Chapter 2

The Time-Independent Schrödinger Equation

Problem 2.1

(a)

$$\Psi(x,t) = \psi(x)e^{-i(E_0+i\Gamma)t/\hbar} = \psi(x)e^{\Gamma t/\hbar}e^{-iE_0 t/\hbar} \Longrightarrow |\Psi|^2 = |\psi|^2 e^{2\Gamma t/\hbar}.$$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = e^{2\Gamma t/\hbar}\int_{-\infty}^{\infty} |\psi|^2 dx.$$

The second term is independent of t, so if the product is to be 1 for all time, the first term ($e^{2\Gamma t/\hbar}$) must also be constant, and hence $\Gamma = 0$. QED

(b) If ψ satisfies Eq. 2.5, $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$, then (taking the complex conjugate and noting that V and E are real): $-\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} + V\psi^* = E\psi^*$, so ψ^* *also* satisfies Eq. 2.5. Now, if ψ_1 and ψ_2 satisfy Eq. 2.5, so too does any linear combination of them ($\psi_3 \equiv c_1\psi_1 + c_2\psi_2$):

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_3}{dx^2} + V\psi_3 &= -\frac{\hbar^2}{2m}\left(c_1\frac{d^2\psi_1}{dx^2} + c_2\frac{d^2\psi_2}{dx^2}\right) + V(c_1\psi_1 + c_2\psi_2) \\ &= c_1\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1\right] + c_2\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2\right] \\ &= c_1(E\psi_1) + c_2(E\psi_2) = E(c_1\psi_1 + c_2\psi_2) = E\psi_3. \end{aligned}$$

Thus, $(\psi + \psi^*)$ and $i(\psi - \psi^*)$ – both of which are *real* – satisfy Eq. 2.5. *Conclusion:* From any complex solution, we can always construct two *real* solutions (of course, if ψ is already real, the second one will be zero). In particular, since $\psi = \frac{1}{2}[(\psi + \psi^*) - i(i(\psi - \psi^*))]$, ψ can be expressed as a linear combination of two real solutions. QED

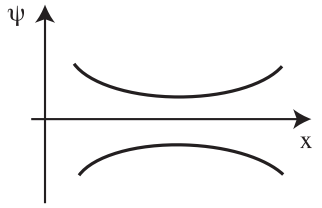
(c) If $\psi(x)$ satisfies Eq. 2.5, then, changing variables $x \to -x$ and noting that $d^2/d(-x)^2 = d^2/dx^2$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2} + V(-x)\psi(-x) = E\psi(-x);$$

so if $V(-x) = V(x)$ then $\psi(-x)$ *also* satisfies Eq. 2.5. It follows that $\psi_+(x) \equiv \psi(x) + \psi(-x)$ (which is *even*: $\psi_+(-x) = \psi_+(x)$) and $\psi_-(x) \equiv \psi(x) - \psi(-x)$ (which is *odd*: $\psi_-(-x) = -\psi_-(x)$) both satisfy Eq. 2.5. But $\psi(x) = \frac{1}{2}(\psi_+(x) + \psi_-(x))$, so any solution can be expressed as a linear combination of even and odd solutions. QED

Problem 2.2

Given $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi$, if $E < V_{\min}$, then ψ'' and ψ always have the same sign: If ψ is positive(negative), then ψ'' is also positive(negative). This means that ψ always curves away from the axis (see Figure). However, it has got to go to zero as $x \to -\infty$ (else it would not be normalizable). At some point it's got to *depart* from zero (if it *doesn't*, it's going to be identically zero *everywhere*), in (say) the positive direction. At this point its slope is positive, and *increasing*, so ψ gets bigger and bigger as x increases. It can't ever "turn over" and head back toward the axis, because that would require a negative second derivative—it always has to bend away from the axis. By the same token, if it starts out heading negative, it just runs more and more negative. In neither case is there any way for it to come back to zero, as it must (at $x \to \infty$) in order to be normalizable. QED



Problem 2.3

Equation 2.23 says $\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$; Eq. 2.26 says $\psi(0) = \psi(a) = 0$. If $E = 0$, $d^2\psi/dx^2 = 0$, so $\psi(x) = A + Bx$; $\psi(0) = A = 0 \Rightarrow \psi = Bx$; $\psi(a) = Ba = 0 \Rightarrow B = 0$, so $\psi = 0$. If $E < 0$, $d^2\psi/dx^2 = \kappa^2\psi$, with $\kappa \equiv \sqrt{-2mE}/\hbar$ real, so $\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$. This time $\psi(0) = A + B = 0 \Rightarrow B = -A$, so $\psi = A(e^{\kappa x} - e^{-\kappa x})$, while $\psi(a) = A\left(e^{\kappa a} - e^{-\kappa a}\right) = 0 \Rightarrow$ either $A = 0$, so $\psi = 0$, or else $e^{\kappa a} = e^{-\kappa a}$, so $e^{2\kappa a} = 1$, so $2\kappa a = \ln(1) = 0$, so $\kappa = 0$, and again $\psi = 0$. In all cases, then, the boundary conditions force $\psi = 0$, which is unacceptable (non-normalizable).

Problem 2.4

$$\langle x\rangle = \int x|\psi|^2 dx = \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)dx. \qquad \text{Let } y \equiv \frac{n\pi}{a}x, \text{ so } dx = \frac{a}{n\pi}dy; \quad y: 0 \to n\pi.$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^2\int_0^{n\pi} y\sin^2 y\,dy = \frac{2a}{n^2\pi^2}\left[\frac{y^2}{4} - \frac{y\sin 2y}{4} - \frac{\cos 2y}{8}\right]\Bigg|_0^{n\pi}$$

$$= \frac{2a}{n^2\pi^2}\left[\frac{n^2\pi^2}{4} - \frac{\cos 2n\pi}{8} + \frac{1}{8}\right] = \boxed{\frac{a}{2}.} \quad \text{(Independent of } n.\text{)}$$

$$\langle x^2\rangle = \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{2}{a}\left(\frac{a}{n\pi}\right)^3\int_0^{n\pi} y^2\sin^2 y\,dy$$

$$= \frac{2a^2}{(n\pi)^3}\left[\frac{y^3}{6} - \left(\frac{y^2}{4} - \frac{1}{8}\right)\sin 2y - \frac{y\cos 2y}{4}\right]_0^{n\pi}$$

$$= \frac{2a^2}{(n\pi)^3}\left[\frac{(n\pi)^3}{6} - \frac{n\pi\cos(2n\pi)}{4}\right] = \boxed{a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right].}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.} \quad (\textit{Note}: \text{Eq. 1.33 is much faster than Eq. 1.35.})$$

$$\langle p^2\rangle = \int \psi_n^* \left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 \psi_n\,dx = -\hbar^2\int \psi_n^*\left(\frac{d^2\psi_n}{dx^2}\right)dx$$
$$= (-\hbar^2)\left(-\frac{2mE_n}{\hbar^2}\right)\int \psi_n^*\psi_n\,dx = 2mE_n = \boxed{\left(\frac{n\pi\hbar}{a}\right)^2.}$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2} - \frac{1}{4}\right) = \frac{a^2}{4}\left(\frac{1}{3} - \frac{2}{(n\pi)^2}\right); \quad \boxed{\sigma_x = \frac{a}{2}\sqrt{\frac{1}{3} - \frac{2}{(n\pi)^2}}.}$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \left(\frac{n\pi\hbar}{a}\right)^2; \quad \boxed{\sigma_p = \frac{n\pi\hbar}{a}.} \quad \therefore \sigma_x\sigma_p = \boxed{\frac{\hbar}{2}\sqrt{\frac{(n\pi)^2}{3} - 2}.}$$

The product $\sigma_x\sigma_p$ is $\boxed{\text{smallest for } n = 1;}$ in that case, $\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3} - 2} = (1.136)\hbar/2 > \hbar/2.$ ✓

Problem 2.5

(a)

$$|\Psi|^2 = \Psi^*\Psi = |A|^2(\psi_1^* + \psi_2^*)(\psi_1 + \psi_2) = |A|^2[\psi_1^*\psi_1 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_2^*\psi_2].$$

$$1 = \int |\Psi|^2 dx = |A|^2\int [|\psi_1|^2 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + |\psi_2|^2]dx = 2|A|^2 \Rightarrow \boxed{A = 1/\sqrt{2}.}$$

(b)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1 e^{-iE_1t/\hbar} + \psi_2 e^{-iE_2t/\hbar}\right] \quad \left(\text{but } \frac{E_n}{\hbar} = n^2\omega\right)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-i4\omega t}\right] = \boxed{\frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi}{a}x\right) + \sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right].}$$

$$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left(e^{-3i\omega t} + e^{3i\omega t}\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right]$$
$$= \boxed{\frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right].}$$

(c)

$$\langle x\rangle = \int x|\Psi(x,t)|^2 dx$$
$$= \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]dx$$

$$\int_0^a x\sin^2\left(\frac{\pi}{a}x\right)dx = \left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi}{a}x\right)}{4\pi/a} - \frac{\cos\left(\frac{2\pi}{a}x\right)}{8(\pi/a)^2}\right]\Bigg|_0^a = \frac{a^2}{4} = \int_0^a x\sin^2\left(\frac{2\pi}{a}x\right)dx.$$

$$\int_0^a x\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)dx = \frac{1}{2}\int_0^a x\left[\cos\left(\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\cos\left(\frac{\pi}{a}x\right) + \frac{ax}{\pi}\sin\left(\frac{\pi}{a}x\right) - \frac{a^2}{9\pi^2}\cos\left(\frac{3\pi}{a}x\right) - \frac{ax}{3\pi}\sin\left(\frac{3\pi}{a}x\right)\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\big(\cos(\pi) - \cos(0)\big) - \frac{a^2}{9\pi^2}\big(\cos(3\pi) - \cos(0)\big)\right] = -\frac{a^2}{\pi^2}\left(1 - \frac{1}{9}\right) = -\frac{8a^2}{9\pi^2}.$$

$$\therefore \langle x\rangle = \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} - \frac{16a^2}{9\pi^2}\cos(3\omega t)\right] = \boxed{\frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right].}$$

Amplitude: $\boxed{\frac{32}{9\pi^2}\left(\frac{a}{2}\right) = 0.3603(a/2);}$ *angular frequency:* $\boxed{3\omega = \frac{3\pi^2\hbar}{2ma^2}.}$

(d)

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\left(\frac{a}{2}\right)\left(-\frac{32}{9\pi^2}\right)(-3\omega)\sin(3\omega t) = \boxed{\frac{8\hbar}{3a}\sin(3\omega t).}$$

(e) You could get either $\boxed{E_1 = \pi^2\hbar^2/2ma^2}$ or $\boxed{E_2 = 2\pi^2\hbar^2/ma^2,}$ with equal probability $\boxed{P_1 = P_2 = 1/2.}$

So $\langle H\rangle = \boxed{\frac{1}{2}(E_1 + E_2) = \frac{5\pi^2\hbar^2}{4ma^2};}$ it's the *average* of E_1 and E_2.

Problem 2.6

From Problem 2.5, we see that

$$\Psi(x,t) = \boxed{\tfrac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\tfrac{\pi}{a}x\right) + \sin\left(\tfrac{2\pi}{a}x\right)e^{-3i\omega t}e^{i\phi}\right];}$$

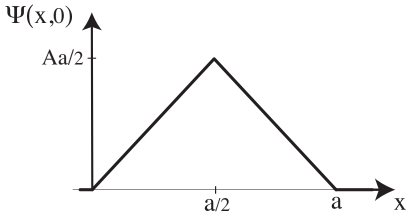
$$|\Psi(x,t)|^2 = \boxed{\tfrac{1}{a}\left[\sin^2\left(\tfrac{\pi}{a}x\right) + \sin^2\left(\tfrac{2\pi}{a}x\right) + 2\sin\left(\tfrac{\pi}{a}x\right)\sin\left(\tfrac{2\pi}{a}x\right)\cos(3\omega t - \phi)\right];}$$

and hence $\boxed{\langle x\rangle = \tfrac{a}{2}\left[1 - \tfrac{32}{9\pi^2}\cos(3\omega t - \phi)\right].}$ This amounts physically to starting the clock at a different time (i.e., shifting the $t = 0$ point).

If $\phi = \frac{\pi}{2}$, so $\Psi(x,0) = A[\psi_1(x) + i\psi_2(x)]$, then $\cos(3\omega t - \phi) = \sin(3\omega t)$; $\langle x\rangle$ starts at $\frac{a}{2}$.

If $\phi = \pi$, so $\Psi(x,0) = A[\psi_1(x) - \psi_2(x)]$, then $\cos(3\omega t - \phi) = -\cos(3\omega t)$; $\langle x\rangle$ starts at $\frac{a}{2}\left(1 + \frac{32}{9\pi^2}\right)$.

Problem 2.7



(a)

$$1 = A^2 \int_0^{a/2} x^2 dx + A^2 \int_{a/2}^{a} (a-x)^2 dx = A^2 \left[\frac{x^3}{3}\Big|_0^{a/2} - \frac{(a-x)^3}{3}\Big|_{a/2}^{a} \right]$$

$$= \frac{A^2}{3}\left(\frac{a^3}{8} + \frac{a^3}{8}\right) = \frac{A^2 a^3}{12} \Rightarrow \boxed{A = \frac{2\sqrt{3}}{\sqrt{a^3}}.}$$

(b)

$$c_n = \sqrt{\frac{2}{a}}\frac{2\sqrt{3}}{a\sqrt{a}}\left[\int_0^{a/2} x \sin\left(\frac{n\pi}{a}x\right)dx + \int_{a/2}^{a} (a-x)\sin\left(\frac{n\pi}{a}x\right)dx\right]$$

$$= \frac{2\sqrt{6}}{a^2}\Bigg\{\left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{a}x\right) - \frac{xa}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]\Bigg|_0^{a/2}$$

$$+ a\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]\Bigg|_{a/2}^{a} - \left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{a}x\right) - \left(\frac{ax}{n\pi}\right)\cos\left(\frac{n\pi}{a}x\right)\right]\Bigg|_{a/2}^{a}\Bigg\}$$

$$= \frac{2\sqrt{6}}{a^2}\Bigg[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) - \cancel{\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)} - \cancel{\frac{a^2}{n\pi}\cos n\pi} + \cancel{\frac{a^2}{n\pi}\cos\left(\frac{n\pi}{2}\right)}$$

$$+ \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{a^2}{n\pi}\cos n\pi} - \cancel{\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)}\Bigg]$$

$$= \frac{2\sqrt{6}}{\cancel{a^2}} 2 \frac{\cancel{a^2}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) = \frac{4\sqrt{6}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ even}, \\ (-1)^{(n-1)/2}\frac{4\sqrt{6}}{(n\pi)^2}, & n \text{ odd}. \end{cases}$$

So $\boxed{\Psi(x,t) = \frac{4\sqrt{6}}{\pi^2}\sqrt{\frac{2}{a}} \sum_{n=1,3,5,\ldots} (-1)^{(n-1)/2}\frac{1}{n^2}\sin\left(\frac{n\pi}{a}x\right)e^{-iE_n t/\hbar},}$ where $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

(c)

$$P_1 = |c_1|^2 = \frac{16\cdot 6}{\pi^4} = \boxed{0.9855.}$$

(d)

$$\langle H\rangle = \sum |c_n|^2 E_n = \frac{96}{\pi^4}\frac{\pi^2\hbar^2}{2ma^2}\Bigg(\underbrace{\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots}_{\pi^2/8}\Bigg) = \frac{48\hbar^2}{\pi^2 ma^2}\frac{\pi^2}{8} = \boxed{\frac{6\hbar^2}{ma^2}.}$$

Problem 2.8

$$A^2 \int_0^{a/2} dx = A^2(a/2) = 1 \Rightarrow A = \sqrt{\frac{2}{a}}.$$

From Eq. 2.37,

$$c_1 = A\sqrt{\frac{2}{a}} \int_0^{a/2} \sin\left(\frac{\pi}{a}x\right) dx = \frac{2}{a}\left[-\frac{a}{\pi}\cos\left(\frac{\pi}{a}x\right)\right]\Big|_0^{a/2} = -\frac{2}{\pi}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = \frac{2}{\pi}.$$

$$P_1 = |c_1|^2 = \boxed{(2/\pi)^2 = 0.4053.}$$

Problem 2.9

$$\hat{H}\Psi(x,0) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[Ax(a-x)\right] = -A\frac{\hbar^2}{2m}\frac{\partial}{\partial x}(a-2x) = A\frac{\hbar^2}{m}.$$

$$\begin{aligned}\int \Psi(x,0)^* \hat{H}\Psi(x,0)\, dx \;&=\; A^2\frac{\hbar^2}{m}\int_0^a x(a-x)\, dx = A^2\frac{\hbar^2}{m}\left(a\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^a \\ &=\; A^2\frac{\hbar^2}{m}\left(\frac{a^3}{2} - \frac{a^3}{3}\right) = \frac{30}{a^5}\frac{\hbar^2}{m}\frac{a^3}{6} = \boxed{\frac{5\hbar^2}{ma^2}}\end{aligned}$$

(same as Example 2.3).

Problem 2.10

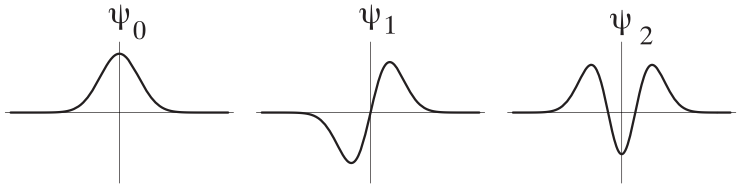
(a) Using Eqs. 2.48 and 2.60,

$$\begin{aligned}a_+\psi_0 &= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \\ &= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(-\frac{m\omega}{2\hbar}\right)2x + m\omega x\right] e^{-\frac{m\omega}{2\hbar}x^2} = \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega x e^{-\frac{m\omega}{2\hbar}x^2}. \\ (a_+)^2\psi_0 &= \frac{1}{2\hbar m\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} 2m\omega\left(-\hbar\frac{d}{dx} + m\omega x\right) x e^{-\frac{m\omega}{2\hbar}x^2} \\ &= \frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(1 - x\frac{m\omega}{2\hbar}2x\right) + m\omega x^2\right] e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right) e^{-\frac{m\omega}{2\hbar}x^2}.\end{aligned}$$

Therefore, from Eq. 2.68,

$$\psi_2 = \frac{1}{\sqrt{2}}(a_+)^2\psi_0 = \boxed{\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right) e^{-\frac{m\omega}{2\hbar}x^2}.}$$

(b)



(c) Since ψ_0 and ψ_2 are even, whereas ψ_1 is odd, $\int \psi_0^* \psi_1 dx$ and $\int \psi_2^* \psi_1 dx$ vanish automatically. The only one we need to check is $\int \psi_2^* \psi_0 \, dx$:

$$\begin{aligned}\int \psi_2^* \psi_0 \, dx &= \frac{1}{\sqrt{2}} \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} \left(\frac{2m\omega}{\hbar} x^2 - 1 \right) e^{-\frac{m\omega}{\hbar} x^2} dx \\ &= -\sqrt{\frac{m\omega}{2\pi\hbar}} \left(\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar} x^2} dx - \frac{2m\omega}{\hbar} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar} x^2} dx \right) \\ &= -\sqrt{\frac{m\omega}{2\pi\hbar}} \left(\sqrt{\frac{\pi\hbar}{m\omega}} - \frac{2m\omega}{\hbar} \frac{\hbar}{2m\omega} \sqrt{\frac{\pi\hbar}{m\omega}} \right) = 0. \checkmark\end{aligned}$$

Problem 2.11

(a) Note that ψ_0 is even, and ψ_1 is odd. In either case $|\psi|^2$ is even, so $\langle x \rangle = \int x |\psi|^2 dx = \boxed{0.}$ Therefore $\langle p \rangle = m d\langle x \rangle / dt = \boxed{0.}$ (These results hold for *any* stationary state of the harmonic oscillator.)

