INTRODUCTION TO LINEAR ALGEBRA
Fourth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

2 $\boldsymbol{v}+\boldsymbol{w}=(2,3)$ and $\boldsymbol{v}-\boldsymbol{w}=(6,-1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0,0)$.

3 This problem gives the diagonals $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ of the parallelogram and asks for the sides: The opposite of Problem 2. In this example $\boldsymbol{v}=(3,3)$ and $\boldsymbol{w}=(2,-2)$.

4 $3\boldsymbol{v}+\boldsymbol{w}=(7,5)$ and $c\boldsymbol{v}+d\boldsymbol{w}=(2c+d,c+2d)$.

5 $\boldsymbol{u}+\boldsymbol{v}=(-2,3,1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}=(0,0,0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w}=$ (add first answers) $=(-2,3,1)$. The vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are in the same plane because a combination gives $(0,0,0)$. Stated another way: $\boldsymbol{u}=-\boldsymbol{v}-\boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero. $c=3$ and $d=9$ give $(3,3,-6)$.

7 The nine combinations $c(2,1)+d(0,1)$ with $c=0,1,2$ and $d=(0,1,2)$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.

8 The other diagonal is $\boldsymbol{v}-\boldsymbol{w}$ (or else $\boldsymbol{w}-\boldsymbol{v}$). Adding diagonals gives $2\boldsymbol{v}$ (or $2\boldsymbol{w}$).

9 The fourth corner can be $(4,4)$ or $(4,0)$ or $(-2,2)$. Three possible parallelograms!

10 $\boldsymbol{i}-\boldsymbol{j}=(1,1,0)$ is in the base (x-y plane). $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}=(1,1,1)$ is the opposite corner from $(0,0,0)$. Points in the cube have $0\le x\le 1, 0\le y\le 1, 0\le z\le 1$.

11 Four more corners $(1,1,0),(1,0,1),(0,1,1),(1,1,1)$. The center point is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Centers of faces are $(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},1)$ and $(0,\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2})$.

12 A four-dimensional cube has $2^4=16$ corners and $2\cdot 4=8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum $=$ zero vector. Sum $=-2{:}00$ vector $=8{:}00$ vector. 2:00 is 30° from horizontal $=(\cos\frac{\pi}{6},\sin\frac{\pi}{6})=(\sqrt{3}/2,1/2)$.

14 Moving the origin to 6:00 adds $\boldsymbol{j}=(0,1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\boldsymbol{j}=(0,12)$.

15 The point $\dfrac{3}{4}\boldsymbol{v}+\dfrac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\dfrac{1}{4}\boldsymbol{v}+\dfrac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u}=\dfrac{1}{2}\boldsymbol{v}+\dfrac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v}+\boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

16 All combinations with $c+d=1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V}=-\boldsymbol{v}+2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v}+c\boldsymbol{w}$ are on the line passing through $(0,0)$ and $\boldsymbol{u}=\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v}+\boldsymbol{w}$ and back beyond $(0,0)$. With $c\ge 0$, half of this line is removed, leaving a *ray* that starts at $(0,0)$.

18 The combinations $c\boldsymbol{v}+d\boldsymbol{w}$ with $0\le c\le 1$ and $0\le d\le 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$ then $c\boldsymbol{v}+d\boldsymbol{w}$ fills the unit square.

19 With $c\ge 0$ and $d\ge 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$, then the cone is the whole quadrant $x\ge 0$, $y\ge 0$. *Question*: What if $\boldsymbol{w}=-\boldsymbol{v}$? The cone opens to a half-space.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = \mathbf{1}$.

21 The sum is $(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{w}) =$ **zero vector**. Those three sides of a triangle are in the same plane!

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

23 *All* vectors are combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u} + d\boldsymbol{v}$ fills a plane, then adding $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$.

24 The combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ fill one plane. The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill another plane. Those planes meet in a *line*: *only the vectors* $c\boldsymbol{v}$ are in both planes.

25 (a) For a line, choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w} =$ any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is in the same plane.

26 Two equations come from the two components: $c + 3d = 14$ and $2c + d = 8$. *The solution is* $c = 2$ *and* $d = 4$. Then $2(1,2) + 4(3,1) = (14,8)$.

27 The combinations of $\boldsymbol{i} = (1,0,0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1,1,0)$ fill the xy plane in xyz space.

28 There are **6** unknown numbers $v_1, v_2, v_3, w_1, w_2, w_3$. The six equations come from the components of $\boldsymbol{v} + \boldsymbol{w} = (4,5,6)$ and $\boldsymbol{v} - \boldsymbol{w} = (2,5,8)$. Add to find $2\boldsymbol{v} = (6,10,14)$ so $\boldsymbol{v} = (3,5,7)$ and $\boldsymbol{w} = (1,0,-1)$.

