

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

We let $x = 0, 1, 2, 3, 4, 5$ to find all possible solutions.
When $x = 0$ or 3, we get $3y = 4$ which clearly has no solution.
When $x = 1$ or 4, we get $2 + 3y = 4 \Rightarrow 3y = 2$ which has no solution.
When $x = 2$ or 5, we get $4 + 3y = 4 \Rightarrow 3y = 0$ has three solutions $y = 0, 2, 4$.

So the six solutions are $\begin{bmatrix} 2\\ 0 \end{bmatrix}, \begin{bmatrix} 2\\ 2 \end{bmatrix}, \begin{bmatrix} 2\\ 4 \end{bmatrix}, \begin{bmatrix} 5\\ 0 \end{bmatrix}, \begin{bmatrix} 5\\ 2 \end{bmatrix}$, and $\begin{bmatrix} 5\\ 4 \end{bmatrix}$.

Exploration: Partial Pivoting

1. (a) Solving $0.00021x = 1$ to 5 significant digits, we have: $x = \dfrac{1}{0.00021} \approx 4761.9$.

(b) Solving $0.0002x = 1$ to 4 significant digits, we have: $x = \dfrac{1}{0.0002} \approx 5000$.

So, the effect of an 0.00001 error is $5000 - 4761.9 = 238.1$.

2. (a) Without partial pivoting:

Pivoting on 0.400 to 3 significant digits,
we first divide the first row by 0.400 ($\Rightarrow 1 \quad 249 \quad 250$),
then multiply it by 75.3 and subtract the result from row 2 $\Rightarrow$

$$\left[\begin{array}{cc|c} 0.4 & 99.6 & 100 \\ 75.3 & -45.3 & 30.0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0.4 & 99.6 & 100 \\ 0 & 18700 & 18800 \end{array}\right] \Rightarrow y = 1.01, \text{ to 3 significant digits.}$$

Back substituting, we get $x = 250 - 249(1.01) = 250 - 251 = -1.00$.

This error was introduced because -45.3 and 30.0 were ignored
when reducing to 3 significant digits.
They were overwhelmed by 18700 and 18800 respectively.

(b) With partial pivoting:

Pivoting on 75.3 to 3 significant digits,
we first divide the first row by 75.3 ($\Rightarrow 1 \quad 0.601 \quad 0.398$),
then multiply it by 0.004 and subtract the result from row 2 $\Rightarrow$

$$\left[\begin{array}{cc|c} 75.3 & -45.3 & 30.0 \\ 0.4 & 99.6 & 100 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 75.3 & -45.3 & 30.0 \\ 0 & 99.8 & 100 \end{array}\right] \Rightarrow y = 1.00, \text{ again to 3 significant digits.}$$

Back substituting, we get $x = \dfrac{30.0 + 45.3}{75.3} = 1.00$.

Pivoting on the largest absolute value reduces the error in our solution.

3. (a) Without partial pivoting:

Pivoting on 0.001 to 3 significant digits,
we first divide the first row by 0.001 ($\Rightarrow 1 \quad 995 \quad 1000$),
then multiply it by -10.2 and subtract the result from row 2 $\Rightarrow$

$$\left[\begin{array}{rr|r} 0.001 & 0.995 & 1.00 \\ -10.2 & 1.00 & -50.0 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 0.001 & 0.995 & 1.00 \\ 0 & -10,100 & -10,200 \end{array}\right] \Rightarrow y = 1.01.$$

Back substituting, we get $x = \dfrac{1.00 - 1.00}{0.001} = 0$.
This error was introduced because 1.00 and -50.0 were ignored when reducing to 3 significant digits.
They were overwhelmed by $-10,100$ and $-10,200$ respectively.

With partial pivoting:

Pivoting on -10.2 to 3 significant digits,
we first divide the first row by -10.2 ($\Rightarrow 1 \quad 0.098 \quad 4.90$),
then multiply it by 0.001 and subtract the result from row 2 $\Rightarrow$

$$\left[\begin{array}{rr|r} -10.2 & 1.00 & -50.0 \\ 0.001 & 0.995 & 1.00 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} -10.2 & 1.00 & -50.0 \\ 0 & 0.995 & 1.00 \end{array}\right] \Rightarrow y = 1.00.$$

Back substituting, we get $x = \dfrac{-50.0 - 1.00}{-10.2} = \dfrac{51}{10.2} = 5$.
Pivoting on the largest absolute value reduces the error in our solution.

(b) Without partial pivoting:

Pivoting on 10 to 3 significant digits,
we first divide the first row by 10 ($\Rightarrow 1 \quad -0.7 \quad 0 \quad 0.7$),
then multiply it by 3 and add the result to row 2 and
also multiply it by 5 and subtract the result from row 3.

$$\left[\begin{array}{rrr|r} 10.0 & -7.00 & 0.00 & 7.00 \\ -3.0 & 2.09 & 6.00 & 3.91 \\ 5.0 & 1.00 & 5.00 & 6.00 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & -0.01 & 6.00 & 6.01 \\ 0.0 & 2.50 & 5.00 & 2.50 \end{array}\right].$$

Pivoting on -0.01 to 3 significant digits,
we first divide the second row by -0.01 ($\Rightarrow 0 \quad 1 \quad -600 \quad -601$),
then multiply it by 2.5 and and subtract the result from row 3.

$$\left[\begin{array}{rrr|r} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & -0.01 & 6.00 & 6.01 \\ 0.0 & 2.50 & 5.00 & 2.50 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 10.0 & -7.00 & 0.00 & 7.00 \\ 0.0 & -0.01 & 6.00 & 6.01 \\ 0.0 & 0 & -1500 & -1500 \end{array}\right] \Rightarrow$$

Back substituting, we get $z = 1.00$, $y = 1.00$, and $x = 0.00$.
Repeat by interchanging rows 2 and 3 to make the second pivot 2.5.

Exploration: An Introduction to the Analysis of Algorithms

1. We will count the number of operations, one step at a time.

$$\left[\begin{array}{rrr|r} 2 & 4 & 6 & 8 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{array}\right].$$

In this step, we performed 3 operations, $\frac{1}{2} \cdot 4 = 2$, $\frac{1}{2} \cdot 6 = 3$, and $\frac{1}{2} \cdot 8 = 4$.

Note, we don't count $\frac{1}{2} \cdot 2 = 1$ because that's automatic once we decide to multiply by $\frac{1}{2}$.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 \\ -1 & 1 & -1 & 1 \end{array}\right].$$

In this step, we performed 3 operations, $3 \cdot 2 = 6$, $3 \cdot 3 = 9$, and $3 \cdot 4 = 12$, for a total of 6.

Note, we don't count the subtractions (or additions) and again we don't count $3 \cdot 1 = 3$ that created the zero because this is automatic once we decide to multiply by 3.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 \\ -1 & 1 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right].$$

In this step, we performed 3 operations, $1 \cdot 2 = 2$, $1 \cdot 3 = 3$, and $1 \cdot 4 = 4$, for a total of 9.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right].$$

In this step, we performed 2 operations, $(\frac{1}{3})(-3) = -1$ and $\frac{1}{3} \cdot 0 = 0$, for a total of 11.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right].$$

In this step, we performed 2 operations, $(-3)(-1) = 3$ and $(-3)(0) = 0$, for a total of 13.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

In this step, we performed 1 operation, $\frac{1}{5} \cdot 5 = 5$, for a total of 14.

Finally, to complete the back substitution, we need only 3 more operations. See that? Namely, $-x_3$ to find x_2, and $2x_2$, $3x_3$ to find x_1, for a total of 17.

2. To the reduced form given below (as computed above) we have 14 operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Now, we create a zero above the 1 in the second column:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

This required two operations, $(-2)(-1)$ and $(-2)(0)$, for a total of 16.

Finally, we create two zeroes above the 1 in the third column:

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

This required two operations, $(1)(1)$ and $(-5)(1)$, for a total of 18.

So, Gauss-Jordan required 1 more operation than Gaussian elimination.
Therefore, Gauss-Jordan is probably less efficient than Gaussian elimination.

3. (a) There are n operations required to create the first leading 1 because we have to divide every entry in the first row (except a_{11}) by a_{11}.

We don't have to divide a_{11} by a_{11} because the resulting 1 results from the choice itself.

There are n operations required to create the first zero in column 1 because we have to multiply every entry in the first row (except the leading 1) by a_{21}.

We don't have to multiply 1 by $a_2 1$ because the resulting 0 results from the choice itself.

There are n operations required to create each zero in column 1 because we have to multiply every entry in the first row (except the leading 1) by a_{k1}.

There are $n-1$ rows (excluding the first row), so $n+(n-1)n$ operations are required.

Recall, the first n operations created the leading 1 in row 1.

(b) As above, we see it takes $n-1$ operations to create the leading 1 in the second row and then $n-1$ operations to create the zeros in column 2.

Now there are only $n-2$ rows left to create zeroes in since rows 1 and 2 are excluded.

So, $(n-1)+(n-2)(n-1)$ operations are required to create the leading 1 in row 2 and all the zeros beneath it.

Continuing this process, we see:
$[n+(n-1)n]+[(n-1)+(n-2)(n-1)]+\cdots+[2+1\cdot 2]+1.$
This simplifies to: $n^2+(n-1)^2+\cdots+2^2+1^2$.

This simplification follows quickly from the following observation:
$n+(n-1)n = n[1+(n-1)] = n\cdot n = n^2$.

(c) There is 1 operation required to find x_{n-1} (involving x_n).
There are 2 operations required to find x_{n-2} (involving x_n and x_{n-1})
Continuing this reasoning, we get the total: $1+2+\cdots+(n-1)$.

(d) Exercises 45 and 46 in Section 2.4 and Appendix B give us the following equations:
$n^2+(n-1)^2+\cdots+2^2+1^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$ and
$1+2+\cdots+(n-1) = \frac{n(n-1)}{2} = \frac{1}{2}n^2-\frac{1}{2}n$.
Adding these together we get the total number of operations is:
$\left(\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n\right)+\left(\frac{1}{2}n^2-\frac{1}{2}n\right) = \frac{1}{3}n^3+n^2-\frac{1}{3}n.$
So, for large values of n the total number of operations required is $T(n)\approx\frac{1}{3}n^3$.

4. As we saw in Exercise 2, we get the same $n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$ operations to create row echelon form.

To create the 1 zero above row 2 requires $1(n-1)$ operations.
To create the 2 zeroes above row 3 requires $2(n-2)$ operations.
Continuing with this reasoning, we see it takes $(n-1)+2(n-2)+\cdots+(n-1)(n-(n-1))$ operations to create all the necessary zeroes. Now note:

$$\begin{aligned}&(n-1) + 2(n-2) + \cdots + (n-1)(n-(n-1))\\ &\quad = (1\cdot n + 2\cdot n + \cdots + (n-1)\cdot n) - (1^2 + 2^2 + \cdots + (n-1)^2)\\ &\quad = n(1 + 2 + \cdots + (n-1)) - (1^2 + 2^2 + \cdots + (n-1)^2).\end{aligned}$$

Similar to Exercise 3, we have the following equations:
$(n-1)^2 + \cdots + 2^2 + 1^2 = \frac{n(n-1)(2n-1)}{6} = \frac{1}{3}\,n^3 - \frac{1}{2}\,n^2 + \frac{1}{6}\,n$ and
$1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} = \frac{1}{2}\,n^2 - \frac{1}{2}\,n$.

Putting these together we get the total number of operations is:
$\left(\frac{1}{3}\,n^3 + \frac{1}{2}\,n^2 + \frac{1}{6}\,n\right) + n\,\left(\frac{1}{2}\,n^2 - \frac{1}{2}\,n\right) - \left(\frac{1}{3}\,n^3 - \frac{1}{2}\,n^2 + \frac{1}{6}\,n\right) = \frac{1}{2}\,n^3 + \frac{1}{2}\,n^2.$

So, for large values of n the total number of operations required is $T(n) \approx \frac{1}{2}\,n^3$.

2.3 Spanning Sets and Linear Independence

1. As in Example 2.18, we want to find scalars x and y such that:

$x\begin{bmatrix}1\\-1\end{bmatrix}+y\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ Expanding, we obtain the system: $\begin{array}{rcrcr} x & + & 2y & = & 1\\ -x & - & y & = & 2\end{array}$

We then row reduce the associated augmented matrix: $\left[\begin{array}{rr|r}1 & 2 & 1\\-1 & -1 & 2\end{array}\right]\longrightarrow\left[\begin{array}{rr|r}1 & 0 & -5\\0 & 1 & 3\end{array}\right]$

So the solution is $x=-5$, $y=2$, and the linear combination is $-5\begin{bmatrix}1\\-1\end{bmatrix}+3\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$.

2. As in Example 2.18, we want to find scalars x and y such that:

$$x\begin{bmatrix}-1\\3\end{bmatrix}+y\begin{bmatrix}2\\-6\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

That is, we want to solve the system of linear equations

$$\begin{aligned}-1x+2y&=1\\3x-6y&=2\end{aligned}$$

Row-reducing the augmented matrix for the system gives

$$\left[\begin{array}{rr|r}-1 & 2 & 1\\3 & -6 & 2\end{array}\right]\xrightarrow{3R_1+R_3}\left[\begin{array}{rr|r}-1 & 2 & 1\\0 & 0 & -1\end{array}\right]$$

Since $0\neq 1$, this system clearly has no solution. So, what do we conclude?
We conclude that $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

3. As in Example 2.18, we want to find scalars x and y such that:

$x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ Expanding, we obtain the system: $\begin{array}{rcrcr} x & & & = & 1\\ x & + & y & = & 2\\ & & y & = & 3\end{array}$

Since $x=1$ and $y=3$ implies $x+y\neq 2$, this system clearly has no solution.
Therefore, $\mathbf{v}$ is not a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

4. As in Example 2.18, we want to find scalars x and y such that:

$$\begin{array}{rcrcr} x & & & = & 3\\ x & + & y & = & 1\\ & & y & = & -2\end{array}$$

Since $x=3$ and $y=-2$ implies $x+y=1$, those values of x and y are clearly the solution.

That is, the solution is $x=3$, $y=-2$, and the linear combination is $\mathbf{v}=3\mathbf{u}_1-2\mathbf{u}_2$.

5. Similar to Example 2.18, we want to find scalars x, y, and z such that:

$$x\begin{bmatrix}1\\1\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\1\end{bmatrix}+z\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$ Expanding, we obtain the system: $$\begin{array}{rcrcrcr} x & & & + & z & = & 1\\ x & + & y & & & = & 2\\ & & y & + & z & = & 3\end{array}$$

Since $z = 1$ and $y = 2$ implies $y + z = 3$, the solution is $x = 0$, $y = 2$, $z = 1$.

Row reduce the augmented matrix to confirm that: $\left[\begin{array}{ccc|c}1&0&1&1\\1&1&0&2\\0&1&1&3\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&2\\0&0&1&1\end{array}\right]$

So the solution is $x = 0$, $y = 2$, $z = 1$, and the linear combination is $2\begin{bmatrix}0\\1\\1\end{bmatrix}+1\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$.

6. Similar to Example 2.18, we want to find scalars x, y, and z such that:

$$\begin{array}{rcrcrcr} 1.0x & + & 2.4y & + & 1.2z & = & 2.2\\ 0.5x & + & 1.2y & - & 2.3z & = & 4.0\\ -0.5x & + & 3.1y & + & 4.8z & = & -2.2\end{array}$$

The obvious solution is $x = 1$, $y = 1$, $z = -1$.

The linear combination is $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3$. That is, $\begin{bmatrix}1.0\\0.5\\-0.5\end{bmatrix}+\begin{bmatrix}2.4\\1.2\\3.1\end{bmatrix}+-\begin{bmatrix}1.2\\-2.3\\4.8\end{bmatrix}=\begin{bmatrix}2.2\\4.0\\-2.2\end{bmatrix}$.

Note: We should always look for an easy or obvious solution first.

7. Applying Theorem 2.4, we check to see if $[A|\mathbf{b}]$ is consistent. Why?
Theorem 2.4 says $[A|\mathbf{b}]$ is consistent $\Leftrightarrow$ $\mathbf{b}$ is a linear combination of the columns of A.
That is exactly what is required for $\mathbf{b}$ to be in the span of the columns of A.

So we row reduce $[A|\mathbf{b}]=\left[\begin{array}{cc|c}1&2&5\\3&4&6\end{array}\right]\longrightarrow\left[\begin{array}{cc|c}1&0&-4\\0&1&\frac{9}{2}\end{array}\right]$ to see that it is consistent.

What do we conclude? The vector $\mathbf{b}$ is in the span of the columns of A.

In particular, the solution tells us the linear combination is $-4\begin{bmatrix}1\\3\end{bmatrix}+\frac{9}{2}\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}5\\6\end{bmatrix}$.

8. Applying Theorem 2.4, we check to see if $[A|\mathbf{b}]$ is consistent. Why?
Theorem 2.4 says $[A|\mathbf{b}]$ is consistent $\Leftrightarrow$ $\mathbf{b}$ is a linear combination of the columns of A.
That is exactly what is required for $\mathbf{b}$ to be in the span of the columns of A.

Row reduce $[A|\mathbf{b}]=\left[\begin{array}{ccc|c}1&2&3&10\\4&5&6&11\\7&8&9&12\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&-1&-\frac{28}{3}\\0&1&2&\frac{29}{3}\\0&0&0&0\end{array}\right]$ to see that it is consistent.

What do we conclude? The vector $\mathbf{b}$ is in the span of the columns of A.

The solution tells us one possible linear combination is $-\frac{28}{3}\begin{bmatrix}1\\4\\7\end{bmatrix}+\frac{29}{3}\begin{bmatrix}2\\5\\8\end{bmatrix}=\begin{bmatrix}10\\11\\12\end{bmatrix}$.

9. As in Example 2.19, we must show $x\begin{bmatrix}1\\1\end{bmatrix}+y\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$ can always be solved.

The augmented matrix is $\left[\begin{array}{cc|c}1&1&a\\1&-1&b\end{array}\right]$, and row reduction produces:

$$\left[\begin{array}{cc|c}1&1&a\\1&-1&b\end{array}\right]\xrightarrow{R_1+R_2}\left[\begin{array}{cc|c}2&0&a+b\\1&-1&b\end{array}\right]\underset{-R_2}{\xrightarrow{1/2R_1}}\left[\begin{array}{cc|c}1&0&(a+b)/2\\0&1&-b\end{array}\right]\xrightarrow{R_2+R_1}\left[\begin{array}{cc|c}1&0&(a+b)/2\\-1&1&(a-b)/2\end{array}\right]$$

We see that $x=(a+b)/2$ and $y=(a-b)/2$, so for any choice of a and b we have

$$\left(\frac{a+b}{2}\right)\begin{bmatrix}1\\1\end{bmatrix}+\left(\frac{a-b}{2}\right)\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}\quad\text{Check this!}$$

10. As in Example 2.19, we must show $x\begin{bmatrix}2\\-1\end{bmatrix}+y\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$ can always be solved.

