Introduction to Cosmology

Instructor's Manual

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Introduction

The purpose of this instructor's manual is primarily to provide worked solutions for the end-of-chapter problems. In writing up the solutions, I didn't try to mimic the solutions that an A+ student will submit; instead, I tried to go into "verbose" mode, giving general solutions that emphasize the physics behind the problem. I also indulge in occasional asides, and recommendations for further problems you can pose for your students. (The solutions were written in some haste, so errors could lurk – let me know if you find any.)

In the worked solutions, I refer to equations both in the textbook *Introduction to Cosmology (2nd edition)* and in this Instructor's Manual. To help keep things clear, when I refer to an equation in the textbook, I use the format "Eq. 6.66"; when I refer to an equation in this Instructor's Manual, I use the format "equation (6.66)".

In addition to worked solutions, I also give a brief summary of the changes that I made in going from the first edition to the second edition. I also give references for some of the assertions made in the text. (I didn't want to clutter up the text with references, but I assume that some of you will be curious about the source of some of my less-obvious assertions.)

In writing the second edition, I took the opportunity to correct the typographical errors present in the first edition of *Introduction to Cosmology*. However, I am sure I introduced some new errors! If you find any, please let me know at `ryden.1@osu.edu`. Future versions of the Instructor's Manual will contain a list of errata.

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Fundamental observations

In the second edition of *Introduction to Cosmology*, Section 2.1 contains a more physically realistic analysis of Olbers' paradox. Instead of treating stars as point light sources, as I did in the first edition, I acknowledge that they are spheres of finite size, and use a mean free path analysis to find the resulting surface brightness of the night sky in an infinite Euclidean universe.

In Section 2.2, I clarify the difference between the Copernican principle and the cosmological principle, which I failed to make clear in the first edition.

Section 2.3 contains a bit more historical background on Hubble's law, placing it in the context of the work by Lemaître and others. My default value of the Hubble constant, taken to be $H_0 = 70 \pm 7\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in the first edition, is $H_0 = 68 \pm 2\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in the second edition. (Note that although this value of H_0 is consistent with the WMAP 9-year results and the Planck 2015 results, it is lower than value found by direct measurements in the local universe. For instance, Riess et al. (2016) find $H_0 = 73.24 \pm 1.74\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. In the textbook, I do not bring up this tension; you may want to use it as a topic of discussion with your students.)

In Section 2.4, I clarify the discussion of the different neutrino mass states, placing upper and lower limits on the sum of the neutrino masses. (These limits reappear in Chapter 11, during the discussion of the difference between hot dark matter and cold dark matter.)

Exercises

2.1 *Assume you are a perfect blackbody at a temperature of* $T = 310\,\mathrm{K}$. *What is the rate, in watts, at which you radiate energy? (For the purposes of this problem, you may assume you are spherical.)*

I have a mass $M_{\mathrm{me}} = 70\,\mathrm{kg}$, and my density is comparable to that of water, $\rho_{\rm me} = 1000\,{\rm kg\,m^{-3}}$. If I curl myself into a ball, my radius is

$$R_{\rm me} = \left(\frac{3M_{\rm me}}{4\pi\rho_{\rm me}}\right)^{1/3} = 0.256\,{\rm m} = 3.65 \times 10^{-10}\,{\rm R}_\odot \ . \tag{2.1}$$

My temperature is

$$T_{\rm me} = 310\,{\rm K} = 0.0534\,{\rm T}_\odot \ . \tag{2.2}$$

(The Sun's radius is given Section 2.1 of the text, and the Sun's effective temperature is given in Section 2.4.) Since the luminosity of a spherical blackbody is $L = 4\pi R^2 \sigma_{\rm sb} T^4$, where $\sigma_{\rm sb}$ is the Stefan-Boltzmann constant, my luminosity is

$$\begin{aligned} L_{\rm me} &= L_\odot \left(\frac{R_{\rm me}}{{\rm R}_\odot}\right)^2 \left(\frac{T_{\rm me}}{{\rm T}_\odot}\right)^4 \\ &= 3.838 \times 10^{26}\,{\rm watts}(3.65 \times 10^{-10})^2 (0.534)^4 = 420\,{\rm watts} \ . \end{aligned} \tag{2.3}$$

(Note that the text never explicitly states that $L \propto R^2 T^4$ for a spherical blackbody; your students may need a prompt, depending on their physics background.)