From Eqs. 2.60 and 2.63, $\psi_0 = \alpha e^{-\xi^2/2}$, $\psi_1 = \sqrt{2} \alpha \xi e^{-\xi^2/2}$. So

<u>$n = 0$</u>**:**

$$\langle x^2 \rangle = \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-\xi^2} dx = \alpha^2 \left(\frac{\hbar}{m\omega} \right)^{3/2} \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2} d\xi = \frac{1}{\sqrt{\pi}} \left(\frac{\hbar}{m\omega} \right) \frac{\sqrt{\pi}}{2} = \boxed{\frac{\hbar}{2m\omega}}.$$

$$\begin{aligned}\langle p^2 \rangle &= \int \psi_0 \left(\frac{\hbar}{i} \frac{d}{dx} \right)^2 \psi_0 \, dx = -\hbar^2 \alpha^2 \sqrt{\frac{m\omega}{\hbar}} \int_{-\infty}^{\infty} e^{-\xi^2/2} \left(\frac{d^2}{d\xi^2} e^{-\xi^2/2} \right) d\xi \\ &= -\frac{m\hbar\omega}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\xi^2 - 1) e^{-\xi^2} d\xi = -\frac{m\hbar\omega}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} - \sqrt{\pi} \right) = \boxed{\frac{m\hbar\omega}{2}}.\end{aligned}$$

<u>$n = 1$</u>**:**

$$\langle x^2 \rangle = 2\alpha^2 \int_{-\infty}^{\infty} x^2 \xi^2 e^{-\xi^2} dx = 2\alpha^2 \left(\frac{\hbar}{m\omega} \right)^{3/2} \int_{-\infty}^{\infty} \xi^4 e^{-\xi^2} d\xi = \frac{2\hbar}{\sqrt{\pi} m\omega} \frac{3\sqrt{\pi}}{4} = \boxed{\frac{3\hbar}{2m\omega}}.$$

$$\begin{aligned}\langle p^2 \rangle &= -\hbar^2 2\alpha^2 \sqrt{\frac{m\omega}{\hbar}} \int_{-\infty}^{\infty} \xi e^{-\xi^2/2} \left[\frac{d^2}{d\xi^2} (\xi e^{-\xi^2/2}) \right] d\xi \\ &= -\frac{2m\omega\hbar}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\xi^4 - 3\xi^2) e^{-\xi^2} d\xi = -\frac{2m\omega\hbar}{\sqrt{\pi}} \left(\frac{3}{4}\sqrt{\pi} - 3\frac{\sqrt{\pi}}{2} \right) = \boxed{\frac{3m\hbar\omega}{2}}.\end{aligned}$$

(b) $\underline{n=0}$**:**

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}};\ \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{m\hbar\omega}{2}};$$

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega\hbar}{2}} = \frac{\hbar}{2}. \quad \text{(Right } at \text{ the uncertainty limit.)}\checkmark$$

$\underline{n=1}$**:**

$$\sigma_x = \sqrt{\frac{3\hbar}{2m\omega}}; \quad \sigma_p = \sqrt{\frac{3m\hbar\omega}{2}}; \quad \sigma_x\sigma_p = 3\frac{\hbar}{2} > \frac{\hbar}{2}.\ \checkmark$$

(c)

$$\langle T\rangle = \frac{1}{2m}\langle p^2\rangle = \boxed{\left\{\begin{array}{l} \frac{1}{4}\hbar\omega\ \ (n=0) \\ \\ \frac{3}{4}\hbar\omega\ \ (n=1) \end{array}\right\}}; \quad \langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \boxed{\left\{\begin{array}{l} \frac{1}{4}\hbar\omega\ \ (n=0) \\ \\ \frac{3}{4}\hbar\omega\ \ (n=1) \end{array}\right\}}.$$

$$\langle T\rangle + \langle V\rangle = \langle H\rangle = \left\{\begin{array}{l} \frac{1}{2}\hbar\omega\ \ (n=0)\ = E_0 \\ \\ \frac{3}{2}\hbar\omega\ \ (n=1)\ = E_1 \end{array}\right\}, \text{ as expected.}$$

Problem 2.12

From Eq. 2.70,

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-), \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-),$$

so

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int \psi_n^*(a_+ + a_-)\psi_n\,dx.$$

But (Eq. 2.67)

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad a_-\psi_n = \sqrt{n}\psi_{n-1}.$$

So

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\int \psi_n^*\psi_{n+1}\,dx + \sqrt{n}\int \psi_n^*\psi_{n-1}\,dx\right] = \boxed{0} \text{ (by orthogonality).}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.} \quad x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right).$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\int \psi_n^*\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right)\psi_n. \quad \text{But}$$

$$\left\{\begin{array}{llll} a_+^2\psi_n & = a_+\left(\sqrt{n+1}\psi_{n+1}\right) & = \sqrt{n+1}\sqrt{n+2}\psi_{n+2} & = \sqrt{(n+1)(n+2)}\psi_{n+2}. \\ a_+a_-\psi_n & = a_+\left(\sqrt{n}\psi_{n-1}\right) & = \sqrt{n}\sqrt{n}\psi_n & = n\psi_n. \\ a_-a_+\psi_n & = a_-\left(\sqrt{n+1}\psi_{n+1}\right) & = \sqrt{n+1)}\sqrt{n+1}\psi_n & = (n+1)\psi_n. \\ a_-^2\psi_n & = a_-\left(\sqrt{n}\psi_{n-1}\right) & = \sqrt{n}\sqrt{n-1}\psi_{n-2} & = \sqrt{(n-1)n}\psi_{n-2}. \end{array}\right.$$

So

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left[0 + n\int |\psi_n|^2 dx + (n+1)\int |\psi_n|^2\,dx + 0\right] = \frac{\hbar}{2m\omega}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}.}$$

$$p^2 = -\frac{\hbar m\omega}{2}(a_+ - a_-)^2 = -\frac{\hbar m\omega}{2}\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right) \Rightarrow$$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\left[0 - n - (n+1) + 0\right] = \frac{\hbar m\omega}{2}(2n+1) = \boxed{\left(n+\frac{1}{2}\right)m\hbar\omega.}$$

$$\langle T\rangle = \langle p^2/2m\rangle = \boxed{\frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega}.$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{\frac{\hbar}{m\omega}}; \quad \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{m\hbar\omega}; \quad \sigma_x\sigma_p = \left(n+\frac{1}{2}\right)\hbar \geq \frac{\hbar}{2}. \checkmark$$

Problem 2.13

(a)

$$1 = \int |\Psi(x,0)|^2 dx = |A|^2 \int \left(9|\psi_0|^2 + 12\psi_0^*\psi_1 + 12\psi_1^*\psi_0 + 16|\psi_1|^2\right) dx$$
$$= |A|^2(9+0+0+16) = 25|A|^2 \Rightarrow \boxed{A = 1/5.}$$

(b)

$$\Psi(x,t) = \frac{1}{5}\left[3\psi_0(x)e^{-iE_0t/\hbar} + 4\psi_1(x)e^{-iE_1t/\hbar}\right] = \boxed{\frac{1}{5}\left[3\psi_0(x)e^{-i\omega t/2} + 4\psi_1(x)e^{-3i\omega t/2}\right].}$$

(Here ψ_0 and ψ_1 are given by Eqs. 2.60 and 2.63; E_0 and E_1 by Eq. 2.62.)

$$|\Psi(x,t)|^2 = \frac{1}{25}\left[9\psi_0^2 + 12\psi_0\psi_1 e^{i\omega t/2}e^{-3i\omega t/2} + 12\psi_0\psi_1 e^{-i\omega t/2}e^{3i\omega t/2} + 16\psi_1^2\right]$$
$$= \boxed{\frac{1}{25}\left[9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1\cos(\omega t)\right].}$$

(With ψ_2 in place of ψ_1 the frequency would be $(E_2 - E_0)/\hbar = [(5/2)\hbar\omega - (1/2)\hbar\omega]/\hbar = 2\omega$.)

(c)

$$\langle x\rangle = \frac{1}{25}\left[9\int x\psi_0^2\,dx + 16\int x\psi_1^2\,dx + 24\cos(\omega t)\int x\psi_0\psi_1\,dx\right].$$

But $\int x\psi_0^2\,dx = \int x\psi_1^2\,dx = 0$ (see Problem 2.11 or 2.12), while

$$\int x\psi_0\psi_1\,dx = \sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{2m\omega}{\hbar}}\int xe^{-\frac{m\omega}{2\hbar}x^2}xe^{-\frac{m\omega}{2\hbar}x^2}dx = \sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}dx$$
$$= \sqrt{\frac{2}{\pi}}\left(\frac{m\omega}{\hbar}\right)2\sqrt{\pi}2\left(\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}\right)^3 = \sqrt{\frac{\hbar}{2m\omega}}.$$

So

$$\langle x\rangle = \boxed{\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t);} \quad \langle p\rangle = m\frac{d}{dt}\langle x\rangle = \boxed{-\frac{24}{25}\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t).}$$

Ehrenfest's theorem says $d\langle p\rangle/dt = -\langle \partial V/\partial x\rangle$. Here

$$\frac{d\langle p\rangle}{dt} = -\frac{24}{25}\sqrt{\frac{m\omega\hbar}{2}}\omega\cos(\omega t), \quad V = \frac{1}{2}m\omega^2x^2 \Rightarrow \frac{\partial V}{\partial x} = m\omega^2 x,$$

so

$$-\langle\frac{\partial V}{\partial x}\rangle = -m\omega^2\langle x\rangle = -m\omega^2\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t) = -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\omega\cos(\omega t),$$

so Ehrenfest's theorem holds.

(d) You could get $\boxed{E_0 = \frac{1}{2}\hbar\omega,}$ with probability $|c_0|^2 = \boxed{9/25,}$ or $\boxed{E_1 = \frac{3}{2}\hbar\omega,}$ with probability $|c_1|^2 = \boxed{16/25.}$

Problem 2.14

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\xi^2/2}, \text{ so } P = 2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{x_0}^{\infty}e^{-\xi^2}dx = 2\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}\int_{\xi_0}^{\infty}e^{-\xi^2}d\xi.$$

Classically allowed region extends out to: $\frac{1}{2}m\omega^2x_0^2 = E_0 = \frac{1}{2}\hbar\omega$, or $x_0 = \sqrt{\frac{\hbar}{m\omega}}$, so $\xi_0 = 1$.

$$P = \frac{2}{\sqrt{\pi}}\int_1^{\infty}e^{-\xi^2}d\xi = 2(1 - F(\sqrt{2})) \text{ (in notation of CRC Table)} = \boxed{0.157.}$$

Problem 2.15

n = 5: $j = 1 \Rightarrow a_3 = \frac{-2(5-1)}{(1+1)(1+2)}a_1 = -\frac{4}{3}a_1; j = 3 \Rightarrow a_5 = \frac{-2(5-3)}{(3+1)(3+2)}a_3 = -\frac{1}{5}a_3 = \frac{4}{15}a_1;\ j = 5 \Rightarrow a_7 = 0$. So $H_5(\xi) = a_1\xi - \frac{4}{3}a_1\xi^3 + \frac{4}{15}a_1\xi^5 = \frac{a_1}{15}(15\xi - 20\xi^3 + 4\xi^5)$. By convention the coefficient of ξ^5 is 2^5, so $a_1 = 15\cdot 8$, and $\boxed{H_5(\xi) = 120\xi - 160\xi^3 + 32\xi^5}$ (which agrees with Table 2.1).

n = 6: $j = 0 \Rightarrow a_2 = \frac{-2(6-0)}{(0+1)(0+2)}a_0 = -6a_0;\ j = 2 \Rightarrow a_4 = \frac{-2(6-2)}{(2+1)(2+2)}a_2 = -\frac{2}{3}a_2 = 4a_0; j = 4 \Rightarrow a_6 = \frac{-2(6-4)}{(4+1)(4+2)}a_4 = -\frac{2}{15}a_4 = -\frac{8}{15}a_0;\ j = 6 \Rightarrow a_8 = 0$. So $H_6(\xi) = a_0 - 6a_0\xi^2 + 4a_0\xi^4 - \frac{8}{15}\xi^6 a_0$. The coefficient of ξ^6 is 2^6, so $2^6 = -\frac{8}{15}a_0 \Rightarrow a_0 = -15\cdot 8 = -120$. $\boxed{H_6(\xi) = -120 + 720\xi^2 - 480\xi^4 + 64\xi^6.}$

Problem 2.16

(a)

$$\frac{d}{d\xi}(e^{-\xi^2}) = -2\xi e^{-\xi^2};\ \left(\frac{d}{d\xi}\right)^2 e^{-\xi^2} = \frac{d}{d\xi}(-2\xi e^{-\xi^2}) = (-2 + 4\xi^2)e^{-\xi^2};$$

$$\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = \frac{d}{d\xi}\left[(-2 + 4\xi^2)e^{-\xi^2}\right] = \left[8\xi + (-2 + 4\xi^2)(-2\xi)\right]e^{-\xi^2} = (12\xi - 8\xi^3)e^{-\xi^2};$$

$$\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = \frac{d}{d\xi}\left[(12\xi - 8\xi^3)e^{-\xi^2}\right] = \left[12 - 24\xi^2 + (12\xi - 8\xi^3)(-2\xi)\right]e^{-\xi^2} = (12 - 48\xi^2 + 16\xi^4)e^{-\xi^2}.$$

$$H_3(\xi) = -e^{\xi^2}\left(\frac{d}{d\xi}\right)^3 e^{-\xi^2} = \boxed{-12\xi + 8\xi^3;}\ H_4(\xi) = e^{\xi^2}\left(\frac{d}{d\xi}\right)^4 e^{-\xi^2} = \boxed{12 - 48\xi^2 + 16\xi^4.}$$

(b)

$$H_5 = 2\xi H_4 - 8H_3 = 2\xi(12 - 48\xi^2 + 16\xi^4) - 8(-12\xi + 8\xi^3) = \boxed{120\xi - 160\xi^3 + 32\xi^5.}$$

$$H_6 = 2\xi H_5 - 10H_4 = 2\xi(120\xi - 160\xi^3 + 32\xi^5) - 10(12 - 48\xi^2 + 16\xi^4) = \boxed{-120 + 720\xi^2 - 480\xi^4 + 64\xi^6.}$$

(c)

$$\frac{dH_5}{d\xi} = 120 - 480\xi^2 + 160\xi^4 = 10(12 - 48\xi^2 + 16\xi^4) = (2)(5)H_4. \checkmark$$

$$\frac{dH_6}{d\xi} = 1440\xi - 1920\xi^3 + 384\xi^5 = 12(120\xi - 160\xi^3 + 32\xi^5) = (2)(6)H_5. \checkmark$$

(d)

$$\frac{d}{dz}(e^{-z^2+2z\xi}) = (-2z + 2\xi)e^{-z^2+2z\xi}; \text{ setting } z = 0, \boxed{H_1(\xi) = 2\xi.}$$

$$\begin{aligned}\left(\frac{d}{dz}\right)^2 (e^{-z^2+2z\xi}) &= \frac{d}{dz}\left[(-2z + 2\xi)e^{-z^2+2z\xi}\right] \\ &= \left[-2 + (-2z + 2\xi)^2\right]e^{-z^2+2z\xi}; \text{ setting } z = 0, \boxed{H_2(\xi) = -2 + 4\xi^2.}\end{aligned}$$

$$\begin{aligned}\left(\frac{d}{dz}\right)^3 (e^{-z^2+2z\xi}) &= \frac{d}{dz}\left\{\left[-2 + (-2z + 2\xi)^2\right]e^{-z^2+2z\xi}\right\} \\ &= \left\{2(-2z + 2\xi)(-2) + \left[-2 + (-2z + 2\xi)^2\right](-2z + 2\xi)\right\}e^{-z^2+2z\xi};\end{aligned}$$

$$\text{setting } z = 0, \; H_3(\xi) = -8\xi + (-2 + 4\xi^2)(2\xi) = \boxed{-12\xi + 8\xi^3.}$$

Problem 2.17

$$\begin{aligned}Ae^{ikx} + Be^{-ikx} &= A(\cos kx + i \sin kx) + B(\cos kx - i \sin kx) = (A + B)\cos kx + i(A - B)\sin kx \\ &= C \cos kx + D \sin kx, \text{ with } \boxed{C = A + B; \; D = i(A - B).}\end{aligned}$$

$$\begin{aligned}C \cos kx + D \sin kx &= C\left(\frac{e^{ikx} + e^{-ikx}}{2}\right) + D\left(\frac{e^{ikx} - e^{-ikx}}{2i}\right) = \frac{1}{2}(C - iD)e^{ikx} + \frac{1}{2}(C + iD)e^{-ikx} \\ &= Ae^{ikx} + Be^{-ikx}, \text{ with } \boxed{A = \frac{1}{2}(C - iD); \; B = \frac{1}{2}(C + iD).}\end{aligned}$$

Problem 2.18

Equation 2.95 says $\Psi = Ae^{i(kx-\frac{\hbar k^2}{2m}t)}$, so

$$J = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{i\hbar}{2m}|A|^2\left[e^{i(kx-\frac{\hbar k^2}{2m}t)}(-ik)e^{-i(kx-\frac{\hbar k^2}{2m}t)} - e^{-i(kx-\frac{\hbar k^2}{2m}t)}(ik)e^{i(kx-\frac{\hbar k^2}{2m}t)}\right]$$
$$= \frac{i\hbar}{2m}|A|^2(-2ik) = \boxed{\frac{\hbar k}{m}|A|^2.}$$

It flows in the positive (x) direction (as you would expect).

Problem 2.19

(a)

$$f(x) = b_0 + \sum_{n=1}^{\infty}\frac{a_n}{2i}\left(e^{in\pi x/a} - e^{-in\pi x/a}\right) + \sum_{n=1}^{\infty}\frac{b_n}{2}\left(e^{in\pi x/a} + e^{-in\pi x/a}\right)$$
$$= b_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{in\pi x/a} + \sum_{n=1}^{\infty}\left(-\frac{a_n}{2i} + \frac{b_n}{2}\right)e^{-in\pi x/a}.$$

Let

$$\boxed{c_0 \equiv b_0;\ c_n = \tfrac{1}{2}\left(-ia_n + b_n\right),\ \text{for } n = 1, 2, 3, \ldots;\ c_n \equiv \tfrac{1}{2}\left(ia_{-n} + b_{-n}\right),\ \text{for } n = -1, -2, -3, \ldots.}$$

Then $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a}$. QED

(b)

$$\int_{-a}^{a} f(x)e^{-im\pi x/a}dx = \sum_{n=-\infty}^{\infty} c_n \int_{-a}^{a} e^{i(n-m)\pi x/a}dx. \quad \text{But for } n \neq m,$$

$$\int_{-a}^{a} e^{i(n-m)\pi x/a}dx = \left.\frac{e^{i(n-m)\pi x/a}}{i(n-m)\pi/a}\right|_{-a}^{a} = \frac{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}{i(n-m)\pi/a} = \frac{(-1)^{n-m} - (-1)^{n-m}}{i(n-m)\pi/a} = 0,$$

whereas for $n = m$,

$$\int_{-a}^{a} e^{i(n-m)\pi x/a}dx = \int_{-a}^{a} dx = 2a.$$

So all terms except $n = m$ are zero, and

$$\int_{-a}^{a} f(x)e^{-im\pi x/a} = 2ac_m,\ \text{so } c_n = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}dx. \quad \text{QED}$$

(c)

$$f(x) = \sum_{n=-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{1}{a}F(k)e^{ikx} = \frac{1}{\sqrt{2\pi}}\sum F(k)e^{ikx}\Delta k,$$

where $\boxed{\Delta k \equiv \frac{\pi}{a}}$ is the increment in k from n to $(n+1)$.

$$F(k) = \sqrt{\frac{2}{\pi}} a \frac{1}{2a} \int_{-a}^{a} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x) e^{-ikx} dx.$$

(d) As $a \to \infty$, k becomes a continuous variable,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk; \; F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

Problem 2.20

(a)

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 2|A|^2 \int_0^{\infty} e^{-2ax} dx = 2|A|^2 \frac{e^{-2ax}}{-2a}\Big|_0^{\infty} = \frac{|A|^2}{a} \Rightarrow A = \boxed{\sqrt{a}.}$$

(b)

$$\phi(k) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx}\, dx = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos kx - i \sin kx) dx.$$

The cosine integrand is even, and the sine is odd, so the latter vanishes and

$$\begin{aligned} \phi(k) &= 2\frac{A}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cos kx\, dx = \frac{A}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \left(e^{ikx} + e^{-ikx}\right) dx \\ &= \frac{A}{\sqrt{2\pi}} \int_0^{\infty} \left(e^{(ik-a)x} + e^{-(ik+a)x}\right) dx = \frac{A}{\sqrt{2\pi}} \left[\frac{e^{(ik-a)x}}{ik-a} + \frac{e^{-(ik+a)x}}{-(ik+a)}\right]\Bigg|_0^{\infty} \\ &= \frac{A}{\sqrt{2\pi}} \left(\frac{-1}{ik-a} + \frac{1}{ik+a}\right) = \frac{A}{\sqrt{2\pi}} \frac{-ik-a+ik-a}{-k^2-a^2} = \boxed{\sqrt{\frac{a}{2\pi}} \frac{2a}{k^2+a^2}.} \end{aligned}$$

(c)

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} 2\sqrt{\frac{a^3}{2\pi}} \int_{-\infty}^{\infty} \frac{1}{k^2+a^2} e^{i(kx - \frac{\hbar k^2}{2m} t)} dk = \boxed{\frac{a^{3/2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{k^2+a^2} e^{i(kx - \frac{\hbar k^2}{2m} t)} dk.}$$

(d) For *large* a, $\Psi(x,0)$ is a sharp narrow spike whereas $\phi(k) \cong \sqrt{2/\pi a}$ is broad and flat; position is well-defined but momentum is ill-defined. For *small* a, $\Psi(x,0)$ is a broad and flat whereas $\phi(k) \cong (\sqrt{2a^3/\pi})/k^2$ is a sharp narrow spike; position is ill-defined but momentum is well-defined.