So, we row-reduce the associated augmented matrix.

$$\left[\begin{array}{cc|c}2&-1&a\\-1&2&b\end{array}\right]\xrightarrow{R_2+R_1}\left[\begin{array}{cc|c}1&1&a+b\\-1&2&b\end{array}\right]\xrightarrow{R_2+R_1}\left[\begin{array}{cc|c}1&1&a+b\\0&3&a+2b\end{array}\right]\underset{\frac{1}{3}R_2}{\xrightarrow{R_1-\frac{1}{3}R_2}}\left[\begin{array}{cc|c}1&0&(2a+b)/3\\0&1&(a+2b)/3\end{array}\right]$$

Therefore, $x=(2a+b)/3$ and $y=(a+2b)/3$.

So, for any choice of a and b, $\left(\frac{2a+b}{3}\right)\begin{bmatrix}2\\-1\end{bmatrix}+\left(\frac{a+2b}{3}\right)\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$. Check this!

11. As in Example 2.19, show $x\begin{bmatrix}1\\0\\1\end{bmatrix}+y\begin{bmatrix}1\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$ can be solved.

The augmented matrix is $\left[\begin{array}{ccc|c}1&1&0&a\\0&1&1&b\\1&0&1&c\end{array}\right]$, and row reduction produces:

$$\left[\begin{array}{ccc|c}1&1&0&a\\0&1&1&b\\1&0&1&c\end{array}\right]\xrightarrow{R_3-R_1+R_2}\left[\begin{array}{ccc|c}1&1&0&a\\0&1&1&b\\0&0&2&-a+b+c\end{array}\right]\xrightarrow{\frac{1}{2}R_3}\left[\begin{array}{ccc|c}1&1&0&a\\0&1&1&b\\0&0&1&(-a+b+c)/2\end{array}\right]$$

$$\xrightarrow{R_2-R_3}\left[\begin{array}{ccc|c}1&1&0&a\\0&1&0&(a+b-c)/2\\0&0&1&(-a+b+c)/2\end{array}\right]\xrightarrow{R_1-R_2}\left[\begin{array}{ccc|c}1&0&0&(a-b+c)/2\\0&1&0&(a+b-c)/2\\0&0&1&(-a+b+c)/2\end{array}\right]$$

We see that $x=(a-b+c)/2$, $y=(a+b-c)/2$, and $z=(-a+b+c)/2$.
So for any choice of a, b, and c we have:

$$\left(\frac{a-b+c}{2}\right)\begin{bmatrix}1\\0\\1\end{bmatrix}+\left(\frac{a+b-c}{2}\right)\begin{bmatrix}1\\1\\0\end{bmatrix}+\left(\frac{-a+b+c}{2}\right)\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}\quad\text{Check this!}$$

12. We want to write any vector as a linear combination of the given vectors.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + z\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

So, we row-reduce the associated augmented matrix:

$$\left[\begin{array}{rrr|c} 1 & 1 & -1 & a \\ 1 & 1 & 1 & b \\ 1 & -1 & 1 & c \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{rrr|r} 1 & 1 & -1 & a \\ 0 & 0 & 2 & b-a \\ 0 & -2 & 2 & c-a \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & 1 & -1 & a \\ 0 & -2 & 2 & c-a \\ 0 & 0 & 2 & b-a \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{rrr|r} 1 & 1 & -1 & a \\ 0 & 1 & -1 & (a-c)/2 \\ 0 & 0 & 1 & (b-a)/2 \end{array}\right] \xrightarrow[R_2+R_3]{R_1-R_2} \left[\begin{array}{rrr|r} 1 & 0 & 0 & (a+c)/2 \\ 0 & 1 & 0 & (b-c)/2 \\ 0 & 0 & 1 & (b-a)/2 \end{array}\right]$$

So, we can write any vector as a linear combination as follows:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{a+c}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{b-c}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{b-a}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

13. We should describe the span of the given vectors (a) geometrically and (b) algebraically.

(a) Geometrically, we can see that the set of all linear combinations of $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is just the line through the origin with $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ as direction vector.

Why do we not have to consider $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$? Because $\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(b) Algebraically, the vector equation of this line is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} t$.

That is just another way of saying that $\begin{bmatrix} x \\ y \end{bmatrix}$ is in the span of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
Suppose we want to obtain the general equation of this line.
One method is to use the system of equations arising from the vector equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} t \Rightarrow \begin{array}{rcr} x & = & -t \\ y & = & 2t \end{array} \text{ So } y = 2(-x) = -2x \Rightarrow 2x + y = 0.$$

14. We should describe the span of the given vectors (a) geometrically and (b) algebraically.

(a) Geometrically, we can see that the set of all linear combinations of $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$ is just the line through the origin with $\begin{bmatrix}3\\4\end{bmatrix}$ as direction vector.

Why do we not have to consider $\begin{bmatrix}0\\0\end{bmatrix}$? Because $\begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}3\\4\end{bmatrix}$.

(b) Algebraically, the vector equation of this line is $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}t$.

That is just another way of saying that $\begin{bmatrix}x\\y\end{bmatrix}$ is in the span of $\begin{bmatrix}3\\4\end{bmatrix}$.
Suppose we want to obtain the general equation of this line.
One method is to use the system of equations arising from the vector equation:

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}t \Rightarrow \begin{array}{rcl} x &=& 3t \\ y &=& 4t \end{array} \quad \text{So } y = 4\left(\tfrac{1}{3}x\right) \Rightarrow 3y = 4x \Rightarrow 4x - 3y = 0.$$

15. We should describe the span of the given vectors (a) geometrically and (b) algebraically.

(a) Geometrically, we can see that the set of all linear combinations of $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}3\\2\\-1\end{bmatrix}$ is just the plane through the origin with $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}3\\2\\-1\end{bmatrix}$ as direction vectors.

(b) Algebraically, the vector equation of this plane is $\begin{bmatrix}x\\y\\z\end{bmatrix} = s\begin{bmatrix}1\\2\\0\end{bmatrix} + t\begin{bmatrix}3\\2\\-1\end{bmatrix}$.

That is just another way of saying that $\begin{bmatrix}x\\y\\z\end{bmatrix}$ is in the span of $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}3\\2\\-1\end{bmatrix}$.
Suppose we want to obtain the general equation of this plane.
One method is to use the system of equations arising from the vector equation:

$$\begin{array}{rcrcl} s &+& 3t &=& x \\ 2s &+& 2t &=& y \\ && -t &=& z \end{array} \Rightarrow \left[\begin{array}{rr|c} 1 & 3 & x \\ 2 & 2 & y \\ 0 & -1 & z \end{array}\right] \longrightarrow \left[\begin{array}{rr|c} 1 & 3 & x \\ 0 & -4 & -2x+y \\ 0 & 0 & (2x-y+4z)/4 \end{array}\right]$$

We know this system is consistent, since $\begin{bmatrix}x\\y\\z\end{bmatrix}$ *is* in the span of $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}3\\2\\-1\end{bmatrix}$.

So, we *must* have $2x - y + 4z = 0$, giving us the general equation we seek.

Note: Both $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}3\\2\\-1\end{bmatrix}$ are orthogonal to $\begin{bmatrix}2\\-1\\4\end{bmatrix}$. Should they be?

16. We should describe the span of the given vectors (a) geometrically and (b) algebraically.

(a) Geometrically, we can see that the set of all linear combinations of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is just the plane through the origin with $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ as direction vectors.

Q: Why do we get to ignore the third vector, $\mathbf{v}_3 = [0, -1, 1]$?
A: Since $\mathbf{v}_3 = -\mathbf{v}_1 + \mathbf{v}_2$, it does not affect the span. Why not?

(b) Algebraically, the vector equation of this plane is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

That is just another way of saying that $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the span of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Suppose we want to obtain the general equation of this plane.
One method is to use the system of equations arising from the vector equation:

$$\begin{array}{rcrcl} s & - & t & = & x \\ & & t & = & y \\ -s & & & = & z \end{array} \Rightarrow \left[\begin{array}{rr|c} 1 & -1 & x \\ 0 & 1 & y \\ -1 & 0 & z \end{array}\right] \longrightarrow \left[\begin{array}{rr|c} 1 & -1 & x \\ 0 & 1 & y \\ 0 & 0 & x+y+z \end{array}\right]$$

We know this system is consistent, since $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *is* in the span of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

So, we *must* have $x + y + z = 0$, giving us the general equation we seek.

Note: Both $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Should they be?

17. Since the three points $(1, 0, 3)$, $(-1, 1, -3)$, and $(0, 0, 0)$ must lie in the plane, $(1, 0, 3)$ and $(-1, 1, -3)$ must satisfy the equation of a plane through the origin $ax + by + cz = 0$.

We substitute the two nonzero points into $ax + by + cz = 0$ to create a homogenous system:

$$\begin{array}{rcrcrcl} a & & & + & 3c & = & 0 \\ -a & + & b & - & 3c & = & 0 \end{array} \Rightarrow \left[\begin{array}{rrr|c} 1 & 0 & 3 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{rrr|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

So, we have $b = 0$ and $a = -3c$.

Letting $c = -1 \Rightarrow a = 3$ and $b = 0$ yields $3x + 0y - z = 3x - z = 0$ is the general equation.

Q: Does the free variable c imply infinitely many planes contain these three points?
A: Hint: We can divide the general solution $-3cx + cz = 0$ by $-c \neq 0 \Rightarrow 3x - z = 0$.

18. To show $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are in span($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$), we must show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ can be written as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Q: Can we simply let $\mathbf{u} = \mathbf{u}$, $\mathbf{v} = \mathbf{v}$, and $\mathbf{w} = \mathbf{w}$?
A: Yes and no. Technically, we have to write the linear combination using all the vectors:

$\mathbf{u} = \mathbf{u} + 0\mathbf{v} + 0\mathbf{w}$, $\mathbf{v} = 0\mathbf{u} + \mathbf{v} + 0\mathbf{w}$, and $\mathbf{w} = 0\mathbf{u} + 0\mathbf{v} + \mathbf{w}$.

In the future, we will only write vectors with nonzero scalars in linear combinations.

19. To show $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are in span($\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{v}+\mathbf{w}$), we must show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ can be written as a linear combination of $\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{v}+\mathbf{w}$.

Q: Can we simply let $\mathbf{u} = \mathbf{u}$, $\mathbf{v} = \mathbf{v}$, and $\mathbf{w} = \mathbf{w}$?
A: No. Why not? These vectors, except for $\mathbf{u}$, are not explicitly listed in the spanning set.

Instead, we need linear combinations of $\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{v}+\mathbf{w}$ that yield $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. So:

$\mathbf{u} = \mathbf{u}$, $\mathbf{v} = -\mathbf{u} + (\mathbf{u}+\mathbf{v})$, and $\mathbf{w} = -(\mathbf{u}+\mathbf{v}) + (\mathbf{u}+\mathbf{v}+\mathbf{w})$.

Note: We have now shown that we can use $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. How?

20. Both span(S) and span(T) are contained in $\mathbb{R}^n$. That is, span(S) $\subseteq \mathbb{R}^n$, span(T) $\subseteq \mathbb{R}^n$.

(a) We need to show that any vector s in span(S) is also in span(T).
The idea is simple: Let the scalars for all the vectors $\mathbf{u}_{k+1}$ through $\mathbf{u}_m$ equal zero.
In symbols, we write:
$\mathbf{s} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k + 0\mathbf{u}_{k+1} + \cdots + 0\mathbf{u}_m$.
Therefore, any vector s in span(S) is in also span(T) as we were to show.

(b) We are told to *deduce* that if $\mathbb{R}^n$ =span(S), then $\mathbb{R}^n$ =span(T).
The word *deduce* tells us to use the basic properties of sets in our proof.
The basic property of sets we will use is: if $V \subseteq W$ and $W \subseteq V$, then $V = W$.
In the statement of the problem, we were told that span(T) $\subseteq \mathbb{R}^n$.
If we can show $\mathbb{R}^n \subseteq$ span(T), then we can deduce $\mathbb{R}^n =$ span(T).
From (a), we have span(S) $\subseteq$ span(T). For (b), we suppose span(S) $= \mathbf{R}^n$.
Therefore, we have span(S) $= \mathbb{R}^n \subseteq$ span(T) and deduce $\mathbb{R}^n =$ span(T).

21. When proving something for n, first let $n = 1$ or 2 to look for the underlying pattern.

Assume that there are only two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ and two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
We are told $\mathbf{w}$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. So: $\mathbf{w} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2$.
We are also told that both $\mathbf{u}_1$ and $\mathbf{u}_2$ are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.
So, we have both: $\mathbf{u}_1 = v_{11}\mathbf{v}_1 + v_{12}\mathbf{v}_2$ and $\mathbf{u}_2 = v_{21}\mathbf{v}_1 + v_{22}\mathbf{v}_2$.
We need to show these assumptions imply $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. How?
Let $\mathbf{u}_1 = u_{1_1}\mathbf{v}_1 + u_{1_2}\mathbf{v}_2$ and $\mathbf{u}_2 = u_{2_1}\mathbf{v}_1 + u_{2_2}\mathbf{v}_2$ in $\mathbf{w} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2$.
This substitution yields: $\mathbf{w} = w_1(u_{1_1}\mathbf{v}_1 + u_{1_2}\mathbf{v}_2) + w_2(u_{2_1}\mathbf{v}_1 + u_{2_2}\mathbf{v}_2)$.
It is now obvious that $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Why?
Observe that this reasoning holds for any n and proceed to the proof.

(a) Let $\mathbf{w} = w_1\mathbf{u}_1 + w_1\mathbf{u}_2 + \cdots + w_1\mathbf{u}_k$,
and assume that each $\mathbf{u}_i$ is a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.
Then each $\mathbf{u}_i = u_{i_1}\mathbf{v}_1 + u_{i_2}\mathbf{v}_2 + \cdots + u_{i_m}\mathbf{v}_m$, and

$$\begin{aligned}
\mathbf{w} &= w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k \\
&= w_1\left(u_{1_1}\mathbf{v}_1 + u_{1_2}\mathbf{v}_2 + \cdots + u_{1_m}\mathbf{v}_m\right) + w_2\left(u_{2_1}\mathbf{v}_1 + u_{2_2}\mathbf{v}_2 + \cdots + u_{2_m}\mathbf{v}_m\right) + \cdots \\
&\qquad \cdots + w_k\left(u_{k_1}\mathbf{v}_1 + u_{k_2}\mathbf{v}_2 + \cdots + u_{k_m}\mathbf{v}_m\right) \\
&= \left(w_1u_{1_1} + w_2u_{2_1} + \cdots + w_ku_{k_1}\right)\mathbf{v}_1 + \left(w_1u_{1_2} + w_2u_{2_2} + \cdots + w_ku_{k_2}\right)\mathbf{v}_2 + \cdots \\
&\qquad \cdots + \left(w_1u_{1_m} + w_2u_{2_m} + \cdots + w_ku_{k_m}\right)\mathbf{v}_m \\
&= w_1'\mathbf{v}_1 + w_2'\mathbf{v}_2 + \cdots + w_m'\mathbf{v}_m.
\end{aligned}$$

So, any vector $\mathbf{w} \in \operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$ is also in $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$,
and $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) \subseteq \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$
.

(b) Suppose that in addition to (a), each $\mathbf{v}_j$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.
Let $\mathbf{w}$ be an arbitrary vector in $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$.
Then $\mathbf{w} = w_1'\mathbf{v}_1 + w_2'\mathbf{v}_2 + \cdots + w_m'\mathbf{v}_m$, but each $\mathbf{v}_j = v_{j_1}\mathbf{u}_1 + v_{j_2}\mathbf{u}_2 + \cdots + v_{j_k}\mathbf{u}_k$, so

$$\begin{aligned}
\mathbf{w} &= w_1'\left(v_{1_1}\mathbf{u}_1 + v_{1_2}\mathbf{u}_2 + \cdots + v_{1_k}\mathbf{u}_k\right) + w_2'\left(v_{2_1}\mathbf{u}_1 + v_{2_2}\mathbf{u}_2 + \cdots + v_{2_k}\mathbf{u}_k\right) + \cdots \\
&\qquad \cdots + w_m'\left(v_{m_1}\mathbf{u}_1 + v_{m_2}\mathbf{u}_2 + \cdots + v_{m_k}\mathbf{u}_k\right) \\
&= \left(w_1'v_{1_1} + w_2'v_{2_1} + \cdots + w_m'v_{m_1}\right)\mathbf{u}_1 + \left(w_1'v_{1_2} + w_2'v_{2_2} + \cdots + w_m'v_{m_2}\right)\mathbf{u}_2 + \cdots \\
&\qquad \cdots + \left(w_1'v_{1_k} + w_2'v_{2_k} + \cdots + w_m'v_{m_k}\right)\mathbf{u}_k \\
&= w_1\mathbf{u}_1 + w_1\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k.
\end{aligned}$$

So, any vector $\mathbf{w} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ is also in $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$,
and $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) \subseteq \operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$.
But we already had $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) \subseteq \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$,
so $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$.

(c) Need only show $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are linear combinations of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

That's obvious since $\mathbf{e}_1 = \mathbf{v}_1$, $\mathbf{e}_2 = \mathbf{v}_2 - \mathbf{v}_1$, and $\mathbf{e}_3 = \mathbf{v}_3 - \mathbf{v}_2$. Why is that enough?
Because then we have $\mathbb{R}^3 = \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

22. The vectors $\begin{bmatrix}3\\1\\4\end{bmatrix}$ and $\begin{bmatrix}-2\\1\\-1\end{bmatrix}$ are not scalar multiples of one another, so are linearly independent.

23. Since there is no obvious dependence relation here, we follow Example 2.23.

Find scalars c_1, c_2, and c_3 such that: $c_1\begin{bmatrix}1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\2\\3\end{bmatrix}+c_3\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Form the linear system, its associated augmented matrix, and row reduce to solve:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0\\ c_1 + 2c_2 - c_3 &= 0\\ c_1 + 3c_2 + 2c_3 &= 0 \end{aligned} \quad\Rightarrow\quad \left[\begin{array}{ccc|c}1&1&1&0\\1&2&-1&0\\1&3&2&0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

Since $c_1 = c_2 = c_3 = 0$ is the unique solution, the vectors are linearly independent.

24. There is no obvious dependence relation, so we follow Example 2.23.

We want scalars c_1, c_2, c_3 so that $c_1\begin{bmatrix}3\\2\\2\end{bmatrix}+c_2\begin{bmatrix}2\\1\\3\end{bmatrix}+c_3\begin{bmatrix}1\\2\\-6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Form the linear system, its associated augmented matrix, and row-reduce to solve:

$$\begin{aligned} 3c_1 + 2c_2 + c_3 &= 0\\ 2c_1 + c_2 + 2c_3 &= 0\\ 2c_1 + 3c_2 - 6c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c}3&2&1&0\\2&1&2&0\\2&3&-6&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&0&3&0\\0&1&-4&0\\0&0&0&0\end{array}\right]$$

This system has a nontrivial solution. So, the vectors are linearly dependent.

