1 Solutions to Odd Numbered Problems *Random Processes for Engineers*

1.1 Simple events (a) $\Omega = \{0,1\}^8$, or $\Omega = \{x_1x_2x_3x_4x_5x_6x_7x_8 : x_i \in \{0,1\}$ for each $i\}$. It is natural to let $\mathcal{F}$ be the set of all subsets of Ω. Finally, let $P(A) = \frac{|A|}{256}$, where $|A|$ denotes the cardinality of a set $|A|$.
(b) $E_1 = \{01010101, 10101010\}$ and $P(E_1) = \frac{2}{256} = \frac{1}{128}$.
$E_2 = \{00110011, 01100110, 11001100, 10011001\}$ and $P(E_2) = 4/256 = 1/64$.
$E_3 = \{x \in \Omega : x_1 + \cdots + x_8 = 4\}$ and $P(E_2) = \binom{8}{4}/256 = 70/256 = 35/128$.
$E_4 = \{11111111, 11111110, 11111101, 10111111, 01111111, 00111111, 01111110, 11111100\}$ and $P(E_4) = 8/256 = 1/32$.
(c) $E_1 \subset E_3$, so $P(E_1|E_3) = |E_1|/|E_3| = 2/70 = 1/35$.
$E_2 \subset E_3$, so $P(E_2|E_3) = |E_2|/|E_3| = 4/70 = 2/35$.

1.3 Ordering of three random variables $P\{X < u < Y\} = P\{X < u\}P\{u < Y\} = (1 - e^{-\lambda u})e^{-\lambda u} = e^{-\lambda u} - e^{-2\lambda u}$. Averaging over the choices of u using the pdf of U yields,

$$P\{X < U < Y\} = \int_0^1 e^{-\lambda u} - e^{-2\lambda u} du = \frac{0.5 - e^{-\lambda} + 0.5e^{-2\lambda}}{\lambda}.$$

1.5 Congestion at output ports (a) One possibility is $\Omega = \{1, 2, \ldots, 8\}^4 = \{(d_1, d_2, d_3, d_4) : 1 \le d_i \le 8 \text{ for } 1 \le i \le 4\}$, where the packets are assumed to be numbered one through four, and d_i is the output port of packet i. Let $\mathcal{F}$ be all the subsets of Ω, and for any event A, let $P(A) = \frac{|A|}{8^4}$.
(b)

$$P\{X_1 = k_1, \ldots, X_8 = k_8\} = \frac{1}{8^4}\binom{4}{k_1 k_2 \cdots k_8}$$

where $\binom{4}{k_1 k_2 \cdots k_8} = \frac{4!}{k_1! k_2! \cdots k_8!}$ is the multinomial coefficient.
(c) One way to do this problem is to note that $X_j = \sum_{i=1}^4 X_{ij}$, where $X_{ij} = 1$ if packet i is routed to output port j, and $X_{ij} = 0$ otherwise. Suppose $j \neq j'$. Then $X_{ij}X_{ij'} \equiv 0$, and so also, $E[X_{ij}X_{ij'}] = 0$. Thus, $\mathrm{Cov}(X_{ij}, X_{ij'}) = 0 - \frac{1}{8}^2 = -\frac{1}{64}$.

Also, $\mathrm{Cov}(X_{ij}, X_{i'j'}) = 0$ if $i \neq i'$. Thus,

$$\begin{aligned}\mathrm{Cov}(X_j, X_{j'}) &= \mathrm{Cov}(\sum_{i=1}^{4} X_{ij}, \sum_{i'=1}^{4} X_{i'j'}) \\ &= \sum_{i=1}^{4}\sum_{i'=1}^{4} \mathrm{Cov}(X_{ij}, X_{i'j'}) \\ &= \sum_{i=1}^{4} \mathrm{Cov}(X_{ij}, X_{ij'}) = 4(-\frac{1}{64}) = -\frac{1}{16}.\end{aligned}$$

(d) Consider the packets one at a time in order. The first packet is routed to a random output port. The second is routed to a different output port with probability $\frac{7}{8}$. Given the first two packets are routed to different output ports, the third packet is routed to yet another output port with probability $\frac{6}{8}$. Similarly, given the first three packets are routed to distinct output ports, the fourth packet is routed to yet another output port with probability $\frac{5}{8}$. The answer is thus $\frac{8\cdot 7\cdot 6\cdot 5}{8^4} = \frac{105}{256} \approx 0.410$.
(e) The event is not true if and only if there are either exactly 3 packets assigned to one output port or all four packets assigned to one output port. There are $4 \cdot 8 \cdot 7$ possibilities for exactly three packets to be assigned to one output port, since there are four choices for which packet is not with the other three, eight choices of output port for the group of three, and given that, seven choices of output port for the fourth packet. There are 8 possibilities for all four packets to be routed to the same output port. Thus, some output port has three or more packets assigned to it with probability $\frac{4\cdot 8\cdot 7+8}{8^4} = \frac{4\cdot 7+1}{8^3} = \frac{29}{512} \approx 0.0566$. Thus, $P\{X_i \leq 2 \text{ for all } i\} = 1 - \frac{29}{512} \approx 0.9434$.