Problem 2.21

(a)

$$1 = |A|^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = |A|^2 \sqrt{\frac{\pi}{2a}}; \quad \boxed{A = \left(\frac{2a}{\pi}\right)^{1/4}.}$$

(b)

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)} dx = \int_{-\infty}^{\infty} e^{-y^2+(b^2/4a)} \frac{1}{\sqrt{a}} dy = \frac{1}{\sqrt{a}} e^{b^2/4a} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}} e^{b^2/4a}.$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}} e^{-k^2/4a} = \frac{1}{(2\pi a)^{1/4}} e^{-k^2/4a}.$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} \underbrace{e^{-k^2/4a} e^{i(kx - \hbar k^2 t/2m)}}_{e^{-[(\frac{1}{4a}+i\hbar t/2m)k^2 - ixk]}} dk$$

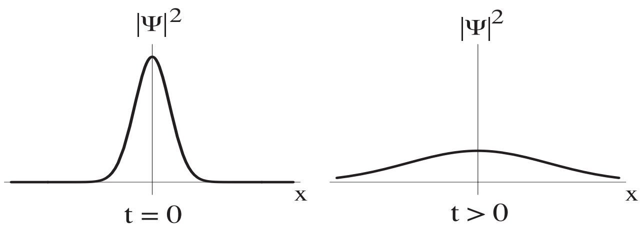
$$= \frac{1}{\sqrt{2\pi}(2\pi a)^{1/4}} \frac{\sqrt{\pi}}{\sqrt{\frac{1}{4a} + i\hbar t/2m}} e^{-x^2/4(\frac{1}{4a}+i\hbar t/2m)} = \boxed{\left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/(1+2i\hbar a t/m)}}{\sqrt{1+2i\hbar a t/m}}.}$$

(c)

Let $\theta \equiv 2\hbar a t/m$. Then $|\Psi|^2 = \sqrt{\frac{2a}{\pi}} \frac{e^{-ax^2/(1+i\theta)} e^{-ax^2/(1-i\theta)}}{\sqrt{(1+i\theta)(1-i\theta)}}$. The exponent is

$$-\frac{ax^2}{(1+i\theta)} - \frac{ax^2}{(1-i\theta)} = -ax^2 \frac{(1-i\theta+1+i\theta)}{(1+i\theta)(1-i\theta)} = \frac{-2ax^2}{1+\theta^2}; \ |\Psi|^2 = \sqrt{\frac{2a}{\pi}} \frac{e^{-2ax^2/(1+\theta^2)}}{\sqrt{1+\theta^2}}.$$

Or, with $w \equiv \sqrt{\frac{a}{1+\theta^2}}$, $\boxed{|\Psi|^2 = \sqrt{\frac{2}{\pi}} w e^{-2w^2x^2}.}$ As t increases, the graph of $|\Psi|^2$ flattens out and broadens.



(d)

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0} \text{ (odd integrand); } \langle p \rangle = m\frac{d\langle x \rangle}{dt} = \boxed{0.}$$

$$\langle x^2 \rangle = \sqrt{\frac{2}{\pi}} w \int_{-\infty}^{\infty} x^2 e^{-2w^2x^2} dx = \sqrt{\frac{2}{\pi}} w \frac{1}{4w^2} \sqrt{\frac{\pi}{2w^2}} = \boxed{\frac{1}{4w^2}.} \quad \langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2} dx.$$

Write $\Psi = Be^{-bx^2}$, where $B \equiv \left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\sqrt{1+i\theta}}$ and $b \equiv \frac{a}{1+i\theta}$.

$$\frac{\partial^2\Psi}{\partial x^2} = B\frac{\partial}{\partial x}\left(-2bxe^{-bx^2}\right) = -2bB(1-2bx^2)e^{-bx^2}.$$

$$\Psi^*\frac{\partial^2\Psi}{d\partial x^2} = -2b|B|^2(1-2bx^2)e^{-(b+b^*)x^2};\ b+b^* = \frac{a}{1+i\theta} + \frac{a}{1-i\theta} = \frac{2a}{1+\theta^2} = 2w^2.$$

$$|B|^2 = \sqrt{\frac{2a}{\pi}}\frac{1}{\sqrt{1+\theta^2}} = \sqrt{\frac{2}{\pi}}w. \quad \text{So } \Psi^*\frac{\partial^2\Psi}{\partial x^2} = -2b\sqrt{\frac{2}{\pi}}w(1-2bx^2)e^{-2w^2x^2}.$$

$$\begin{aligned}\langle p^2\rangle &= 2b\hbar^2\sqrt{\frac{2}{\pi}}w\int_{-\infty}^{\infty}(1-2bx^2)e^{-2w^2x^2}\,dx \\ &= 2b\hbar^2\sqrt{\frac{2}{\pi}}w\left(\sqrt{\frac{\pi}{2w^2}} - 2b\frac{1}{4w^2}\sqrt{\frac{\pi}{2w^2}}\right) = 2b\hbar^2\left(1-\frac{b}{2w^2}\right).\end{aligned}$$

But $1 - \frac{b}{2w^2} = 1 - \left(\frac{a}{1+i\theta}\right)\left(\frac{1+\theta^2}{2a}\right) = 1 - \frac{(1-i\theta)}{2} = \frac{1+i\theta}{2} = \frac{a}{2b}$, so

$$\langle p^2\rangle = 2b\hbar^2\frac{a}{2b} = \boxed{\hbar^2 a.} \qquad \boxed{\sigma_x = \frac{1}{2w};} \qquad \boxed{\sigma_p = \hbar\sqrt{a}.}$$

(e)

$$\sigma_x\sigma_p = \frac{1}{2w}\hbar\sqrt{a} = \frac{\hbar}{2}\sqrt{1+\theta^2} = \frac{\hbar}{2}\sqrt{1+(2\hbar at/m)^2} \geq \frac{\hbar}{2}. \checkmark$$

Closest at $\boxed{t=0,}$ at which time it is right *at* the uncertainty limit.

Problem 2.22

(a)

$$(-2)^3 - 3(-2)^2 + 2(-2) - 1 = -8 - 12 - 4 - 1 = \boxed{-25.}$$

(b)

$$\cos(3\pi) + 2 = -1 + 2 = \boxed{1.}$$

(c)

$\boxed{0}$ ($x = 2$ is outside the domain of integration).

Problem 2.23

(a) Let $y \equiv cx$, so $dx = \frac{1}{c}dy$. $\left\{ \begin{array}{l} \text{If } c > 0,\ y : -\infty \to \infty. \\ \text{If } c < 0,\ y : \infty \to -\infty. \end{array} \right\}$

$$\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \begin{cases} \frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\delta(y)dy = \frac{1}{c}f(0) \quad (c>0);\ \text{or} \\ \frac{1}{c}\int_{\infty}^{-\infty} f(y/c)\delta(y)dy = -\frac{1}{c}\int_{-\infty}^{\infty} f(y/c)\delta(y)dy = -\frac{1}{c}f(0)\ (c<0). \end{cases}$$

In either case, $\int_{-\infty}^{\infty} f(x)\delta(cx)dx = \frac{1}{|c|}f(0) = \int_{-\infty}^{\infty} f(x)\frac{1}{|c|}\delta(x)dx$. So $\delta(cx) = \frac{1}{|c|}\delta(x)$. ✓

(b)

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f\theta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df}{dx}\theta dx \quad \text{(integration by parts)}$$

$$= f(\infty) - \int_0^{\infty}\frac{df}{dx}dx = f(\infty) - f(\infty) + f(0) = f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx.$$

So $d\theta/dx = \delta(x)$. ✓ [Makes sense: The θ function is constant (so derivative is zero) except at $x = 0$, where the derivative is infinite.]

Problem 2.24

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2} = \frac{\sqrt{m\alpha}}{\hbar}\begin{cases} e^{-m\alpha x/\hbar^2}, & (x \ge 0), \\ e^{m\alpha x/\hbar^2}, & (x \le 0). \end{cases}$$

$$\langle x\rangle = 0 \text{ (odd integrand).}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\psi|^2dx = 2\frac{m\alpha}{\hbar^2}\int_0^{\infty} x^2e^{-2m\alpha x/\hbar^2}dx = \frac{2m\alpha}{\hbar^2}2\left(\frac{\hbar^2}{2m\alpha}\right)^3 = \frac{\hbar^4}{2m^2\alpha^2}; \quad \sigma_x = \frac{\hbar^2}{\sqrt{2}m\alpha}.$$

$$\frac{d\psi}{dx} = \frac{\sqrt{m\alpha}}{\hbar}\left\{\begin{array}{ll} -\frac{m\alpha}{\hbar^2}e^{-m\alpha x/\hbar^2}, & (x \ge 0) \\ \frac{m\alpha}{\hbar^2}e^{m\alpha x/\hbar^2}, & (x \le 0) \end{array}\right\} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3\left[-\theta(x)e^{-m\alpha x/\hbar^2} + \theta(-x)e^{m\alpha x/\hbar^2}\right].$$

$$\frac{d^2\psi}{dx^2} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3\left[-\delta(x)e^{-m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2}\theta(x)e^{-m\alpha x/\hbar^2} - \delta(-x)e^{m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2}\theta(-x)e^{m\alpha x/\hbar^2}\right]$$
$$= \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3\left[-2\delta(x) + \frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2}\right].$$

In the last step I used the fact that $\delta(-x) = \delta(x)$ (Eq. 2.145), $f(x)\delta(x) = f(0)\delta(x)$ (Eq. 2.115), and $\theta(-x) + \theta(x) = 1$ (Eq. 2.146). Since $d\psi/dx$ is an odd function, $\langle p\rangle = 0$.

$$\langle p^2\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi\frac{d^2\psi}{dx^2}dx = -\hbar^2\frac{\sqrt{m\alpha}}{\hbar}\left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3\int_{-\infty}^{\infty}e^{-m\alpha|x|/\hbar^2}\left[-2\delta(x) + \frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2}\right]dx$$
$$= \left(\frac{m\alpha}{\hbar}\right)^2\left[2 - 2\frac{m\alpha}{\hbar^2}\int_0^{\infty}e^{-2m\alpha x/\hbar^2}dx\right] = 2\left(\frac{m\alpha}{\hbar}\right)^2\left[1 - \frac{m\alpha}{\hbar^2}\frac{\hbar^2}{2m\alpha}\right] = \left(\frac{m\alpha}{\hbar}\right)^2.$$

Evidently

$$\sigma_p = \frac{m\alpha}{\hbar}, \quad \text{so} \quad \sigma_x\sigma_p = \frac{\hbar^2}{\sqrt{2}m\alpha}\frac{m\alpha}{\hbar} = \sqrt{2}\frac{\hbar}{2} > \frac{\hbar}{2}.\ ✓$$

Problem 2.25

$\langle\psi_{\text{bound}}|\psi_{\text{scattering}}\rangle$

$$= \frac{\sqrt{m\alpha}}{\hbar}\left[\int_{-\infty}^{0} e^{m\alpha x/\hbar^2}\left(Ae^{ikx}+Be^{-ikx}\right)dx + \int_{0}^{\infty} e^{-m\alpha x/\hbar^2}\left(Fe^{ikx}+Ge^{-ikx}\right)dx\right]$$
$$= \frac{\sqrt{m\alpha}}{\hbar}\left[A\int_{-\infty}^{0} e^{\left(\frac{m\alpha}{\hbar^2}+ik\right)x}dx + B\int_{-\infty}^{0} e^{\left(\frac{m\alpha}{\hbar^2}-ik\right)x}dx + F\int_{0}^{\infty} e^{\left(-\frac{m\alpha}{\hbar^2}+ik\right)x}dx + G\int_{0}^{\infty} e^{\left(-\frac{m\alpha}{\hbar^2}-ik\right)x}dx\right]$$
$$= \frac{\sqrt{m\alpha}}{\hbar}\left[A\left.\frac{e^{\left(\frac{m\alpha}{\hbar^2}+ik\right)x}}{\frac{m\alpha}{\hbar^2}+ik}\right|_{-\infty}^{0} + B\left.\frac{e^{\left(\frac{m\alpha}{\hbar^2}-ik\right)x}}{\frac{m\alpha}{\hbar^2}-ik}\right|_{-\infty}^{0} + F\left.\frac{e^{\left(-\frac{m\alpha}{\hbar^2}+ik\right)x}}{-\frac{m\alpha}{\hbar^2}+ik}\right|_{0}^{\infty} + G\left.\frac{e^{\left(-\frac{m\alpha}{\hbar^2}-ik\right)x}}{-\frac{m\alpha}{\hbar^2}-ik}\right|_{0}^{\infty}\right]$$
$$= \frac{\sqrt{m\alpha}}{\hbar}\left[\frac{A}{\frac{m\alpha}{\hbar^2}+ik} + \frac{B}{\frac{m\alpha}{\hbar^2}-ik} - \frac{F}{-\frac{m\alpha}{\hbar^2}+ik} - \frac{G}{-\frac{m\alpha}{\hbar^2}-ik}\right] = \frac{\sqrt{m\alpha}}{\hbar}\left[\frac{A+G}{\frac{m\alpha}{\hbar^2}+ik} + \frac{B+F}{\frac{m\alpha}{\hbar^2}-ik}\right]$$
$$= \frac{\sqrt{m\alpha}}{\hbar}\left[\frac{\left(\frac{m\alpha}{\hbar^2}-ik\right)(A+G)+\left(\frac{m\alpha}{\hbar^2}+ik\right)(B+F)}{\left(\frac{m\alpha}{\hbar^2}\right)^2+k^2}\right] = \frac{\sqrt{m\alpha}}{\hbar}\left[\frac{\frac{m\alpha}{\hbar^2}(A+G+B+F)+ik(B+F-A-G)}{\left(\frac{m\alpha}{\hbar^2}\right)^2+k^2}\right]$$

But Equation 2.136 says $(A+G+B+F) = 2(A+B)$, and Equation 2.137 says $ik(B+F-A-G) = -(2m\alpha/\hbar^2)(A+B)$, so

$$\langle\psi_{\text{bound}}|\psi_{\text{scattering}} =\rangle \frac{\sqrt{m\alpha}}{\hbar}\left[\frac{\frac{m\alpha}{\hbar^2}2(A+B) - \frac{2m\alpha}{\hbar^2}(A+B)}{\left(\frac{m\alpha}{\hbar^2}\right)^2+k^2}\right] = 0. \ \checkmark$$

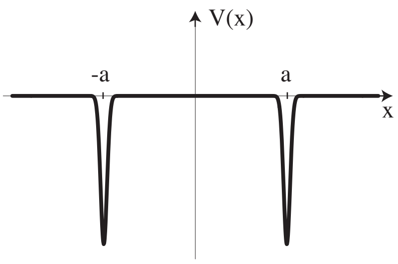
Problem 2.26

Put $f(x)=\delta(x)$ into Eq. 2.103: $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-ikx}dx = \boxed{\frac{1}{\sqrt{2\pi}}}.$

$$\therefore f(x) = \delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}dk. \quad \text{QED}$$

Problem 2.27

(a)



(b) From Problem 2.1(c) the solutions are even or odd. Look first for *even solutions*:

$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x>a), \\ B(e^{\kappa x}+e^{-\kappa x}) & (-a<x<a), \\ Ae^{\kappa x} & (x<-a). \end{cases}$$

Continuity at a: $Ae^{-\kappa a} = B(e^{\kappa a}+e^{-\kappa a})$, or $A = B(e^{2\kappa a}+1)$.

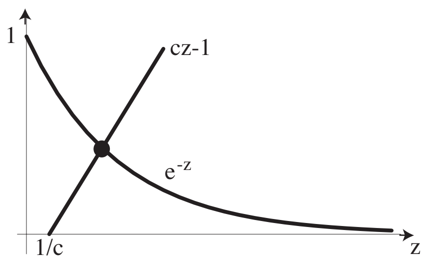
Discontinuous derivative at a, $\Delta\frac{d\psi}{dx} = -\frac{2m\alpha}{\hbar^2}\psi(a)$:

$$-\kappa A e^{-\kappa a} - B(\kappa e^{\kappa a} - \kappa e^{-\kappa a}) = -\frac{2m\alpha}{\hbar^2}Ae^{-\kappa a} \Rightarrow A + B(e^{2\kappa a} - 1) = \frac{2m\alpha}{\hbar^2\kappa}A; \text{ or}$$

$$B(e^{2\kappa a} - 1) = A\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = B(e^{2\kappa a} + 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) \Rightarrow e^{2\kappa a} - 1 = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) + \frac{2m\alpha}{\hbar^2\kappa} - 1.$$

$$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 + \frac{2m\alpha}{\hbar^2\kappa}e^{-2\kappa a}; \quad \frac{\hbar^2\kappa}{m\alpha} = 1 + e^{-2\kappa a}, \text{ or } \boxed{e^{-2\kappa a} = \frac{\hbar^2\kappa}{m\alpha} - 1.}$$

This is a transcendental equation for κ (and hence for E). I'll solve it graphically: Let $z \equiv 2\kappa a$, $c \equiv \frac{\hbar^2}{2am\alpha}$, so $e^{-z} = cz - 1$. Plot both sides and look for intersections:



From the graph, noting that c and z are both positive, we see that there is one (and only one) solution (for even ψ). If $\alpha = \frac{\hbar^2}{2ma}$, so $c = 1$, the calculator gives $z = 1.278$, so $\kappa^2 = -\frac{2mE}{\hbar^2} = \frac{z^2}{(2a)^2} \Rightarrow E = -\frac{(1.278)^2}{8}\left(\frac{\hbar^2}{ma^2}\right) = -0.204\left(\frac{\hbar^2}{ma^2}\right)$.

Now look for *odd solutions:*

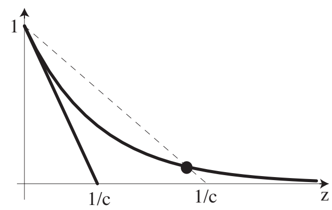
$$\psi(x) = \begin{cases} Ae^{-\kappa x} & (x > a), \\ B(e^{\kappa x} - e^{-\kappa x}) & (-a < x < a), \\ -Ae^{\kappa x} & (x < -a). \end{cases}$$

Continuity at a : $Ae^{-\kappa a} = B(e^{\kappa a} - e^{-\kappa a})$, or $A = B(e^{2\kappa a} - 1)$.