In fact, from the reduced matrix we see one solution is $-3\begin{bmatrix}3\\2\\2\end{bmatrix}+4\begin{bmatrix}2\\1\\3\end{bmatrix}+\begin{bmatrix}1\\2\\-6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Q: Are there other solutions? Why or why not?

25. The vectors $\mathbf{v}_1=\begin{bmatrix}0\\1\\2\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}2\\1\\3\end{bmatrix}$ and $\mathbf{v}_3=\begin{bmatrix}2\\0\\1\end{bmatrix}$ are linearly dependent.

This can be determined by inspection because $\mathbf{v}_1-\mathbf{v}_2+\mathbf{v}_3=\mathbf{0}$.

26. There is no obvious dependence relation, so follow Example 2.23.

We want scalars c_1, c_2, c_3, c_4 such that

$$c_1 \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Form the linear system and its associated augmented matrix, and row-reduce to solve:

$$\begin{array}{rcrcrcrcl} 2c_1 & - & 5c_2 & + & 4c_3 & + & 3c_4 & = & 0 \\ -3c_1 & + & c_2 & + & 3c_3 & + & c_4 & = & 0 \\ 7c_1 & + & c_2 & & & + & 5c_4 & = & 0 \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 2 & -5 & 4 & 3 & 0 \\ -3 & 1 & 3 & 1 & 0 \\ 7 & 1 & 0 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 100/151 & 0 \\ 0 & 1 & 0 & 55/151 & 0 \\ 0 & 0 & 1 & 132/151 & 0 \end{array}\right]$$

Since the system has a nontrivial solution, the vectors are linearly dependent.

In fact, from the reduced matrix we can read off the fact that

$$100 \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} + 55 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} + 132 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} - 151 \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

27. The vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are linearly dependent.

This can be determined by inspection because $\mathbf{v}_3$ is the zero vector. Why is that enough?
Because $0\mathbf{v}_1 + 0\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$.
Any set of vectors containing the zero vector is linearly dependent. Why?

28. There is no obvious dependence relation, so follow Example 2.23.
We want scalars c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 4 \\ 3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Form the linear system and its associated augmented matrix, and row-reduce to solve:

$$\begin{array}{rcrcrcl} c_1 & + & 2c_2 & + & 3c_3 & = & 0 \\ 4c_1 & + & 3c_2 & + & 2c_3 & = & 0 \\ 3c_1 & + & 4c_2 & + & 2c_3 & = & 0 \\ & & 5c_2 & + & 4c_3 & = & 0 \end{array} \Rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 3 & 0 \\ 4 & 3 & 2 & 0 \\ 3 & 4 & 2 & 0 \\ 0 & 5 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The only solution to the system is thus $c_1 = c_2 = c_3 = 0$.
So, these vectors are linearly independent.

29. Since there is no obvious dependence relation here, we follow Example 2.23.

Find scalars c_1, c_2, c_3, c_4 such that: $c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Form the linear system, its associated augmented matrix, and row reduce to solve:

$$\begin{array}{rcrcrcrcl} c_1 & - & c_2 & + & c_3 & & & = & 0 \\ -c_1 & + & c_2 & & & + & c_4 & = & 0 \\ c_1 & & & + & c_3 & - & c_4 & = & 0 \\ & & c_2 & - & c_3 & + & c_4 & = & 0 \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Since $c_1 = c_2 = c_3 = c_4 = 0$ is the unique solution, the vectors are linearly independent.

30. The vectors $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ are linearly independent.

This can be determined by inspection. How?
To create a 0 in the first component, the coefficient of $\mathbf{v}_4$ must be 0. Why?
Given that, a 0 in the second component forces the coefficient of $\mathbf{v}_3$ to be 0.
Given those two facts, a 0 in the third component forces the coefficient of $\mathbf{v}_2$ to be 0.
And finally, given all that, a 0 in the fourth component forces the coefficient of $\mathbf{v}_1$ to be 0.

To verify this argument, we follow Example 2.23.

Find scalars c_1, c_2, c_3, and c_4 such that: $c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Form the linear system, its associated augmented matrix, and row reduce to solve:

$$\begin{array}{rcrcrcrcl} & & & & & & c_4 & = & 0 \\ & & & & 2c_3 & + & c_4 & = & 0 \\ & & 3c_2 & + & 2c_3 & + & c_4 & = & 0 \\ 4c_1 & + & 3c_2 & + & 2c_3 & + & c_4 & = & 0 \end{array} \Rightarrow \left[\begin{array}{rrrr|r} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Since $c_1 = c_2 = c_3 = c_4 = 0$ is the unique solution, the vectors are linearly independent.

31. The vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$ are linearly dependent.

This can be determined by inspection because $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$.

32. We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

Construct a matrix A with these vectors as its rows and row-reduce it to echelon form.

$$\begin{bmatrix} 3 & 1 & 4 \\ -2 & 1 & -1 \end{bmatrix} \xrightarrow{R_2+R_2} \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -1 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \end{bmatrix}$$

Now, A is in row-echelon form and has no zero rows. So, A has rank 2.
Therefore, the vectors are linearly independent.

33. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How? In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \underset{\longrightarrow}{\overset{R_2'=R_2-R_1}{R_3'=R_3-R_1}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{R_3''=R_3'+2R_2'} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

We can stop. Why? We have put A into row echelon form. How can we tell?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) = 3$.
What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. How?

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.
But that implies the following: If $\text{rank}(A) \geq m$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

In this case, therefore, we argue as follows:
Since $\text{rank}(A) \geq 3$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Does the agree with the solution we found in Exercise 23? It should.
Which method was easier for this Exercise? Why?

34. Construct a matrix with these vectors as its rows and row-reduce it to echelon form.

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & -6 \end{bmatrix} \xrightarrow{R_1'=R_1-R_2} \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \\ 1 & 2 & -6 \end{bmatrix} \xrightarrow[R_3'=R_3-R_1']{R_2'=R_2-2R_1'} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & 1 & -5 \end{bmatrix} \xrightarrow{R_3''=R_2'+R_3'} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

We can stop. Why? We have created a zero row. What conclusion may we draw?
We may conclude $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. Why?

We have

$$0 = R_3'' = R_2' + R_3' = R_2 - 2R_1' + R_3 - R_1' = R_2 + R_3 - 3(R_1 - R_2) = -3R_1 + 4R_2 + R_3$$

So, in terms of the original vectors, we have:

$$-3[3,\ 2,\ 2] + 4[2,\ 1,\ 3] + [1,\ 2,\ -6] = [0,\ 0,\ 0]$$

Compare this result to Exercise 24. Does it agree?

35. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How?
In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form.
Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow[R_2'=R_1]{R_1'=R_2} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3'=R_3-R_1'} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{R_3''=R_3'+R_2'} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We can stop. Why? We have created a zero row. What does that tell us?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) = 2$.
What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. How?

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.
So, since $\text{rank}(A) = 2 < 3$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

From the row reduction, we see: $\mathbf{0} = R_3'' = R_3' + R_2' = (R_3' - R_1') + R_2' = R_3 - R_2 + R_1$.

This equation yields a dependence relation among the original vectors:

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Compare this result to Exercise 25. Does it agree?

36. Construct a matrix with these vectors as its rows and row-reduce it to echelon form.

$$\begin{bmatrix} 2 & -3 & 7 \\ -5 & 1 & 1 \\ 4 & 3 & 0 \\ 3 & 1 & 5 \end{bmatrix} \xrightarrow{R_2'=R_2+\frac{5}{2}R_1} \xrightarrow{R_3'=R_3-2R_1} \xrightarrow{R_4'=R_4-\frac{3}{2}R_1} \begin{bmatrix} 2 & -3 & 7 \\ 0 & -13/2 & 37/2 \\ 0 & 9 & -14 \\ 0 & 11/2 & -11/2 \end{bmatrix}$$

$$\xrightarrow{R_3''=R_3'+\frac{18}{13}R_2'} \xrightarrow{R_4''=R_4'+\frac{11}{13}R_2'} \begin{bmatrix} 2 & -3 & 7 \\ 0 & -13/2 & 37/2 \\ 0 & 0 & 151/13 \\ 0 & 0 & 132/13 \end{bmatrix} \xrightarrow{R_4'''=R_4''-\frac{132}{151}R_3''} \begin{bmatrix} 2 & -3 & 7 \\ 0 & -13/2 & 37/2 \\ 0 & 0 & 151/13 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echelon form.
Since it has a zero row, the original vectors are linearly dependent:

$$\begin{aligned} 0 = R_4''' &= R_4'' - \frac{132}{151}R_3'' \\ &= R_4' + \frac{11}{13}R_2' - \frac{132}{151}R_3' - \frac{132}{151}\cdot\frac{18}{13}R_2' \\ &= R_4' - \frac{132}{151}R_3' - \frac{55}{151}R_2' \\ &= R_4 - \frac{3}{2}R_1 - \frac{132}{151}R_3 + \frac{264}{151}R_1 - \frac{55}{151}R_2 - \frac{55\cdot 5}{151\cdot 2}R_1 \\ &= -\frac{100}{151}R_1 - \frac{55}{151}R_2 - \frac{132}{151}R_3 + R_4 \end{aligned}$$

So:

$$-\frac{100}{151}[2,\ -3,\ 7] - \frac{55}{151}[-5,\ 1,\ 1] - \frac{132}{151}[4,\ 3,\ 0] + [3,\ 1,\ 5] = [0,\ 0,\ 0],\ \text{or}$$
$$100[2,\ -3,\ 7] + 55[-5,\ 1,\ 1] + 132[4,\ 3,\ 0] - 151[3,\ 1,\ 5] = [0,\ 0,\ 0]$$

Q: Does this answer agree with Exercise 26?
Note: Linear dependence follows immediately from Theorem 2.8. Why?

37. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How? In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

We can stop. Why? The matrix A has a zero row. What does that tell us?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) \le 2$.
What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. How?

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.
So, since $\text{rank}(A) \le 2 < 3$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

Furthermore, we have the obvious dependence relation among the original vectors:

$$0\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Compare this result to Exercise 37. Does it agree?

This can be determined by inspection because $\mathbf{v}_3$ is the zero vector. Why is that enough?
Any set of vectors containing the zero vector is linearly dependent. Why?

38. Construct a matrix with these vectors as its rows and row-reduce it to echelon form.

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 2 & 3 & 4 & 5 \\ 3 & 2 & 2 & 4 \end{bmatrix} \xrightarrow[R_3'=R_3-3R_1]{R_2'=R_2-2R_1} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & -5 & -2 & 5 \\ 0 & -10 & -7 & 4 \end{bmatrix} \xrightarrow{R_3''=R_3'-2R_2'} \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & -5 & -2 & 5 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

This matrix is in row-echelon form and has no zero rows, so has rank 3.
Thus the vectors are linearly independent.

Does the agree with the solution we found in Exercise 28? It should.
Which method was easier for this Exercise? Why?

39. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How? In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_3'=R_3-R_1]{R_2'=R_2+R_1} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_4'=R_4-R_3'} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow[R_3''=R_2']{R_2''=R_3'} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_4''=R_4'+R_3'} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

We can stop. Why? We have put A into row echelon form. How can we tell?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) = 4$.
What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent.

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.
But that implies the following: If $\text{rank}(A) \geq m$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

In this case, therefore, we argue as follows:
Since $\text{rank}(A) \geq 4$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent.

Does the agree with the solution we found in Exercise 29? It should.
Which method was easier for this Exercise? Why?

40. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How? In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{\substack{R_1'=R_4 \\ R_2'=R_3 \\ R_3'=R_2 \\ R_4'=R_1}} \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can stop. Why? We have put A into row echelon form. How can we tell?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) = 4$.
What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent.

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.
But that implies the following: If $\text{rank}(A) \geq m$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

In this case, therefore, we argue as follows:
Since $\text{rank}(A) \geq 4$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly independent.

Does the agree with the solution we found in Exercise 30? It should.
Which method was easier for this Exercise? Why?

41. Exercises 32 through 41 provide a check on our solutions to Exercises 22 through 31. How? In these exercises the directions tell us to follow Example 2.25 and apply Theorem 2.7:

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 & -1 \\ -1 & 3 & 1 & -1 \\ 1 & 1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{bmatrix} \xrightarrow[R_3'=R_1]{R_1'=R_3} \begin{bmatrix} 1 & 1 & 3 & 1 \\ -1 & 3 & 1 & -1 \\ 3 & -1 & 1 & -1 \\ -1 & -1 & 1 & 3 \end{bmatrix} \xrightarrow[R_4'=R_4+R_1']{R_2'=R_2+R_1' \quad R_3''=R_3'-3R_1'} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & -4 & -8 & -4 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3'''=R_3''+R_2'} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \xrightarrow{R_4''=R_4'+R_3'''} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can stop. Why? We have created a zero row. What does that tell us?
Since the rank of a matrix is the number of nonzero rows in its row echelon form, $\text{rank}(A) = 3$.
What do we conclude? We conclude $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly dependent.

Theorem 2.7 states that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.
So, since $\text{rank}(A) = 3 < 4$, Theorem 2.7 implies $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_3$ are linearly dependent.

From the row reduction above, we see: $\mathbf{0} = R_4'' = R_4' + R_3'''$. Substituting, we have:

$$\begin{aligned} \mathbf{0} &= (R_4 + R_1') + (R_3'' + R_2') = (R_4 + R_1') + (R_3' - 3R_1') + (R_2 + R_1') \\ &= R_4 + R_3' + R_2 - R_1' = R_4 + R_1 + R_2 - R_3. \end{aligned}$$

This equation yields a dependence relation among the original vectors:

$$\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$ Does this agree with Exercise 31?

42. We will use the theorems of Sections 2.2 and 2.3 to support the conclusions below.

(a) Given $A = [\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n]$, we will show that $\text{rank}(A) = n$.
Theorem 2.6 of Section 2.3 implies:
Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if and only if the only solution of $[A \mid \mathbf{0}] = [\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n \mid \mathbf{0}]$ is the trivial solution.
Therefore, the number of free variables in the associated system is 0.
So, Theorem 2.2 of Section 2.2 (The Rank Theorem) implies:
number of free variables $= 0 = n - \text{rank}(A) \Rightarrow \text{rank}(A) = n$, as claimed.

(b) Given $A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$ we will show that $\text{rank}(A) = n$.
Theorem 2.7 of Section 2.3 implies:
Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if and only if the rank of $A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$ is greater than or equal to n. That is, $\text{rank}(A) \geq n$.
But A has n rows therefore we have $n \leq \text{rank}(A) \leq n \Rightarrow \text{rank}(A) = n$, as claimed.

43. We apply the definition of linear independence and Examples 2.23 and 2.25 to prove our claims.

(a) We will show that $\mathbf{u}+\mathbf{v}$, $\mathbf{v}+\mathbf{w}$, and $\mathbf{u}+\mathbf{w}$ are linearly independent.
Given $c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{v}+\mathbf{w}) + c_3(\mathbf{u}+\mathbf{w}) = 0$, we will show $c_1 = c_2 = c_3 = 0$.
Multiplying and gathering like terms yields: $(c_1+c_3)\mathbf{u} + (c_1+c_2)\mathbf{v} + (c_2+c_3)\mathbf{w} = 0$.
Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, $c_1+c_3 = c_1+c_2 = c_2+c_3 = 0$.
We create the matrix of coefficients A and row reduce to determine its rank:

$$\begin{aligned} c_1 + \quad\ \ c_3 &= 0 \\ c_1 + c_2 \qquad &= 0 \\ c_2 + c_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since $\text{rank}(A) = 3$ the only solution is the trivial one, so $c_1 = c_2 = c_3 = 0$.

(b) We will show that $\mathbf{u}-\mathbf{v}$, $\mathbf{v}-\mathbf{w}$, and $\mathbf{u}-\mathbf{w}$ are linearly dependent.
Given $c_1(\mathbf{u}-\mathbf{v}) + c_2(\mathbf{v}-\mathbf{w}) + c_3(\mathbf{u}-\mathbf{w}) = 0$, we will show $c_1 = c_2 = -c_3$.
Multiplying and gathering like terms yields: $(c_1+c_3)\mathbf{u} + (-c_1+c_2)\mathbf{v} + (-c_2-c_3)\mathbf{w} = 0$.
Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, $c_1+c_3 = -c_1+c_2 = -c_2-c_3 = 0$.
We form the augmented matrix and row reduce to solve:

$$\begin{aligned} c_1 + \quad\ \ c_3 &= 0 \\ -c_1 + c_2 \qquad &= 0 \\ -c_2 - c_3 &= 0 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This clearly has the solution $c_1 = c_2 = -c_3$ as we were to show.

44. We will consider the case when one of the vectors the zero vector, $\mathbf{0}$, separately.

Case 1 Given $\mathbf{v}_1 = \mathbf{0}$. We will show $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent and multiples.
They are linearly dependent since $\mathbf{v}_1 + 0\mathbf{v}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$.
They are multiples since $\mathbf{v}_1 = 0\mathbf{v}_2 = \mathbf{0}$.

Case 2 Assume both $\mathbf{v}_1 \neq \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$.
To show two vectors are linearly dependent if and only if they are multiples,
we must prove two separate conditions:
(a) if two vectors are multiples of each other, then they are linearly dependent.
(b) if two vectors are linearly dependent, then they are multiples of each other.

(a) Assume $\mathbf{v}_1 = c\mathbf{v}_2$. We will show they are linearly dependent.
Since $\mathbf{v}_1 = c\mathbf{v}_2$ implies $\mathbf{v}_1 - c\mathbf{v}_2 = \mathbf{0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent.

(b) Assume $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. We will show $\mathbf{v}_1 = c\mathbf{v}_2$.
Since $\mathbf{v}_1 \neq \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$ are linearly dependent,
we have $c_1 \neq 0$ and $c_2 \neq 0$ such that: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$.
So, $\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2 = c\mathbf{v}_2$, as we were to show.

45. We will follow the proof of Theorem 2.8 and use the assumption that $n < m$.

PROOF: Let A be the $m \times n$ matrix with vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $\mathbb{R}^n$ as its rows.

By Theorem 2.7, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.
By definition, $\text{rank}(A) =$ number of nonzero rows its its row echelon form. But, the definition of row echelon form implies that $\text{rank}(A)$ is always $\leq$ number of columns in $A = n$. Why?

So, we have $\text{rank}(A) \leq n < m$, as required to show $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent.

46. Idea: Given any linear combination of a subset that sums to the zero vector,
extend it to the entire set by including missing vectors with a coefficient of 0.

PROOF: Let $A = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of linearly independent vectors.
By definition, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ if and only if $c_i = 0$ for all i.

Given any subset of $B = \{\ldots, \mathbf{v}_j, \ldots, \mathbf{v}_k, \ldots\}$, we must show:
$\cdots + c_j\mathbf{v}_j + \cdots + c_k\mathbf{v}_k + \cdots = \mathbf{0}$ if and only if $c_i = 0$ for all i.