1.7 Conditional probability of failed device given failed attempts (a) P(first attempt fails)=0.2+(0.8)(0.1)=0.28
(b) P(server is working | first attempt fails) =
P(server working, first attempt fails)/P(first attempt fails) =(0.8)(0.1)/0.28≈ 0.286
(c) P(second attempt fails | first attempt fails) =P(first two attempts fail)/P(first attempt fails) = $[0.2 + (0.8)(0.1)^2]/0.28$ ≈0.783
(d) P(server is working | first and second attempts fail) =P(server is working and first two attempts fail)/P(first two attempts fail) = $(0.8)(0.1)^2/[0.2 + (0.8)(0.1)^2]$ ≈0.0385

1.9 Conditional lifetimes; memoryless property of the geometric distribution
(a) $P\{X > 3\} = 1 - p(3) = 0.8$, $P(X > 8|X > 5) = \frac{P(\{X>8\}\cap\{X>5\})}{P\{X>5\}} = \frac{P\{X>8\}}{P\{X>5\}} = \frac{0}{0.40} = 0$.
(So a five year old working battery is not equivalent to a new one!)
(b) $P\{Y > 3\} = P(\text{miss first three shots}) = (1-p)^3$. On the other hand,

$$P(Y > 8|Y > 5) = \frac{P(\{Y > 8\} \cap \{Y > 5\})}{P\{Y > 5\}} = \frac{P\{Y > 8\}}{P\{Y > 5\}} = \frac{(1-p)^8}{(1-p)^5} = (1-p)^3.$$

(A player that has missed five shots is equivalent to a player just starting to take shots.)
(c) Y has a geometric distribution. (Part (b) illustrates the fact that the geometric distribution is the memoryless lifetime distribution on the positive integers. The exponential distribution is the continuous type distribution with the same property.)
1.11 Distribution of the flow capacity of a network One way to solve this problem is to compute X for each of the 32 outcomes for the links. Another is to use divide and conquer by conditioning on the state of a key link, such as link 4.

$$\begin{aligned} P\{X=0\} &= P(((F_1F_3)\cup(F_2F_5))F_4^c) + P((F_1\cup F_2)(F_3\cup F_5)F_4) \\ &= ((0.2)^2+(0.2)^2-(0.2)^4)(0.8) + (0.2+0.2-(0.2)^2)^2(0.2) = 0.08864. \end{aligned}$$

$$\begin{aligned} P\{X=10\} &= P(F_1^cF_3^c(F_2F_5)^cF_4^c) + P(F_1^cF_2^cF_3^cF_5^cF_4) \\ &= (0.8)^3(1-(0.2)^2) + (0.8)^4(0.2) = 0.57344. \end{aligned}$$

$P\{X=5\} = 1 - P\{X=0\} - P\{X=10\} = 0.33792.$
1.13 A CDF of mixed type (a) $F_X(0.8) = 0.5$.
(b) There is a half unit of probability mass at zero and a density of value 0.5 between 1 and 2. Thus, $E[X] = 0\times 0.5 + \int_1^2 x(0.5)dx = 3/4$ and,
(c) $E[X^2] = 0^2\times 0.5 + \int_1^2 x^2(0.5)dx = 7/6$. So $\mathrm{Var}(X) = 7/6 - (3/4)^2 = 29/48$
.
1.15 Poisson and geometric random variables with conditioning
(a) $P\{Y<Z\} = \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty}\frac{e^{-\mu}\mu^i}{i!}p(1-p)^{j-1} = \sum_{i=0}^{\infty}\frac{e^{-\mu}[\mu(1-p)]^i}{i!} = e^{-\mu p}$
(b) $P(Y<Z|Z=i) = P(Y<i|Z=i) = P\{Y<i\} = \sum_{j=0}^{i-1}\frac{e^{-\mu}\mu^j}{j!}$
(c) $P(Y=i|Y<Z) = P\{Y=i<Z\}/P\{Y<Z\} = \left(\frac{e^{-\mu}\mu^i}{i!}(1-p)^i\right)/e^{-\mu p} = \frac{e^{-\mu(1-p)}[\mu(1-p)]^i}{i!}$, which is the Poisson distribution with mean $\mu(1-p)$
(d) $\mu(1-p)$
1.17 Transformation of a random variable (a) Observe that Y takes values in the interval $[1,+\infty)$.