Discontinuity in ψ' : $-\kappa Ae^{-\kappa a} - B(\kappa e^{\kappa a} + \kappa e^{-\kappa a}) = -\frac{2m\alpha}{\hbar^2}Ae^{-\kappa a} \Rightarrow B(e^{2\kappa a} + 1) = A\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right)$,

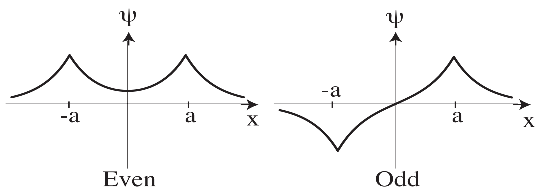
$$e^{2\kappa a} + 1 = (e^{2\kappa a} - 1)\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) = e^{2\kappa a}\left(\frac{2m\alpha}{\hbar^2\kappa} - 1\right) - \frac{2m\alpha}{\hbar^2\kappa} + 1,$$

$$1 = \frac{2m\alpha}{\hbar^2\kappa} - 1 - \frac{2m\alpha}{\hbar^2\kappa}e^{-2\kappa a}; \quad \frac{\hbar^2\kappa}{m\alpha} = 1 - e^{-2\kappa a}, \quad \boxed{e^{-2\kappa a} = 1 - \frac{\hbar^2\kappa}{m\alpha},} \text{ or } e^{-z} = 1 - cz.$$



This time there may or may not be a solution. Both graphs have their y-intercepts at 1, but if c is too large (α too small), there may be no intersection (solid line), whereas if c is smaller (dashed line) there will be. (Note that $z = 0 \Rightarrow \kappa = 0$ is *not* a solution, since ψ is then non-normalizable.) The slope of e^{-z} (at $z = 0$) is -1; the slope of $(1 - cz)$ is $-c$. So there is an *odd* solution $\Leftrightarrow c < 1$, or $\alpha > \hbar^2/2ma$.

Conclusion: $\boxed{\textit{One} \text{ bound state if } \alpha \le \hbar^2/2ma;\ \textit{two} \text{ if } \alpha > \hbar^2/2ma.}$



$$\alpha = \frac{\hbar^2}{ma} \Rightarrow c = \frac{1}{2}.\begin{cases} \textit{Even: } e^{-z} = \frac{1}{2}z - 1 \Rightarrow z = 2.21772, \\ \textit{Odd: } \ e^{-z} = 1 - \frac{1}{2}z \Rightarrow z = 1.59362. \end{cases}$$

$$\boxed{E = -0.615(\hbar^2/ma^2);\ E = -0.317(\hbar^2/ma^2).}$$

$$\alpha = \frac{\hbar^2}{4ma} \Rightarrow c = 2.\ \textit{Only even: } e^{-z} = 2z - 1 \Rightarrow z = 0.738835;\ \boxed{E = -0.0682(\hbar^2/ma^2).}$$

(c) (i) There is *one* bound state (even); c is huge, so z is small, so $e^{-z} \approx 1 = cz - 1$, which means $z = 2/c$, or $2\kappa a = 2(2am\alpha/\hbar^2) \Rightarrow \kappa = (2m\alpha/\hbar^2)$.

$$E = -\frac{\hbar^2\kappa^2}{2m} = \boxed{-\frac{2m\alpha^2}{\hbar^2}.}$$

This makes sense: the two delta-functions coincide, so there is really just *one* delta-function, with "strength" 2α. Putting this into Equation 2.130 we recover the answer in the box.

(ii) There are two bound states, one even and one odd; c is small, so z is huge, and $e^{-z} \approx 0$. For the even case, $0 = cz - 1 \Rightarrow z = 1/c \Rightarrow \kappa = (m\alpha/\hbar^2)$. For the odd case, $0 = 1 - cz$, which leads to the same result: the two states are degenerate, each with energy $\boxed{-\frac{m\alpha^2}{2\hbar^2}.}$ Any linear combination of the two will be an eigenstate (with the same energy); the *sum* (properly normalized) would represent a particle in the delta-function out at large *positive* x, and the difference would be a particle in the delta-function at large *negative* x (the other—distant—delta-function becomes irrelevant), so it makes sense that we get two states, each with the energy of a particle in a single delta-function well.

Problem 2.28

$$\psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < -a) \\ Ce^{ikx} + De^{-ikx} & (-a < x < a) \\ Fe^{ikx} & (x > a) \end{array} \right\}.\ \text{Impose boundary conditions:}$$

(1) Continuity at $-a$: $Ae^{-ika} + Be^{ika} = Ce^{-ika} + De^{ika} \Rightarrow \beta A + B = \beta C + D$, where $\beta \equiv e^{-2ika}$.

(2) Continuity at $+a$: $Ce^{ika} + De^{-ika} = Fe^{ika} \Rightarrow F = C + \beta D$.

(3) Discontinuity in ψ' at $-a$: $ik(Ce^{-ika} - De^{ika}) - ik(Ae^{-ika} - Be^{ika}) = -\frac{2m\alpha}{\hbar^2}(Ae^{-ika} + Be^{ika})$
$\Rightarrow \beta C - D = \beta(\gamma+1)A + B(\gamma-1)$, where $\gamma \equiv i2m\alpha/\hbar^2 k$.

(4) Discontinuity in ψ' at $+a$: $ikFe^{ika} - ik(Ce^{ika} - De^{-ika}) = -\frac{2m\alpha}{\hbar^2}(Fe^{ika})$
$\Rightarrow C - \beta D = (1-\gamma)F$.

To solve for C and D, $\begin{cases} \text{add (2) and (4) :} & 2C = F + (1-\gamma)F \Rightarrow 2C = (2-\gamma)F. \\ \text{subtract (2) and (4) :} & 2\beta D = F - (1-\gamma)F \Rightarrow 2D = (\gamma/\beta)F. \end{cases}$

$$\begin{cases} \text{add (1) and (3) :} & 2\beta C = \beta A + B + \beta(\gamma+1)A + B(\gamma-1) \Rightarrow 2C = (\gamma+2)A + (\gamma/\beta)B. \\ \text{subtract (1) and (3) :} & 2D = \beta A + B - \beta(\gamma+1)A - B(\gamma-1) \Rightarrow 2D = -\gamma\beta A + (2-\gamma)B. \end{cases}$$

$$\begin{cases} \text{Equate the two expressions for } 2C: & (2-\gamma)F = (\gamma+2)A + (\gamma/\beta)B. \\ \text{Equate the two expressions for } 2D: & (\gamma/\beta)F = -\gamma\beta A + (2-\gamma)B. \end{cases}$$

Solve these for F and B, in terms of A. Multiply the first by $\beta(2-\gamma)$, the second by γ, and subtract:

$$\left[\beta(2-\gamma)^2 F = \beta(4-\gamma^2)A + \gamma(2-\gamma)B\right]; \quad \left[(\gamma^2/\beta)F = -\beta\gamma^2 A + \gamma(2-\gamma)B\right].$$

$$\Rightarrow \left[\beta(2-\gamma)^2 - \gamma^2/\beta\right] F = \beta\left[4 - \gamma^2 + \gamma^2\right] A = 4\beta A \Rightarrow \frac{F}{A} = \frac{4}{(2-\gamma)^2 - \gamma^2/\beta^2}.$$

Let $g \equiv i/\gamma = \dfrac{\hbar^2 k}{2m\alpha}$; $\phi \equiv 4ka$, so $\gamma = \dfrac{i}{g}$, $\beta^2 = e^{-i\phi}$. Then: $\dfrac{F}{A} = \dfrac{4g^2}{(2g-i)^2 + e^{i\phi}}$.

Denominator: $4g^2 - 4ig - 1 + \cos\phi + i\sin\phi = (4g^2 - 1 + \cos\phi) + i(\sin\phi - 4g)$.

$$\begin{aligned} |\text{Denominator}|^2 &= (4g^2 - 1 + \cos\phi)^2 + (\sin\phi - 4g)^2 \\ &= 16g^4 + 1 + \cos^2\phi - 8g^2 - 2\cos\phi + 8g^2\cos\phi + \sin^2\phi - 8g\sin\phi + 16g^2 \\ &= 16g^4 + 8g^2 + 2 + (8g^2 - 2)\cos\phi - 8g\sin\phi. \end{aligned}$$

$$T = \left|\frac{F}{A}\right|^2 = \boxed{\frac{8g^4}{(8g^4 + 4g^2 + 1) + (4g^2 - 1)\cos\phi - 4g\sin\phi}, \text{ where } g \equiv \frac{\hbar^2 k}{2m\alpha} \text{ and } \phi \equiv 4ka.}$$

Problem 2.29

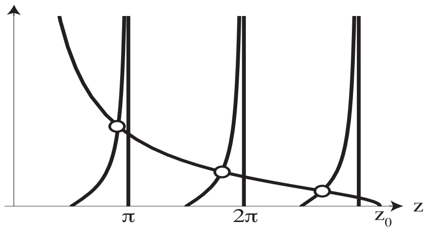
In place of Eq. 2.154, we have: $\psi(x) = \left\{ \begin{array}{ll} Fe^{-\kappa x} & (x > a) \\ D\sin(lx) & (0 < x < a) \\ -\psi(-x) & (x < 0) \end{array} \right\}.$

Continuity of ψ : $Fe^{-\kappa a} = D\sin(la)$; continuity of ψ' : $-F\kappa e^{-\kappa a} = Dl\cos(la)$.

Divide: $-\kappa = l\cot(la)$, or $-\kappa a = la\cot(la) \Rightarrow \sqrt{z_0^2 - z^2} = -z\cot z$, or $\boxed{-\cot z = \sqrt{(z_0/z)^2 - 1}.}$

Wide, deep well: Intersections are at $\pi, 2\pi, 3\pi$, etc. Same as Eq. 2.160, but now for n *even*. This fills in the rest of the states for the infinite square well.

Shallow, narrow well: If $z_0 < \pi/2$, there is *no* odd bound state. The corresponding condition on V_0 is $\boxed{V_0 < \dfrac{\pi^2\hbar^2}{8ma^2} \Rightarrow \textit{no} \text{ odd bound state.}}$



Problem 2.30

$$1 = 2\int_0^\infty |\psi|^2 dx = 2\left(|D|^2 \int_0^a \cos^2 lx\, dx + |F|^2 \int_a^\infty e^{-2\kappa x} dx\right)$$
$$= 2\left[|D|^2 \left(\frac{x}{2} + \frac{1}{4l}\sin 2lx\right)\Big|_0^a + |F|^2\left(-\frac{1}{2\kappa}e^{-2\kappa x}\right)\Big|_a^\infty\right] = 2\left[|D|^2\left(\frac{a}{2} + \frac{\sin 2la}{4l}\right) + |F|^2\frac{e^{-2\kappa a}}{2\kappa}\right].$$

But $F = De^{\kappa a}\cos la$ (Eq. 2.152), so $1 = |D|^2\left(a + \frac{\sin(2la)}{2l} + \frac{\cos^2(la)}{\kappa}\right)$.

Furthermore $\kappa = l\tan(la)$ (Eq. 2.157), so

$$1 = |D|^2\left(a + \frac{2\sin la\cos la}{2l} + \frac{\cos^3 la}{l\sin la}\right) = |D|^2\left[a + \frac{\cos la}{l\sin la}(\sin^2 la + \cos^2 la)\right]$$
$$= |D|^2\left(a + \frac{1}{l\tan la}\right) = |D|^2\left(a + \frac{1}{\kappa}\right). \quad \boxed{D = \frac{1}{\sqrt{a + 1/\kappa}}}, \quad \boxed{F = \frac{e^{\kappa a}\cos la}{\sqrt{a+1/\kappa}}}.$$

Problem 2.31

Equation 2.158 $\Rightarrow z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$. We want $\alpha =$ *area of potential* $= 2aV_0$ held constant as $a \to 0$. Therefore $V_0 = \frac{\alpha}{2a}$; $z_0 = \frac{a}{\hbar}\sqrt{2m\frac{\alpha}{2a}} = \frac{1}{\hbar}\sqrt{m\alpha a} \to 0$. So z_0 is *small*, and the intersection in Fig. 2.17 occurs at very small z. Solve Eq. 2.159 for very small z, by expanding $\tan z$:

$$\tan z \cong z = \sqrt{(z_0/z)^2 - 1} = (1/z)\sqrt{z_0^2 - z^2}.$$

Now (from Eqs. 2.149, 2.151 and 2.158) $z_0^2 - z^2 = \kappa^2 a^2$, so $z^2 = \kappa a$. But $z_0^2 - z^2 = z^4 \ll 1 \Rightarrow z \cong z_0$, so $\kappa a \cong z_0^2$. But we found that $z_0 \cong \frac{1}{\hbar}\sqrt{m\alpha a}$ here, so $\kappa a = \frac{1}{\hbar^2}m\alpha a$, or $\kappa = \frac{m\alpha}{\hbar^2}$. (At this point the a's have canceled, and we can go to the limit $a \to 0$.)

$$\frac{\sqrt{-2mE}}{\hbar} = \frac{m\alpha}{\hbar^2} \Rightarrow -2mE = \frac{m^2\alpha^2}{\hbar^2}. \quad \boxed{E = -\frac{m\alpha^2}{2\hbar^2}} \text{ (which agrees with Eq. 2.132).}$$

In Eq. 2.172, $V_0 \gg E \Rightarrow T^{-1} \cong 1 + \frac{V_0^2}{4EV_0}\sin^2\left(\frac{2a}{\hbar}\sqrt{2mV_0}\right)$. But $V_0 = \frac{\alpha}{2a}$, so the argument of the sine is small, and we can replace $\sin\epsilon$ by ϵ: $T^{-1} \cong 1 + \frac{V_0}{4E}\left(\frac{2a}{\hbar}\right)^2 2mV_0 = 1 + (2aV_0)^2\frac{m}{2\hbar^2 E}$. But $2aV_0 = \alpha$, so $T^{-1} = 1 + \frac{m\alpha^2}{2\hbar^2 E}$, in agreement with Eq. 2.144.

Problem 2.32

Multiply Eq. 2.168 by $\sin la$, Eq. 2.169 by $\frac{1}{l}\cos la$, and add:

$$\left.\begin{aligned} C\sin^2 la + D\sin la\cos la &= Fe^{ika}\sin la \\ C\cos^2 la - D\sin la\cos la &= \tfrac{ik}{l}Fe^{ika}\cos la \end{aligned}\right\} \quad C = Fe^{ika}\left[\sin la + \frac{ik}{l}\cos la\right].$$

Multiply Eq. 2.168 by $\cos la$, Eq. 2.169 by $\frac{1}{l}\sin la$, and subtract:

$$\left.\begin{aligned} C\sin la\cos la + D\cos^2 la &= Fe^{ika}\cos la \\ C\sin la\cos la - D\sin^2 la &= \tfrac{ik}{l}Fe^{ika}\sin la \end{aligned}\right\} \quad D = Fe^{ika}\left[\cos la - \frac{ik}{l}\sin la\right].$$

Put these into Eq. 2.166:

$$\begin{aligned} (\mathbf{1})\ Ae^{-ika} + Be^{ika} &= -Fe^{ika}\left[\sin la + \frac{ik}{l}\cos la\right]\sin la + Fe^{ika}\left[\cos la - \frac{ik}{l}\sin la\right]\cos la \\ &= Fe^{ika}\left[\cos^2 la - \frac{ik}{l}\sin la\cos la - \sin^2 la - \frac{ik}{l}\sin la\cos la\right] \\ &= Fe^{ika}\left[\cos(2la) - \frac{ik}{l}\sin(2la)\right]. \end{aligned}$$

Likewise, from Eq. 2.167:

$$\begin{aligned} (\mathbf{2})\ Ae^{-ika} - Be^{ika} &= -\frac{il}{k}Fe^{ika}\left[\left(\sin la + \frac{ik}{l}\cos la\right)\cos la + \left(\cos la - \frac{ik}{l}\sin la\right)\sin la\right] \\ &= -\frac{il}{k}Fe^{ika}\left[\sin la\cos la + \frac{ik}{l}\cos^2 la + \sin la\cos la - \frac{ik}{l}\sin^2 la\right] \\ &= -\frac{il}{k}Fe^{ika}\left[\sin(2la) + \frac{ik}{l}\cos(2la)\right] = Fe^{ika}\left[\cos(2la) - \frac{il}{k}\sin(2la)\right]. \end{aligned}$$

Add **(1)** and **(2)**: $2Ae^{-ika} = Fe^{ika}\left[2\cos(2la) - i\left(\frac{k}{l} + \frac{l}{k}\right)\sin(2la)\right]$, or:

$F = \dfrac{e^{-2ika}A}{\cos(2la) - i\frac{\sin(2la)}{2kl}(k^2+l^2)}$ (confirming Eq. 2.171). Now subtract **(2)** from **(1)**:

$2Be^{ika} = Fe^{ika}\left[i\left(\frac{l}{k} - \frac{k}{l}\right)\sin(2la)\right] \Rightarrow B = i\dfrac{\sin(2la)}{2kl}(l^2 - k^2)F$ (confirming Eq. 2.170).

$$T^{-1} = \left|\frac{A}{F}\right|^2 = \left|\cos(2la) - i\frac{\sin(2la)}{2kl}(k^2+l^2)\right|^2 = \cos^2(2la) + \frac{\sin^2(2la)}{(2lk)^2}(k^2+l^2)^2.$$

But $\cos^2(2la) = 1 - \sin^2(2la)$, so

$$T^{-1} = 1 + \sin^2(2la)\Bigg[\underbrace{\frac{(k^2+l^2)^2}{(2lk)^2} - 1}_{\frac{1}{(2kl)^2}[k^4+2k^2l^2+l^4-4k^2l^2] = \frac{1}{(2kl)^2}[k^4-2k^2l^2+l^4] = \frac{(k^2-l^2)^2}{(2kl)^2}.}\Bigg] = 1 + \frac{(k^2-l^2)^2}{(2kl)^2}\sin^2(2la).$$

But $k = \dfrac{\sqrt{2mE}}{\hbar}$, $l = \dfrac{\sqrt{2m(E+V_0)}}{\hbar}$; so $(2la) = \dfrac{2a}{\hbar}\sqrt{2m(E+V_0)}$; $k^2 - l^2 = -\dfrac{2mV_0}{\hbar^2}$, and

$$\frac{(k^2-l^2)^2}{(2kl)^2} = \frac{\left(\frac{2m}{\hbar^2}\right)^2 V_0^2}{4\left(\frac{2m}{\hbar^2}\right)^2 E(E+V_0)} = \frac{V_0^2}{4E(E+V_0)}.$$

$\therefore T^{-1} = 1 + \dfrac{V_0^2}{4E(E+V_0)}\sin^2\left(\dfrac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$, confirming Eq. 2.172.

Problem 2.33

$\underline{\mathbf{E < V_0}}$. $\quad \psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < -a) \\ Ce^{\kappa x} + De^{-\kappa x} & (-a < x < a) \\ Fe^{ikx} & (x > a) \end{array} \right\} \quad k = \frac{\sqrt{2mE}}{\hbar}; \; \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$

(1) Continuity of ψ at $-a$: $Ae^{-ika} + Be^{ika} = Ce^{-\kappa a} + De^{\kappa a}$.

(2) Continuity of ψ' at $-a$: $ik(Ae^{-ika} - Be^{ika}) = \kappa(Ce^{-\kappa a} - De^{\kappa a})$.

$$\Rightarrow 2Ae^{-ika} = \left(1 - i\frac{\kappa}{k}\right) Ce^{-\kappa a} + \left(1 + i\frac{\kappa}{k}\right) De^{\kappa a}.$$

(3) Continuity of ψ at $+a$: $Ce^{\kappa a} + De^{-\kappa a} = Fe^{ika}$.