Extend $\cdots + c_j\mathbf{v}_j + \cdots + c_k\mathbf{v}_k + \cdots = \mathbf{0}$ to all of A
by letting $c_i = 0$ for any vector in set A that is not in subset B.

Then we have: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$
which is possible if and only if $c_i = 0$ for all i, as we were to show.

47. As suggested in the hint, we will use the result of Exercise 21(b).

PROOF:

Since $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}\}$, Exercise 21(b) implies $\text{span}(S') \subseteq \text{span}(S)$. So, we need only show $\text{span}(S) \subseteq \text{span}(S')$ to prove $\text{span}(S) = \text{span}(S')$.

Since $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, Exercise 21(b) implies $\text{span}(S) \subseteq \text{span}(S')$. Therefore, $\text{span}(S) = \text{span}(S')$.

48. We need to show $B = \{\mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of linearly independent vectors. That is, $b_1\mathbf{v} + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k = \mathbf{0}$ if and only if $b_i = 0$ for all i.

The key to the proof below is the fact that $c_1 \neq 0$.

PROOF: We are given $A = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of linearly independent vectors.
That is, $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$ if and only if $a_i = 0$ for all i.
Furthermore, we are told $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ with $c_1 \neq 0$.

When we substitute this expression for $\mathbf{v}$ in $b_1\mathbf{v} + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k = \mathbf{0}$, we have:

$$b_1(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k = \mathbf{0}.$$

Now distribute and combine like terms: $b_1c_1\mathbf{v}_1 + (b_1c_2 + b_2)\mathbf{v}_2 + \cdots + (b_1c_k + b_k)\mathbf{v}_k = 0$.

Since all the coefficients of $\mathbf{v}_i$ must be 0, we get the following equations:

$$b_1c_1 = 0 \qquad b_1c_2 + b_2 = 0 \qquad \ldots \qquad b_1c_k + b_k = 0$$

Since $c_1 \neq 0$ implies $b_1 = 0$, we have $b_i = 0$ for all i, as we were to show.

2.4 Applications

1. Let x_1, x_2, and x_3 be the number of bacteria of strains I, II, and III, respectively. Then from the consumption of A, B, and C, we get the following system:

$$\begin{aligned} x_1 + 2x_2 \quad &= 400 \\ 2x_1 + x_2 + x_3 &= 600 \\ x_1 + x_2 + 2x_2 &= 600 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 400 \\ 2 & 1 & 1 & 600 \\ 1 & 1 & 2 & 600 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 160 \\ 0 & 1 & 0 & 120 \\ 0 & 0 & 1 & 160 \end{array}\right].$$

So, 160, 120, and 160 bacteria of strains I, II, and III respectively can coexist.

2. Let x_1, x_2, and x_3 be the number of bacteria of strains I, II, and III, respectively. Then from the consumption of A, B, and C, we get the system:

$$\begin{aligned} x_1 + 2x_2 \quad &= 400 \\ 2x_1 + x_2 + 3x_3 &= 500 \\ x_1 + x_2 + x_2 &= 600 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 400 \\ 2 & 1 & 3 & 500 \\ 1 & 1 & 1 & 600 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

This is an inconsistent system, so the bacteria cannot coexist.

3. Let x_1, x_2, and x_3 be the number of small, medium, and large arrangements. Then from the consumption of flowers in each arrangement we get:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 24 \\ 3x_1 + 4x_2 + 8x_3 &= 50 \\ 3x_1 + 6x_2 + 6x_2 &= 48 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 24 \\ 3 & 4 & 8 & 50 \\ 3 & 6 & 6 & 48 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

So, 2 small, 3 medium, and 4 large arrangements were sold that day.

4. (a) Let $x_1 =$ number of nickels, $x_2 =$ number of dimes, $x_3 =$ number of quarters. So, we get $x_1 + x_2 + x_3 = 20$, $2x_1 - x_2 = 0$, $5x_1 + 10x_2 + 25x_2 = 300$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 2 & -1 & 0 & 0 \\ 5 & 10 & 25 & 300 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 8 \end{array}\right] \Rightarrow \text{There are 4 nickels, 8 dimes, and 8 quarters.}$$

(b) Similar to (a), but the constraint $2x_1 - x_2 = 0$ does not apply.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 5 & 10 & 25 & 300 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -20 \\ 0 & 1 & 4 & 40 \end{array}\right] \Rightarrow x_1 = -20 + 3t,\ x_2 = 40 - 4t,\ x_3 = t.$$

Summarizing, we have: $t \geq \frac{20}{3} \Rightarrow t \geq 7$ and $0 \leq t \leq 10$.
So with $7 \leq t \leq 10$, the possible combinations of 20 nickels, dimes, and quarters is $[x_1, x_2, x_3] \in \{[1, 12, 7], [4, 8, 8], [7, 4, 9], [10, 0, 10]\}$.

5. Let x_1, x_2, and x_3 be the number of house, special, and gourmet blends. Then from the consumption of beans in each blend we get

$$\begin{aligned} 300x_1 + 200x_2 + 100x_3 &= 30,000 \\ 200x_2 + 200x_3 &= 15,000 \\ 200x_1 + 100x_2 + 200x_3 &= 25,000 \end{aligned}$$

Form augmented matrix and reduce it: $\left[\begin{array}{ccc|c} 300 & 200 & 100 & 30,000 \\ 0 & 200 & 200 & 15,000 \\ 200 & 100 & 200 & 25,000 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 65 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 45 \end{array}\right].$

The merchant should make 65 house blend, 30 special blend, and 45 gourmet blend.

6. We let x_1, x_2, and x_3 be the number of house, special, and gourmet blends respectively. Then from the consumption of beans in each blends we get

$$\begin{aligned} 300x_1 + 200x_2 + 100x_3 &= 30,000 \\ 50x_1 + 200x_2 + 350x_3 &= 15,000 \\ 150x_1 + 100x_2 + 50x_3 &= 15,000 \end{aligned}$$

Form matrix and reduce it: $\left[\begin{array}{ccc|c} 300 & 200 & 100 & 30,000 \\ 50 & 200 & 350 & 15,000 \\ 150 & 100 & 50 & 15,000 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 60 \\ 0 & 1 & 2 & 60 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$

$x_1 = 60 + t$, $x_2 = 60 - 2t$, $x_3 = t$ with $t \geq -60$, $t \leq 30$, $t \geq 0 \Rightarrow 0 \leq t \leq 30$.
But we also need to maximize the profit $P \Rightarrow$
$\frac{1}{2}x_1 + \frac{3}{2}x_2 + 2x_3 = P \Rightarrow \frac{1}{2}(60+t) + \frac{3}{2}(60-2t) + 2t = P \Rightarrow 120 - \frac{1}{2}t = P$.
Since $0 \leq t \leq 30$, the profit P is maximized if $t = 0$, in which case $x_1 = x_2 = 60$ and $x_3 = t$.
Therefore the merchant should make 60 house and special blends, and no gourmet blends.

The maximum profit is \$120.

7. Let x, y, z, and w be the number of FeS_2, O_2, Fe_2O_3, and SO_2 molecules respectively. Then, compare the number of iron, sulfur, and oxygen atoms in reactants and products:

$$\begin{array}{lrcl} \text{Iron}: & x &=& 2z \\ \text{Sulfur}: & 2x &=& w \\ \text{Oxygen}: & 2y &=& 3z + 2w \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -3 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{11}{8} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & 0 \end{array}\right].$$

Thus $z = \frac{1}{4}w$, $y = \frac{11}{8}w$, and $x = \frac{1}{2}w$.
The smallest positive value of w that will produce integer values for all four variables is the least common denominator of $\frac{1}{2}$, $\frac{11}{8}$, $\frac{1}{2} \Rightarrow w = 8$, $x = 4$, $y = 11$, and $z = 2$.
Therefore, the balanced chemical equation is $4FeS_2 + 11O_2 \longrightarrow 2Fe_2O_3 + 8SO_2$.

8. $6CO_2 + 6H_2O \longrightarrow C_6H_{12}O_6 + 6O_2$

9. $2C_4H_{10} + 13O_2 \longrightarrow 8CO_2 + 10H_2O$

10. $2C_7H_6O_2 + 15O_2 \longrightarrow 6H_2O + 14CO_2$

11. $2C_5H_{11}OH + 15O_2 \longrightarrow 12H_2O + 10CO_2$

12. $12HClO_4 + P_4O_{10} \longrightarrow 4H_3PO_4 + 6Cl_2O_7$

13. $Na_2CO_3 + 4C + N_2 \longrightarrow 2NaCN + 3CO$

14. $2C_2H_2Cl_4 + Ca(OH)_2 \longrightarrow 2C_2HCl_3 + CaCl_2 + 2H_2O$

15. (a) By applying the conservation of flow rule to each node we obtain the system of equations

$$f_1 + f_2 = 20 \qquad f_2 - f_3 = -10 \qquad f_1 + f_3 = 30$$

We form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 20 \\ 0 & 1 & -1 & -10 \\ 1 & 0 & 1 & 30 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 30 \\ 0 & 1 & -1 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Letting $f_3 = t$, the possible flows are $f_1 = 30 - t$, $f_2 = t - 10$, $f_3 = t$.

(b) In this case $f_2 = 5$, but $f_2 = t - 10$ so $t = 15$. Then the other flows are $f_1 = f_3 = 15$.

(c) Each flow must be nonnegative $\Rightarrow t \leq 30$, $t \geq 10$, $t \geq 0$
Thus $10 \leq t \leq 30$, so $0 \leq f_1 \leq 20$, $0 \leq f_2 \leq 20$, $10 \leq f_3 \leq 30$.

(d) A negative flow, if allowed, would indicate a transport in the opposite direction.
For example, a negative flow into a node is the same as a positive flow out of a node.
So, if $f_2 < 0$, then the arrow on f_2 could be changed and the flow taken as positive.

16. (a) By applying the conservation of flow rule to each node we obtain the system of equations

$$f_1 + f_2 = 20 \qquad f_1 + f_3 = 25 \qquad f_2 + f_4 = 25 \qquad f_3 + f_4 = 30$$

We form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 1 & 0 & 25 \\ 0 & 1 & 0 & 1 & 25 \\ 0 & 0 & 1 & 1 & 30 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -5 \\ 0 & 1 & 0 & 1 & 25 \\ 0 & 0 & 1 & 1 & 30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Letting $f_4 = t$, the possible flows are $f_1 = -5 + t$, $f_2 = 25 - t$, $f_3 = 30 - t$, $f_4 = t$.

(b) If $f_4 = t = 10$, then average flows on other streets will be $f_1 = 5$, $f_2 = 15$, and $f_3 = 20$.

(c) Each flow must be nonnegative so the set of solutions gives the four constraints
$t \geq 5$, $t \leq 25$, $t \leq 30$, $t \geq 0$. To satisfy all four constraints t must satisfy $5 \leq t \leq 25$.
Combining the constraint on t with the four solutions,
we see that $0 \leq f_1 \leq 20$, $0 \leq f_2 \leq 20$, $5 \leq f_3 \leq 25$, $5 \leq f_4 \leq 25$.

(d) Reversing all directions would have no effect on the solution since this would be equivalent to multiplying each row of the augmented matrix by -1. But this is an elementary row operation so the new matrix would have the same reduced row echelon form, and the solutions will be the same.

17. (a) By applying the conservation of flow rule to each node we obtain the system of equations

$$f_1 + f_2 = 100 \qquad f_2 + f_3 = f_4 + 150 \qquad f_4 + f_5 = 150 \qquad f_1 + 200 = f_3 + f_5$$

We rearrange the equations, and perform Gauss-Jordan elimination

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 100 \\ 0 & 1 & 1 & -1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 1 & 0 & -1 & 0 & -1 & -200 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -1 & -200 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We parameterize the solution by letting $f_5 = t$, and $f_3 = s$, so the possible flows are $f_1 = -200 + s + t,\ f_2 = 300 - s - t,\ f_3 = s,\ f_4 = 150 - t,\ f_5 = t$.

(b) If DC is closed then $f_5 = t = 0$, so the flow through DB must be $200 \le f_3 \le 300\ \frac{liters}{day}$.

(c) If DB were closed, then f_5 must carry away at least $200\ \frac{liters}{day}$.
But node C has a maximum outflow of $150\ \frac{liters}{day}$,
so, it will not be able to handle the inflow from f_5.
Thus DB cannot be closed.

From the solution in (a), with DB closed $f_3 = s = 0$, and the solution becomes $f_1 = -200 + t,\ f_2 = 300 - t,\ f_3 = 0,\ f_4 = 150 - t,\ f_5 = t$.
But each flow must be positive, so we get the following constraints on t:
$t \ge 200,\ t \le 300,\ t \le 150,\ t \ge 0$.

But this gives rise to a contradiction since it demands $t \le 150$ and $t \ge 200$, which is impossible, and again we see that DB cannot be closed.

(d) Each flow must be nonnegative, so the solutions give the following constraints:
$s \ge 200 - t,\ s \le 300 - t,\ s \ge 0,\ t \le 150,\ t \ge 0$.
From these we see that $0 \le t \le 150$ and $50 \le s \le 300$.
Combining the constraints on s and t with the five solutions,
we see that $0 \le f_1 \le 100,\ 0 \le f_2 \le 100,\ 50 \le f_3 \le 300,\ 0 \le f_4 \le 150,\ 0 \le f_5 \le 150$.

18. (a) By applying the conservation of flow rule to each node we obtain:

$$\begin{aligned} f_3+200 &= f_1+100 \qquad & f_1+150=f_2+f_4 \qquad & f_2+f_5=300 \\ f_6+100 &= f_3+200 \qquad & f_4+f_7=f_6+100 \qquad & f_5+f_7=250 \end{aligned}$$

We rearrange the equations, and perform Gauss-Jordan elimination

$$\left[\begin{array}{ccccccc|c} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 100 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -150 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \end{array}\right] \longrightarrow \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 50 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -100 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We parameterize the solution by letting $f_7 = t$, and $f_6 = s$, so the possible flows are $f_1 = s$, $f_2 = 50 + t,\ f_3 = -100 + s,\ f_4 = 100 + s - t,\ f_5 = 250 - t,\ f_6 = s,\ f_7 = t$.

(b) From the solution in (a), we see that it is not possible for $f_1 = 100$ and $f_6 = 150$.
From the diagram, if $f_1 = 100$ we see that node A demands that $f_3 = 0$.
Then, node D demands $f_6 = 100$, so, it is impossible for $f_1 = 100$ and $f_6 = 150$.

(c) If $f_4 = 0$ then $100 + s - t = 0 \Rightarrow t = 100 + s$, so the possible flows are
$f_1 = s,\ f_2 = 150 + s,\ f_3 = -100 + s,\ f_4 = 0,\ f_5 = 150 - s,\ f_6 = s,\ f_7 = 100 + s$.
If we demand that the flows be positive, then we see that $100 \le s \le 150$,
and the flows will be restricted to $100 \le f_1 = 150$, $250 \le f_2 = 300$,
$0 \le f_3 \le 50$, $0 \le f_4 = 5$, $50 \le f_5 \le 100$, $100 \le f_6 \le 150$, $200 \le f_7 \le 250$.

19. Applying the current law to node A gives the equation $I_1 + I_3 = I_2$ or $I_1 - I_2 + I_3 = 0$.
Applying the voltage law to the top circuit (circuit $ABCA$) gives $-I_2 - I_1 + 8 = 0$.
Similarly, for the circuit $ABDA$ we obtain $-I_2 + 13 - 4I_3 = 0$.
We get the following system:

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ I_1 + I_2 &= 8 \\ I_2 + 4I_3 &= 13 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 8 \\ 0 & 1 & 4 & 13 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow$$

$I_1 = 3$ amps, $I_2 = 5$ amps, and $I_3 = 2$ amps.

20. Applying the current and voltage laws, we have:
$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ I_1 + 2I_2 + 2I_3 &= 5 \\ 0I_1 + 2I_2 + 4I_3 &= 8 \end{aligned} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 5 \\ 0 & 2 & 4 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow I_1 = 1 \text{ amp},\ I_2 = 2 \text{ amps, and } I_3 = 1 \text{ amp.}$$

21. (a) Applying the current and voltage laws to the circuit gives the system:

$$\begin{array}{ll} \text{Node B}: & I = I_1 + I_4 \\ \text{Node C}: & I_1 = I_1 + I_4 \\ \text{Node D}: & I_2 + I_5 = I \\ \text{Node E}: & I_3 + I_4 = I_5 \end{array} \Rightarrow \begin{array}{lrcl} \text{Circuit ABEDA}: & -2I_4 - I_5 + 14 &=& 0 \\ \text{Circuit BCEB}: & -I_1 - I_3 + 2I_4 &=& 0 \\ \text{Circuit CDEC}: & -2I_2 + I_5 + I_3 &=& 0 \end{array} \Rightarrow$$

Gauss-Jordan elimination gives

$$\left[\begin{array}{rrrrrr|r} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, the currents are $I = 10$, $I_1 = 6$, $I_2 = 4$, $I_3 = 2$, $I_4 = 4$, and $I_5 = 6$ amps.

(b) From Ohm's Law, the effective resistance is found to be $R_{eff} = \frac{V}{I} = \frac{14}{10} = \frac{7}{5}$ ohms.

(c) In this case we force $I_3 = 0$, and assign r to the resistance in branch BC.
We then get the following system of equations:

$$\begin{array}{ll} \text{Node B}: & I = I_1 + I_4 \\ \text{Node C}: & I_1 = I_2 \\ \text{Node D}: & I_2 + I_5 = I \\ \text{Node E}: & I_4 = I_5 \end{array} \Rightarrow \begin{array}{lrcl} \text{Circuit ABEDA}: & -2I_4 - I_5 + 14 &=& 0 \\ \text{Circuit BCEB}: & -rI_1 + 2I_4 &=& 0 \\ \text{Circuit CDEC}: & -2I_2 + I_5 &=& 0 \end{array} \Rightarrow$$

Partial row reduction gives

$$\left[\begin{array}{rrrrrr|r} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & r & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrrrr|r} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 14/3 \\ 0 & r & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, substitution $\Rightarrow I_4 = I_5 = \frac{14}{3}$ and $I_1 = I_2 = \frac{7}{3} \Rightarrow \frac{7}{3}r - 2\,\frac{14}{3} = 0 \Rightarrow r = 4$.
So, if we let the resistance in $BC = r = 4$, then the current in $CE = 0$.

22. (a) Applying the voltage law to the first diagram of Figure 2.23 gives us: $IR_1 + IR_2 = E$. But from Ohm's Law, $E = IR_{\text{eff}}$, so the previous equation becomes $IR_1 + IR_2 = IR_{\text{eff}} \Rightarrow R_{\text{eff}} = R_1 + R_2$.