$$F_Y(c) = P\{\exp(X)\le c\} = \begin{cases} P\{X\le \ln c\} = 1-\exp(-\lambda\ln c) = 1-c^{-\lambda} & c\ge 1 \\ 0 & c<1 \end{cases}$$

Differentiate to obtain

$$f_Y(c) = \begin{cases} \lambda c^{-(1+\lambda)} & c\ge 1 \\ 0 & c<1 \end{cases}$$

(b) Observe that Z takes values in the interval $[0,3]$.

$$F_Z(c) = P\{\min\{X,3\}\le c\} = \begin{cases} 0 & c<0 \\ P\{X\le c\} = 1-\exp(-\lambda c) & 0\le c<3 \\ 1 & c\ge 3 \end{cases}$$

The random variable Z is neither discrete nor continuous type. Rather it is a mixture, having a density over the interval $[0, 3)$ and a discrete mass at the point 3.

1.19 Moments and densities of functions of a random variable

$E[C] = 2E[L] + 2E[W] = 2$ $\quad \mathrm{Var}(C) = 4\mathrm{Var}(L) + 4\mathrm{Var}(W) = \frac{2}{3}$ $\quad$ The pdf of C is the convolution of the pdf of $2L$ with the pdf of $2W$. But $2L$ and $2W$ are each uniformly distributed over the interval $[0, 2]$, so their pdfs are rectangular pulse functions. The convolution of such a function with itself is a triangular pulse function. The base of the triangle, equal to the support of f_C, is the interval $[0, 4]$. The peak of the triangle is at the midpoint, and must have height $1/2$ in order that the area of the triangle be one. Therefore, $f_C(x) = \begin{cases} x/4 & 0 \le x \le 2 \\ \frac{4-x}{4} & 2 \le x \le 4 \\ 0 & \text{else} \end{cases}$

$E[A] = E[L]E[W] = (\frac{1}{2})^2 = \frac{1}{4}$ $\quad E[A^2] = E[L^2]E[W^2] = (\frac{1}{3})^2 = \frac{1}{9}$ $\quad$ so $\mathrm{Var}(A) = \frac{1}{9} - (\frac{1}{4})^2 = \frac{7}{144}$.

For $0 \le c \le 1$, $P\{A \le c\}$ =area of $\{(x, y) \in [0, 1]^2 : xy \le c\} = c + \int_c^1 \frac{c}{x} dx = c(1 - \ln c)$, so $f_A(c) = \begin{cases} -\ln(c) & 0 \le c \le 1 \\ 0 & \text{else} \end{cases}$

1.21 Using the Gaussian Q function (a) $P\{X \ge 16\} = P\{\frac{X-10}{3} > \frac{16-10}{3}\} = Q(\frac{16-10}{3}) = Q(2)$.

(b) $P\{X^2 \ge 16\} = P\{X \ge 4\} + P\{X \le -4\} = Q(\frac{4-10}{3}) + 1 - Q(\frac{-4-10}{3}) = Q(-2) + 1 - Q(-\frac{14}{3}) = 1 - Q(2) + Q(\frac{14}{3})$.

(c) Z is $N(0, 5)$ so $P\{|Z| > 1\} = P\{Z > 1\} + P\{Z < -1\} = Q(\frac{1}{\sqrt{5}}) + 1 - Q(-\frac{1}{\sqrt{5}}) = 2Q(\frac{1}{\sqrt{5}})$.

1.23 Correlation of histogram values (a) X_1 is Bernoulli($\frac{1}{6}$), so $E[X_1] = \frac{1}{6}$ and $\mathrm{Var}(X_1) = \frac{1}{6}(1 - \frac{1}{6}) = \frac{5}{36}$.

(b) $E[X] = nE[X_1] = \frac{n}{6}$ and $\mathrm{Var}(X) = n\mathrm{Var}(X_1) = \frac{5n}{36}$.

(c) We begin by computing $\mathrm{Cov}(X_1, Y_1)$. Since $X_1Y_1 = 0$ with probability one, $E[X_1Y_1] = 0$. Therefore $\mathrm{Cov}(X_1, Y_1) = E[X_1Y_1] - E[X_1]E[Y_1] = 0 - \frac{1}{6}\frac{1}{6} = \frac{-1}{36}$. So $\mathrm{Cov}(X_i, Y_i) = \frac{-1}{36}$ for any i. On the other hand, if $i \ne j$ then X_i is independent of X_j. So

$$\mathrm{Cov}(X_i, Y_j) = \begin{cases} \frac{-1}{36} & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

(d)

$$\mathrm{Cov}(X, Y) = \sum_i \sum_j \mathrm{Cov}(X_i, Y_j) = \sum_i \mathrm{Cov}(X_i, Y_i) = n\mathrm{Cov}(X_1, Y_1) = \frac{-n}{36}.$$

and

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{-1}{5}.$$

(e) Given that x of the dice show a 1, each of the remaining dice is equally likely to show 2,3,4,5, or 6. Thus, each of the remaining $n - x$ dice shows a 2 with conditional probability $\frac{1}{5}$. Therefore $E[Y|X = x] = \frac{n-x}{5}$.