29 Two combinations out of infinitely many that produce $\boldsymbol{b} = (0,1)$ are $-2\boldsymbol{u} + \boldsymbol{v}$ and $\frac{1}{2}\boldsymbol{w} - \frac{1}{2}\boldsymbol{v}$. **No**, three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the x-y plane could fail to produce $\boldsymbol{b}$ if all three lie on a line that does not contain $\boldsymbol{b}$. *Yes*, if one combination produces $\boldsymbol{b}$ then two (and infinitely many) combinations will produce $\boldsymbol{b}$. This is true even if $\boldsymbol{u} = \mathbf{0}$; the combinations can have different $c\boldsymbol{u}$.

30 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0,0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$.

31 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rrrl} 2c & -d & & = 1 \\ -c & +2d & -e & = 0 \\ & -d & +2e & = 0 \end{array} \qquad \begin{array}{l} \text{So } d = 2e \\ \text{then } c = 3e \\ \text{then } 4e = 1 \end{array} \qquad \begin{array}{l} c = 3/4 \\ d = 2/4 \\ e = 1/4 \end{array}$$

Problem Set 1.2, page 19

1 $\boldsymbol{u} \cdot \boldsymbol{v} = -1.8 + 3.2 = 1.4$, $\boldsymbol{u} \cdot \boldsymbol{w} = -4.8 + 4.8 = 0$, $\boldsymbol{v} \cdot \boldsymbol{w} = 24 + 24 = 48 = \boldsymbol{w} \cdot \boldsymbol{v}$.

2 $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 10$. Then $1.4 < (1)(5)$ and $48 < (5)(10)$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{24}{25}$. The vectors $\boldsymbol{w}, \boldsymbol{u}, -\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$.

4 (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = -1$ (b) $(\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = 0$ so $\theta = 90°$ (notice $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$) (c) $(\boldsymbol{v} - 2\boldsymbol{w}) \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 4\boldsymbol{w} \cdot \boldsymbol{w} = 1 - 4 = -3$.

5 $\boldsymbol{u}_1 = \boldsymbol{v}/\|\boldsymbol{v}\| = (3,1)/\sqrt{10}$ and $\boldsymbol{u}_2 = \boldsymbol{w}/\|\boldsymbol{w}\| = (2,1,2)/3$. $\boldsymbol{U}_1 = (1,-3)/\sqrt{10}$ is perpendicular to $\boldsymbol{u}_1$ (and so is $(-1,3)/\sqrt{10}$). $\boldsymbol{U}_2$ could be $(1,-2,0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to $\boldsymbol{u}_2$, and a whole circle of unit vectors in that plane.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. All vectors (x, y, z) with $x+y+z = 0$ lie on a *plane*. All vectors perpendicular to $(1,1,1)$ and $(1,2,3)$ lie on a *line*.

7 (a) $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 1/(2)(1)$ so $\theta = 60°$ or $\pi/3$ radians (b) $\cos\theta = 0$ so $\theta = 90°$ or $\pi/2$ radians (c) $\cos\theta = 2/(2)(2) = 1/2$ so $\theta = 60°$ or $\pi/3$ (d) $\cos\theta = -1/\sqrt{2}$ so $\theta = 135°$ or $3\pi/4$.

8 (a) False: $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors in the plane perpendicular to $\boldsymbol{u}$ (b) True: $\boldsymbol{u}\cdot(\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + 2\boldsymbol{u}\cdot\boldsymbol{w} = 0$ (c) True, $\|\boldsymbol{u}-\boldsymbol{v}\|^2 = (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})$ splits into $\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = \mathbf{2}$ when $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} = 0$.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1 + v_2w_2 = \boldsymbol{v}\cdot\boldsymbol{w} = 0$: perpendicular!

10 Slopes $2/1$ and $-1/2$ multiply to give -1: then $\boldsymbol{v}\cdot\boldsymbol{w} = 0$ and the vectors (the directions) are perpendicular.

11 $\boldsymbol{v}\cdot\boldsymbol{w} < 0$ means angle $> 90°$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1,1)$ perpendicular to $(1,5) - c(1,1)$ if $6 - 2c = 0$ or $c = 3$; $\boldsymbol{v}\cdot(\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v}\cdot\boldsymbol{w}/\boldsymbol{v}\cdot\boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to perpendicular vectors.

13 The plane perpendicular to $(1,0,1)$ contains all vectors $(c, d, -c)$. In that plane, $\boldsymbol{v} = (1,0,-1)$ and $\boldsymbol{w} = (0,1,0)$ are perpendicular.