(4) Continuity of ψ' at $+a$: $\kappa(Ce^{\kappa a} - De^{-\kappa a}) = ikFe^{ika}$.

$$\Rightarrow 2Ce^{\kappa a} = \left(1 + \frac{ik}{\kappa}\right) Fe^{ika}; \quad 2De^{-\kappa a} = \left(1 - \frac{ik}{\kappa}\right) Fe^{ika}.$$

$$\begin{aligned} 2Ae^{-ika} &= \left(1 - \frac{i\kappa}{k}\right)\left(1 + \frac{ik}{\kappa}\right) Fe^{ika}\frac{e^{-2\kappa a}}{2} + \left(1 + \frac{i\kappa}{k}\right)\left(1 - \frac{ik}{\kappa}\right) Fe^{ika}\frac{e^{2\kappa a}}{2} \\ &= \frac{Fe^{ika}}{2}\left\{\left[1 + i\left(\frac{k}{\kappa} - \frac{\kappa}{k}\right) + 1\right] e^{-2\kappa a} + \left[1 + i\left(\frac{\kappa}{k} - \frac{k}{\kappa}\right) + 1\right] e^{2\kappa a}\right\} \\ &= \frac{Fe^{ika}}{2}\left[2\left(e^{-2\kappa a} + e^{2\kappa a}\right) + i\frac{(\kappa^2 - k^2)}{k\kappa}\left(e^{2\kappa a} - e^{-2\kappa a}\right)\right]. \end{aligned}$$

But $\sinh x \equiv \frac{e^x - e^{-x}}{2}$, $\cosh x \equiv \frac{e^x + e^{-x}}{2}$, so

$$\begin{aligned} &= \frac{Fe^{ika}}{2}\left[4\cosh(2\kappa a) + i\frac{(\kappa^2 - k^2)}{k\kappa} 2\sinh(2\kappa a)\right] \\ &= 2Fe^{ika}\left[\cosh(2\kappa a) + i\frac{(\kappa^2 - k^2)}{2k\kappa}\sinh(2\kappa a)\right]. \end{aligned}$$

$$T^{-1} = \left|\frac{A}{F}\right|^2 = \cosh^2(2\kappa a) + \frac{(\kappa^2 - k^2)^2}{(2\kappa k)^2}\sinh^2(2\kappa a). \quad \text{But } \cosh^2 = 1 + \sinh^2, \text{ so}$$

$$T^{-1} = 1 + \underbrace{\left[1 + \frac{(\kappa^2 - k^2)^2}{(2\kappa k)^2}\right]}_{\bigstar}\sinh^2(2\kappa a) = \boxed{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right)},$$

where $\bigstar = \dfrac{4\kappa^2 k^2 + k^4 + \kappa^4 - 2\kappa^2 k^2}{(2\kappa k)^2} = \dfrac{(\kappa^2 + k^2)^2}{(2\kappa k)^2} = \dfrac{\left(\frac{2mE}{\hbar^2} + \frac{2m(V_0 - E)}{\hbar^2}\right)^2}{4\frac{2mE}{\hbar^2}\frac{2m(V_0 - E)}{\hbar^2}} = \dfrac{V_0^2}{4E(V_0 - E)}$.

(You can also get this from Eq. 2.172 by switching the sign of V_0 and using $\sin(i\theta) = i\sinh\theta$.)

$\underline{\mathbf{E = V_0}}$. $\quad \psi = \left\{ \begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < -a) \\ C + Dx & (-a < x < a) \\ Fe^{ikx} & (x > a) \end{array} \right\}$

(In central region $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + V_0\psi = E\psi \Rightarrow \dfrac{d^2\psi}{dx^2} = 0$, so $\psi = C + Dx$.)

(1) Continuous ψ at $-a$: $Ae^{-ika} + Be^{ika} = C - Da$.

(2) Continuous ψ at $+a$: $Fe^{ika} = C + Da$.

$\Rightarrow$ **(2.5)** $2Da = Fe^{ika} - Ae^{-ika} - Be^{ika}$.

(3) Continuous ψ' at $-a$: $ik\left(Ae^{-ika} - Be^{ika}\right) = D$.

(4) Continuous ψ' at $+a$: $ikFe^{ika} = D$.

$\Rightarrow$ **(4.5)** $Ae^{-2ika} - B = F$.

Use **(4)** to eliminate D in **(2.5)**: $Ae^{-2ika} + B = F - 2aikF = (1 - 2iak)F$, and add to **(4.5)**:

$$2Ae^{-2ika} = 2F(1 - ika), \text{ so } T^{-1} = \left|\frac{A}{F}\right|^2 = 1 + (ka)^2 = \boxed{1 + \frac{2mE}{\hbar^2}a^2.}$$

(You can also get this from Eq. 2.172 by changing the sign of V_0 and taking the limit $E \to V_0$, using $\sin\epsilon \cong \epsilon$.)

$\underline{\mathbf{E > V_0}}$. This case is identical to the one in the book, only with $V_0 \to -V_0$. So

$$\boxed{T^{-1} = 1 + \frac{V_0^2}{4E(E - V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E - V_0)}\right).}$$

Problem 2.34

(a)

$$\psi = \left\{\begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{-\kappa x} & (x > 0) \end{array}\right\} \text{ where } k = \frac{\sqrt{2mE}}{\hbar};\ \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}.$$

(1) Continuity of ψ: $A + B = F$.

(2) Continuity of ψ': $ik(A - B) = -\kappa F$.

$$\Rightarrow A + B = -\frac{ik}{\kappa}(A - B) \Rightarrow A\left(1 + \frac{ik}{\kappa}\right) = -B\left(1 - \frac{ik}{\kappa}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \frac{|(1 + ik/\kappa)|^2}{|(1 - ik/\kappa)|^2} = \frac{1 + (k/\kappa)^2}{1 + (k/\kappa)^2} = \boxed{1.}$$

Although the wave function penetrates into the barrier, it is eventually all reflected.

(b)

$$\psi = \left\{\begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < 0) \\ Fe^{ilx} & (x > 0) \end{array}\right\} \text{ where } k = \frac{\sqrt{2mE}}{\hbar};\ l = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

(1) Continuity of ψ: $A + B = F$.

(2) Continuity of ψ': $ik(A - B) = ilF$.

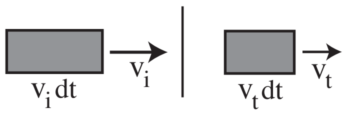
$$\Rightarrow A+B=\frac{k}{l}(A-B);\ A\left(1-\frac{k}{l}\right)=-B\left(1+\frac{k}{l}\right).$$

$$R=\left|\frac{B}{A}\right|^2=\frac{(1-k/l)^2}{(1+k/l)^2}=\frac{(k-l)^2}{(k+l)^2}=\frac{(k-l)^4}{(k^2-l^2)^2}.$$

Now $k^2-l^2=\frac{2m}{\hbar^2}(E-E+V_0)=\left(\frac{2m}{\hbar^2}\right)V_0;\ k-l=\frac{\sqrt{2m}}{\hbar}[\sqrt{E}-\sqrt{E-V_0}]$, so

$$\boxed{R=\frac{(\sqrt{E}-\sqrt{E-V_0})^4}{V_0^2}.}$$

(c)



From the diagram, $T=P_t/P_i=|F|^2v_t/|A|^2v_i$, where P_i is the probability of finding the incident particle in the box corresponding to the time interval dt, and P_t is the probability of finding the transmitted particle in the associated box to the *right* of the barrier.

But $\frac{v_t}{v_i}=\frac{\sqrt{E-V_0}}{\sqrt{E}}$ (from Eq. 2.98). So $T=\sqrt{\frac{E-V_0}{E}}\left|\frac{F}{A}\right|^2$. Alternatively, from Problem 2.18:

$$J_i=\frac{\hbar k}{m}|A|^2;\quad J_t=\frac{\hbar l}{m}|F|^2;\quad T=\frac{J_t}{J_i}=\left|\frac{F}{A}\right|^2\frac{l}{k}=\left|\frac{F}{A}\right|^2\sqrt{\frac{E-V_0}{E}}.$$

For $E<V_0$, of course, $\boxed{T=0.}$

(d)

$$\text{For } E>V_0,\ F=A+B=A+A\frac{(\frac{k}{l}-1)}{(\frac{k}{l}+1)}=A\frac{2k/l}{(\frac{k}{l}+1)}=\frac{2k}{k+l}A.$$

$$T=\left|\frac{F}{A}\right|^2\frac{l}{k}=\left(\frac{2k}{k+l}\right)^2\frac{l}{k}=\frac{4kl}{(k+l)^2}=\frac{4kl(k-l)^2}{(k^2-l^2)^2}=\boxed{\frac{4\sqrt{E}\sqrt{E-V_0}(\sqrt{E}-\sqrt{E-V_0})^2}{V_0^2}}.$$

$$T+R=\frac{4kl}{(k+l)^2}+\frac{(k-l)^2}{(k+l)^2}=\frac{4kl+k^2-2kl+l^2}{(k+l)^2}=\frac{k^2+2kl+l^2}{(k+l)^2}=\frac{(k+l)^2}{(k+l)^2}=1.\ \checkmark$$

Problem 2.35

(a)

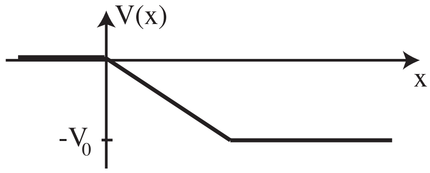
$$\psi(x)=\left\{\begin{array}{ll}Ae^{ikx}+Be^{-ikx} & (x<0)\\ Fe^{ilx} & (x>0)\end{array}\right\}\ \text{where } k\equiv\frac{\sqrt{2mE}}{\hbar},\ l\equiv\frac{\sqrt{2m(E+V_0)}}{\hbar}.$$

$$\left.\begin{array}{l}\text{Continuity of }\psi\ \Rightarrow A+B=F\\ \text{Continuity of }\psi'\ \Rightarrow ik(A-B)=ilF\end{array}\right\}\Longrightarrow$$

$$A+B=\frac{k}{l}(A-B);\quad A\left(1-\frac{k}{l}\right)=-B\left(1+\frac{k}{l}\right);\quad \frac{B}{A}=-\left(\frac{1-k/l}{1+k/l}\right).$$

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{l-k}{l+k}\right)^2 = \left(\frac{\sqrt{E+V_0}-\sqrt{E}}{\sqrt{E+V_0}+\sqrt{E}}\right)^2$$
$$= \left(\frac{\sqrt{1+V_0/E}-1}{\sqrt{1+V_0/E}+1}\right)^2 = \left(\frac{\sqrt{1+3}-1}{\sqrt{1+3}+1}\right)^2 = \left(\frac{2-1}{2+1}\right)^2 = \boxed{\frac{1}{9}}.$$

(b) The cliff is *two*-dimensional, and even if we pretend the car drops straight down, the potential *as a function of distance along the* (crooked, but now one-dimensional) *path* is $-mgx$ (with x the vertical coordinate), as shown.



(c) Here $V_0/E = 12/4 = 3$, the same as in part (a), so $R = 1/9$, and hence $T = \boxed{8/9 = 0.8889.}$

Problem 2.36

Start with Eq. 2.25: $\psi(x) = A\sin kx + B\cos kx$. This time the boundary conditions are $\psi(a) = \psi(-a) = 0$:

$$A\sin ka + B\cos ka = 0; \quad -A\sin ka + B\cos ka = 0.$$

$$\begin{cases} \textit{Subtract}: & A\sin ka = 0 \Rightarrow ka = j\pi \text{ or } A = 0, \\ \textit{Add}: & B\cos ka = 0 \Rightarrow ka = (j-\frac{1}{2})\pi \text{ or } B = 0, \end{cases}$$

(where $j = 1, 2, 3, \ldots$).

If $B = 0$ (so $A \neq 0$), $k = j\pi/a$. In this case let $n \equiv 2j$ (so n is an *even* integer); then $k = n\pi/2a$, $\psi = A\sin(n\pi x/2a)$. Normalizing: $1 = |A|^2\int_{-a}^{a}\sin^2(n\pi x/2a)\,dx = |A|^2 a \Rightarrow A = 1/\sqrt{a}$.

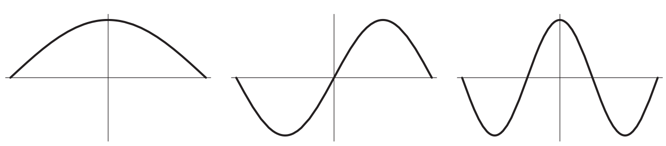
If $A = 0$ (so $B \neq 0$), $k = (j-\frac{1}{2})\pi/a$. In this case let $n \equiv 2j-1$ (n is an *odd* integer); again $k = n\pi/2a$, $\psi = B\cos(n\pi x/2a)$. Normalizing: $1 = |B|^2\int_{-a}^{a}\cos^2(n\pi x/2a)dx = |B|^2 a \Rightarrow B = 1/\sqrt{a}$.

In either case Eq. 2.24 yields $E = \frac{\hbar^2k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$ (in agreement with Eq. 2.30 for a well of width $2a$).

The substitution $x \to (x+a)/2$ takes Eq. 2.31 to

$$\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}\frac{(x+a)}{2}\right) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{2a}+\frac{n\pi}{2}\right) = \begin{cases} (-1)^{n/2}\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{2a}\right) & (n \text{ even}), \\ (-1)^{(n-1)/2}\sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{2a}\right) & (n \text{ odd}). \end{cases}$$

So (apart from normalization) we recover the results above. The graphs are the same as Figure 2.2, except that some are upside down (different normalization).



cos(πx/2a) sin(2πx/2a) cos(3πx/2a)

Problem 2.37

Use the trig identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ to write

$$\sin^3\left(\frac{\pi x}{a}\right) = \frac{3}{4}\sin\left(\frac{\pi x}{a}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{a}\right). \quad \text{So (Eq. 2.31): } \Psi(x,0) = A\sqrt{\frac{a}{2}}\left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right].$$

Normalize using Eq. 2.20: $|A|^2\frac{a}{2}\left(\frac{9}{16}+\frac{1}{16}\right) = \frac{5}{16}a|A|^2 = 1 \Rightarrow \boxed{A = \frac{4}{\sqrt{5a}}.}$

So $\Psi(x,0) = \frac{1}{\sqrt{10}}\left[3\psi_1(x) - \psi_3(x)\right]$, and hence (Eq. 2.17)

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{10}}\left[3\psi_1(x)e^{-iE_1t/\hbar} - \psi_3(x)e^{-iE_3t/\hbar}\right].}$$

$$|\Psi(x,t)|^2 = \frac{1}{10}\left[9\psi_1^2 + \psi_3^2 - 6\psi_1\psi_3\cos\left(\frac{E_3-E_1}{\hbar}t\right)\right]; \text{ so}$$

$$\langle x\rangle = \int_0^a x|\Psi(x,t)|^2dx = \frac{9}{10}\langle x\rangle_1 + \frac{1}{10}\langle x\rangle_3 - \frac{3}{5}\cos\left(\frac{E_3-E_1}{\hbar}t\right)\int_0^a x\psi_1(x)\psi_3(x)dx,$$

where $\langle x\rangle_n = a/2$ is the expectation value of x in the nth stationary state. The remaining integral is

$$\frac{2}{a}\int_0^a x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)dx = \frac{1}{a}\int_0^a x\left[\cos\left(\frac{2\pi x}{a}\right) - \cos\left(\frac{4\pi x}{a}\right)\right]dx$$

$$= \frac{1}{a}\left[\left(\frac{a}{2\pi}\right)^2\cos\left(\frac{2\pi x}{a}\right) + \left(\frac{xa}{2\pi}\right)\sin\left(\frac{2\pi x}{a}\right) - \left(\frac{a}{4\pi}\right)^2\cos\left(\frac{4\pi x}{a}\right) - \left(\frac{xa}{4\pi}\right)\sin\left(\frac{4\pi x}{a}\right)\right|_0^a = 0.$$

Evidently then,

$$\langle x\rangle = \frac{9}{10}\left(\frac{a}{2}\right) + \frac{1}{10}\left(\frac{a}{2}\right) = \boxed{\frac{a}{2}.}$$

Using Eq. 2.21,

$$\langle H\rangle = |c_1|^2E_1 + |c_3|^2E_3 = \left(\frac{9}{10}\right)\frac{\pi^2\hbar^2}{2ma^2} + \left(\frac{1}{10}\right)\frac{9\pi^2\hbar^2}{2ma^2} = \boxed{\frac{9\pi^2\hbar^2}{10ma^2}.}$$

Problem 2.38

(a) According to Eq. 2.39, the most general solution to the time-dependent Schrödinger equation for the infinite square well is

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-i(n^2\pi^2\hbar/2ma^2)t}.$$

Now $\frac{n^2\pi^2\hbar}{2ma^2}T = \frac{n^2\pi^2\hbar}{2ma^2}\frac{4ma^2}{\pi\hbar} = 2\pi n^2$, so $e^{-i(n^2\pi^2\hbar/2ma^2)(t+T)} = e^{-i(n^2\pi^2\hbar/2ma^2)t}e^{-i2\pi n^2}$, and since n^2 is an integer, $e^{-i2\pi n^2} = 1$. Therefore $\Psi(x,t+T) = \Psi(x,t)$. QED

(b) The classical revival time is the time it takes the particle to go down and back: $T_c = 2a/v$, with the velocity given by

$$E = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2E}{m}} \Rightarrow \boxed{T_c = a\sqrt{\frac{2m}{E}}.}$$

(c) The two revival times are equal if

$$\frac{4ma^2}{\pi\hbar} = a\sqrt{\frac{2m}{E}}, \quad \text{or} \quad \boxed{E = \frac{\pi^2\hbar^2}{8ma^2} = \frac{E_1}{4}.}$$

Problem 2.39

(a)

$$\frac{d\Psi}{dx} = \frac{2\sqrt{3}}{a\sqrt{a}} \cdot \left\{ \begin{array}{ll} 1, & (0 < x < a/2) \\ -1, & (a/2 < x < a) \end{array} \right\} = \boxed{\frac{2\sqrt{3}}{a\sqrt{a}}\left[1 - 2\theta\left(x - \frac{a}{2}\right)\right].}$$

(b)

$$\frac{d^2\Psi}{dx^2} = \frac{2\sqrt{3}}{a\sqrt{a}}\left[-2\delta\left(x - \frac{a}{2}\right)\right] = \boxed{-\frac{4\sqrt{3}}{a\sqrt{a}}\delta\left(x - \frac{a}{2}\right).}$$

(c)

$$\langle H\rangle = -\frac{\hbar^2}{2m}\left(-\frac{4\sqrt{3}}{a\sqrt{a}}\right)\int \Psi^*\delta\left(x - \frac{a}{2}\right)dx = \frac{2\sqrt{3}\hbar^2}{ma\sqrt{a}}\underbrace{\Psi^*\left(\frac{a}{2}\right)}_{\sqrt{3/a}} = \frac{2\cdot 3\cdot\hbar^2}{m\cdot a\cdot a} = \boxed{\frac{6\hbar^2}{ma^2}.} \checkmark$$

Problem 2.40

(a) In the standard notation $\xi \equiv \sqrt{m\omega/\hbar}\,x,\ \alpha \equiv (m\omega/\pi\hbar)^{1/4}$,

$$\Psi(x,0) = A(1-2\xi)^2 e^{-\xi^2/2} = A(1 - 4\xi + 4\xi^2)e^{-\xi^2/2}.$$

It can be expressed as a linear combination of the first three stationary states (Eq. 2.60 and 2.63, and Problem 2.10):

$$\psi_0(x) = \alpha e^{-\xi^2/2}, \qquad \psi_1(x) = \sqrt{2}\,\alpha\xi e^{-\xi^2/2}, \qquad \psi_2(x) = \frac{\alpha}{\sqrt{2}}(2\xi^2 - 1)e^{-\xi^2/2}.$$

So $\Psi(x,0) = c_0\psi_0 + c_1\psi_1 + c_2\psi_2 = \alpha(c_0 + \sqrt{2}\xi c_1 + \sqrt{2}\xi^2 c_2 - \frac{1}{\sqrt{2}}c_2)e^{-\xi^2/2}$ with (equating like powers)

$$\begin{cases} \alpha\sqrt{2}c_2 = 4A & \Rightarrow c_2 = 2\sqrt{2}A/\alpha, \\ \alpha\sqrt{2}c_1 = -4A & \Rightarrow c_1 = -2\sqrt{2}A/\alpha, \\ \alpha(c_0 - c_2/\sqrt{2}) = A & \Rightarrow c_0 = (A/\alpha) + c_2/\sqrt{2} = (1+2)A/\alpha = 3A/\alpha. \end{cases}$$

Normalizing: $1 = |c_0|^2 + |c_1|^2 + |c_2|^2 = (8+8+9)(A/\alpha)^2 = 25(A/\alpha)^2 \Rightarrow A = \alpha/5 = \boxed{\frac{1}{5}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.}$

$$\boxed{c_0 = \frac{3}{5}, \quad c_1 = -\frac{2\sqrt{2}}{5}, \quad c_2 = \frac{2\sqrt{2}}{5}.}$$

(b) You could get $\boxed{\frac{1}{2}\hbar\omega, \text{ probability } \frac{9}{25}; \frac{3}{2}\hbar\omega, \text{ probability } \frac{8}{25}; \frac{5}{2}\hbar\omega, \text{ probability } \frac{8}{25}.}$

$$\langle H\rangle = \frac{9}{25}\left(\frac{1}{2}\hbar\omega\right) + \frac{8}{25}\left(\frac{3}{2}\hbar\omega\right) + \frac{8}{25}\left(\frac{5}{2}\hbar\omega\right) = \frac{\hbar\omega}{50}(9+24+40) = \boxed{\frac{73}{50}\hbar\omega.}$$

(c)

$$\Psi(x,t) = \frac{3}{5}\psi_0 e^{-i\omega t/2} - \frac{2\sqrt{2}}{5}\psi_1 e^{-3i\omega t/2} + \frac{2\sqrt{2}}{5}\psi_2 e^{-5i\omega t/2} = e^{-i\omega t/2}\left[\frac{3}{5}\psi_0 - \frac{2\sqrt{2}}{5}\psi_1 e^{-i\omega t} + \frac{2\sqrt{2}}{5}\psi_2 e^{-2i\omega t}\right].$$

To change the sign of the middle term we need $e^{-i\omega T} = -1$ (then $e^{-2i\omega T} = 1$); evidently $\omega T = \pi$, or $\boxed{T = \pi/\omega.}$

Problem 2.41

Everything in Section 2.3.2 still applies, except that there is an additional boundary condition: $\psi(0) = 0$. This eliminates all the *even* solutions ($n = 0, 2, 4, \ldots$), leaving only the odd solutions. So

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \; n = 1, 3, 5, \ldots.}$$

Problem 2.42

(a) Normalization is the same as before: $A = \left(\frac{2a}{\pi}\right)^{1/4}$.