1.25 A function of jointly distributed random variables The square has unit area so that the joint density is unit valued within the square. The range of X is the interval $[0, 1]$, so fix c in $[0, 1]$ and consider the event $\{UV \leq c\}$. The probability of this event is the area of the square minus the upper right region above the curve $v = c/u$. This area is one minus the area of the region inside the square above the curve $v = c/u$. Therefore,

$$F_X(c) = \begin{cases} 0 & c \leq 0 \\ 1 - \int_c^1 (1 - \frac{c}{u})du = c - c \ln c & 0 \leq c \leq 1 \\ 1 & c \geq 1 \end{cases}$$

Differentiating yields

$$f_X(c) = \begin{cases} -\ln c & 0 < c \leq 1 \\ 0 & \text{else} \end{cases}$$

1.27 Working with a two dimensional density (a) The parallelogram has base and height one, and thus area one, so that the density is one on the region.

(b) By inspection, $f_X(x) = \begin{cases} 0.5x & 0 \leq x \leq 1 \\ 0.5 & 1 \leq x \leq 2 \\ 0.5(3 - x) & 2 \leq x \leq 3 \\ 0 & else \end{cases}$

(c) Since the density of X is symmetric about 1.5 and the mean exists, $E[X] = 1.5$. $E[X^2] = 0.5[\int_0^1 x^3 dx + \int_1^2 x^2 dx + \int_2^3 x^2(3 - x)dx] = 0.5[\frac{1}{4} + \frac{7}{3} + \frac{11}{4}] = \frac{8}{3}$, so $\text{Var}(X) = \frac{8}{3} - (\frac{3}{2})^2 = \frac{5}{12}$. A slicker way to find the variance is to observe the X has the same distribution as $U_1 + 2U_2$, where U_1 and U_2 are independent and uniformly distributed over $[0, 1]$, so $\text{Var}(X) = \text{Var}(U_1) + 4\text{Var}(U_2) = \frac{5}{12}$.

(d) If $0 \leq x \leq 1$, the conditional density of Y given $X = x$ is the uniform density over the interval $[0, \frac{x}{2}]$. That is, for $0 < x \leq 1$: $f_{Y|X}(y|x) = \begin{cases} \frac{2}{x} & 0 \leq y \leq \frac{x}{2} \\ 0 & \text{else} \end{cases}$

(e) By inspection, if $1 \leq x \leq 2$, the conditional density of Y given $X = x$ is the uniform density over the interval $[\frac{x-1}{2}, \frac{x}{2}]$. That is, for $1 < x \leq 2$: $f_{Y|X}(y|x) = \begin{cases} 2 & \frac{x-1}{2} \leq y \leq \frac{x}{2} \\ 0 & \text{else} \end{cases}$

(f) $E[Y|X = x]$ is well defined over the support of f_X, namely, over the interval $[0, 3]$. For each X in this interval, the conditional density of Y give $X = x$ is a uniform density, so the conditional mean is the midpoint of the interval.

Therefore. $E[Y|X = x] = \begin{cases} x/4 & 0 \leq x \leq 1 \\ (x - 0.5)/2 & 1 \leq x \leq 2 \\ (x + 1)/4 & 2 \leq x \leq 3 \\ \text{undefined} & x \notin [0, 3] \end{cases}$

1.29 Uniform density over a union of two square regions (a) Region has area 2 so the density function is 1/2 in the region and zero outside.

(b) $f_X(x) = \begin{cases} 0.5 & \text{if } |x| \leq 1 \\ 0 & \text{else} \end{cases}$

(c) If $0 < a \leq 1$, $f_{Y|X}(y|a) = \begin{cases} 1 & \text{if } 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$

(d) If $-1 \leq a < 0$, $f_{Y|X}(y|a) = \begin{cases} 1 & \text{if } -1 \leq y \leq 0 \\ 0 & \text{else} \end{cases}$

(e) $E[Y|X = a] = \begin{cases} -0.5 & \text{if } -1 \leq a < 0 \\ 0.5 & \text{if } 0 < a < 1 \end{cases}$

(f) $E[X] = E[Y] = 0$, $\mathrm{Var}(X) = E[X^2] = 1/3$, $\mathrm{Var}(Y) = 1/3$, $E[XY] = \frac{1}{2}\int_0^1\int_0^1 xydxdy + \frac{1}{2}\int_{-1}^0\int_{-1}^0 xydxdy = \int_0^1\int_0^1 xydxdy = 1/4$. So $\rho_{XY} = \frac{1/4}{\sqrt{1/3 \times 1/3}} = \frac{3}{4}$.