(b) Applying the current and voltage laws to the second circuit of Figure 2.23 gives the system of equations $I = I_1 + I_2$, $-I_1R_1 + I_2R_2 = 0$, $-I_1R_1 + E = 0$. Gauss-Jordan elimination gives

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -R_1 & R_2 & 0 \\ 0 & -R_1 & 0 & -E \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{R_1+R_2}{R_1}\frac{E}{R_2} \\ 0 & 1 & 0 & \frac{1}{R_1}E \\ 0 & 0 & 1 & \frac{E}{R_2} \end{array}\right] \Rightarrow I = \frac{R_1+R_2}{R_1}\frac{E}{R_2},\ I_1 = \frac{E}{R_1},\ I_2 = \frac{E}{R_2}.$$

But $E = IR_{\text{eff}}$, which we plug into the expression for the I current to get:
$I = \frac{R_1+R_2}{R_1}\frac{IR_{\text{eff}}}{R_2} \Rightarrow R_{\text{eff}} = \frac{R_1R_2}{R_1+R_2} = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$.

23. The matrix for this system (with inputs on the left, outputs along the top) is

	Farming	Manufacturing
Farming	1/2	1/3
Manufacturing	1/2	2/3

Let the output of the farm sector be f and the output of the manufacturing sector be m.

Then:

$$\begin{aligned} \tfrac{1}{2}f + \tfrac{1}{3}m &= f \\ \tfrac{1}{2}f + \tfrac{2}{3}m &= m \end{aligned}$$

Simplifying and row-reducing the associated augmented matrix gives

$$\begin{aligned} -\tfrac{1}{2}f + \tfrac{1}{3}m &= 0 \\ \tfrac{1}{2}f - \tfrac{1}{3}m &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} -1/2 & 1/3 & 0 \\ 1/2 & -1/3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, $f = \frac{2}{3}m$.

So, the farming and manufacturing must be in a 2 : 3 ratio.

24. The matrix for this system (with needed resources on the left, outputs along the top) is

	Coal	Steel
Coal	0.3	0.8
Steel	0.7	0.2

Let the coal output be c and the steel output be s. Then

$$\begin{array}{rcrcl} 0.3c & + & 0.8s & = & c \\ 0.7c & + & 0.2s & = & s \end{array}$$

Simplifying and row-reducing the associated augmented matrix gives

$$\begin{array}{rcrcl} -0.7c & + & 0.8s & = & 0 \\ 0.7c & + & -0.8s & = & 0 \end{array} \Rightarrow \left[\begin{array}{rr|r} -0.7 & 0.8 & 0 \\ 0.7 & -0.8 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -8/7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, $c = \frac{8}{7}s$. So, the coal and steel must be in an $8 : 7$ ratio.
For the total output of \$20 million,

$$\text{Coal is } \frac{8}{15} \cdot \$20 = \$\frac{32}{3} \approx \$10.667 \text{ million}$$
$$\text{Steel is } \frac{7}{15} \cdot \$20 = \$\frac{28}{3} \approx \$9.333 \text{ million}$$

25. Let x be the painter's rate, y the plumber's rate, and z the electrician's rate.
From the matrix given, we get the linear system

$$\begin{array}{rcrcrcl} 2x & + & y & + & 5z & = & 10x \\ 4x & + & 5y & + & z & = & 10y \\ 4x & + & 4y & + & 4z & = & 10z \end{array}$$

Simplifying and row-reducing the associated augmented matrix gives

$$\begin{array}{rcrcrcl} -8x & + & y & + & 5z & = & 0 \\ 4x & - & 5y & + & z & = & 0 \\ 4x & + & 4y & - & 6z & = & 0 \end{array} \Rightarrow \left[\begin{array}{rrr|r} -8 & 1 & 5 & 0 \\ 4 & -5 & 1 & 0 \\ 4 & 4 & -6 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -13/18 & 0 \\ 0 & 1 & -7/9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so that $x = \frac{13}{18}z$, $y = \frac{7}{9}z$.

Let $z = 54$; then $x = 39$ and $y = 42$.

The painter should charge \$39 per hour, the plumber \$42 per hour,
and the electrician \$54 per hour.

26. From the given matrix, we get the linear system

$$\begin{array}{rcrcrcrcl} & & \frac{1}{4}L & + & \frac{1}{8}T & + & \frac{1}{6}Z & = & B \\ \frac{1}{2}B & + & \frac{1}{4}L & + & \frac{1}{4}T & + & \frac{1}{6}Z & = & L \\ \frac{1}{4}B & + & \frac{1}{4}L & + & \frac{1}{2}T & + & \frac{1}{3}Z & = & T \\ \frac{1}{4}B & + & \frac{1}{4}L & + & \frac{1}{8}T & + & \frac{1}{3}Z & = & Z \end{array}$$

Simplify and row-reduce the associated augmented matrix:

$$\begin{array}{rcrcrcrcl} -B & + & \frac{1}{4}L & + & \frac{1}{8}T & + & \frac{1}{6}Z & = & 0 \\ \frac{1}{2}B & - & \frac{3}{4}L & + & \frac{1}{4}T & + & \frac{1}{6}Z & = & 0 \\ \frac{1}{4}B & + & \frac{1}{4}L & - & \frac{1}{2}T & + & \frac{1}{3}Z & = & 0 \\ \frac{1}{4}B & + & \frac{1}{4}L & + & \frac{1}{8}T & - & \frac{2}{3}Z & = & 0 \end{array} \Rightarrow \left[\begin{array}{rrrr|r} -1 & 1/4 & 1/8 & 1/6 & 0 \\ 1/2 & -3/4 & 1/4 & 1/6 & 0 \\ 1/4 & 1/4 & -1/2 & 1/3 & 0 \\ 1/4 & 1/4 & 1/8 & -2/3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -2/3 & 0 \\ 0 & 1 & 0 & -6/5 & 0 \\ 0 & 0 & 1 & -8/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So, $B = \frac{2}{3}Z$, $L = \frac{6}{5}Z$, and $T = \frac{8}{5}Z$.
Furthermore, Z must be a multiple of 15 and $\frac{2}{3}Z$ must be greater than 50. Why?
So, if we let $Z = 90$, $B = 60$, $L = 108$, and $T = 144$.

27. The input/output matrix is (outputs on the top, inputs on the left)

	Coal	Steel
Coal	0.15	0.25
Steel	0.2	0.1

(a) Since 45 million of external demand is required for coal and 124 million for steel:

$$\begin{array}{rcrcrcr} c & = & 0.15c & + & 0.25s & + & 45 \\ s & = & 0.2c & + & 0.1s & + & 124 \end{array}$$

Simplify and row-reduce the associated augmented matrix:

$$\begin{array}{rcrcr} 0.85c & - & 0.25s & = & 45 \\ -0.2c & + & 0.9s & = & 124 \end{array} \Rightarrow \left[\begin{array}{rr|r} 0.85 & -0.25 & 45 \\ -0.2 & 0.9 & 124 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 100 \\ 0 & 1 & 160 \end{array}\right]$$

So, the coal industry should produce \$100 million and the steel industry should produce \$160 million.

(b) The new external demands for coal and steel are 40 and 130 million respectively. So:

$$\begin{array}{rcrcrcr} c & = & 0.15c & + & 0.25s & + & 40 \\ s & = & 0.2c & + & 0.1s & + & 130 \end{array}$$

Simplify and row-reduce the associated augmented matrix:

$$\begin{array}{rcrcr} 0.85c & - & 0.25s & = & 40 \\ -0.2c & + & 0.9s & = & 130 \end{array} \Rightarrow \left[\begin{array}{rr|r} 0.85 & -0.25 & 40 \\ -0.2 & 0.9 & 130 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 95.8 \\ 0 & 1 & 165.73 \end{array}\right]$$

So, the coal industry should reduce production by \$4.2 million.
and the steel industry should increase production by \$5.73 million.

28. This is an open system with external demand given by the other city departments. From the given matrix, we get the system:

$$\begin{aligned} 1 + 0.2A + 0.1H + 0.2T &= A \\ 1.2 + 0.1A + 0.1H + 0.2T &= H \\ 0.8 + 0.2A + 0.4H + 0.3T &= T \end{aligned}$$

Simplifying and row-reducing the associated augmented matrix gives:

$$\begin{aligned} 0.8A - 0.1H - 0.2T &= 1 \\ -0.1A + 0.9H - 0.2T &= 1.2 \\ -0.2A - 0.4H + 0.7T &= 0.8 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 0.8 & -0.1 & -0.2 & 1 \\ -0.1 & 0.9 & -0.2 & 1.2 \\ -0.2 & -0.4 & 0.7 & 0.8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2.31 \\ 0 & 1 & 0 & 2.28 \\ 0 & 0 & 1 & 3.11 \end{array}\right]$$

So, the Administrative department should produce \$2.31 million of services, the Health department \$2.28 million, and the Transportation department \$3.11 million.

29. (a) Over $\mathbb{Z}_2$, we need to solve $x_1\mathbf{a} + x_2\mathbf{b} + \cdots + x_5\mathbf{e} = \mathbf{t} + \mathbf{s}$:

$$\mathbf{s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{t} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, x_5 is a free variable; hence there are exactly two solutions.
Solving for the other variables, we get: $x_1 = 1 + x_5$, $x_2 = 1 + x_5$, $x_3 = 1$, $x_4 = x_5 \Rightarrow$

$x_5 = 0$ and $x_5 = 1 \Rightarrow$ the solutions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

So, push switches 1, 2, and 3 or switches 3, 4, and 5.

(b) In this case, $\mathbf{t} = \mathbf{e}_2$ over $\mathbb{Z}_2 \Rightarrow$:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

This shows that there is no solution in this case.
That is, it is impossible to start with all lights off and turn only the second light on.

30. (a) With $\mathbf{s} = \mathbf{e}_4$, $\mathbf{t} = \mathbf{e}_2 + \mathbf{e}_4$ over $\mathbb{Z}_2 \Rightarrow$:

$$\left[\begin{array}{ccccc|c} 1&1&0&0&0&0\\ 1&1&1&0&0&1\\ 0&1&1&1&0&0\\ 0&0&1&1&1&0\\ 0&0&0&1&1&0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1&0&0&0&1&0\\ 0&1&0&0&-1&0\\ 0&0&1&0&0&0\\ 0&0&0&1&1&0\\ 0&0&0&0&0&1 \end{array}\right]$$

This shows that there is no solution in this case; that is, it is impossible to start with only the fourth light on and end up with only the second and fourth lights on.

(b) In this case $\mathbf{t} = \mathbf{e}_2$ over $\mathbf{Z}_2 \Rightarrow$:

$$\left[\begin{array}{ccccc|c} 1&1&0&0&0&0\\ 1&1&1&0&0&1\\ 0&1&1&1&0&0\\ 0&0&1&1&1&1\\ 0&0&0&1&1&0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&0&0&1\\ 0&0&0&1&1&0\\ 0&0&0&0&0&0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1\\x_2\\x_3\\x_4\\x_5 \end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix}.$$

31. The possible configurations are

$$\begin{bmatrix} x_1\\x_2\\x_3\\x_4\\x_5 \end{bmatrix} \in \left\{ \begin{array}{c} \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix} \\ \begin{bmatrix}1\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \end{array} \right\}$$

32. (a) Over $\mathbb{Z}_3$, we need to solve $x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} = \mathbf{t}$.

$$\mathbf{t} = \begin{bmatrix}0\\2\\1\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1&1&0&0\\ 1&1&1&2\\ 0&1&1&1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1&0&0&1\\ 0&1&0&2\\ 0&0&1&2 \end{array}\right] \Rightarrow x_1 = 1,\ x_2 = 2, \text{ and } x_3 = 2.$$

In other words, we must push switch A once, and the other two switches twice each.

(b) $$\mathbf{t} = \begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1&1&0&1\\ 1&1&1&0\\ 0&1&1&1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1&0&0&2\\ 0&1&0&2\\ 0&0&1&2 \end{array}\right] \Rightarrow x_1 = 2,\ x_2 = 2,\ x_3 = 2.$$

In other words, we must push each of the switches twice.

(c) Let x, y, and z be the final states of the three lights in the system:

$$\left[\begin{array}{ccc|c} 1&1&0&x\\ 1&1&1&y\\ 0&1&1&z \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1&0&0&y+2z\\ 0&1&0&x+2y+z\\ 0&0&1&2x+y \end{array}\right] \Rightarrow \begin{array}{c}\text{if the lights}\\ \text{are in the final configuration}\\ x,\ y, \text{ and } z,\end{array}$$

we can push switch A $y + 2z$ times, switch B $x + 2y + z$ times, and switch C $2x + y$ times (in $\mathbb{Z}_3$) in order to reach that state.

33. In this situation the switches correspond to the vectors

$$\mathbf{a}=\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix},\mathbf{b}=\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix},\mathbf{c}=\begin{bmatrix}0\\1\\1\\1\\0\end{bmatrix},\mathbf{d}=\begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix},\mathbf{e}=\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}$$

In $\mathbb{Z}_3^5$, with $\mathbf{s}=\mathbf{0}$, we need to solve $x_1\mathbf{a}+x_2\mathbf{b}+\cdots+x_5\mathbf{e}=\mathbf{t}$.

$$\mathbf{t}=\begin{bmatrix}2\\1\\2\\1\\2\end{bmatrix}\Rightarrow\left[\begin{array}{ccccc|c}1&1&0&0&0&2\\1&1&1&0&0&1\\0&1&1&1&0&2\\0&0&1&1&1&1\\0&0&0&1&1&2\end{array}\right]\longrightarrow\left[\begin{array}{ccccc|c}1&0&0&0&1&1\\0&1&0&0&2&1\\0&0&1&0&0&2\\0&0&0&1&1&2\\0&0&0&0&0&0\end{array}\right]$$

So, x_5 is free and there are exactly three solutions ($x_5=0,1,2$).
Solving for the other variables in terms of x_5 (over $\mathbb{Z}_3$), we get:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1\\1\\2\\2\\0\end{bmatrix},\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}0\\2\\2\\1\\1\end{bmatrix},\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}2\\0\\2\\0\\2\end{bmatrix}.$$

34. The possible configurations are $\begin{bmatrix}1&1&0&0&0\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\\0&0&0&1&1\end{bmatrix}\begin{bmatrix}s_1\\s_2\\s_3\\s_4\\s_5\end{bmatrix}=\begin{bmatrix}s_1+s_2\\s_1+s_2+s_3\\s_2+s_3+s_4\\s_3+s_4+s_5\\s_4+s_5\end{bmatrix}$,
where $s_i\in\{0,1,2\}$ is the number of times that switch i is thrown.

35. (a) The matrix representing actions of touching the squares is $S=\begin{bmatrix}\mathbf{a}&\mathbf{b}&\mathbf{c}&\mathbf{d}&\mathbf{e}&\mathbf{f}&\mathbf{g}&\mathbf{h}&\mathbf{i}\end{bmatrix}$.
Over $\mathbb{Z}_2$, we need to solve $\mathbf{s}+x_1\mathbf{a}+x_2\mathbf{b}+\cdots+x_5\mathbf{e}=\mathbf{t}$ or $x_1\mathbf{a}+x_2\mathbf{b}+\cdots+x_9\mathbf{i}=\mathbf{t}-\mathbf{s}=\mathbf{s}$.

$$\mathbf{s}=\begin{bmatrix}0\\1\\1\\1\\0\\1\\1\\1\\0\end{bmatrix},\mathbf{t}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}\Rightarrow\left[\begin{array}{ccccccccc|c}1&1&0&1&0&0&0&0&0&0\\1&1&1&0&1&0&0&0&0&1\\0&1&1&0&0&1&0&0&0&1\\1&0&0&1&1&0&1&0&0&1\\1&0&1&0&1&0&1&0&1&0\\0&0&1&0&1&1&0&0&1&1\\0&0&0&1&0&0&1&1&0&1\\0&0&0&0&1&0&1&1&1&1\\0&0&0&0&0&1&0&1&1&0\end{array}\right]\longrightarrow\left[\begin{array}{ccccccccc|c}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&1\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\end{array}\right]$$

showing that touching the third and seventh squares will turn all nine squares black.

(b) Since $S\longrightarrow I_9$, we can always find a solution to the system of equations
$x_1\mathbf{a}+x_2\mathbf{b}+\cdots+x_9\mathbf{i}=\mathbf{s}$ in part (a).

36. Over $\mathbb{Z}_3$, we need to go from $\mathbf{s}$ to $\mathbf{t}$.

$$\mathbf{s}=\begin{bmatrix}2\\2\\0\\1\\1\\0\\2\\0\\1\end{bmatrix},\ \mathbf{t}=\begin{bmatrix}2\\2\\2\\2\\2\\2\\2\\2\\2\end{bmatrix}\Rightarrow\left[\begin{array}{ccccccccc|c}1&1&0&1&0&0&0&0&0&0\\1&1&1&0&1&0&0&0&0&0\\0&1&1&0&0&1&0&0&0&2\\1&0&0&1&1&0&1&0&0&1\\1&0&1&0&1&0&1&0&1&1\\0&0&1&0&1&1&0&0&1&2\\0&0&0&1&0&0&1&1&0&0\\0&0&0&0&1&0&1&1&1&2\\0&0&0&0&0&1&0&1&1&1\end{array}\right]\longrightarrow\left[\begin{array}{ccccccccc|c}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&2\\0&0&0&0&1&0&0&0&0&2\\0&0&0&0&0&1&0&0&0&1\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&1\\0&0&0&0&0&0&0&0&1&2\end{array}\right]$$

showing that a solution exists.

37. Let Grace's and Hans' ages be g and h. Then we have $g = 3h$ and $g+5 = 2(h+5) \Leftrightarrow g = 2(h+5)-5$, so $3h = 2h+5 \Leftrightarrow h = 5$ and $g = 15$. So, Hans is 5 and Grace is 15.

38. Let the ages be a, b, and c. Then we have $a+b+c = 60$, $a-b = b-c$, $a+(a-b) = 3c$. We can rewrite these as

$$\begin{aligned}a+b+c &= 60\\ a-2b+c &= 0\\ 2a-b-3c &= 0\end{aligned}\quad \text{So, row reduction} \Rightarrow \left[\begin{array}{ccc|c}1&1&1&60\\1&-2&1&0\\2&-1&-3&0\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&0&28\\0&1&0&20\\0&0&1&12\end{array}\right].$$

Thus, Annie is 28, Bert is 20, and Chris is 12.

39. Let the areas be a and b. We have the following equations:

$$\begin{aligned}a+b &= 1800\\ \frac{2}{3}a+\frac{1}{2}b &= 1100\end{aligned}$$

We solve these to find that $a = 1200$ square yards and $b = 600$ square yards.

40. Let x_1, x_2, x_3 be the number of bundles of the first, second, and third types of corn. Then from the number of bundles in each given measure we get the following system:

$$\begin{aligned}3x_1+2x_2+x_3 &= 39\\ 2x_1+3x_2+x_3 &= 34\\ x_1+2x_2+3x_2 &= 26\end{aligned}$$

We form the augmented matrix and row reduce it into reduced row echelon form:

$$\left[\begin{array}{ccc|c}3&2&1&39\\2&3&1&34\\1&2&3&26\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&0&\frac{37}{4}\\0&1&0&\frac{17}{4}\\0&0&1&\frac{11}{4}\end{array}\right]$$

Therefore there are 9.25 measures of corn in a bundle of the first type, 4.25 measures of corn in a bundle of the second type, and 2.75 measures of corn in a bundle of the third type.