14 One possibility among many: $\boldsymbol{u} = (1,-1,0,0)$, $\boldsymbol{v} = (0,0,1,-1)$, $\boldsymbol{w} = (1,1,-1,-1)$ and $(1,1,1,1)$ are perpendicular to each other. "We can rotate those $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in their 3D hyperplane."

15 $\frac{1}{2}(x + y) = (2+8)/2 = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.

16 $\|\boldsymbol{v}\|^2 = 1 + 1 + \cdots + 1 = 9$ so $\|\boldsymbol{v}\| = 3$; $\boldsymbol{u} = \boldsymbol{v}/3 = (\frac{1}{3}, \ldots, \frac{1}{3})$ is a unit vector in 9D; $\boldsymbol{w} = (1,-1,0,\ldots,0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to $\boldsymbol{v}$.

17 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v}$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

18 $\|\boldsymbol{v}\|^2 = 4^2 + 2^2 = 20$ and $\|\boldsymbol{w}\|^2 = (-1)^2 + 2^2 = 5$. Pythagoras is $\|(3,4)\|^2 = 25 = 20 + 5$.

19 Start from the rules (1), (2), (3) for $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$ and $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})$ and $(c\boldsymbol{v})\cdot\boldsymbol{w}$. Use rule (2) for $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{v} + (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{w}$. By rule (1) this is $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w}) + \boldsymbol{w}\cdot(\boldsymbol{v}+\boldsymbol{w})$. Rule (2) again gives $\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}$. Notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$! The main point is to be free to open up parentheses.

20 We know that $(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$ for $\boldsymbol{v}\cdot\boldsymbol{w}$. When $\theta < 90°$ this $\boldsymbol{v}\cdot\boldsymbol{w}$ is positive, so in this case $\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w}$ is larger than $\|\boldsymbol{v}-\boldsymbol{w}\|^2$.

21 $2\boldsymbol{v}\cdot\boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

22 $v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \le v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$ which is $(v_1w_2 - v_2w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta - a) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = v_1w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta - \alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

25 The cosine of θ is $x/\sqrt{x^2 + y^2}$, near side over hypotenuse. Then $|\cos\theta|^2$ is not greater than 1: $x^2/(x^2 + y^2) \le 1$.

26 The vectors $\boldsymbol{w} = (x, y)$ with $(1,2)\cdot\boldsymbol{w} = x + 2y = 5$ lie on a line in the xy plane. The shortest $\boldsymbol{w}$ on that line is $(1, 2)$. (The Schwarz inequality $\|\boldsymbol{w}\| \ge \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\| = \sqrt{5}$ is an equality when $\cos\theta = 0$ and $\boldsymbol{w} = (1, 2)$ and $\|\boldsymbol{w}\| = \sqrt{5}$.)

27 The length $\|\boldsymbol{v} - \boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v}\cdot\boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

28 Three vectors in the plane could make angles greater than 90° with each other: for example $(1, 0), (-1, 4), (-1, -4)$. Four vectors could *not* do this (360° total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$? Ben Harris and Greg Marks showed me that the answer is $n + 1$. The vectors from the center of a regular simplex in $\mathbf{R}^n$ to its $n + 1$ vertices all have negative dot products. If $n+2$ vectors in $\mathbf{R}^n$ had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n + 1$ vectors in $\mathbf{R}^{n-1}$ with negative dot products. Keep going to 4 vectors in $\mathbf{R}^2$: no way!

29 For a specific example, pick $\boldsymbol{v} = (1, 2, -3)$ and then $\boldsymbol{w} = (-3, 1, 2)$. In this example $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2$ and $\theta = 120°$. This always happens when $x + y + z = 0$:

$$\boldsymbol{v}\cdot\boldsymbol{w} = xz + xy + yz = \frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$$

This is the same as $\boldsymbol{v}\cdot\boldsymbol{w} = 0 - \frac{1}{2}\,\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\cos\theta = \frac{1}{2}$.

30 Wikipedia gives this proof of geometric mean $G = \sqrt[3]{xyz} \le$ arithmetic mean $A = (x + y + z)/3$. First there is equality in case $x = y = z$. Otherwise A is somewhere between the three positive numbers, say for example $z < A < y$.

Use the known inequality $g \le a$ for the *two* positive numbers x and $y + z - A$. Their mean $a = \frac{1}{2}(x + y + z - A)$ is $\frac{1}{2}(3A - A) =$ same as A! So $a \ge g$ says that $A^3 \ge g^2 A = x(y + z - A)A$. But $(y + z - A)A = (y - A)(A - z) + yz > yz$. Substitute to find $A^3 > xyz = G^3$ as we wanted to prove. Not easy!