(g) No, because $f_{XY}(x,y)$ doesn't factor into the product of a function of x and a function of y.

(h) The range of Z is $[-2, 2]$. $f_Z(z) = \begin{cases} |z|/2 & \text{if } -0 \leq |z| \leq 1 \\ 1 - |z|/2 & \text{if } 1 \leq |z| \leq 2 \\ 0 & \text{else} \end{cases}$ (Shape is two triangles.)

1.31 Transformation of densities (a) $\int_0^1\int_0^1 (u-v)^2 dudv = \int_0^1\int_0^1 (u^2 - 2uv + v^2)dudv = \frac{1}{6}$, so $c = 6$.

(b) The map from the u, v plane to the x, y plane given by $x = u^2$ and $y = u^2v^2$ maps the unit square $[0,1] \times [0,1]$ into the triangular region $0 \leq y \leq x \leq 1$ in one-to-one fashion. The inverse mapping is given by $u = v^{1/2}$ and $v = (y/x)^{1/2}$. Also, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{array}{cc} 2u & 0 \\ 2uv^2 & 2u^2v \end{array}\right| = 4u^3v = 4xy^{1/2}$. Therefore,

$$\begin{aligned} f_{XY}(x,y) &= f_{UV}(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^{-1} \\ &= \begin{cases} 6(x^{1/2} - (y/x)^{1/2})^2 \frac{1}{4xy^{1/2}} & \text{if } 0 \leq y \leq x \leq 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

1.33 Transformation of joint densities To be definite, assume $\binom{X}{Y}$ takes values in the positive quadrant of the $u - v$ plane and $\binom{W}{Z}$ takes values in the $\alpha - \beta$ plane. We have $\binom{W}{Z} = g\binom{X}{Y}$ where the transformation g is given by $\alpha = u - v$ and $\beta = u^2 + u - v$. The transformation is invertible. In fact, we see that $u = \sqrt{\beta - \alpha}$ and $v = \sqrt{\beta - \alpha} - \alpha$, for (α, β) in the range of g, which is the set $\{(\alpha, \beta) : \beta > \alpha + (\max\{0, \alpha\})^2\}$. (To understand the geometry of the function better, note as u varies over $u > 0$ with $v = 0$ the function $g(u,v)$ traces out the curve $\beta = \alpha^2 + \alpha$ for $\alpha > 0$. Then for any u fixed with $u > 0$, the function $g(u,v)$ traces out a half line of slope one as v ranges over $v > 0$.) The determinant of the Jacobian of g is given by

$$\det(J) = \det\begin{pmatrix} 1 & -1 \\ 2u & -1 \end{pmatrix} = u = \sqrt{\beta - \alpha}.$$

Therefore,

$$f_{W,Z}(\alpha,\beta) = \frac{f_{X,Y}(u.v)}{\det J} = \begin{cases} \frac{\exp\left(-\lambda\left(2\sqrt{\beta-\alpha}-\alpha\right)\right)}{\sqrt{\beta-\alpha}} & \beta > \alpha + (\max\{0,\alpha\})^2 \\ 0 & else \end{cases}$$

1.35 Conditional densities and expectations
(a)

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^u uv(4u^2)dvdu \\ &= \int_0^1 4u^3 \left(\int_0^u vdv\right) du \\ &= \int_0^1 2u^5 du = \frac{1}{3}. \end{aligned}$$

(b)

$$f_Y(v) = \int_v^1 4u^2 \; du = \begin{cases} \frac{4}{3}(1-v^3), & 0 \le v \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(c)

$$f_{X|Y}(u|v) = \begin{cases} 0, & 0 < v < 1, \;\; 0 < u < v \\ \frac{4u^2}{\frac{4}{3}(1-v^3)} = \frac{3u^2}{1-v^3}, & 0 < v < 1, \;\; v < u < 1 \\ \text{undefined}, & v < 0 \text{ or } v > 1 \end{cases}$$

(d) For $0 < v < 1$, $E[X^2|Y=v] = \int_v^1 \; u^2 \; \frac{3u^2}{1-v^3} \; du = \frac{3}{5}\frac{1-v^5}{1-v^3}$