41. (a) From the addition table we get: $a + c = 2$, $a + d = 4$, $b + c = 3$, $b + d = 5$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So we see that d is a free variable. Solving for the other variables in terms of d we obtain $a = 4 - d$, $b = 5 - d$, and $c = -2 + d$. Hence there are an infinite number of solutions.

(b) We have: $a + c = 3$, $a + d = 4$, $b + c = 6$, and $b + d = 5$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

So we see that this is an inconsistent system.

42. From the addition table we get the system of equations $a + c = w$, $a + d = y$, $b + c = x$, and $b + d = z$. We form the augmented matrix and perform Gaussian elimination to get

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & w \\ 1 & 0 & 0 & 1 & y \\ 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & z \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & w \\ 0 & 1 & 1 & 0 & x \\ 0 & 0 & -1 & 1 & y - w \\ 0 & 0 & 0 & 0 & z - x - y + w \end{array}\right]$$

These solutions $\Rightarrow$ to get a valid addition table we require $w - x - y + z = 0$.

43. (a) From the addition table we get the following system of equations:

$$\begin{array}{ccc} a+d=3 & b+d=2 & c+d=1 \\ a+e=5 & b+e=4 & c+e=3 \\ a+f=4 & b+f=3 & c+f=1 \end{array}$$

We form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{cccccc|c} 1&0&0&1&0&0&3 \\ 1&0&0&0&1&0&5 \\ 1&0&0&0&0&1&4 \\ 0&1&0&1&0&0&2 \\ 0&1&0&0&1&0&4 \\ 0&1&0&0&0&1&3 \\ 0&0&1&1&0&0&1 \\ 0&0&1&0&1&0&3 \\ 0&0&1&0&0&1&1 \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} 1&0&0&0&0&1&0 \\ 0&1&0&0&0&1&0 \\ 0&0&1&0&0&1&0 \\ 0&0&0&1&0&-1&0 \\ 0&0&0&0&1&-1&0 \\ 0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \end{array}\right]$$

So, we see that this is an inconsistent system.

(b) From the addition table we get the following system of equations:

$$\begin{array}{ccc} a+d=1 & b+d=2 & c+d=3 \\ a+e=3 & b+e=4 & c+e=5 \\ a+f=4 & b+f=5 & c+f=6 \end{array}$$

We form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{cccccc|c} 1&0&0&1&0&0&1 \\ 1&0&0&0&1&0&3 \\ 1&0&0&0&0&1&4 \\ 0&1&0&1&0&0&2 \\ 0&1&0&0&1&0&4 \\ 0&1&0&0&0&1&5 \\ 0&0&1&1&0&0&3 \\ 0&0&1&0&1&0&5 \\ 0&0&1&0&0&1&6 \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} 1&0&0&0&0&1&4 \\ 0&1&0&0&0&1&5 \\ 0&0&1&0&0&1&6 \\ 0&0&0&1&0&-1&-3 \\ 0&0&0&0&1&-1&-1 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \end{array}\right]$$

We see that f is a free variable, and there are an infinite number of solutions of the form $a = 4 - f$, $b = 5 - f$, $c = 6 - f$, $d = -3 + f$, $e = -1 + f$.

44. We generalize the 3×3 addition table to

$+$	a	b	c
d	m	n	o
e	p	q	r
f	s	t	u

From the addition table we get the system of equations

$$\begin{array}{ccc} a+d=m & b+d=n & c+d=o \\ a+e=p & b+e=q & c+e=r \\ a+f=s & b+f=t & c+f=u \end{array}$$

We form the augmented matrix and perform Gaussian elimination to get

$$\left[\begin{array}{cccccc|c} 1&0&0&1&0&0&m \\ 1&0&0&0&1&0&p \\ 1&0&0&0&0&1&s \\ 0&1&0&1&0&0&n \\ 0&1&0&0&1&0&q \\ 0&1&0&0&0&1&t \\ 0&0&1&1&0&0&o \\ 0&0&1&0&1&0&r \\ 0&0&1&0&0&1&u \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} 1&0&0&1&0&0&m \\ 0&1&0&1&0&0&n \\ 0&0&1&1&0&0&o \\ 0&0&0&1&-1&0&m-p \\ 0&0&0&0&1&-1&n+p-m-t \\ 0&0&0&0&0&0&m-n-p+q \\ 0&0&0&0&0&0&n-o-q+r \\ 0&0&0&0&0&0&p-q-s+t \\ 0&0&0&0&0&0&q-r-t+u \end{array}\right]$$

For a valid table we need: $m+q=n+p$, $n+r=o+q$, $p+t=q+s$, $q+u=r+t$.
That is, the sum of the two diagonal entries must equal the sum of the two off-diagonal entries.

45. (a) We know that the three points $(0,1)$, $(-1,4)$, and $(2,1)$ must satisfy the equation $y=ax^2+bx+c$. Substitution $\Rightarrow c=1$, $a-b+c=4$, and $4a+2b+c=1$. So:

$$\left[\begin{array}{ccc|c} 0&0&1&1 \\ 1&-1&1&4 \\ 4&2&1&1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1&0&0&1 \\ 0&1&0&-2 \\ 0&0&1&1 \end{array}\right]$$

Thus $a=1$, $b=-2$,and $c=1$, and the equation of the parabola is $y=x^2-2x+1$.

(b) We know that the three points $(-3,1)$, $(-2,2)$, and $(-1,5)$ must satisfy the equation $y=ax^2+bx+c$. Substitution $\Rightarrow 9a-3b+c=1$, $4a-2b+c=2$, and $a-b+c=5$. So:

$$\left[\begin{array}{ccc|c} 9&-3&1&1 \\ 4&-2&1&2 \\ 1&1&1&5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1&0&0&1 \\ 0&1&0&6 \\ 0&0&1&10 \end{array}\right]$$

Thus $a=1$, $b=6$, and $c=10$, and the equation of the parabola is $y=x^2+6x+10$.

46. (a) We know that the three points $(0,1)$, $(-1,4)$, and $(2,1)$ must satisfy the equation $x^2+y^2+ax+by+c=0$. Plugging in these points, we get

$$\begin{aligned} 1+b+c &= 0 \\ 1+16-a+4b+c &= 0 \\ 4+1+2a+b+c &= 0 \end{aligned}$$

We form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 1 & -4 & -1 & 17 \\ 2 & 1 & 1 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Thus $a=-2$, $b=-6$, $c=5$, and the equation of the circle is $x^2+y^2-2x-6y+5=0$. By completing the square and simplifying we get the equation $(x-1)^2+(y-3)^2=5$. Thus the center of this circle is at $(1,3)$ and the radius is $r=\sqrt{5}$.

(b) We know that the three points $(-3,1)$, $(-2,2)$, and $(-1,5)$ must satisfy the equation $x^2+y^2+ax+by+c=0$. By plugging in these points we get the system of equations

$$\begin{aligned} 9+1-3a+b+c &= 0 \\ 4+4-2a+2b+c &= 0 \\ 1+25-a+5b+c &= 0 \end{aligned}$$

We form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 10 \\ 2 & -2 & -1 & 8 \\ 1 & -5 & -1 & 26 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 36 \end{array}\right]$$

$\Rightarrow a=12$, $b=-10$,and $c=36 \Rightarrow$ the equation is $x^2+y^2+12x-10y+36=0$. By completing the square and simplifying we get the equation $(x+6)^2+(y-5)^2=25$. The center of this circle is at $(-6,5)$ and the radius is $r=5$.

47. We have: $\frac{3x+1}{x^2+2x-3}=\frac{A}{x-1}+\frac{B}{x+3} \Leftrightarrow (x+3)\,A+(x-1)\,B=3x+1 \Leftrightarrow$
$x\,(A+B)+(3A-B)=3x+1.$

Equating the coefficients of x and constants we get:

$A+B=3,\ 3A-B=1 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow A=1,\ B=2 \Rightarrow$

The partial fraction decomposition is $\frac{3x+1}{x^2+2x-3}=\frac{1}{x-1}+\frac{2}{x+3}$.

48. We have: $\frac{x^2-3x+3}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Leftrightarrow$

$$\left(x^2+2x+1\right)A + \left(x^2+x\right)B + (x)\,C = x^2 - 3x + 3 \Leftrightarrow$$
$$x^2(A+B) + x(2A+B+C) + (A) = x^2 - 3x + 3.$$

Equating coefficients, we get $A+B=1$, $2A+B+C=-3$, $A=3$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & -3 \\ 1 & 0 & 0 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -7 \end{array}\right] \Rightarrow A=3,\ B=-2,\ C=-7 \Rightarrow$$

The partial fraction decomposition is $\frac{x^2-3x+3}{x^3+2x^2+x} = \frac{3}{x} + \frac{-2}{x+1} + \frac{-7}{(x+1)^2}$.

49. We have: $\frac{x-1}{(x+1)(x^2+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4} \Leftrightarrow$

$$\left(x^4+5x^2+4\right)A + \left(x^3+4x+x^2+4\right)(Bx+C) + \left(x^3+x+x^2+1\right)(Dx+E) = x-1 \Leftrightarrow$$
$$x^4(A+B+D) + x^3(B+C+D+E) + x^2(5A+4B+C+D+E)$$
$$+x(4C+4B+E+D) + (4A+4C+E) = x-1$$

Equating coefficients, we get

$$\begin{aligned} A+B+D &= 0 \\ B+C+D+E &= 0 \\ 5A+4B+C+D+E &= 0 \\ 4B+4C+D+E &= 1 \\ 4A+4C+E &= -1 \end{aligned}$$

From these, we form the augmented matrix and perform Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 5 & 4 & 1 & 1 & 1 & 0 \\ 0 & 4 & 4 & 1 & 1 & 1 \\ 4 & 0 & 4 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{2}{15} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{5} \end{array}\right]$$

So, $A = -\frac{1}{5}$, $B = \frac{1}{3}$, $C = 0$, $D = -\frac{2}{15}$, and $E = -\frac{1}{5}$, and we have

$$\frac{x-1}{(x+1)\left(x^2+1\right)\left(x^2+4\right)} = -\frac{\frac{1}{5}}{x+1} + \frac{\frac{1}{3}x}{x^2+1} - \frac{\frac{1}{15}(2x+3)}{x^2+4}.$$

50. We have the equation

$$\frac{x^3+x+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

We multiply both sides of the above equation by $x(x-1)\left(x^2+x+1\right)\left(x^2+1\right)^3$, simplify, and equate the coefficients of x^n to get the system:

$$\begin{array}{rr} A+B+C+E=0 & -3A+3B-3C+3D-2E+F-G+H+J=0 \\ B-C+D+F=0 & A+4B+C-3D-2F-H=1 \\ 3A+4B+3C-D+2E+G=0 & -3A+B-C+D-E-G-I=0 \\ -A+3B-3C+3D-E+2F+H=0 & B-D-F-H-J=1 \\ 3A+6B+3C-3D+E-F+G+I=0 & -A=1 \end{array}$$

From these we form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{cccccccccc|c} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 3 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -3 & 3 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 6 & 3 & -3 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ -3 & 3 & -3 & 3 & -2 & 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 4 & 1 & -3 & 0 & -2 & 0 & -1 & 0 & 0 & 1 \\ -3 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{15}{8} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{9}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{7}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

We find that the partial fraction decomposition is

$$\frac{x^3+x+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = -\frac{1}{x} + \frac{\frac{1}{8}}{x-1} - \frac{x}{x^2+x+1} + \frac{\frac{3}{8}(5x-3)}{x^2+1} + \frac{\frac{1}{4}(7x-1)}{(x^2+1)^2} + \frac{\frac{1}{2}(x+1)}{(x^2+1)^3}$$

51. Assume $1+2+\cdots+n = an^2+bn+c$, and let $n = 0, 1, 2$ to get

$$\begin{aligned} c &= 0 \\ a+b+c &= 1 \\ 4a+2b+c &= 3 \end{aligned}$$

From these we form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 0$, and we find that $1+2+\cdots+n = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1)$.

52. Assume $1^2+2^2+\cdots+n^2=an^3+bn^2+cn+d$, and let $n=0,1,2,3$ to get

$$\begin{aligned} d &= 0 \\ a+b+c+d &= 1 \\ 8a+4b+2c+d &= 5 \\ 27a+9b+3c+d &= 14 \end{aligned}$$

From these we form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 5 \\ 27 & 9 & 3 & 1 & 14 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Thus $a=\frac{1}{3}$, $b=\frac{1}{2}$, $c=\frac{1}{6}$, $d=0$, and we find that

$$1^2+2^2+\cdots+n^2=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n=\frac{n(n+1)(2n+1)}{6}.$$

53. Assume $1^3+2^3+\cdots+n^3=an^4+bn^3+cn^2+dn+e$, and let $n=0,1,2,3,4$ to get

$$\begin{aligned} e &= 0 \\ a+b+c+d+e &= 1 \\ 16a+8b+4c+2d+e &= 9 \\ 81a+27b+9c+3d+e &= 36 \\ 256a+64b+16c+4d+e &= 100 \end{aligned}$$

From these we form the augmented matrix and perform Gauss-Jordan elimination to get

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 & 9 \\ 81 & 27 & 9 & 3 & 1 & 36 \\ 256 & 64 & 16 & 4 & 1 & 100 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Thus $a=\frac{1}{4}$, $b=\frac{1}{2}$, $c=\frac{1}{4}$, $d=0$, $e=0$, and we find that

$$1^3+2^3+\cdots+n^3=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2=\frac{1}{4}n^2(n+1)^2=\left(\frac{n(n+1)}{2}\right)^2.$$

2.5 Iterative Methods for Solving Linear Systems

1. Begin by solving the first equation for x_1 and the second equation for x_2 to obtain

$$x_1 = \frac{6}{7} + \frac{1}{7}x_2$$
$$x_2 = \frac{4}{5} + \frac{1}{5}x_1$$

Using the initial vector $[x_1, x_2] = [0, 0]$, we get a sequence of approximations:

n	0	1	2	3	4	5
x_1	0	0.857	0.971	0.996	0.999	1.000
x_2	0	0.800	0.971	0.994	0.999	0.999

The exact solution to this system is $[x_1, x_2] = [1, 1]$.

2.

n	0	1	2	3	4	5	6	7	8
x_1	0	2.5	3.0	1.75	1.5	2.125	2.25	1.9375	1.875
x_2	0	−1.0	1.5	2.0	0.75	0.5	1.125	1.25	0.9375

n	9	10	11	12	13	14	15	16
x_1	2.031	2.063	1.984	1.969	2.008	2.016	1.996	1.992
x_2	0.8750	1.031	1.063	0.984	0.969	1.008	1.016	0.996

n	17	18	19	20	21	22
x_1	2.002	2.004	1.999	1.998	2.001	2.000
x_2	0.9922	1.002	1.003	0.999	1.001	1.001

The exact solution is $[x_1, x_2] = [2, 1]$.

3.

n	0	1	2	3	4	5
x_1	0	0.2222	0.2540	0.2610	0.2620	0.2623
x_2	0	0.2857	0.3492	0.3583	0.3603	0.3606

The exact solution is $x = \frac{16}{61}$, $y = \frac{22}{61}$.

4.

n	0	1	2	3	4	5
x_1	0	0.8500	1.005	1.003	1.001	1.000
x_2	0	−1.300	−1.035	−0.9980	−0.9993	−1.000
x_3	0	1.800	2.015	2.004	2.000	2.000

The exact solution is $[x_1, x_2, x_3] = [1, -1, 2]$.

5.

n	0	1	2	3	4	5	6	7	8
x_1	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997
x_2	0	0.2500	0.08337	0.1250	0.09722	0.1042	0.09957	0.1008	0.09997
x_3	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997

The exact solution is $[x_1, x_2, x_3] = [0.3, 0.1, 0.3]$

6.

n	0	1	2	3	4	5
x_1	0	0.3333	0.3333	0.4074	0.4198	0.4403
x_2	0	0	0.2222	0.2593	0.3210	0.3333
x_3	0	0.3333	0.4444	0.5556	0.5803	0.6132
x_4	0	0.3333	0.4444	0.4815	0.5185	0.5268

n	6	7	8	9	10	11
x_1	0.4444	0.4504	0.4516	0.4533	0.4537	0.4542
x_2	0.3512	0.3548	0.3600	0.3611	0.3626	0.3629
x_3	0.6200	0.6296	0.6316	0.6344	0.6350	0.6358
x_4	0.5377	0.5400	0.5432	0.5439	0.5448	0.5450

The exact solution is $[x_1, x_2, x_3, x_4] = \left[\frac{5}{11}, \frac{4}{11}, \frac{7}{11}, \frac{6}{11}\right]$.

7.

n	0	1	2	3	4
x_1	0	0.8571	0.9959	0.9999	1.000
x_2	0	0.9714	0.9992	0.9999	1.000

The Gauss-Seidel method takes 4 steps while the Jacobi method takes 5.

8.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
x_1	0	2.5	1.75	2.125	1.938	2.031	1.984	2.008	1.996	2.002	1.999	2.001	2.000
x_2	0	1.5	0.75	1.125	0.938	1.031	0.984	1.008	0.996	1.002	0.999	1.001	1.000

The Gauss-Seidel method takes 12 steps while the Jacobi method takes 22.

9.

n	0	1	2	3	4
x_1	0	0.2222	0.2610	0.2623	0.2623
x_2	0	0.3492	0.3603	0.3607	0.3607

The Gauss-Seidel method takes 4 steps while the Jacobi method takes 5.

10.

n	0	1	2	3	4
x_1	0	0.85	1.011	1.000	1.000
x_2	0	-1.215	-0.998	-1.000	-1.000
x_3	0	2.007	2.001	2.000	2.000

The Gauss-Seidel method takes 4 steps while the Jacobi method takes 5.

11.

n	0	1	2	3	4	5
x_1	0	0.3333	0.2778	0.2963	0.2994	0.2999
x_2	0	0.1667	0.1111	0.1019	0.1003	0.1001
x_3	0	0.2778	0.2963	0.2994	0.2999	0.2999

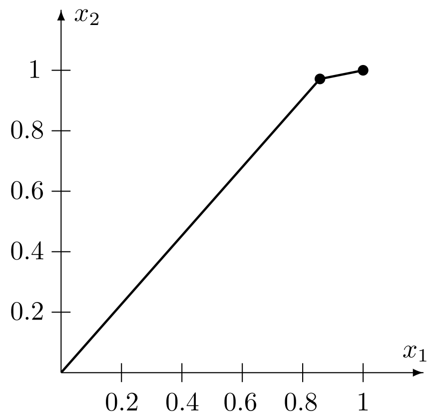
The Gauss-Seidel method takes 5 steps while the Jacobi method takes 8.