2.2 *Since you are made mostly of water, you are very efficient at absorbing microwave photons. If you were in intergalactic space, how many CMB photons would you absorb per second? (The assumption that you are spherical will be useful.) What is the rate, in watts, at which you would absorb radiative energy from the CMB?*

Since my radius is $R_{\rm me} = 0.256\,{\rm m}$ (from the previous problem), my geometric cross-section is

$$\sigma_{\rm me} = \pi R_{\rm me}^2 = 0.205\,{\rm m}^2 \ . \tag{2.4}$$

The number density of CMB photons is (from Eq. 2.35 of the text),

$$n_\gamma = 4.107 \times 10^8\,{\rm m}^{-3} \ . \tag{2.5}$$

At what rate will I absorb CMB photons? This is easiest to compute if I adopt the fiction that all the CMB photons are moving in the same direction. In that case, during a time interval dt, I would sweep up all the photons in a cylinder of length cdt and cross-sectional area $\sigma_{\rm me}$. That is, the number dN of photons absorbed during time interval dt is

$$dN = cdt\sigma_{\rm me} n_\gamma \ , \tag{2.6}$$

or

$$\frac{dN}{dt} = c\sigma_{\rm me} n_\gamma = 2.52 \times 10^{16}\,{\rm s}^{-1} \ . \tag{2.7}$$

The average energy of a CMB photon is $E_{\rm mean} = 6.344 \times 10^{-4}\,{\rm eV} = 1.016 \times 10^{-22}\,{\rm J}$, from Eq. 2.36 of the text. Thus, the heating rate from absorbing CMB photons will be

$$G_{\rm me} = \frac{dN}{dt} E_{\rm mean} = 2.6 \times 10^{-6}\,{\rm J\,s^{-1}} = 2.6 \times 10^{-6}\,{\rm watts}\ . \qquad (2.8)$$

[A student wise in the ways of the Stefan-Boltzmann law might make the following alternative argument: If I had a temperature equal to that of the CMB, I would be in an equilibrium state, with the rate at which I absorbed energy from the CMB being exactly equal to the rate at which I emitted energy from my surface. Thus, I can state that the rate at which I absorb energy from the CMB is

$$G_{\rm me} = 4\pi R_{\rm me}^2 \sigma_{\rm sb} T_0^4\ , \qquad (2.9)$$

where $T_0 = 2.7255\,{\rm K}$ is the temperature of the CMB. Using $R_{\rm me} = 0.256\,{\rm m}$ and $\sigma_{\rm sb} = 5.670 \times 10^{-8}\,{\rm watts\,m^{-2}\,K^{-4}}$, this works out to

$$G_{\rm me} = 2.58 \times 10^{-6}\,{\rm watts}\ , \qquad (2.10)$$

from which I can work backward to find dN/dt, the rate at which CMB photons are absorbed.]

2.3 *Suppose that intergalactic space pirates toss you out the airlock of your spacecraft without a spacesuit. Combining the results of the two previous questions, at what rate would your temperature change? (Assume your heat capacity is that of pure water, $C = 4200\,{\rm J\,kg^{-1}\,K^{-1}}$.) Would you be most worried about overheating, freezing, or asphyxiating?*

From the previous two problems, I know that since my temperature ($T_{\rm me} \approx 310\,{\rm K}$) is much greater than that of the CMB ($T_0 = 2.7255\,{\rm K}$), I will *lose* energy rather than gain it, at a net rate

$$L_{\rm net} = L_{\rm me} - G_{\rm me} = 420\,{\rm watts} - 0.000003\,{\rm watts} = 420\,{\rm watts}\ . \qquad (2.11)$$

My temperature will then drop at the rate

$$\frac{dT_{\rm me}}{dt} = -\frac{L_{\rm net}}{CM_{\rm me}} = -\frac{420\,{\rm J\,s^{-1}}}{2.94 \times 10^5\,{\rm J\,K^{-1}}} = -1.4 \times 10^{-3}\,{\rm K\,s^{-1}}\ . \qquad (2.12)$$

It takes about 12 minutes for my temperature to drop by one degree; even if I hyperventilate with panic at the prospect of being tossed out the airlock, I will asphyxiate before I freeze.