2.1 Limits and infinite sums for deterministic sequences (a) Before beginning the proof we observe that $|\cos(\theta)| \le 1$, so $|\theta(1+\cos(\theta))| \le 2|\theta|$. Now, for the proof. Given an arbitrary ϵ with $\epsilon > 0$, let $\delta = \epsilon/2$. For any θ with $|\theta - 0| \le \delta$, the following holds: $|\theta(1+\cos(\theta)) - 0| \le 2|\theta| \le 2\delta = \epsilon$. Since ϵ was arbitrary the convergence is proved.
(b) Before beginning the proof we observe that if $0 < \theta < \pi/2$, then $\cos(\theta) \ge 0$ and $\frac{1+\cos(\theta)}{\theta} \ge 1/\theta$. Now, for the proof. Given an arbitrary positive number K, let $\delta = \min\{\frac{\pi}{2}, \frac{1}{K}\}$. For any θ with $0 < \theta < \delta$, the following holds: $\frac{1+\cos(\theta)}{\theta} \ge 1/\theta \ge 1/\delta \ge K$. Since K was arbitrary the convergence is proved.
(c) The sum is by definition equal to $\lim_{N\to\infty} s_N$ where $s_N = \sum_{n=1}^N \frac{1+\sqrt{n}}{1+n^2}$. The sequence S_N is increasing in N. Note that the $n=1$ term of the sum is 1 and for any $n \ge 1$ the n^{th} term of the sum can be bounded as follows:

$$\frac{1+\sqrt{n}}{1+n^2} \le \frac{2\sqrt{n}}{n^2} = 2n^{-3/2}.$$

Therefore, comparing the partial sum with an integral, yields

$$s_N \le 1 + \sum_{n=2}^N 2n^{-3/2} \le 1 + \int_1^N 2x^{-3/2}dx = 5 - 4N^{-1/2} \le 5.$$

In summary, the sequence $(S_N : N \geq 1)$ is an increasing, bounded sequence, and it thus has a finite limit.

2.3 The reciprocal of the limit is the limit of the reciprocal Let $\epsilon > 0$. Let $\epsilon' = \min\{\frac{|x_\infty|}{2}, \frac{\epsilon x_\infty^2}{2}\}$. By the hypothesis, there exists a value of n_o so large that for all $n \geq n_o$, $|x_n - x_\infty| \leq \epsilon'$. This condition implies that $|x_n| \geq |x_\infty|/2$, because of the choice of ϵ'. Therefore, for all $n \geq n_o$,

$$\left| \frac{1}{x_n} - \frac{1}{x_\infty} \right| = \frac{|x_n - x_\infty|}{|x_n||x_\infty|} \leq \frac{2\epsilon'}{x_\infty^2} \leq \epsilon,$$

which, by definition, shows that $(1/x_n)$ converges to $1/x_\infty$.

2.5 On convergence of deterministic sequences and functions (a) Note that $x_n - \frac{8}{3} = \frac{1}{3n}$. Thus, given any $\epsilon > 0$, let $n_\epsilon = \lceil \frac{1}{3\epsilon} \rceil$. Then for any $n \geq n_\epsilon$, $|X_n - \frac{8}{3}| \leq \frac{1}{3n} \leq \frac{1}{3n_\epsilon} \leq \epsilon$. Thus, by definition, $\lim_{n\to\infty} x_n = \frac{8}{3}$.
(b) Let $\epsilon = 1/3$ and let $x_n = (2/3)^{1/n}$ for $n \geq 1$. Note that $x_n \in [0, 1)$ and $f_n(x_n) = \frac{2}{3}$. Thus, there is no positive integer n such that $|f_n(x) - 0| \leq \epsilon$ for all $x \in [0, 1)$. So it is impossible to select n_ϵ with the property required for uniform convergence. Therefore f_n does not converge uniformly to zero.
(c) Let $c < \sup_D f$. Then there is an $x \in D$ so that $c \leq f(x)$. Therefore,
$c \leq f(x) - g(x) + g(x) \leq \sup_D |f - g| + \sup_D g$. Thus, $c < \sup_D f$ implies $c < \sup_D |f - g| + \sup_D g$. Equivalently, $\sup_D f \leq \sup_D |f - g| + \sup_D g$, or $\sup_D f - \sup_D g \leq \sup_D |f - g|$. Exchanging the roles of f and g yields $\sup_D g - \sup_D f \leq \sup_D |f - g|$. Combining yields the desired inequality, $|\sup_D f - \sup_D g| \leq \sup_D |f - g|$. As an application, suppose $f_n \to f$ uniformly on D. Then given any $\epsilon > 0$, there exists an n_ϵ so large, that $\sup_D |f_n - f| \leq \epsilon$, whenever $n \geq n_\epsilon$. But then by the inequality proved, $|\sup_D f_n - \sup_D f| \leq \sup_D |f_n - f| \leq \epsilon$, whenever $n \geq n_\epsilon$. Thus, by definition, $\sup_D f_n \to \sup_D f$ as $n \to \infty$.