12.

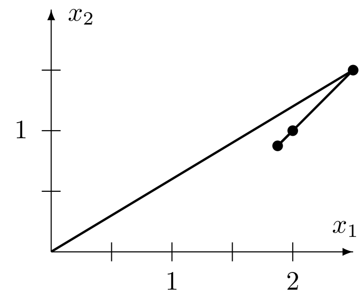
n	0	1	2	3	4	5	6	7
x_1	0	0.3333	0.3704	0.4156	0.4426	0.4511	0.4535	0.4543
x_2	0	0.1111	0.2469	0.3279	0.3532	0.3606	0.3628	0.3634
x_3	0	0.3704	0.5679	0.6168	0.6307	0.6347	0.6359	0.6362
x_4	0	0.4568	0.5226	0.5389	0.5436	0.5449	0.5453	0.5454

The Gauss-Seidel method takes 7 steps while the Jacobi method takes 11.

13.



14.



15. Applying the Gauss-Seidel method to $x_1 - 2x_2 = 3$, $3x_1 + 2x_2 = 1$ gives:

n	0	1	2	3	4
x_1	0	3	−5	19	−53
x_2	0	−4	8	−28	80

which evidently diverges. If, however, we swap the two equations to get $3x_1+2x_2 = 1$, $x_1-2x_2 = 3$ and use the Gauss-Seidel method on this system we get the table:

n	0	1	2	3	4	5	6	7	8	9
x_1	0	0.333	1.222	0.926	1.025	0.992	1.003	0.999	1.000	1.000
x_2	0	−1.334	−0.889	−1.037	−0.988	−1.004	−0.999	−1.000	−1.000	−1.000

Thus the solution to the system of equations is approximately $[x_1, x_2] = [1.000, -1.000]$. The exact solution is $[1, -1]$.

16. Applying the Gauss-Seidel method to the system of equations

$$\begin{aligned} x_1 - 4x_2 + 2x_3 &= 2 \\ 2x_2 + 4x_3 &= 1 \\ 6x_1 - x_2 - 2x_3 &= 1 \end{aligned}$$

gives the following table of iterations:

n	0	1	2	3	4
x_1	0	2	-6.5	-8	203.5
x_2	0	0.5	-10	30.5	80
x_3	0	5.250	-15	-39.75	570

which evidently diverges. If, however, we rearrange the equations to make them diagonally dominant we get the following system of equations:

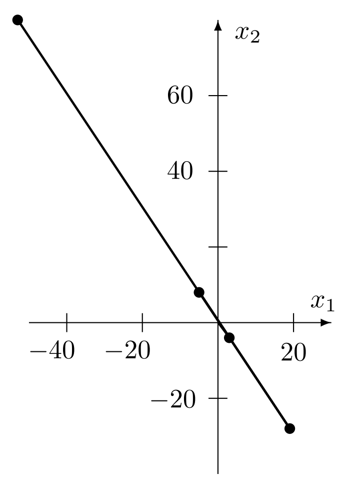
$$\begin{aligned} 6x_1 - x_2 - 2x_3 &= 1 \\ x_1 - 4x_2 + 2x_3 &= 2 \\ 2x_2 + 4x_3 &= 1 \end{aligned}$$

Using the Gauss-Seidel method on this system we get the following table of iterations:

n	0	1	2	3	4	5	6	7
x_1	0	0.167	0.250	0.250	0.250	0.250	0.250	0.250
x_2	0	-0.458	-0.198	-0.263	-0.246	-0.251	-0.250	-0.250
x_3	0	0.479	0.349	0.382	0.373	0.376	0.375	0.375

Thus the solution to the system of equations is approximately $[x_1, x_2, x_3] = [0.250, -0.250, 0.375]$. The exact solution is $\left[\frac{1}{4}, -\frac{1}{4}, \frac{3}{8}\right]$.

17.



18. Applying the Gauss-Seidel method to the system of equations

$$\begin{aligned} -4x_1 + 5x_2 &= 14 \\ x_1 - 3x_2 &= -7 \end{aligned}$$

gives the following table of iterations:

n	0	1	2	3	4	5	6	7	8	9
x_1	0	−3.5	−2.041	−1.434	−1.181	−1.075	−1.031	−1.013	−1.006	−1.002
x_2	0	1.167	1.653	1.855	1.940	1.975	1.990	1.996	1.998	1.999

Thus the solution to the system of equations is approximately $[x_1, x_2] = [-1.01, 1.99]$. The exact solution is $[-1, 2]$.

19. Applying the Gauss-Seidel method to the system of equations

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -8 \\ x_1 + 4x_2 - 4x_3 &= 102 \\ -2x_1 - 2x_2 + 4x_3 &= -90 \end{aligned}$$

gives the following table of iterations:

n	0	1	2	3	4	5	6
x_1	0	−1.60	14.97	8.55	10.74	9.84	10.12
x_2	0	25.9	11.41	14.05	11.62	11.72	11.25
x_3	0	−10.35	−9.31	−11.20	−11.32	−11.72	−11.82

n	7	8	9	10	11	12	13
x_1	9.99	10.02	10.00	10.01	10.00	10.00	10.00
x_2	11.18	11.08	11.05	11.02	11.01	11.00	11.00
x_3	−11.91	−11.95	−11.98	−11.99	−12.00	−12.00	−12.00

Thus the solution to the system of equations is approximately $[x_1, x_2, x_3] = [10.00, 11.00, -12.00]$. The exact solution is $[10, 11, -12]$.

20. Continuing the iterations of Exercise 18 to achieve a solution accurate to within 0.001 gives the following table:

n	0	1	···	9	10	12	13
x_1	0	−3.5	···	−1.002	−1.001	−1.000	−1.000
x_2	0	1.167	···	1.999	2.000	2.000	2.000

21. Continuing iterations of Exercise 19 to achieve a solution accurate to within 0.001:

n	0	1	···	13	14	15	16	17
x_1	0	−1.60	···	10.001	10.000	10.000	10.000	10.000
x_2	0	25.9	···	11.004	11.002	11.001	11.000	11.000
x_3	0	−10.35	···	−11.998	−11.999	−12.000	−12.000	−12.000

22. Let the equilibrium temperatures of the interior points be t_1, t_2, and t_3 as shown. Thus, by the temperature-averaging property, we have

$$t_1 = \tfrac{1}{4}(0 + 40 + 40 + t_2)$$
$$t_2 = \tfrac{1}{4}(0 + t_1 + t_3 + 5)$$
$$t_3 = \tfrac{1}{4}(t_2 + 40 + 40 + 5)$$

The Gauss-Seidel method gives the following with initial approximation $t_1 = t_2 = t_3 = 0$:

n	0	1	2	3	4	5	6	7	8
t_1	0	20.000	21.563	23.086	23.277	23.300	23.303	23.304	23.304
t_2	0	6.250	12.344	13.106	13.201	13.213	13.214	13.215	13.215
t_3	0	22.813	24.336	24.527	24.550	24.553	24.554	24.554	24.554

Thus the equilibrium temperatures at the interior points are $t_1 = 23.304$, $t_2 = 13.314$, and $t_3 = 24.544$ (to an accuracy of 0.001).

23. From the temperature-averaging property we get the system of four equations

$$t_1 = \frac{1}{4}(t_2 + t_3)$$
$$t_2 = \frac{1}{4}(t_1 + t_4)$$
$$t_3 = \frac{1}{4}(t_1 + t_4 + 200)$$
$$t_4 = \frac{1}{4}(t_2 + t_3 + 200)$$

upon which we apply the Gauss-Seidel method. With $t_1 = t_2 = t_3 = t_4 = 0$ we get:

n	0	1	2	3	4	5	6	7	8	9	10
t_1	0	0	12.5	21.876	24.220	24.806	24.952	24.988	24.998	25.000	25.000
t_2	0	0	18.75	23.438	24.610	24.904	24.976	24.994	25.000	25.000	25.000
t_3	0	50	68.75	73.438	74.610	74.904	74.976	74.994	75.000	75.000	75.000
t_4	0	62.5	71.876	74.220	74.806	74.952	74.988	74.998	75.000	75.000	75.000

Thus the equilibrium temperatures at the interior points are $t_1 = 25.000$, $t_2 = 25.000$, $t_4 = 75.000$ (to an accuracy of 0.001).

24. As in the previous exercises we approach the problem using the Gauss-Seidel method. The equations are

$$
\begin{aligned}
t_1 &= \frac{1}{4}(t_2 + t_3) \\
t_2 &= \frac{1}{4}(t_1 + t_4 + 40) \\
t_3 &= \frac{1}{4}(t_1 + t_4 + 80) \\
t_4 &= \frac{1}{4}(t_2 + t_3 + 200)
\end{aligned}
$$

which gives the following table of iterations with an initial approximation $t_1 = t_2 = t_3 = 0$:

n	0	1	2	3	4	5	6	7	8	9	10
t_1	0	0	7.5	15.625	17.656	18.164	18.292	18.324	18.332	18.334	18.334
t_2	0	10	26.25	30.312	31.328	31.582	31.646	31.662	31.666	31.668	31.668
t_3	0	20	36.25	40.312	41.328	41.582	41.646	41.662	41.666	41.668	41.668
t_4	0	57.5	65.625	67.656	68.164	68.292	68.324	68.332	68.334	68.334	68.334

Thus the equilibrium temperatures at the interior points are $t_1 = 18.333$, $t_2 = 31.666$, $t_3 = 41.666$, and $t_4 = 68.333$ (to an accuracy of 0.001).

25. Here the equations are

$$
\begin{aligned}
t_1 &= \frac{1}{4}(t_2 + 80)\\
t_2 &= \frac{1}{4}(t_1 + t_3 + t_4)\\
t_3 &= \frac{1}{4}(t_2 + t_5 + 80)\\
t_4 &= \frac{1}{4}(t_2 + t_5 + 5)\\
t_5 &= \frac{1}{4}(t_3 + t_4 + t_6 + 5)\\
t_6 &= \frac{1}{4}(t_5 + 85)
\end{aligned}
$$

Following the same procedure as in the previous exercises, we get the following table:

n	0	1	2	3	4	5	6
t_1	0	20	21.250	22.813	23.330	23.660	23.773
t_2	0	5	11.250	13.321	14.639	15.093	15.237
t_3	0	21.25	24.609	26.988	27.731	27.963	28.035
t_4	0	2.5	5.859	8.238	8.981	9.213	9.285
t_5	0	7.188	14.629	16.283	16.758	16.904	16.949
t_6	0	23.047	24.907	25.321	25.440	25.476	25.487

n	7	8	9	10	11	12
t_1	23.809	23.821	23.824	23.825	23.826	23.826
t_2	15.282	15.297	15.301	15.302	15.304	15.304
t_3	28.058	28.065	28.067	28.068	28.069	28.069
t_4	9.308	9.315	9.317	9.318	9.319	9.319
t_5	16.963	16.968	16.969	16.970	16.970	16.970
t_6	25.491	25.492	25.492	25.493	25.493	25.493

So the equilibrium temperatures at the interior points are found to be about $t_1 = 23.826$, $t_2 = 15.304$,
$t_3 = 28.069$, $t_4 = 9.319$, $t_5 = 16.970$, and $t_6 = 25.493$.

26. Here the equations are

$$t_1 = \tfrac{1}{4}(t_2 + t_5) \qquad t_9 = \tfrac{1}{4}(t_5 + t_{10} + t_{13} + 40)$$
$$t_2 = \tfrac{1}{4}(t_1 + t_3 + t_6) \qquad t_{10} = \tfrac{1}{4}(t_6 + t_9 + t_{11} + t_{14})$$
$$t_3 = \tfrac{1}{4}(t_2 + t_4 + t_7 + 20) \qquad t_{11} = \tfrac{1}{4}(t_7 + t_{10} + t_{12} + t_{15})$$
$$t_4 = \tfrac{1}{4}(t_3 + t_8 + 40) \qquad t_{12} = \tfrac{1}{4}(t_8 + t_{11} + t_{16} + 100)$$
$$t_5 = \tfrac{1}{4}(t_1 + t_6 + t_9) \qquad t_{13} = \tfrac{1}{4}(t_9 + t_{14} + 80)$$
$$t_6 = \tfrac{1}{4}(t_2 + t_5 + t_7 + t_{10}) \qquad t_{14} = \tfrac{1}{4}(t_{10} + t_{13} + t_{15} + 40)$$
$$t_7 = \tfrac{1}{4}(t_3 + t_6 + t_8 + t_{11}) \qquad t_{15} = \tfrac{1}{4}(t_{11} + t_{14} + t_{16} + 100)$$
$$t_8 = \tfrac{1}{4}(t_4 + t_7 + t_{12} + 20) \qquad t_{16} = \tfrac{1}{4}(t_{12} + t_{15} + 200)$$

Following the same procedure as in the previous exercises, we get the following tables:

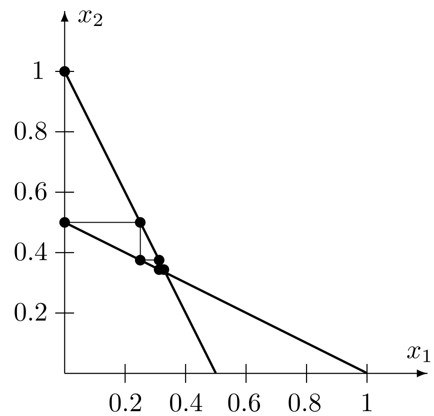
n	0	1	2	3	4	5	6	7	8	9	10
t_1	0	0	0	0.938	1.934	2.8529	3.996	5.194	6.143	6.817	7.275
t_2	0	0	1.25	2.813	4.444	6.6605	9.035	10.925	12.270	13.186	13.795
t_3	0	5	8.438	10.450	13.425	16.637	19.082	20.781	21.924	22.682	23.180
t_4	0	11.25	14.141	16.753	19.533	21.544	22.893	23.782	24.366	24.748	24.999
t_5	0	0	2.5	4.922	6.968	9.324	11.742	13.646	14.996	15.912	16.522
t_6	0	0	1.875	5.391	10.364	15.506	19.422	22.157	24.002	25.225	26.031
t_7	0	1.25	4.844	12.501	20.356	25.747	29.307	31.644	33.175	34.176	34.832
t_8	0	8.125	16.563	24.708	29.538	32.489	34.346	35.539	36.311	36.814	37.143
t_9	0	10	16.875	20.547	24.077	27.467	29.966	31.683	32.831	33.590	34.089
t_{10}	0	2.5	8.9845	17.544	25.683	31.162	34.749	37.094	38.627	39.630	40.286
t_{11}	0	0.938	17.598	32.929	41.308	46.235	49.286	51.230	52.485	53.300	53.832
t_{12}	0	27.266	49.575	58.264	62.662	65.183	66.727	67.704	68.333	68.741	69.007
t_{13}	0	22.5	28.281	31.797	34.859	36.959	38.335	39.233	39.819	40.203	40.453
t_{14}	0	16.25	26.641	35.359	40.368	43.372	45.247	46.445	47.220	47.724	48.052
t_{15}	0	29.297	52.095	60.927	65.369	67.904	69.452	70.430	71.060	71.468	71.734
t_{16}	0	64.141	75.418	79.798	82.008	83.272	84.045	84.534	84.848	85.052	85.186

n	21	22	23	24	25	26	27	28	29	30
t_1	8.156	8.159	8.161	8.162	8.163	8.163	8.164	8.164	8.164	8.164
t_2	14.954	14.957	14.960	14.962	14.963	14.963	14.964	14.964	14.964	14.964
t_3	24.119	24.123	24.125	24.127	24.127	24.127	24.127	24.127	24.127	24.127
t_4	25.469	25.471	25.472	25.473	25.473	25.473	25.473	25.473	25.473	25.473
t_5	17.681	17.685	17.687	17.689	17.690	17.691	17.691	17.691	17.691	17.691
t_6	27.551	27.555	27.559	27.561	27.563	27.564	27.564	27.564	27.564	27.564
t_7	36.063	36.067	36.070	36.072	36.073	36.073	36.073	36.073	36.073	36.073
t_8	37.759	37.762	37.763	37.763	37.764	37.764	37.764	37.764	37.764	37.764
t_9	35.028	35.031	35.033	35.034	35.037	35.037	35.037	35.037	35.037	35.037
t_{10}	41.517	41.521	41.523	41.526	41.527	41.527	41.527	41.527	41.527	41.527
t_{11}	54.829	54.832	54.835	54.836	54.836	54.836	54.836	54.836	54.836	54.836
t_{12}	69.506	69.508	69.509	69.509	69.509	69.509	69.509	69.509	69.509	69.509
t_{13}	40.923	40.925	40.925	40.927	40.927	40.927	40.927	40.927	40.927	40.927
t_{14}	48.668	48.669	48.671	48.673	48.673	48.673	48.673	48.673	48.673	48.673
t_{15}	72.233	72.234	72.236	72.236	72.236	72.236	72.236	72.236	72.236	72.236
t_{16}	85.435	85.436	85.436	85.436	85.436	85.436	85.436	85.436	85.436	85.436

Column 30 gives the equilibrium temperatures to an accuracy of 0.001.

27. (a) Let x_1 correspond to the left end of the paper and x_2 to the right end, and let n be the number of folds. Then the first six values of $[x_1, x_2]$ are

n	0	1	2	3	4	5	6
x_1	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{11}{32}$	$\frac{11}{32}$



(b) Two linear equations that determine the new values of the endpoints at each iteration are $x_2 = -\frac{1}{2}x_1 + \frac{1}{2}$ and $x_1 = -\frac{1}{2}x_2 + \frac{1}{2}$. These two lines are plotted in part (a).

(c) Switching to decimal representation, we continue applying the Gauss-Seidel method to approximate the point of convergence, giving rise to the sequence of endpoints:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
x_1	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$	0.328	0.332	0.332	0.333	0.333	0.333
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{11}{32}$	$\frac{11}{32}$	0.336	0.336	0.334	0.334	0.333	0.333

(d) We have the system of equations $x_2 = -\frac{1}{2}x_1 + \frac{1}{2}$, $x_1 = -\frac{1}{2}x_2 + \frac{1}{2} \Rightarrow$

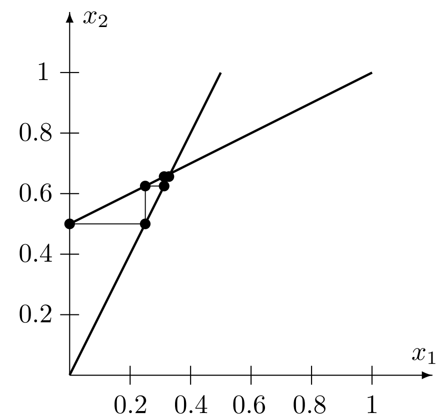
$$\left[\begin{array}{cc|c} \frac{1}{2} & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{array}\right]$$

Hence the exact solution to the system of equations is $[x_1, x_2] = \left[\frac{1}{3}, \frac{1}{3}\right]$.
The ends of the paper converge at $\frac{1}{3}$.