(b) Equation 2.104 says

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{\pi}\right)^{1/4}\int_{-\infty}^{\infty} e^{-ax^2} e^{ilx} e^{-ikx} dx \quad [\text{same as before, only } k \to k - l] = \frac{1}{(2\pi a)^{1/4}} e^{-(k-l)^2/4a}.$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}}\int_{-\infty}^{\infty} e^{-(k-l)^2/4a} e^{i(kx - \hbar k^2 t/2m)}\, dk$$

Let $u \equiv k - l$, so $k = u + l$ and $dk = du$:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}}\int_{-\infty}^{\infty} e^{-u^2/4a} e^{i[ux + lx - (\hbar t/2m)(u^2 + 2ul + l^2)]}\, du$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi a)^{1/4}} e^{il(x - \frac{\hbar l t}{2m})}\int_{-\infty}^{\infty} e^{-u^2\left(\frac{1}{4a} + i\frac{\hbar t}{2m}\right) + iu\left(x - \frac{\hbar l t}{m}\right)}\, du.$$

Using the hint in Problem 2.21, the integral becomes

$$\frac{1}{\sqrt{\frac{1}{4a}+i\frac{\hbar t}{2m}}} e^{\left(x-\frac{\hbar l t}{m}\right)^2/4\left(\frac{1}{4a}+i\frac{\hbar t}{2m}\right)} \int_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{2\sqrt{a}}{\gamma} e^{-a\left(x-\frac{\hbar l t}{m}\right)^2/\gamma^2}\sqrt{\pi},$$

so

$$\Psi(x,t) = \boxed{\left(\frac{2a}{\pi}\right)^{1/4} \frac{1}{\gamma} e^{-a\left(x-\frac{\hbar l t}{m}\right)^2/\gamma^2} e^{il\left(x-\frac{\hbar l t}{2m}\right)}.}$$

The gaussian envelope (the first exponential) travels at speed $\boxed{\hbar l/m;}$ the sinusoidal wave (the second exponential) travels at speed $\boxed{\hbar l/2m}$.

(c)

$$|\Psi(x,t)|^2 = \sqrt{\frac{2a}{\pi}} \frac{1}{|\gamma|^2} e^{a\left(x-\frac{\hbar l t}{m}\right)^2\left[\frac{1}{\gamma^2}+\frac{1}{(\gamma^*)^2}\right]}.$$

The term in square brackets simplifies:

$$\left[\frac{1}{\gamma^2}+\frac{1}{(\gamma^*)^2}\right] = \frac{1}{|\gamma|^4}[(\gamma^*)^2+\gamma^2] = \frac{1}{|\gamma|^4}\left(1-\frac{2i\hbar t}{m}+1+\frac{2i\hbar t}{m}\right) = \frac{2}{|\gamma|^4}.$$

$$|\gamma|^2 = \sqrt{\left(1+\frac{2ia\hbar t}{m}\right)\left(1-\frac{2ia\hbar t}{m}\right)} = \sqrt{1+\theta^2},$$

where (as before) $\theta \equiv 2\hbar a t/m$. So

$$|\Psi(x,t)|^2 = \sqrt{\frac{2a}{\pi}} \frac{1}{\sqrt{1+\theta^2}} e^{2a\left(x-\frac{\hbar l t}{m}\right)^2/(1+\theta^2)} = \boxed{\sqrt{\frac{2}{\pi}} w e^{-2w^2\left(x-\frac{\hbar l t}{m}\right)^2}.}$$

where (as before) $w \equiv \sqrt{a/(1+\theta^2)}$. The result is the same as in Problem 2.21, except that $x \to \left(x-\frac{\hbar l}{m}t\right)$, so $|\Psi|^2$ has the same (flattening gaussian) shape – only this time the center moves at constant speed $v = \hbar l/m$.

(d)

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx. \qquad \text{Let } y \equiv x - vt, \text{ so } x = y + vt.$$

$$= \int_{-\infty}^{\infty} (y+vt)\sqrt{\frac{2}{\pi}} w e^{-2w^2y^2} dy = vt = \boxed{\frac{\hbar l}{m}t.}$$

(The first integral is trivially zero; the second is 1 by normalization.)

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{\hbar l.}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} (y+vt)^2\sqrt{\frac{2}{\pi}} w e^{-2w^2y^2} dy = \frac{1}{4w^2} + 0 + (vt)^2 = \boxed{\frac{1}{4w^2}+\left(\frac{\hbar l t}{m}\right)^2.}$$

(The first integral is same as in Problem 2.21).

$$\langle p^2\rangle = -\hbar^2 \int_{-\infty}^{\infty} \Psi^* \frac{\partial^2\Psi}{\partial x^2}\,dx; \quad \frac{\partial\Psi}{\partial x} = \left[-\frac{2a}{\gamma^2}\left(x-\frac{\hbar l t}{m}\right)+il\right]\Psi,$$

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{2a}{\gamma^2}\Psi + \left[-\frac{2a}{\gamma^2}\left(x-\frac{\hbar l t}{m}\right)+il\right]^2\Psi = \left[Ax^2+Bx+C\right]\Psi,$$

where

$$A \equiv \left(\frac{2a}{\gamma^2}\right)^2, \quad B \equiv -\left(\frac{2a}{\gamma^2}\right)^2 \frac{2\hbar lt}{m} - \frac{4ial}{\gamma^2} = -\frac{4ial}{\gamma^4},$$

$$C \equiv -\frac{2a}{\gamma^2} + \left(\frac{2a}{\gamma^2}\right)^2 \left(\frac{\hbar lt}{m}\right)^2 + \left(\frac{4ial}{\gamma^2}\right)\left(\frac{\hbar lt}{m}\right) - l^2 = -\frac{1}{\gamma^4}(2a\gamma^2 + l^2).$$

$$\begin{aligned}
\langle p^2\rangle &= -\hbar^2 \int_{-\infty}^{\infty} \Psi^*[Ax^2 + Bx + C]\Psi\, dx = -\hbar^2[A\langle x^2\rangle + B\langle x\rangle + C] \\
&= -\frac{\hbar^2}{\gamma^4}\left[4a^2\left(\frac{1}{4w^2} + \left(\frac{\hbar lt}{m}\right)^2\right) - 4ial\left(\frac{\hbar lt}{m}\right) - (2a\gamma^2 + l^2)\right] \\
&= -\frac{\hbar^2}{\gamma^4}\left\{\left[a + a\left(\frac{2\hbar at}{m}\right)^2 - 2a - \frac{4ia^2\hbar t}{m}\right] + l^2\left[-1 - \frac{4ia\hbar t}{m} + 4\left(\frac{\hbar at}{m}\right)^2\right]\right\} \\
&= -\frac{\hbar^2}{\gamma^4}(-a\gamma^4 - l^2\gamma^4) = \boxed{\hbar^2(a + l^2).}
\end{aligned}$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{4w^2} + \left(\frac{\hbar lt}{m}\right)^2 - \left(\frac{\hbar lt}{m}\right)^2 = \frac{1}{4w^2} \Rightarrow \boxed{\sigma_x = \frac{1}{2w};}$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \hbar^2 a + \hbar^2 l^2 - \hbar^2 l^2 = \hbar^2 a, \text{ so } \boxed{\sigma_p = \hbar\sqrt{a}.}$$

(e) σ_x and σ_p are same as before, so the uncertainty principle still holds.

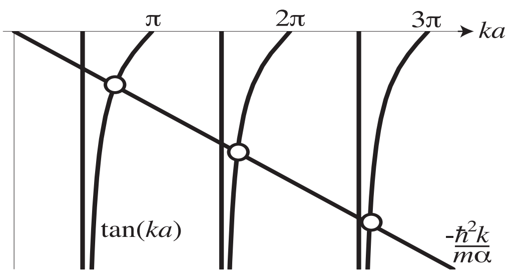
Problem 2.43

Equation 2.25 $\Rightarrow \psi(x) = A\sin kx + B\cos kx,\ 0 \le x \le a,$ with $k = \sqrt{2mE}/\hbar^2$.

Even solutions: $\psi(x) = \psi(-x) = A\sin(-kx) + B\cos(-kx) = -A\sin kx + B\cos kx \quad (-a \le x \le 0).$

Boundary conditions
$$\begin{cases} \psi \text{ continuous at } 0: B = B \text{ (no new condition).} \\ \psi' \text{ discontinuous (Eq. 2.128 with sign of } \alpha \text{ switched): } Ak + Ak = \frac{2m\alpha}{\hbar^2}B \Rightarrow B = \frac{\hbar^2 k}{m\alpha}A. \\ \psi \to 0 \text{ at } x = a: \ A\sin(ka) + \frac{\hbar^2 k}{m\alpha}A\cos(ka) = 0 \Rightarrow \tan(ka) = -\frac{\hbar^2 k}{m\alpha}. \end{cases}$$

$$\boxed{\psi(x) = A\left(\sin kx + \frac{\hbar^2 k}{m\alpha}\cos kx\right) \ (0 \le x \le a);\ \psi(-x) = \psi(x).}$$



From the graph, the allowed energies are slightly above

$$ka = \frac{n\pi}{2}\ (n = 1, 3, 5, \ldots) \ \text{so} \ \boxed{E_n \gtrsim \frac{n^2\pi^2\hbar^2}{2m(2a)^2}\ (n = 1, 3, 5, \ldots).}$$

These energies are somewhat higher than the corresponding energies for the infinite square well (Eq. 2.30, with $a \to 2a$). As $\alpha \to 0$, the straight line $(-\hbar^2 k/m\alpha)$ gets steeper and steeper, and the intersections get closer to $n\pi/2$; the energies then reduce to those of the ordinary infinite well. As $\alpha \to \infty$, the straight line approaches horizontal, and the intersections are at $n\pi$ $(n = 1, 2, 3, \ldots)$, so $E_n \to \frac{n^2\pi^2\hbar^2}{2ma^2}$ – these are the allowed energies for the infinite square well *of width* a. At this point the barrier is impenetrable, and we have two isolated infinite square wells.

Odd solutions: $\psi(x) = -\psi(-x) = -A\sin(-kx) - B\cos(-kx) = A\sin(kx) - B\cos(kx) \quad (-a \le x \le 0).$

$$\text{Boundary conditions} \begin{cases} \psi \text{ continuous at } 0: \ B = -B \Rightarrow B = 0. \\ \psi' \text{ discontinuous: } Ak - Ak = \frac{2m\alpha}{\hbar^2}(0) \text{ (no new condition).} \\ \psi(a) = 0 \Rightarrow A\sin(ka) = 0 \Rightarrow ka = \frac{n\pi}{2}\ (n = 2, 4, 6, \ldots). \end{cases}$$

$$\boxed{\psi(x) = A\sin(kx),\ (-a < x < a); \quad E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}\ (n = 2, 4, 6, \ldots).}$$

These are the *exact* (even n) energies (and wave functions) for the infinite square well (of width $2a$). The point is that the *odd* solutions (even n) are *zero* at the origin, so they never "feel" the delta function at all.

Problem 2.44

$$\left.\begin{array}{l} -\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dx^2} + V\psi_1 = E\psi_1 \Rightarrow -\dfrac{\hbar^2}{2m}\psi_2\dfrac{d^2\psi_1}{dx^2} + V\psi_1\psi_2 = E\psi_1\psi_2 \\ -\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_2}{dx^2} + V\psi_2 = E\psi_2 \Rightarrow -\dfrac{\hbar^2}{2m}\psi_1\dfrac{d^2\psi_2}{dx^2} + V\psi_1\psi_2 = E\psi_1\psi_2 \end{array}\right\} \Rightarrow -\frac{\hbar^2}{2m}\left[\psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2}\right] = 0.$$

But $\dfrac{d}{dx}\left[\psi_2\dfrac{d\psi_1}{dx} - \psi_1\dfrac{d\psi_2}{dx}\right] = \dfrac{d\psi_2}{dx}\dfrac{d\psi_1}{dx} + \psi_2\dfrac{d^2\psi_1}{dx^2} - \dfrac{d\psi_1}{dx}\dfrac{d\psi_2}{dx} - \psi_1\dfrac{d^2\psi_2}{dx^2} = \psi_2\dfrac{d^2\psi_1}{dx^2} - \psi_1\dfrac{d^2\psi_2}{dx^2}$. Since this is zero, it follows that $\psi_2\dfrac{d\psi_1}{dx} - \psi_1\dfrac{d\psi_2}{dx} = K$ (a constant). But $\psi \to 0$ at ∞ so the constant must be zero. Thus $\psi_2\dfrac{d\psi_1}{dx} = \psi_1\dfrac{d\psi_2}{dx}$, or $\dfrac{1}{\psi_1}\dfrac{d\psi_1}{dx} = \dfrac{1}{\psi_2}\dfrac{d\psi_2}{dx}$, so $\ln\psi_1 = \ln\psi_2 +$ constant, or $\psi_1 = (\text{constant})\psi_2$. QED

Problem 2.45

(a)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + V\psi_n = E_n\psi_n \ \Rightarrow \ \frac{d^2\psi_n}{dx^2} = -\frac{2m}{\hbar^2}(E_n - V)\psi_n; \quad \frac{d^2\psi_m}{dx^2} = -\frac{2m}{\hbar^2}(E_m - V)\psi_m.$$

$$\frac{d}{dx}\left(\frac{d\psi_m}{dx}\psi_n - \psi_m\frac{d\psi_n}{dx}\right) = \frac{d^2\psi_m}{dx^2}\psi_n + \cancel{\frac{d\psi_m}{dx}\frac{d\psi_n}{dx}} - \cancel{\frac{d\psi_m}{dx}\frac{d\psi_n}{dx}} - \psi_m\frac{d^2\psi_n}{dx^2}$$
$$= -\frac{2m}{\hbar^2}\left[(E_m - V)\psi_m\psi_n - \psi_m(E_n - V)\psi_n\right] = -\frac{2m}{\hbar^2}(E_m - E_n)\psi_m\psi_n. \quad \checkmark$$

(b) Integrate both sides:

$$\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{d\psi_m}{dx}\psi_n - \psi_m\frac{d\psi_n}{dx}\right)dx = [\psi_m'(x_2)\psi_n(x_2) - \cancel{\psi_m(x_2)}\psi_n'(x_2) - \psi_m'(x_1)\psi_n(x_1) + \cancel{\psi_m(x_1)}\psi_n'(x_1)]$$

$$= \psi_m'(x_2)\psi_n(x_2) - \psi_m'(x_1)\psi_n(x_1) = \frac{2m}{\hbar^2}(E_n - E_m)\int_{x_1}^{x_2}\psi_m\psi_n\,dx. \quad [\bigstar]$$

(c) Because x_1 and x_2 are *adjacent* nodes of ψ_m, $\psi_m(x)$ must either be posiitive or negative throughout the interval. We might as well make it positive (if it's not, multiply ψ_m by -1). Then

$$\psi_m(x) \geq 0 \text{ for } x_1 \leq x \leq x_2; \quad \psi'(x_1) \geq 0 \text{ and } \psi'(x_2) \leq 0.$$

If $\psi_n(x)$ has no nodes between x_1 and x_2 then it *too* must have the same sign throughout the interval (and we may as well choose it to be positive):

$$\psi_n(x) \geq 0 \text{ for } x_1 \leq x \leq x_2.$$

In that case $\psi_m'(x_2)\psi_n(x_2) - \psi_m'(x_1)\psi_n(x_1) \leq 0$, but $(E_n - E_m)\int_{x_1}^{x_2}\psi_m\psi_n\,dx > 0$, in contradiction to Equation [★]. *Conclusion:* $\psi_n(x)$ must have at least one node between x_1 and x_2.

Problem 2.46

$-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} = E\psi$ (where x is measured around the circumference), or $\dfrac{d^2\psi}{dx^2} = -k^2\psi$, with $k \equiv \dfrac{\sqrt{2mE}}{\hbar}$, so

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

But $\psi(x+L) = \psi(x)$, since $x + L$ is the same point as x, so

$$Ae^{ikx}e^{ikL} + Be^{-ikx}e^{-ikL} = Ae^{ikx} + Be^{-ikx},$$

and this is true for *all* x. In particular, for $x = 0$:

(1) $\quad Ae^{ikL} + Be^{-ikL} = A + B.\quad$ And for $x = \dfrac{\pi}{2k}$:

$Ae^{i\pi/2}e^{ikL} + Be^{-i\pi/2}e^{-ikL} = Ae^{i\pi/2} + Be^{-i\pi/2}$, or $iAe^{ikL} - iBe^{-ikL} = iA - iB$, so

(2) $\quad Ae^{ikL} - Be^{-ikL} = A - B.\quad$ Add (1) and (2): $2Ae^{ikL} = 2A$.