2.7 On the Dirichlet criterion for convergence of a series
(a) Let $R_n = \sum_{k=0}^n a_k$. By assumption, the sequence (R_n) has a finite limit, so it is a Cauchy sequence, i.e. $\lim_{m,n\to\infty} |R_m - R_n| = 0$. Now for $n < m$,
$|S_m - S_n| = |\sum_{k=n+1}^m d_k| \leq \sum_{k=n+1}^m |d_k| \leq \sum_{k=n+1}^m La_k = L|R_m - R_n|$. Therefore,
$\lim_{m,n\to\infty} |S_m - S_n| = 0$. That is, (S_n) is also a Cauchy sequence, and hence also has a finite limit.

(b)

$$
\begin{aligned}
S_n &= \sum_{k=0}^{n} A_k B_k - \sum_{k=1}^{n} A_k B_{k-1} \quad \text{since } B_{-1} = 0 \\
&= \sum_{k=0}^{n} A_k B_k - \sum_{k=0}^{n-1} A_{k+1} B_k \\
&= \left(\sum_{k=0}^{n} (A_k - A_{k+1}) B_k \right) - A_{n+1} B_n \\
&= \left(\sum_{k=0}^{n} a_k B_k \right) - A_{n+1} B_n .
\end{aligned}
$$

(c) Since $|a_k B_k| \leq L a_k$ for all k, the sequence of sums $\sum_{k=0}^{n} a_k B_k$ is convergent by the result of part (a). Also, $|A_{n+1} B_n| \leq L A_{n+1} \to 0$ as $n \to \infty$. Thus, by part (b), S_n has a finite limit.

2.9 Convergence of a random sequence (a) The sequence $X_n(\omega)$ is monotone nondecreasing in n for each ω. Also, by induction on n, $X_n(\omega) \leq 1$ for all n and ω. Since bounded monotone sequences have finite limits, $\lim_{n\to\infty} X_n$ exists in the a.s. sense and the limit is less than or equal to one with probability one.
(b) Since a.s. convergence of bounded sequences implies m..s. convergence, $\lim_{n\to\infty} X_n$ also exists in the m.s. sense.
(c) Since (X_n) converges a.s., it also converges in probability to the same random variable, so $Z = \lim_{n\to\infty} X_n$ a.s. It can be shown that $P\{Z = 1\} = 1$. Here is one of several proofs. Let $0 < \epsilon < 1$. Let $a_0 = 0$ and $a_k = \frac{a_{k-1}+1-\epsilon}{2}$ for $k \geq 1$. By induction, $a_k = (1-\epsilon)(1-2^{-k})$. Consider the sequence of events: $\{U_i \geq 1-\epsilon\}$ for $i \geq 1$. These events are independent and each has probability ϵ. So with probability one, for any $k \geq 1$, the probability that at least k of these events happens is one. If at least k of these events happen, then $Z \geq a_k$. So, $P\{(1-\epsilon)(1-2^{-k}) \leq Z \leq 1\} = 1$. Since ϵ can be arbitrarily close to zero and k can be arbitrarily large, it follows that $P\{Z = 1\} = 1$.
ANOTHER APPROACH is to calculate that $E[X_n|X_{n-1} = v] = v + \frac{(1-v)^2}{2}$. Thus, $E[X_n] = E[X_{n-1}] + \frac{E[(1-X_{n-1})^2]}{2} \geq E[X_{n-1}] + \frac{(1-E[X_{n-1}])^2}{2}$. Since $E[X_n] \to E[Z]$, it follows that $E[Z] \geq E[Z] + \frac{(1-E[Z])^2}{2}$. So $E[Z] = 1$. In view of the fact $P\{Z \leq 1\} = 1$, it follows that $P\{Z = 1\} = 1$.

2.11 Convergence of some sequences of random variables (a)For each fixed ω., $\frac{V(\omega)}{n} \to 0$ so $X_n(\omega) \to 1$. Thus, $X_n \to 1$ in the a.s sense, and hence also in the p. and d. senses. Since the random variables X_n are uniformly bounded (specifically, $|X_n| \leq 1$ for all n), the convergence in p. sense implies convergence in m.s. sense as well. So $X_n \to 1$ in all four senses.
(b)To begin we note that $P\{V \geq 0\} = 1$ with $P\{V > 1\} = e^{-3} > 0$. For any ω such that $V(\omega) < 1$, $Y_n(\omega) \to 0$, and for any ω such that $V(\omega) > 1$, $Y_n(\omega) \to +\infty$, so (Y_n) does not converge in the a.s. sense to a finite random variable.