2.4 *A hypothesis once used to explain the Hubble relation is the "tired light hypothesis." The tired light hypothesis states that the universe is not expanding, but that photons simply lose energy as they move through* *space (by some unexplained means), with the energy loss per unit distance being given by the law*

$$\frac{dE}{dr} = -kE, \tag{2.13}$$

where k is a constant. Show that this hypothesis gives a distance-redshift relation that is linear in the limit $z \ll 1$. What must the value of k be in order to yield a Hubble constant of $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$?

If a photon starts with energy E_0 at $r = 0$, then the "tired light" equation

$$\frac{dE}{dr} = -kE \tag{2.14}$$

can be integrated to find the solution

$$E(r) = E_0 e^{-kr} \ . \tag{2.15}$$

Since a photon's energy E is related to wavelength λ by the equation

$$E = hf = \frac{hc}{\lambda} \ , \tag{2.16}$$

the redshift z of the light can be written as

$$z \equiv \frac{\lambda - \lambda_0}{\lambda_0} = \frac{1/E - 1/E_0}{1/E_0} = \frac{E_0 - E}{E} \ . \tag{2.17}$$

Substituting from equation 2.15 above, we find the distance-redshift relation

$$z = \frac{1 - e^{-kr}}{e^{-kr}} = e^{kr} - 1 \ . \tag{2.18}$$

In the limit $kr \ll 1$, we can use the expansion $\exp(kr) \approx 1 + kr$, and thus

$$z \approx kr \ . \tag{2.19}$$

This is the Hubble law, with $k = H_0/c$. To yield $H_0 = 68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, the "tired light" parameter must be

$$k = \frac{H_0}{c} = \frac{68\,\mathrm{km\,s^{-1}\,Mpc^{-1}}}{3 \times 10^5\,\mathrm{km\,s^{-1}}} = 2.3 \times 10^{-4}\,\mathrm{Mpc^{-1}} \ . \tag{2.20}$$

2.5 *Consider blackbody radiation at a temperature T. Show that for an energy threshold $E_0 \gg kT$, the fraction of the blackbody photons that have energy $hf > E_0$ is*

$$\frac{n(hf > E_0)}{n_\gamma} \approx 0.42 \left(\frac{E_0}{kT}\right)^2 \exp\left(-\frac{E_0}{kT}\right) . \tag{2.21}$$

The cosmic background radiation is currently called the "cosmic microwave background." However, photons with $\lambda < 1\,\mathrm{mm}$ actually lie in the far infrared range of the electromagnetic spectrum. It's time for truth in advertising: what fraction of the photons in today's "cosmic microwave background" are actually far infrared photons?

At photon energies $hf \gg kT$, the number density of photons as a function of frequency is (from Eq. 2.30 of the text),

$$n(f) = \frac{8\pi}{c^3} \frac{f^2}{\exp(hf/kT) - 1} \approx \frac{8\pi}{c^3} f^2 \exp(-hf/kT) \ . \qquad (2.22)$$

Thus, the number density of photons with energy greater than some threshold energy $E_0 = hf_0 \gg kT$ is

$$n(hf > E_0) \approx \frac{8\pi}{c^3} \int_{f_0}^{\infty} f^2 \exp(-hf/kT) df \ . \qquad (2.23)$$

Making the substitution $x = hf/kT$, this becomes

$$n(hf > E_0) \approx 8\pi \left(\frac{kT}{hc}\right)^3 \int_{x_0}^{\infty} x^2 e^{-x} dx \ , \qquad (2.24)$$

where $x_0 = hf_0/kT = E_0/kT \gg 1$. The total number density of photons is, from Eqs. 2.31 and 2.32 of the text,

$$n_\gamma = \frac{2.4041}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 = 2.4041(8\pi) \left(\frac{kT}{hc}\right)^3 \ . \qquad (2.25)$$