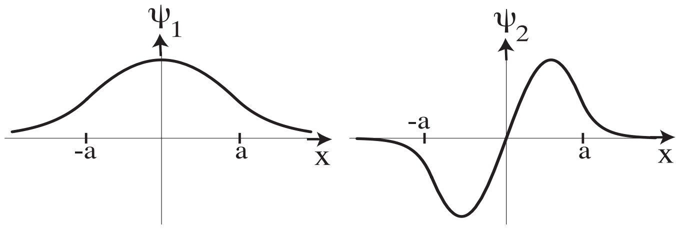
Either $A = 0$, or else $e^{ikL} = 1$, in which case $kL = 2n\pi$ $(n = 0, \pm1, \pm2, \ldots)$. But if $A = 0$, then $Be^{-ikL} = B$, leading to the same conclusion. So for every positive n there are *two* solutions: $\psi_n^+(x) = Ae^{i(2n\pi x/L)}$ and $\psi_n^-(x) = Be^{-i(2n\pi x/L)}$ ($n = 0$ is ok too, but in that case there is just *one* solution). Normalizing: $\int_0^L |\psi_\pm|^2 dx = 1 \Rightarrow A = B = 1/\sqrt{L}$. Any *other* solution (with the same energy) is a linear combination of these.

$$\boxed{\psi_n^\pm(x) = \frac{1}{\sqrt{L}}e^{\pm i(2n\pi x/L)};\ E_n = \frac{2n^2\pi^2\hbar^2}{mL^2}\ (n = 0, 1, 2, 3, \ldots).}$$

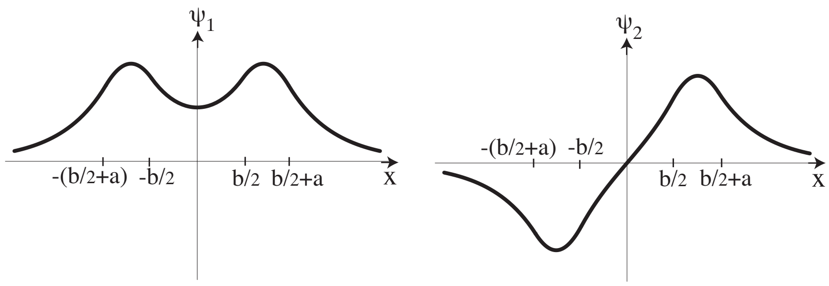
The theorem fails because here ψ does *not* go to zero at ∞; x is restricted to a finite range, and we are unable to determine the constant K (in Problem 2.44).

Problem 2.47

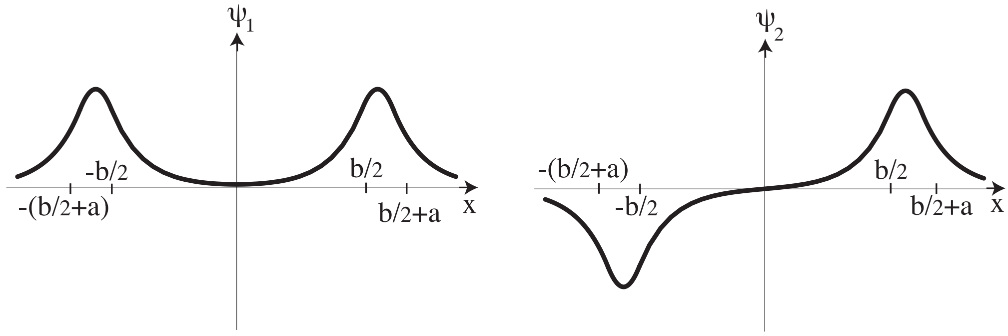
(a) (i) $b = 0 \Rightarrow$ ordinary finite square well. Exponential decay outside; sinusoidal inside (cos for ψ_1, sin for ψ_2). No nodes for ψ_1, one node for ψ_2.



(ii) Ground state is *even*. Exponential decay outside, sinusoidal inside the wells, hyperbolic cosine in barrier. First excited state is *odd* – hyperbolic sine in barrier. No nodes for ψ_1, one node for ψ_2.



(iii) For $b \gg a$, same as (ii), but wave function very small in barrier region. Essentially two isolated finite square wells; ψ_1 and ψ_2 are degenerate (in energy); they are even and odd linear combinations of the ground states of the two separate wells.

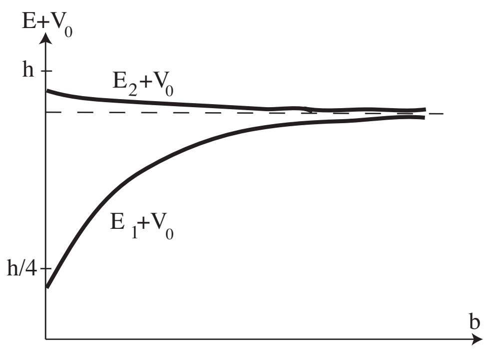


(b) From Eq. 2.160 we know that for $b = 0$ the energies fall slightly below

$$\left.\begin{array}{l} E_1 + V_0 \approx \frac{\pi^2\hbar^2}{2m(2a)^2} = \frac{h}{4} \\ E_2 + V_0 \approx \frac{4\pi^2\hbar^2}{2m(2a)^2} = h \end{array}\right\} \text{ where } h \equiv \frac{\pi^2\hbar^2}{2ma^2}.$$

For $b \gg a$, the width of each (isolated) well is a, so

$$E_1 + V_0 \approx E_2 + V_0 \approx \frac{\pi^2\hbar^2}{2ma^2} = h \text{ (again, slightly } \textit{below} \text{ this).}$$



[Within each well, $\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(V_0 + E)\psi$, so the more *curved* the wave function, the higher the energy.]

(c) In the (even) ground state the energy is *lowest* in configuration (i), with $b \to 0$, so the electron tends to draw the nuclei $\boxed{\text{together,}}$ promoting *bonding* of the atoms. In the (odd) first excited state, by contrast, the electron drives the nuclei $\boxed{\text{apart.}}$

Problem 2.48

(a) Let $V_0 \equiv 32\hbar^2/ma^2$. This is just like the *odd* bound states for the finite square well, since they are the ones that go to zero at the origin. Referring to the solution to Problem 2.29, the wave function is

$$\psi(x) = \begin{cases} D\sin lx, & l \equiv \sqrt{2m(E+V_0)}/\hbar \quad (0 < x < a), \\ Fe^{-\kappa x}, & \kappa \equiv \sqrt{-2mE}/\hbar \quad (x > a), \end{cases}$$

and the boundary conditions at $x = a$ yield

$$-\cot z = \sqrt{(z_0/z)^2 - 1}$$

with

$$z_0 = \frac{\sqrt{2mV_0}}{\hbar}a = \frac{\sqrt{2m(32\hbar^2/ma^2)}}{\hbar}a = 8.$$

Referring to the figure (Problem 2.29), and noting that $(5/2)\pi = 7.85 < z_0 < 3\pi = 9.42$, we see that there are $\boxed{\text{three bound states.}}$

(b) Let

$$I_1 \equiv \int_0^a |\psi|^2 dx = |D|^2 \int_0^a \sin^2 lx\, dx = |D|^2 \left[\frac{x}{2} - \frac{1}{2l}\sin lx \cos lx\right]\Bigg|_0^a = |D|^2 \left[\frac{a}{2} - \frac{1}{2l}\sin la \cos la\right];$$

$$I_2 \equiv \int_a^\infty |\psi|^2 dx = |F|^2 \int_a^\infty e^{-2\kappa x}\, dx = |F|^2 \left[-\frac{e^{-2\kappa x}}{2\kappa}\right]\Bigg|_a^\infty = |F|^2 \frac{e^{-2\kappa a}}{2\kappa}.$$

But continuity at $x = a \Rightarrow Fe^{-\kappa a} = D\sin la$, so $I_2 = |D|^2 \frac{\sin^2 la}{2\kappa}$.

Normalizing:

$$1 = I_1 + I_2 = |D|^2 \left[\frac{a}{2} - \frac{1}{2l}\sin la \cos la + \frac{\sin^2 la}{2\kappa}\right] = \frac{1}{2\kappa}|D|^2 \left[\kappa a - \frac{\kappa}{l}\sin la \cos la + \sin^2 la\right]$$

But (referring again to Problem 2.29) $\kappa/l = -\cot la$, so

$$= \frac{1}{2\kappa}|D|^2 \left[\kappa a + \cot la \sin la \cos la + \sin^2 la\right] = |D|^2 \frac{(1+\kappa a)}{2\kappa}.$$

So $|D|^2 = 2\kappa/(1+\kappa a)$, and the probability of finding the particle outside the well is

$$P = I_2 = \frac{2\kappa}{1+\kappa a}\frac{\sin^2 la}{2\kappa} = \frac{\sin^2 la}{1+\kappa a}.$$

We can express this in terms of $z \equiv la$ and z_0: $\kappa a = \sqrt{z_0^2 - z^2}$ (page 80),

$$\sin^2 la = \sin^2 z = \frac{1}{1+\cot^2 z} = \frac{1}{1+(z_0/z)^2 - 1} = \left(\frac{z}{z_0}\right)^2 \Rightarrow P = \frac{z^2}{z_0^2(1+\sqrt{z_0^2-z^2})}.$$

So far, this is correct for *any* bound state. In the present case $z_0 = 8$ and z is the third solution to $-\cot z = \sqrt{(8/z)^2 - 1}$, which occurs somewhere in the interval $7.85 < z < 8$. Mathematica gives $z = 7.9573$ and $\boxed{P = 0.54204.}$

```
FindRoot[Cot[z] == -√((8 / z)^2 - 1), {z, 7.9}]

{z → 7.95732}

z^2 / (64 (1 + √(64 - z^2)))

       z²
──────────────────
64 (1 + √(64 - z²))

% /. z -> 7.957321523328964`

0.542041
```

Problem 2.49

(a)

$$\frac{\partial \Psi}{\partial t} = \left(-\frac{m\omega}{2\hbar}\right)\left[\frac{x_0^2}{2}\left(-2i\omega e^{-2i\omega t}\right) + \frac{i\hbar}{m} - 2x_0 x(-i\omega)e^{-i\omega t}\right]\Psi, \text{ so}$$

$$i\hbar\frac{\partial \Psi}{\partial t} = \left[-\frac{1}{2}m x_0^2\omega^2 e^{-2i\omega t} + \frac{1}{2}\hbar\omega + m x_0 x \omega^2 e^{-i\omega t}\right]\Psi.$$

$$\frac{\partial \Psi}{\partial x} = \left[\left(-\frac{m\omega}{2\hbar}\right)\left(2x - 2x_0 e^{-i\omega t}\right)\right]\Psi = -\frac{m\omega}{\hbar}\left(x - x_0 e^{-i\omega t}\right)\Psi;$$

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{m\omega}{\hbar}\Psi - \frac{m\omega}{\hbar}\left(x - x_0 e^{-i\omega t}\right)\frac{\partial \Psi}{\partial x} = \left[-\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2\left(x - x_0 e^{-i\omega t}\right)^2\right]\Psi.$$

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{2}m\omega^2x^2\Psi &= -\frac{\hbar^2}{2m}\left[-\frac{m\omega}{\hbar}+\left(\frac{m\omega}{\hbar}\right)^2\left(x-x_0e^{-i\omega t}\right)^2\right]\Psi+\frac{1}{2}m\omega^2x^2\Psi \\
&= \left[\frac{1}{2}\hbar\omega-\frac{1}{2}m\omega^2\left(x^2-2x_0xe^{-i\omega t}+x_0^2e^{-2i\omega t}\right)+\frac{1}{2}m\omega^2x^2\right]\Psi \\
&= \left[\frac{1}{2}\hbar\omega+mx_0x\omega^2e^{-i\omega t}-\frac{1}{2}m\omega^2x_0^2e^{-2i\omega t}\right]\Psi \\
&= i\hbar\frac{\partial\Psi}{\partial t}\ \text{(comparing second line above).}\quad \checkmark
\end{aligned}
$$

(b)

$$
\begin{aligned}
|\Psi|^2 &= \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{2\hbar}\left[\left(x^2+\frac{x_0^2}{2}\left(1+e^{2i\omega t}\right)-\frac{i\hbar t}{m}-2x_0xe^{i\omega t}\right)+\left(x^2+\frac{x_0^2}{2}\left(1+e^{-2i\omega t}\right)+\frac{i\hbar t}{m}-2x_0xe^{-i\omega t}\right)\right]} \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{2\hbar}\left[2x^2+x_0^2+x_0^2\cos(2\omega t)-4x_0x\cos(\omega t)\right]}.\quad \text{But } x_0^2[1+\cos(2\omega t)]=2x_0^2\cos^2\omega t,\ \text{so} \\
&= \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{\hbar}\left[x^2-2x_0x\cos(\omega t)+x_0^2\cos^2(\omega t)\right]} = \boxed{\sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega}{\hbar}(x-x_0\cos\omega t)^2}.}
\end{aligned}
$$

The wave packet is a *Gaussian* of fixed shape, whose *center* oscillates back and forth sinusoidally, with amplitude x_0 and angular frequency ω.

(c) Note that this wave function *is* correctly normalized (compare Eq. 2.60). Let $y \equiv x - x_0\cos\omega t$:

$$\langle x\rangle = \int x|\Psi|^2dx = \int(y+x_0\cos\omega t)|\Psi|^2dy = 0 + x_0a\cos\omega t\int|\Psi|^2dy = \boxed{x_0\cos\omega t.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{-mx_0\omega\sin\omega t.}\quad \frac{d\langle p\rangle}{dt} = -mx_0\omega^2\cos\omega t.\quad V=\frac{1}{2}m\omega^2x^2 \implies \frac{dV}{dx}=m\omega^2x.$$

$$\langle -\frac{dV}{dx}\rangle = -m\omega^2\langle x\rangle = -m\omega^2x_0\cos\omega t = \frac{d\langle p\rangle}{dt},\ \text{so Ehrenfest's theorem } \textit{is} \text{ satisfied.}$$

Problem 2.50

(a)

$$\frac{\partial\Psi}{\partial t} = \left[-\frac{m\alpha}{\hbar^2}\frac{\partial}{\partial t}|x-vt| - i\frac{(E+\frac{1}{2}mv^2)}{\hbar}\right]\Psi;\quad \frac{\partial}{\partial t}|x-vt| = \left\{\begin{array}{ll}-v, & \text{if } x-vt>0\\ v, & \text{if } x-vt<0\end{array}\right\}.$$

We can write this in terms of the θ-function (Eq. 2.146):

$$2\theta(z)-1 = \left\{\begin{array}{ll}1, & \text{if } z>0\\ -1, & \text{if } z<0\end{array}\right\},\ \text{so}\ \frac{\partial}{\partial t}|x-vt| = -v[2\theta(x-vt)-1].$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \left\{i\frac{m\alpha v}{\hbar}[2\theta(x-vt)-1]+E+\frac{1}{2}mv^2\right\}\Psi.\qquad [\bigstar]$$

$$\frac{\partial\Psi}{\partial x} = \left[-\frac{m\alpha}{\hbar^2}\frac{\partial}{\partial x}|x-vt| + \frac{imv}{\hbar}\right]\Psi$$

$$\frac{\partial}{\partial x}|x-vt| = \{1, \text{if } x > vt;\ -1, \text{if } x < vt\} = 2\theta(x-vt) - 1.$$

$$= \left\{-\frac{m\alpha}{\hbar^2}[2\theta(x-vt)-1] + \frac{imv}{\hbar}\right\}\Psi.$$

$$\frac{\partial^2\Psi}{\partial x^2} = \left\{-\frac{m\alpha}{\hbar^2}[2\theta(x-vt)-1] + \frac{imv}{\hbar}\right\}^2\Psi - \frac{2m\alpha}{\hbar^2}\left[\frac{\partial}{\partial x}\theta(x-vt)\right]\Psi.$$

But (from Problem 2.23(b)) $\frac{\partial}{\partial x}\theta(x-vt) = \delta(x-vt)$, so

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - \alpha\delta(x-vt)\Psi$$

$$= \left(-\frac{\hbar^2}{2m}\left\{-\frac{m\alpha}{\hbar^2}[2\theta(x-vt)-1] + \frac{imv}{\hbar}\right\}^2 + \alpha\delta(x-vt) - \alpha\delta(x-vt)\right)\Psi$$

$$= -\frac{\hbar^2}{2m}\left\{\frac{m^2\alpha^2}{\hbar^4}\underbrace{[2\theta(x-vt)-1]^2}_{1} - \frac{m^2v^2}{\hbar^2} - 2i\frac{mv}{\hbar}\frac{m\alpha}{\hbar^2}[2\theta(x-vt)-1]\right\}\Psi$$

$$= \left\{-\frac{m\alpha^2}{2\hbar^2} + \frac{1}{2}mv^2 + i\frac{mv\alpha}{\hbar}[2\theta(x-vt)-1]\right\}\Psi = i\hbar\frac{\partial\Psi}{\partial t} \text{ (compare [}\bigstar\text{]).} \quad \checkmark$$

(b)

$$|\Psi|^2 = \frac{m\alpha}{\hbar^2}e^{-2m\alpha|y|/\hbar^2} \quad (y \equiv x - vt).$$

Check normalization: $2\frac{m\alpha}{\hbar^2}\int_0^\infty e^{-2m\alpha y/\hbar^2}dy = \frac{2m\alpha}{\hbar^2}\frac{\hbar^2}{2m\alpha} = 1.$ ✓

$$\langle H\rangle = \int_{-\infty}^{\infty}\Psi^* H\Psi dx. \quad \text{But } H\Psi = i\hbar\frac{\partial\Psi}{\partial t}, \text{ which we calculated above [}\bigstar\text{].}$$

$$= \int\left\{\frac{im\alpha v}{\hbar}[2\theta(y)-1] + E + \frac{1}{2}mv^2\right\}|\Psi|^2 dy = \boxed{E + \frac{1}{2}mv^2.}$$

(Note that $[2\theta(y)-1]$ is an *odd* function of y.) *Interpretation:* The wave packet is dragged along (at speed v) with the delta-function. The total energy is the energy it *would* have in a stationary delta-function (E), plus *kinetic* energy due to the motion ($\frac{1}{2}mv^2$).

Problem 2.51

$$\Psi_0 = \left(\frac{2a}{\pi}\right)^{1/4}\frac{1}{\gamma}e^{-a(x+\frac{1}{2}gt^2)^2/\gamma^2}; \quad \gamma = \sqrt{1 + 2ia\hbar t/m}.$$

$$\frac{\partial\Psi}{\partial t} = \left[\frac{\partial\Psi_0}{\partial t} + \Psi_0\left(-\frac{img}{\hbar}\right)\left(x + \frac{1}{2}gt^2\right)\right]\exp\left[-i\frac{mgt}{\hbar}\left(x + \frac{1}{6}gt^2\right)\right],$$

$$\frac{\partial \Psi_0}{\partial t} = \left(\frac{2a}{\pi}\right)^{1/4} \left\{-\frac{1}{\gamma^2}\left(\frac{d\gamma}{dt}\right) + \frac{1}{\gamma}\left[-\frac{2a}{\gamma^2}\left(x+\frac{1}{2}gt^2\right)gt - a\left(x+\frac{1}{2}gt^2\right)^2\left(-\frac{2}{\gamma^3}\frac{d\gamma}{dt}\right)\right]\right\} e^{-a(x+\frac{1}{2}gt^2)^2/\gamma^2}$$
$$= \left[-\frac{1}{\gamma}\frac{d\gamma}{dt} - \frac{2agt}{\gamma^2}\left(x+\frac{1}{2}gt^2\right) + \frac{2a}{\gamma^3}\left(x+\frac{1}{2}gt^2\right)^2\frac{d\gamma}{dt}\right]\Psi_0.$$

But $\displaystyle\frac{d\gamma}{dt} = \frac{1}{2\gamma}\left(\frac{2ia\hbar}{m}\right) = \frac{ia\hbar}{\gamma m}$, so

$$\frac{\partial \Psi_0}{\partial t} = \left[\frac{-ia\hbar}{\gamma^2 m} - \frac{2agt}{\gamma^2}\left(x+\frac{1}{2}gt^2\right) + \frac{2ia^2\hbar}{\gamma^4 m}\left(x+\frac{1}{2}gt^2\right)^2\right]\Psi_0, \quad \text{and hence}$$
$$\frac{\partial \Psi}{\partial t} = \left[-\frac{ia\hbar}{\gamma^2 m} - \frac{2agt}{\gamma^2}\left(x+\frac{1}{2}gt^2\right) + \frac{2ia^2\hbar}{\gamma^4 m}\left(x+\frac{1}{2}gt^2\right)^2 - \frac{img}{\hbar}\left(x+\frac{1}{2}gt^2\right)\right]\Psi. \qquad [\bigstar]$$

Meanwhile

$$\frac{\partial \Psi}{\partial x} = \left[\frac{\partial \Psi_0}{\partial x} - \Psi_0\frac{imgt}{\hbar}\right]\exp\left[-\frac{imgt}{\hbar}\left(x+\frac{1}{6}gt^2\right)\right]$$
$$\frac{\partial \Psi_0}{\partial x} = -2a\left[\left(x+\frac{1}{2}gt^2\right)/\gamma^2\right]\Psi_0, \quad \text{so}$$
$$\frac{\partial \Psi}{\partial x} = \left[-2a\left(x+\frac{1}{2}gt^2\right)/\gamma^2 - \frac{imgt}{\hbar}\right]\Psi, \quad \text{and hence}$$
$$\frac{\partial^2 \Psi}{\partial x^2} = \left\{-\frac{2a}{\gamma^2}\Psi + \left[-2a\left(x+\frac{1}{2}gt^2\right)/\gamma^2 - \frac{imgt}{\hbar}\right]\frac{\partial \Psi}{\partial x}\right\} = \left\{-\frac{2a}{\gamma^2} + \left[-2a\left(x+\frac{1}{2}gt^2\right)/\gamma^2 - \frac{imgt}{\hbar}\right]^2\right\}\Psi.$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = \left\{\frac{\hbar^2 a}{m\gamma^2} - \frac{\hbar^2}{2m}\left[-2a\left(x+\frac{1}{2}gt^2\right)/\gamma^2 - \frac{imgt}{\hbar}\right]^2 + mgx\right\}\Psi. \qquad [\bigstar\bigstar]$$

So the time-dependent Schrödinger equation is satisfied if $i\hbar$ times the square bracket in Equation [★] is equal to the curly bracket in Equation [★★]:

$$\cancel{\frac{a\hbar^2}{\gamma^2 m}} - \frac{2ia\hbar gt}{\gamma^2}\left(x+\frac{1}{2}gt^2\right) - \frac{2a^2\hbar^2}{\gamma^4 m}\left(x+\frac{1}{2}gt^2\right)^2 + mg\left(\cancel{x}+\frac{1}{2}gt^2\right)$$
$$\stackrel{?}{=} \cancel{\frac{\hbar^2 a}{m\gamma^2}} - \frac{\hbar^2}{2m}\left[-2a\left(x+\frac{1}{2}gt^2\right)/\gamma^2 - \frac{imgt}{\hbar}\right]^2 + \cancel{mgx}.$$

I have cancelled the first terms on either side, and also the mgx terms. This leaves

$$-\frac{2ia\hbar gt}{\gamma^2}\left(x+\frac{1}{2}gt^2\right) - \frac{2a^2\hbar^2}{\gamma^4 m}\left(x+\frac{1}{2}gt^2\right)^2 + \frac{mg^2t^2}{2}$$
$$\stackrel{?}{=} -\frac{\hbar^2}{2m}\left[\frac{4a^2}{\gamma^4}\left(x+\frac{1}{2}gt^2\right)^2 + \frac{4iamgt}{\hbar\gamma^2}\left(x+\frac{1}{2}gt^2\right) - \frac{m^2g^2t^2}{\hbar^2}\right].$$

The terms quadratic in $\left(x+\frac{1}{2}gt^2\right)$ cancel, as do the linear terms, and so do those of zeroth order. This confirms that Ψ satisfies the Schrödinger equation.