Let us show Y_n does not converge in d. sense. For any $c > 0$ $\lim_{n\to\infty} F_n(c) = \lim_{n\to\infty} P\{Y_n \le c\} = P\{V < 1\} = 1 - e^{-3}$. The limit exists but the limit function F satisfies $F(c) = e^{-1}$ for all $c > 0$, so the limit is not a valid CDF. Thus, (Y_n) does not converge in the d. sense (to a finite limit random variable), and hence does not converge in any of the four senses to a finite limit random variable.

(c)For each ω fixed, $Z_n(\omega) \to e^{V(\omega)}$. So $Z_n \to e^V$ in the a.s. sense, and hence also in the p. and d. senses. Using the inequality $1+u \le e^u$ shows that $Z_n \le e^V$ for all n so that $|Z_n| \le e^V$ for all n. Note that $E[(e^V)^2] = E[e^{2V}] = \int_0^\infty e^{2u} 3e^{-3u} du = 3 < \infty$. Therefore, the sequence (Z_n) is dominated by a single random variable with finite second moment (namely, e^V), so the convergence of (Z_n) in the p. sense to e^V implies that (Z_n) converges to e^V in the m.s. sense as well. So $Z_n \to e^V$ in all four senses.

2.13 On the maximum of a random walk with negative drift (a) By the strong law of large numbers, $P\{S_n/n \to -1\} = 1$. Therefore, with probability one, $S_n/n \le 0$ for all sufficiently large n. That is, with probability one, $S_n > 0$ only finitely many times. The random variable Z, with probability one, is thus the maximum of only finitely many nonnegative numbers. So Z is finite with probability one.

(b) Suppose $P\{X_1 = c-1\} = P\{X_1 = -c-1\} = 0.5$ for a constant $c > 0$. Then X_1 has mean -1 as required. Following the hint, for $c \ge 1$, we have $E[Z] \ge E[\max\{0, X_1\}] = (c-1)/2$. Observe that $E[Z]$ can be made arbitrarily large by taking c arbitrarily large. So the answer to the question is no. (Note: More can be said about $E[Z]$ if the variance of X_1 is known. A celebrated bound of J.F.C. Kingman is that $E[Z] \le \frac{\mathrm{Var}(X_1)}{-2E[X_1]}$.)

2.15 Convergence in distribution to a nonrandom limit Suppose $P\{X = c\} = 1$ and $\lim_{n\to\infty} X_n = X$ $d.$ Let $\epsilon > 0$. It suffices to prove that $P\{X_n - X| \le \epsilon\} \to 1$ as $n \to \infty$. Note that $P\{|X_n - X| \le \epsilon\} \ge P\{c - \epsilon < X_n \le c + \epsilon\} = F_n(c+\epsilon) - F_n(c-\epsilon)$. Since $c - \epsilon$ is a continuity point of F_X and $F_X(c-\epsilon) = 0$, it follows that $F_n(c-\epsilon) \to 0$. Similarly, $F_n(c+\epsilon) \to 1$. Thus $F_n(c+\epsilon) - F_n(c-\epsilon) \to 1$, so that $P\{|X_n - X| \le \epsilon\} \to 1$. Therefore convergence in probability holds.

Note: A slightly different approach would be to prove that for any $\epsilon > 0$, there is an n_ϵ so large that $P\{|X_n - c| \le \epsilon\} \ge 1 - \epsilon$.

2.17 Convergence of a product (a) Examine $S_n = \ln X_n$. The sequence S_n, $n \ge 1$ is the sequence of partial sums of the independent and identically distributed random variables $\ln U_k$. Observe that $E[\ln U_k] = \int_0^2 \ln(u)\frac{1}{2} du = \frac{1}{2}(x \ln x - x)|_0^2 = \ln 2 - 1 \approx -0.306$. Therefore, by the strong law of large numbers, $\lim_{n\to\infty} \frac{S_n}{n} = \ln 2 - 1$ a.s. This means that, given an $\epsilon > 0$, there is an a.s. finite random variable N_ϵ so large that $|\frac{S_n}{n} - (\ln 2 - 1)| \le \epsilon$ for all $n \ge N_\epsilon$. Equivalently,

$$\left(\frac{2(1-\epsilon)}{e}\right)^n \le X_n \le \left(\frac{2(1+\epsilon)}{e}\right)^n \qquad \text{for} \quad n \ge N_\epsilon.$$

Conclude that $\lim_{n\to\infty} X_n = 0$ a.s., which implies that also $\lim_{n\to\infty} X_n = 0$ p.