28. The key is the ant always goes halfway to one of the original endpoints, 0 or 1.

(a) Let x_1 record the positions of the left-hand endpoints of the line segments and x_2 their right-hand endpoints at the end of each walk. Then the first six values of $[x_1, x_2]$ are

n	0	1	2	3	4	5	6
x_1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$	$\frac{21}{64}$
x_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{21}{32}$	$\frac{21}{32}$	$\frac{85}{128}$



(b) Two linear equations that determine the new values of the endpoints at each iteration are $x_2 = \frac{1}{2}x_1 + \frac{1}{2}$ and $x_1 = \frac{1}{2}x_2$. These two lines are plotted in part (a).

(c) Switching to decimal representation, we continue applying the Gauss-Seidel method to approximate the point of convergence, giving rise to the sequence of endpoints presented in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11
x_1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$	$\frac{21}{64}$	0.332	0.332	0.333	0.333	0.333
x_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{21}{32}$	$\frac{21}{32}$	$\frac{85}{128}$	0.664	0.666	0.666	0.666	0.666

(d) We have the system of equations $x_2 = \frac{1}{2}x_1 + \frac{1}{2}$, $x_1 = \frac{1}{2}x_2 \Rightarrow$

$$\left[\begin{array}{rr|r} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{array}\right]$$

Hence the exact solution to the system of equations is $[x_1, x_2] = \left[\frac{1}{3}, \frac{2}{3}\right]$.
So, the ant eventually oscillates between $\frac{1}{3}$ and $\frac{2}{3}$.

Chapter 2 Review

1. We will explain and give counter examples to justify our answers below.

(a) ***False***. In Section 2.1, see the definition of an *inconsistent* system.
A useful counter example is parallel lines in $\mathbb{R}^2$, like $x + y = 0$ and $x + y = 1$.

(b) ***True***. In Section 2.2, see remarks prior to the definition of an *homogenous* system.
Q: Why does a homogenous system guarantee the associated lines intersect?
A: Because all the associated lines pass through the origin.

(c) ***False***. In Section 2.2, see Theorem 2.6 (system must be homogenous).
When there are fewer conditions than variables,
we can solve a homogenous system.

(d) ***False***. In Section 2.2, see remarks under *Homogenous Systems* on p78.
When a system has more equations than variables,
it can either have a unique solution, infinitely many solutions, or no solution.

(e) ***True***. In Section 2.3, see Theorem 2.4 ($[A\,|\,\mathbf{b}]$ is consistent $\Leftrightarrow \mathbf{b} = \sum c_i\mathbf{a}_i$).
Q: How might we state an informal proof of this theorem?
A: The fact that $\mathbf{b} = \sum c_i\mathbf{a}_i$ says $\mathbf{x} = [c_i]$ is a solution of $[A\,|\,\mathbf{b}]$.

(f) ***False***. We need an additional condition to make this true. Which one?
Q: If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v}$ are linearly dependent, what is $\text{span}(\mathbf{u}, \mathbf{v})$?
A: Then $\text{span}(\mathbf{u}, \mathbf{v}) = \text{span}(\mathbf{u}) = c\mathbf{u}$, a line through the origin.
Q: If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, what is $\text{span}(\mathbf{u}, \mathbf{v})$?
A: Then $\text{span}(\mathbf{u}, \mathbf{v}) = c\mathbf{u} + d\mathbf{v}$, a plane a line through the origin.

(g) ***True***. Show this by proving the *contrapositive* (See Example 9 of Appendix A).
Q: What is the *contrapositive* of this statement?
A: If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then they are linearly dependent.
Q: How might we prove this statement is true?
A: If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\mathbf{v} = c\mathbf{u} \Rightarrow -c\mathbf{u} + \mathbf{v} = 0$.
Q: Why does the fact that $\mathbf{u}$ and $\mathbf{v}$ are parallel imply $\mathbf{v} = c\mathbf{u}$?
A: Vectors are defined to be *parallel* if they are scalar multiples of each other.
Q: Why does $-c\mathbf{u} + \mathbf{v} = 0$ imply $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent?
A: Two vectors are linearly dependent if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$ (one $c_i \neq 0$).

(h) ***True***. Why? A *closed* path means there has been no displacement.
Q: Since there is no displacement, what do know about the sum of the vectors?
A: Since there is no displacement, we have $\sum \mathbf{v}_i = \mathbf{0}$.
Q: Why does $\sum \mathbf{v}_i = \mathbf{0}$ imply $\mathbf{v}_i$ are linearly dependent?
A: Vectors are linearly dependent if $\sum c_i\mathbf{v}_i = \mathbf{0}$ (at least one $c_i \neq 0$).

(i) ***False***. This pairwise condition is much *weaker* than linear independence.
Consider this counter example: $\mathbf{u} = [1, 0, 0]$, $\mathbf{v} = [0, 1, 0]$, and $\mathbf{w} = [1, 1, 0]$.
Geometrically, consider 3 lines in the same plane none of which are parallel.

(j) ***True***. In Section 2.3, see Thm 2.8 (m vectors in $\mathbb{R}^n$ are linearly dependent if $m > n$).
Q: What is one way of stating Theorem 2.8 informally in our own words?
A: When there are more vectors than entries, we can solve $\sum c_i\mathbf{v}_i = \mathbf{0}$.

2. In Section 2.2, *rank* is defined to be the number of nonzero rows in row echelon form. So, we row reduce A using Gaussian Elimination to determine the number of nonzero rows.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 2 \\ 3 & -1 & 1 & 3 & 4 \\ 3 & 4 & 2 & -3 & 2 \\ 0 & -5 & -1 & 6 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -2 & 0 & 3 & 2 \\ 0 & -5 & -1 & 6 & 2 \\ 3 & 4 & 2 & -3 & 2 \\ 3 & -1 & 1 & 3 & 4 \end{bmatrix} \xrightarrow[R_4 - 3R_1 + R_2]{R_3 - 3R_1 + 2R_2} \begin{bmatrix} 1 & -2 & 0 & 3 & 2 \\ 0 & -5 & -1 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the reduced echelon form of A has 2 nonzero rows, the rank of A is 2.

3. As in Example 2.12 of Section 2.2, we form the augmented matrix and row reduce to solve.

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 1 & 3 & -1 & 7 \\ 2 & 1 & -5 & 7 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & -1 & -1 & -1 \end{array}\right] \xrightarrow{-2R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 2 & 2 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 - 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

4. As in Example 2.11 of Section 2.2, we form the augmented matrix and row reduce to solve.

$$\left[\begin{array}{cccc|c} 3 & 8 & -18 & 1 & 35 \\ 1 & 2 & -4 & 0 & 11 \\ 1 & 3 & -7 & 1 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

So, $z = -4$, $x - 3y = 3$ and $w + 2y = 5$. Setting $y = t$ yields $x = 3 + 3t$ and $w = 5 - 2t$.

Therefore, the solution is $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$

5. As in Example 2.16 of Section 2.2, we form the augmented matrix and row reduce over $\mathbb{Z}_7$. Since we are using modular arithmetic, we need only add and multiply. Why?

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow{R_2 + 3R_1} \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 4 & 1 \end{array}\right] \xrightarrow{2R_2} \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 + 4R_2} \left[\begin{array}{cc|c} 2 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{4R_1} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 2 \end{array}\right]$$

So, the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$. We can check this: $\begin{array}{rcccl} 2(6) + 3(2) & = & 18 & = & 4 \\ 6 + 2(2) & = & 10 & = & 3 \end{array}$ in $\mathbb{Z}_7$.

6. As in Example 2.16 of Section 2.2, we form the augmented matrix and row reduce over $\mathbb{Z}_5$. Since we are using modular arithmetic, we need only add and multiply. Why?

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 4 & 2 \end{array}\right] \xrightarrow{R_2+3R_1} \left[\begin{array}{cc|c} 3 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

We now find all the solutions by setting $x = 0, 1, 2, 3, 4$ and solving for y.

When $x = 0$, $0 + 4y = 2 \Rightarrow y = 3$, so the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

When $x = 1$, $1 + 4y = 2 \Rightarrow y = 4$, so the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

When $x = 2$, $2 + 4y = 2 \Rightarrow y = 0$, so the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

When $x = 3$, $3 + 4y = 2 \Rightarrow y = 1$, so the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

When $x = 4$, $4 + 4y = 2 \Rightarrow y = 2$, so the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Q: Each possible value for y occurs exactly one time. Is this what we should expect?

7. As in Exercise 40 of Section 2.2, we row reduce to find the restrictions on k.
Note: The system has no solution when A has a zero row with corresponding constant $\neq 0$.

$$\left[\begin{array}{cc|c} k & 2 & 1 \\ 1 & 2k & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & 2k & 1 \\ k & 2 & 1 \end{array}\right] \xrightarrow{R_2 - kR_1} \left[\begin{array}{cc|c} 1 & 2k & 1 \\ 0 & 2-2k^2 & 1-k \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 2k & 1 \\ 0 & 2(k-1)(k+1) & k-1 \end{array}\right]$$

So, the only value of k that creates a zero row with corresponding constant $\neq 0$ is $k = -1$.
That is, the only value of k for which this system is inconsistent is $k = -1$.

8. As in Example 2.14 of Section 2.2, we form the augmented matrix and row reduce to solve.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right] \xrightarrow{R_2-5R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

So, $x - z = -2$, $y + 2z = 3$. Setting $z = t$ yields $x = -2 + t$ and $y = 3 - 2t$.

We get the line with parametric equation: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

9. As in Example 2.15 of Section 2.2, we need to determine the point of intersection. We want to find an $\mathbf{x} = [x, y, z]$ that satisfies both equations simultaneously. That is, we want $\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$ or $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Substituting the given $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$, and $\mathbf{v}$ into $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$, we obtain the equations:

$$s\begin{bmatrix}1\\-1\\2\end{bmatrix} - t\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\-2\\-4\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} \Rightarrow \begin{array}{rcrcr} s & + & t & = & 4\\ -s & - & t & = & -4\\ 2s & - & t & = & -7\end{array}$$

We form the augmented matrix and row reduce to find values for s and t.

$$\left[\begin{array}{rr|r}1 & 2 & 4\\-1 & -1 & -4\\2 & -1 & -7\end{array}\right] \longrightarrow \left[\begin{array}{rr|r}1 & 0 & -1\\0 & 0 & 0\\0 & 1 & 5\end{array}\right] \Rightarrow \text{So, } s = -1 \text{ and } t = 5.$$

Therefore, the point of intersection is: $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} - 1\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\3\\1\end{bmatrix}$.

Check that substituting $t = 5$ into the other equation gives the same point.

10. As in Example 2.18 of Section 2.3, we want to find scalars x and y such that:

$$x\begin{bmatrix}1\\1\\3\end{bmatrix} + y\begin{bmatrix}1\\2\\-2\end{bmatrix} = \begin{bmatrix}3\\5\\-1\end{bmatrix} \Rightarrow \begin{array}{rcrcr} x & + & y & = & 3\\ x & + & 2y & = & 5\\ 3x & - & 2y & = & -1\end{array}$$

We form the augmented matrix and row reduce to find values for x and y.

$$\left[\begin{array}{rr|r}1 & 1 & 3\\1 & 2 & 5\\3 & -2 & -1\end{array}\right] \longrightarrow \left[\begin{array}{rr|r}1 & 0 & 1\\0 & 1 & 2\\0 & 0 & 0\end{array}\right] \Rightarrow \text{So } x = 1 \text{ and } y = 2.$$

Since $\begin{bmatrix}3\\5\\-1\end{bmatrix} = \begin{bmatrix}1\\1\\3\end{bmatrix} + 2\begin{bmatrix}1\\2\\-2\end{bmatrix}$, we conclude $\begin{bmatrix}3\\5\\-1\end{bmatrix}$ is in the span of $\begin{bmatrix}1\\1\\3\end{bmatrix}$ and $\begin{bmatrix}1\\2\\-2\end{bmatrix}$.

11. As in Example 2.21 of Section 2.3, the equation of the plane we are looking for is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{rcrcl} s & + & 3t & = & x \\ s & + & 2t & = & y \\ s & + & t & = & z \end{array}$$

We form the augmented matrix and row reduce to find conditions for x, y, and z.

$$\left[\begin{array}{cc|c} 1 & 3 & x \\ 1 & 2 & y \\ 1 & 1 & z \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & -1 & -x+y \\ 0 & -2 & -x+z \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & x-y \\ 0 & -2 & -x+z \end{array}\right] \xrightarrow{R_3 + 2R_2} \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & 1 & x-y \\ 0 & 0 & x-2y+z \end{array}\right]$$

By assumption the system is consistent so $x - 2y + z = 0$, the equation of the plane we sought.

Q: How can we verify that both these vectors lie in the plane?
A: By computing $1 - 2(1) + 1 = 0$ and $3 - 2(2) + 1 = 0$.

Q: What is the cross product of the given vectors?
A: $[-1, 2, -1]$. Should this agree with the normal of our plane? Does it?

12. As in Example 2.23 of Section 2.3, we want to find scalars c_1, c_2, and c_3 such that:

$$c_1 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We form the augmented matrix and row reduce to find values for c_1, c_2, and c_3.

$$\left[\begin{array}{rrr|r} 2 & 1 & 3 & 0 \\ 1 & -1 & 9 & 0 \\ -3 & -2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since $c_1 = -4c_3$, $c_2 = 5c_3$ is a solution, the vectors are linearly dependent.

Setting $c_3 = -1$ yields the dependence relation: $4 \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

13. We use Exercise 21 of Section 2.3 to determine whether or not $\mathbb{R}^3 = \text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

(a) We need only show $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
Why is that enough? Because then we have $\mathbb{R}^3 = \text{span}(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3) = \text{span}(\mathbf{u},\ \mathbf{v},\ \mathbf{w})$.
We begin with $\mathbf{e}_1$. We want to find scalars x, y, and z such that:

$$x\begin{bmatrix}1\\1\\0\end{bmatrix} + y\begin{bmatrix}1\\0\\1\end{bmatrix} + z\begin{bmatrix}0\\1\\1\end{bmatrix} = \mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

We form the augmented matrix and row reduce to find values for x, y, and z.

$$\left[\begin{array}{ccc|c}1&1&0&1\\1&0&1&0\\0&1&1&0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&0&\frac{1}{2}\\0&1&0&\frac{1}{2}\\0&0&1&-\frac{1}{2}\end{array}\right] \Rightarrow \mathbf{e}_1 = \tfrac{1}{2}\mathbf{u} + \tfrac{1}{2}\mathbf{v} - \tfrac{1}{2}\mathbf{w}.$$

Likewise for $\mathbf{e}_2$, we form the augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c}1&1&0&0\\1&0&1&1\\0&1&1&0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&0&\frac{1}{2}\\0&1&0&-\frac{1}{2}\\0&0&1&\frac{1}{2}\end{array}\right] \Rightarrow \mathbf{e}_1 = \tfrac{1}{2}\mathbf{u} - \tfrac{1}{2}\mathbf{v} + \tfrac{1}{2}\mathbf{w}.$$

Finally for $\mathbf{e}_3$, we form the augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c}1&1&0&0\\1&0&1&0\\0&1&1&1\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&0&-\frac{1}{2}\\0&1&0&\frac{1}{2}\\0&0&1&\frac{1}{2}\end{array}\right] \Rightarrow \mathbf{e}_1 = -\tfrac{1}{2}\mathbf{u} + \tfrac{1}{2}\mathbf{v} + \tfrac{1}{2}\mathbf{w}.$$

What do we conclude? $\mathbb{R}^3 = \text{span}(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3) = \text{span}(\mathbf{u},\ \mathbf{v},\ \mathbf{w})$.
We could also have used our intuition to solve this problem. How?

$$\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}. \text{ So } \tfrac{1}{2}\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} = \mathbf{e}_3 \ldots$$

(b) These are clearly linearly dependent because $\mathbf{w} = \mathbf{u} + \mathbf{v}$.
Therefore, $\mathbb{R}^3$ is not equal to $\text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.
In order for a set to span $\mathbb{R}^3$ is must have 3 linearly independent vectors.

14. By Exercise 13, if $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly independent, then $A \longrightarrow I_3$.

Note I_3 is the 3×3 identity matrix. That is $I_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$.

Furthermore, it is obvious that conditions (a), (b), and (c) are equivalent.
$A \longrightarrow I_3 \Rightarrow \text{rank}(A) = 3 \Rightarrow$ the system $[A \mid \mathbf{b}]$ has a unique solution.
Finally, if $[A \mid \mathbf{b}]$ has a unique solution, then $A \longrightarrow I_3$.

15. Since $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly dependent, $\text{rank}(A) \le 2 < 3$.
Since $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are not zero, $0 < \text{rank}(A)$.
Combining these two conditions, we see that $\text{rank}(A)$ must be 1 or 2.

16. By Theorem 2.8 of Section 2.3, the rank of any 5×3 matrix can be at most 3. Why?
Since the rows are vectors in $\mathbb{R}^3$, any set of 4 or more of them are linearly dependent.
What does that tell us? When we row reduce the matrix, we will create at least 2 zero rows.

There is no minimum rank. Why? Any matrix whose entries are all 0 has a rank of 0.

17. As in Exercise 43 of Section 2.3, we apply the definition of linear independence.

We will show that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.
Given $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = 0$, we will show $c_1 = c_2 = 0$.

Multiplying and gathering like terms yields: $(c_1 + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} = 0$.
Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, $c_1 + c_2 = c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = 0$.

Also we could create the matrix of coefficients A and row reduce to determine its rank:

$$\begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ c_1 & - & c_2 & = & 0 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $\text{rank}(A) = 2$ the only solution is the trivial one, so $c_1 = c_2 = 0$.

18. We need only show that $\mathbf{v}$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$.
But this is obvious since $\mathbf{v} = -1(\mathbf{u}) + 1(\mathbf{u} + \mathbf{v})$.

Why is this enough?

What do we conclude?

We conclude that $\text{span}(\mathbf{u}, \mathbf{v}) = \text{span}(\mathbf{u}, \mathbf{u} + \mathbf{v})$.

19. In order for $[A \mid \mathbf{b}]$ to be consistent, $\text{rank}(A)$ must equal $\text{rank}([A \mid \mathbf{b}])$. Why?
A system has no solution when A has a zero row with corresponding constant $\neq 0$.

Note that $\text{rank}([A \mid \mathbf{b}])$ cannot be less than $\text{rank}(A)$. Why not?

20. By the definition in Section 2.2, A and B are row equivalent if $A \longrightarrow B$.
In this case, we note that $A \longrightarrow I_3$ and $B \longrightarrow I_3$. What does that imply?
Since A and B are row equivalent to a common matrix (in this case, I_3),
we conclude that they are row equivalent to each other.

Why?

The steps we take when we row reduce are completely reversible.

So, we have $A \longrightarrow I_3 \longrightarrow B$.