There are many proofs of $G = (x_1x_2\cdots x_n)^{1/n} \le A = (x_1 + x_2 + \cdots + x_n)/n$. In calculus you are maximizing G on the plane $x_1 + x_2 + \cdots + x_n = n$. The maximum occurs when all x's are equal.

31 The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

32 The commands $V =$ **randn** $(3, 30)$; $D =$ **sqrt** (**diag** $(V' * V)$); $U = V\backslash D$; will give 30 random unit vectors in the columns of U. Then $u' * U$ is a row matrix of 30 dot products whose average absolute value may be close to $2/\pi$.

Problem Set 1.3, page 29

1 $2\boldsymbol{s}_1 + 3\boldsymbol{s}_2 + 4\boldsymbol{s}_3 = (2, 5, 9)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (2, 3, 4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row }1)\cdot\boldsymbol{x} \\ (\text{row }2)\cdot\boldsymbol{x} \\ (\text{row }2)\cdot\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2 The solutions are $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ (right side = column 1) and $y_1 = 1$, $y_2 = 3$, $y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

3 $$\begin{aligned} y_1 &= B_1 \\ y_1 + y_2 &= B_2 \\ y_1 + y_2 + y_3 &= B_3 \end{aligned} \quad \text{gives} \quad \begin{aligned} y_1 &= B_1 \\ y_2 &= -B_1 + B_2 \\ y_3 &= -B_2 + B_3 \end{aligned} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: independent columns in A and S!

4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\boldsymbol{w}_1 - \boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

6 $c = 3$ $\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & \mathbf{3} \end{bmatrix}$ has column 3 = 2 (column 1) + column 2

$c = -1$ $\begin{bmatrix} 1 & 0 & \mathbf{-1} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has column 3 = − column 1 + column 2

$c = 0$ $\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$ has column 3 = 3 (column 1) − column 2

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

8 $$\begin{aligned} x_1 - 0 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \\ x_4 - x_3 &= b_4 \end{aligned} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \\ x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\boldsymbol{b}$$

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

10 $\begin{aligned} z_2 - z_1 &= b_1 \\ z_3 - z_2 &= b_2 \\ 0 - z_3 &= b_3 \end{aligned}$ $\quad \begin{aligned} z_1 &= -b_1 - b_2 - b_3 \\ z_2 &= \quad\;\; -b_2 - b_3 \\ z_3 &= \qquad\quad\;\; -b_3 \end{aligned}$ $\quad = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1}\boldsymbol{b}$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \qquad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ -x_4 &= b_5 \end{aligned}$$

Add equations 1, 3, 5
The left side of the sum is zero
The right side is $b_1 + b_3 + b_5$
There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

Problem Set 2.1, page 40

1 The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

3 The solution is not changed! The second plane and row 2 of the matrix and all columns of the matrix (vectors in the column picture) are changed.

4 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

5 If x, y, z satisfy the first two equations they also satisfy the third equation. The line $\mathbf{L}$ of solutions contains $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ and all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$.

6 Equation 1 + equation 2 − equation 3 is now $0 = -4$. Line misses plane; *no solution*.

7 Column 3 = Column 1 makes the matrix singular. Solutions $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$; $\boldsymbol{b} = (4, 6, c)$ needs $c = 10$ for solvability (then $\boldsymbol{b}$ lies in the plane of the columns).

8 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3, 3, 3, 2)$ is $\boldsymbol{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$. The equations are $x + y + z + t = 3$, $y + z + t = 3$, $z + t = 3$, $t = 2$.

9 (a) $A\boldsymbol{x} = (18, 5, 0)$ and (b) $A\boldsymbol{x} = (3, 4, 5, 5)$.

10 Multiplying as linear combinations of the columns gives the same $A\boldsymbol{x}$. By rows or by columns: **9** separate multiplications for 3 by 3.

11 $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

12 $A\boldsymbol{x}$ equals (z, y, x) and $(0, 0, 0)$ and $(3, 3, 6)$.

13 (a) $\boldsymbol{x}$ has n components and $A\boldsymbol{x}$ has m components (b) Planes from each equation in $A\boldsymbol{x} = \boldsymbol{b}$ are in n-dimensional space, but the columns are in m-dimensional space.

14 $2x + 3y + z + 5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [2 \ \ 3 \ \ 1 \ \ 5]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

15 (a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 90° rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, 180° rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

17 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces (y, z, x) and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers (x, y, z). Q is the inverse of P.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

19 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\boldsymbol{v} = (3, 4, 8)$ and $E^{-1}E\boldsymbol{v}$ recovers $(3, 4, 5)$.