Thus, the fraction of the photons with $hf > E_0 \gg kT$ is

$$F(hf > E_0) = \frac{n(hf > E_0)}{n_\gamma} \approx \frac{1}{2.4041} \int_{x_0}^{\infty} x^2 e^{-x} dx \ . \qquad (2.26)$$

Doing the integral, we find that

$$F(hf > E_0) = 0.416 e^{-x_0} [x_0^2 + 2x_0 + 2] \ , \qquad (2.27)$$

where $x_0 = E_0/kT$. Since we have already made the approximation $x_0 \gg 1$, equation (2.27) can be adequately approximated as

$$F(hf > E_0) \approx 0.416 x_0^2 e^{-x_0} \approx 0.42 \left(\frac{E_0}{kT}\right)^2 \exp\left(-\frac{E_0}{kT}\right) \ . \qquad (2.28)$$

A photon with wavelength $\lambda_0 = 1\,\mathrm{mm}$, at the threshold of the far-infrared range, has an energy $E_0 = hc/\lambda_0 = 1.240 \times 10^{-3}\,\mathrm{eV}$. For the CMB, $kT_0 = 2.349 \times 10^{-4}\,\mathrm{eV}$, yielding $x_0 = E_0/kT_0 = 5.28$, which I declare (slightly rashly) to be much larger than one. The fraction of CMB photons in the far-infrared range, with $hf > E_0$, is then

$$F(hf > 1.24\,\mathrm{meV}) \approx 0.42(5.28)^2 e^{-5.28} \approx 0.06\ . \tag{2.29}$$

2.6 *Show that for an energy threshold $E_0 \ll kT$, the fraction of blackbody photons that have energy $hf < E_0$ is*

$$\frac{n(hf < E_0)}{n_\gamma} \approx 0.21 \left(\frac{E_0}{kT}\right)^2 . \tag{2.30}$$

Microwave (and far infrared) photons with a wavelength $\lambda < 3\,\mathrm{cm}$ are strongly absorbed by H_2O and O_2 molecules. What fraction of the photons in today's cosmic microwave background have $\lambda > 3\,\mathrm{cm}$, and thus are capable of passing through the Earth's atmosphere and being detected on the ground? At photon energies $hf \ll kT$, the number density of photons as a function of frequency is (from Eq. 2.30 of the text),

$$n(f) = \frac{8\pi}{c^3}\frac{f^2}{\exp(hf/kT)-1} \approx \frac{8\pi}{c^3}\frac{f^2}{hf/kT} \approx \frac{8\pi kT}{hc^3} f\ . \tag{2.31}$$

Thus, the number density of photons with energy less than some threshold energy $E_0 = hf_0 \ll kT$ is

$$\begin{aligned} n(hf < E_0) &\approx \frac{8\pi kT}{hc^3}\int_0^{f_0} f\,df \\ &\approx \frac{4\pi kT f_0^2}{hc^3} \approx \frac{4\pi kT E_0^2}{(hc)^3}\ . \end{aligned} \tag{2.32}$$

The total number density of photons is, from Eqs. 2.31 and 2.32 of the text,

$$n_\gamma = \frac{2.4041}{\pi^2}\left(\frac{kT}{hc}\right)^3 = 2.4041(8\pi)\left(\frac{kT}{hc}\right)^3 . \tag{2.33}$$

Combining the two above results, we find that the fraction of photons with $hf < E_0 \ll kT$ is

$$F(hf < E_0) = \frac{n(hf < E_0)}{n_\gamma} \approx 0.208\left(\frac{E_0}{kT}\right)^2 . \tag{2.34}$$

A wavelength $\lambda_0 = 3\,\mathrm{cm}$ corresponds to a photon energy $E_0 = hc/\lambda_0 = 4.133\times10^{-5}\,\mathrm{eV}$. For the CMB, $kT_0 = 2.349\times10^{-4}\,\mathrm{eV}$, yielding $E_0/kT_0 = 0.176$, which I declare, somewhat rashly, to be much smaller than one. The fraction of CMB photons with $\lambda > 3\,\mathrm{cm}$ is then

$$F(hf < 41.33\,\mu\mathrm{eV}) \approx 0.2080(0.176)^2 \approx 0.006\ . \tag{2.35}$$