20 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the x-axis and $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the y-axis. $\boldsymbol{v} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ has $P_1\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $P_2P_1\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

21 $R = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ rotates all vectors by 45°. The columns of R are the results from rotating $(1, 0)$ and $(0, 1)$!

22 The dot product $A\boldsymbol{x} = [1 \ \ 4 \ \ 5]\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$ (1 by 3)(3 by 1) is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 $A = [1 \ \ 2 \ \ ; \ \ 3 \ \ 4]$ and $\boldsymbol{x} = [5 \ \ -2]'$ and $\boldsymbol{b} = [1 \ \ 7]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

24 $A * \boldsymbol{v} = [3 \ \ 4 \ \ 5]'$ and $\boldsymbol{v}' * \boldsymbol{v} = 50$. But $\boldsymbol{v} * A$ gives an error message from 3 by 1 times 3 by 3.

25 **ones**$(4, 4) *$ **ones**$(4, 1) = [4 \ \ 4 \ \ 4 \ \ 4]'$; $B * \boldsymbol{w} = [10 \ \ 10 \ \ 10 \ \ 10]'$.

26 The row picture has two lines meeting at the solution $(4, 2)$. The column picture will have $4(1, 1) + 2(-2, 1) = 4(\text{column } 1) + 2(\text{column } 2) = \text{right side } (0, 6)$.

27 The row picture shows **2 planes** in **3-dimensional space**. The column picture is in **2-dimensional space**. The solutions normally lie on a ***line***.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ and $\boldsymbol{u}_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components add to 1. They are always positive. $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to $(.6, .4)$. Their components still add to 1.

30 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$ = *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15,15,15)$; $M_4(1,1,1,1) = (34,34,34,34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution*.

33 $\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x}$ = sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 51

1 Multiply by $\ell_{21} = \frac{10}{2} = 5$ and subtract to find $2x + 3y = 14$ and $-6y = 6$. The pivots to circle are 2 and -6.

2 $-6y = 6$ gives $y = -1$. Then $2x + 3y = 1$ gives $x = 2$. Multiplying the right side $(1, 11)$ by 4 will multiply the solution by 4 to give the new solution $(x, y) = (8, -4)$.

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. (The two lines in the row picture are the same line, containing all solutions).

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line).

10 The equation $y = 1$ comes from elimination (subtract $x + y = 5$ from $x + 2y = 6$). Then $x = 4$ and $5x - 4y = c = 16$.

11 (a) Another solution is $\frac{1}{2}(x + X, y + Y, z + Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

12 Elimination leads to an upper triangular system; then comes back substitution.

$$\begin{aligned} 2x + 3y + z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \end{aligned} \quad \text{gives} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 1 \end{aligned}$$

If a zero is at the start of row 2 or 3, that avoids a row operation.

13

$$\begin{aligned} 2x - 3y \qquad &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned} \quad \text{gives} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ 2y + 3z &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ -5z &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 3 \\ y &= 1 \\ z &= 0 \end{aligned}$$

Subtract $2 \times$ row 1 from row 2, subtract $1 \times$ row 1 from row 3, subtract $2 \times$ row 2 from row 3

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y-z = 2$. Equation (3) is $y-z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 The second pivot position will contain $-2 - b$. If $b = -2$ we exchange with row 3. If $b = -1$ (singular case) the second equation is $-y - z = 0$. A solution is $(1, 1, -1)$.

16 (a) **Example of 2 exchanges**

$$\begin{aligned} 0x + 0y + 2z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$

(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then break down**

$$\begin{aligned} 0x + 3y + 4z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$$

(rows 1 and 3 are not consistent)

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

18 *Example* $x + 2y + 3z = 0$, $4x + 8y + 12z = 0$, $5x + 10y + 15z = 0$ has 9 different coefficients but rows 2 and 3 become $0 = 0$: infinitely many solutions.

19 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular—no third pivot. Then if $t = 5$ the third equation is $0 = 0$. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1+2 =$ row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution.

21 (a) Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x + y = 0$, $\frac{3}{2}y + z = 0$, $\frac{4}{3}z + t = 0$, $\frac{5}{4}t = 5$ after elimination. Back substitution gives $t = 4, z = -3, y = 2, x = -1$. (b) If the off-diagonal entries change from $+1$ to -1, the pivots are the same. The solution is $(1, 2, 3, 4)$ instead of $(-1, 2, -3, 4)$.

22 The fifth pivot is $\frac{6}{5}$ for both matrices (1's or -1's off the diagonal). The nth pivot is $\frac{n+1}{n}$.