To calculate the expectation value of x, first note that

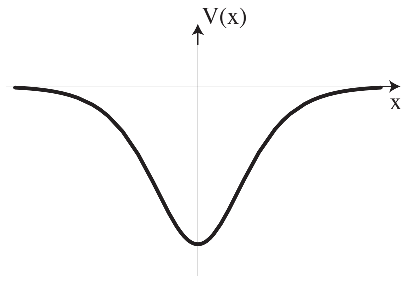
$|\Psi|^2 = \Psi_0|^2 = \sqrt{\frac{2a}{\pi}}\frac{1}{|\gamma|^2}e^{-\left(x+\frac{1}{2}gt^2\right)^2\left(\frac{1}{\gamma^2}+\frac{1}{(\gamma^*)^2}\right)}$. But $\frac{1}{\gamma^2}+\frac{1}{(\gamma^*)^2} = \frac{(\gamma^*)+\gamma^2}{|\gamma|^4} = \frac{2}{|\gamma|^4}$, so (letting $y \equiv x+\frac{1}{2}gt^2$)

$$\langle x\rangle = \sqrt{\frac{2a}{\pi}}\frac{1}{|\gamma|^2}\int_{-\infty}^{\infty} xe^{-2a\left(x+\frac{1}{2}gt^2\right)^2/|\gamma|^4}\,dx = \sqrt{\frac{2a}{\pi}}\frac{1}{|\gamma|^2}\int_{-\infty}^{\infty}\left(y-\frac{1}{2}gt^2\right)e^{-2ay^2/|\gamma|^4}\,dy$$
$$= \sqrt{\frac{2a}{\pi}}\frac{1}{|\gamma|^2}\left(-\frac{1}{2}gt^2\right)\int_{-\infty}^{\infty}e^{-2ay^2/|\gamma|^4}\,dy.$$

The integral is $\sqrt{\pi/2a}\,|\gamma|^2$, so $\boxed{\langle x\rangle = -\frac{1}{2}gt^2.}$ This is precisely the *classical* motion under free fall—as we should have anticipated from Ehrenfest's theorem.

Problem 2.52

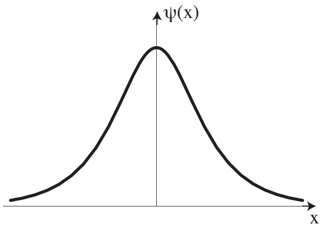
(a)



(b) $\frac{d\psi_0}{dx} = -Aa\,\text{sech}(ax)\tanh(ax);\quad \frac{d^2\psi_0}{dx^2} = -Aa^2\left[-\text{sech}(ax)\tanh^2(ax)+\text{sech}(ax)\,\text{sech}^2(ax)\right].$

$$\begin{aligned}H\psi_0 &= -\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2} - \frac{\hbar^2a^2}{m}\text{sech}^2(ax)\psi_0\\
&= \frac{\hbar^2}{2m}Aa^2\left[-\text{sech}(ax)\tanh^2(ax)+\text{sech}^3(ax)\right] - \frac{\hbar^2a^2}{m}A\,\text{sech}^3(ax)\\
&= \frac{\hbar^2a^2A}{2m}\left[-\text{sech}(ax)\tanh^2(ax)+\text{sech}^3(ax)-2\,\text{sech}^3(ax)\right]\\
&= -\frac{\hbar^2a^2}{2m}A\,\text{sech}(ax)\left[\tanh^2(ax)+\text{sech}^2(ax)\right].\\
&\quad\text{But } (\tanh^2\theta+\text{sech}^2\theta) = \frac{\sinh^2\theta}{\cosh^2\theta}+\frac{1}{\cosh^2\theta} = \frac{\sinh^2\theta+1}{\cosh^2\theta} = 1, \text{ so}\\
&= -\frac{\hbar^2a^2}{2m}\psi_0. \quad \text{QED} \quad \text{Evidently } \boxed{E = -\frac{\hbar^2a^2}{2m}.}\end{aligned}$$

$$1 = |A|^2\int_{-\infty}^{\infty}\text{sech}^2(ax)dx = |A|^2\,\frac{1}{a}\tanh(ax)\Big|_{-\infty}^{\infty} = \frac{2}{a}|A|^2 \Longrightarrow \boxed{A = \sqrt{\frac{a}{2}}.}$$



(c)

$$\frac{d\psi_k}{dx} = \frac{A}{ik+a}\left[(ik - a\tanh ax)ik - a^2\,\mathrm{sech}^2\,ax\right]e^{ikx}.$$

$$\frac{d^2\psi_k}{dx^2} = \frac{A}{ik+a}\left\{ik\left[(ik - a\tanh ax)ik - a^2\,\mathrm{sech}^2\,ax\right] - a^2ik\,\mathrm{sech}^2\,ax + 2a^3\,\mathrm{sech}^2\,ax\tanh ax\right\}e^{ikx}.$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_k}{dx^2} + V\psi_k = \frac{A}{ik+a}\left\{\frac{-\hbar^2 ik}{2m}\left[-k^2 - iak\tanh ax - a^2\,\mathrm{sech}^2\,ax\right] + \frac{\hbar^2a^2}{2m}ik\,\mathrm{sech}^2\,ax\right.$$
$$\left. - \frac{\hbar^2a^3}{m}\mathrm{sech}^2\,ax\tanh ax - \frac{\hbar^2a^2}{m}\mathrm{sech}^2\,ax(ik - a\tanh ax)\right\}e^{ikx}$$
$$= \frac{Ae^{ikx}}{ik+a}\frac{\hbar^2}{2m}\left(ik^3 - ak^2\tanh ax + ia^2k\,\mathrm{sech}^2\,ax + ia^2k\,\mathrm{sech}^2\,ax\right.$$
$$\left. - 2a^3\,\mathrm{sech}^2\,ax\tanh ax - 2ia^2k\,\mathrm{sech}^2\,ax + 2a^3\,\mathrm{sech}^2\,ax\tanh ax\right)$$
$$= \frac{Ae^{ikx}}{ik+a}\frac{\hbar^2}{2m}k^2(ik - a\tanh ax) = \frac{\hbar^2k^2}{2m}\psi_k = E\psi_k.\quad \text{QED}$$

As $x \to +\infty$, $\tanh ax \to +1$, so $\boxed{\psi_k(x) \to A\left(\frac{ik-a}{ik+a}\right)e^{ikx},}$ which represents a transmitted wave.

$$\boxed{R = 0.} \qquad T = \left|\frac{ik-a}{ik+a}\right|^2 = \left(\frac{-ik-a}{-ik+a}\right)\left(\frac{ik-a}{ik+a}\right) = \boxed{1.}$$

Problem 2.53

(a) (1) From Eq. 2.136: $F + G = A + B$.

(2) From Eq. 2.138: $F - G = (1+2i\beta)A - (1-2i\beta)B$, where $\beta = m\alpha/\hbar^2 k$.

Subtract: $2G = -2i\beta A + 2(1-i\beta)B \Rightarrow B = \frac{1}{1-i\beta}(i\beta A + G)$. Multiply (1) by $(1-2i\beta)$ and add:

$2(1-i\beta)F - 2i\beta G = 2A \Rightarrow F = \frac{1}{1-i\beta}(A + i\beta G)$. $\boxed{\mathsf{S} = \frac{1}{1-i\beta}\begin{pmatrix} i\beta & 1 \\ 1 & i\beta\end{pmatrix}.}$

(b) For an *even* potential, $V(-x) = V(x)$, scattering from the right is the same as scattering from the left, with $x \leftrightarrow -x$, $A \leftrightarrow G$, $B \leftrightarrow F$ (see Fig. 2.21): $F = S_{11}G + S_{12}A$, $B = S_{21}G + S_{22}A$. So $S_{11} = S_{22}$, $S_{21} = S_{12}$. (Note that the delta-well S matrix in (a) has this property.) In the case of the finite square well, Eqs. 2.170 and 2.171 give

$$S_{21} = \frac{e^{-2ika}}{\cos 2la - i\frac{(k^2+l^2)}{2kl}\sin 2la}; \qquad S_{11} = \frac{i\frac{(l^2-k^2)}{2kl}\sin 2la\, e^{-2ika}}{\cos 2la - i\frac{(k^2+l^2)}{2kl}\sin 2la}. \quad \text{So}$$

$$\boxed{\mathsf{S} = \frac{e^{-2ika}}{\cos 2la - i\frac{(k^2+l^2)}{2kl}\sin 2la}\begin{pmatrix} i\frac{(l^2-k^2)}{2kl}\sin 2la & 1 \\ 1 & i\frac{(l^2-k^2)}{2kl}\sin 2la \end{pmatrix}.}$$

Problem 2.54

(a)

$$B = S_{11}A + S_{12}G \Rightarrow G = \frac{1}{S_{12}}(B - S_{11}A) = M_{21}A + M_{22}B \Rightarrow M_{21} = -\frac{S_{11}}{S_{12}},\ M_{22} = \frac{1}{S_{12}}.$$

$$F = S_{21}A + S_{22}G = S_{21}A + \frac{S_{22}}{S_{12}}(B - S_{11}A) = -\frac{(S_{11}S_{22} - S_{12}S_{21})}{S_{12}}A + \frac{S_{22}}{S_{12}}B = M_{11}A + M_{12}B.$$

$$\Rightarrow M_{11} = -\frac{\det S}{S_{12}},\ M_{12} = \frac{S_{22}}{S_{12}}.\quad \boxed{\mathsf{M} = \frac{1}{S_{12}}\begin{pmatrix} -\det(\mathsf{S}) & S_{22} \\ -S_{11} & 1 \end{pmatrix}.}\quad \text{Conversely:}$$

$$G = M_{21}A + M_{22}B \Rightarrow B = \frac{1}{M_{22}}(G - M_{21}A) = S_{11}A + S_{12}G \Rightarrow S_{11} = -\frac{M_{21}}{M_{22}};\ S_{12} = \frac{1}{M_{22}}.$$

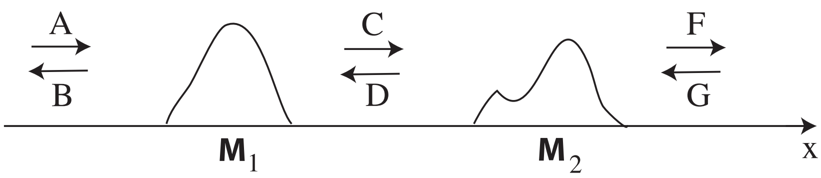
$$F = M_{11}A + M_{12}B = M_{11}A + \frac{M_{12}}{M_{22}}(G - M_{21}A) = \frac{(M_{11}M_{22} - M_{12}M_{21})}{M_{22}}A + \frac{M_{12}}{M_{22}}G = S_{21}A + S_{22}G.$$

$$\Rightarrow S_{21} = \frac{\det M}{M_{22}};\ S_{22} = \frac{M_{12}}{M_{22}}.\quad \boxed{\mathsf{S} = \frac{1}{M_{22}}\begin{pmatrix} -M_{21} & 1 \\ \det(\mathsf{M}) & M_{12} \end{pmatrix}.}$$

[It happens that the time-reversal invariance of the Schrödinger equation, plus conservation of probability, requires $M_{22} = M_{11}^*,\ M_{21} = M_{12}^*$, and $\det(\mathsf{M}) = 1$, but I won't use this here. See Merzbacher's *Quantum Mechanics*. Similarly, for *even* potentials $S_{11} = S_{22},\ S_{12} = S_{21}$ (Problem 2.53).]

$$R_l = |S_{11}|^2 = \boxed{\left|\frac{M_{21}}{M_{22}}\right|^2},\ T_l = |S_{21}|^2 = \boxed{\left|\frac{\det(\mathsf{M})}{M_{22}}\right|^2},\ R_r = |S_{22}|^2 = \boxed{\left|\frac{M_{12}}{M_{22}}\right|^2},\ T_r = |S_{12}|^2 = \boxed{\frac{1}{|M_{22}|^2}}.$$

(b)



$$\begin{pmatrix} F \\ G \end{pmatrix} = \mathsf{M}_2\begin{pmatrix} C \\ D \end{pmatrix},\ \begin{pmatrix} C \\ D \end{pmatrix} = \mathsf{M}_1\begin{pmatrix} A \\ B \end{pmatrix},\ \text{so}\ \begin{pmatrix} F \\ G \end{pmatrix} = \mathsf{M}_2\mathsf{M}_1\begin{pmatrix} A \\ B \end{pmatrix} = \mathsf{M}\begin{pmatrix} A \\ B \end{pmatrix},\ \text{with}\ \mathsf{M} = \mathsf{M}_2\mathsf{M}_1.\quad \text{QED}$$

(c)

$$\psi(x) = \left\{\begin{array}{ll} Ae^{ikx} + Be^{-ikx} & (x < a) \\ Fe^{ikx} + Ge^{-ikx} & (x > a) \end{array}\right\}.$$

$$\begin{cases} \text{Continuity of } \psi: & Ae^{ika} + Be^{-ika} = Fe^{ika} + Ge^{-ika} \\ \text{Discontinuity of } \psi': & ik\left(Fe^{ika} - Ge^{-ika}\right) - ik\left(Ae^{ika} - Be^{-ika}\right) = -\frac{2m\alpha}{\hbar^2}\psi(a) = -\frac{2m\alpha}{\hbar^2}\left(Ae^{ika} + Be^{-ika}\right). \end{cases}$$

(1) $Fe^{2ika} + G = Ae^{2ika} + B.$

(2) $Fe^{2ika} - G = Ae^{2ika} - B + i\frac{2m\alpha}{\hbar^2 k}\left(Ae^{2ika} + B\right).$

Add (1) and (2):

$$2Fe^{2ika} = 2Ae^{2ika} + i\frac{2m\alpha}{\hbar^2 k}\left(Ae^{2ika} + B\right) \Rightarrow F = \left(1 + i\frac{m\alpha}{\hbar^2 k}\right)A + i\frac{m\alpha}{\hbar^2 k}e^{-2ika}B = M_{11}A + M_{12}B.$$

So $M_{11} = (1 + i\beta);\ M_{12} = i\beta e^{-2ika};\ \beta \equiv \dfrac{m\alpha}{\hbar^2 k}.$

Subtract (2) from (1):

$$2G = 2B - 2i\beta e^{2ika}A - 2i\beta B \Rightarrow G = (1 - i\beta)B - i\beta e^{2ika}A = M_{21}A + M_{22}B.$$

So $M_{21} = -i\beta e^{2ika};\ M_{22} = (1 - i\beta).$ $\boxed{\mathsf{M} = \begin{pmatrix} (1+i\beta) & i\beta e^{-2ika} \\ -i\beta e^{2ika} & (1-i\beta) \end{pmatrix}.}$

(d)

$$\mathsf{M}_2 = \begin{pmatrix} (1+i\beta) & i\beta e^{-2ika} \\ -i\beta e^{2ika} & (1-i\beta) \end{pmatrix};\ \text{to get } \mathsf{M}_1,\ \text{just switch the sign of } a\text{: } \mathsf{M}_1 = \begin{pmatrix} (1+i\beta) & i\beta e^{2ika} \\ -i\beta e^{-2ika} & (1-i\beta) \end{pmatrix}.$$

$$\mathsf{M} = \mathsf{M}_2\mathsf{M}_1 = \boxed{\begin{pmatrix} [1 + 2i\beta + \beta^2(e^{-4ika} - 1)] & 2i\beta[\cos 2ka - \beta\sin 2ka] \\ -2i\beta[\cos 2ka - \beta\sin 2ka] & [1 - 2i\beta + \beta^2(e^{4ika} - 1)] \end{pmatrix}.}$$

$$T = T_l = T_r = \frac{1}{|M_{22}|^2} \Rightarrow$$

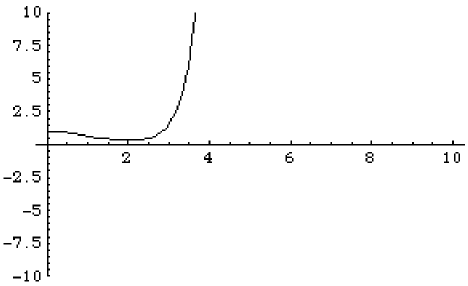
$$\begin{aligned} T^{-1} &= [1 + 2i\beta + \beta^2(e^{-4ika} - 1)][1 - 2i\beta + \beta^2(e^{4ika} - 1)] \\ &= 1 - 2i\beta + \beta^2 e^{4ika} - \beta^2 + 2i\beta + 4\beta^2 + 2i\beta^3 e^{4ika} - 2i\beta^3 + \beta^2 e^{-4ika} \\ &\quad - \beta^2 - 2i\beta^3 e^{-4ika} + 2i\beta^3 + \beta^4(1 - e^{-4ika} - e^{4ika} + 1) \\ &= 1 + 2\beta^2 + \beta^2(e^{-4ika} + e^{4ika}) - 2i\beta^3(e^{-4ika} - e^{4ika}) + 2\beta^4 - \beta^4(e^{-4ika} + e^{4ika}) \\ &= 1 + 2\beta^2 + 2\beta^2\cos 4ka + 2i\beta^3 2i\sin 4ka + 2\beta^4 - 2\beta^4\cos 4ka \\ &= 1 + 2\beta^2(1 + \cos 4ka) - 4\beta^3\sin 4ka + 2\beta^4(1 - \cos 4ka) \\ &= 1 + 4\beta^2\cos^2 2ka - 8\beta^3\sin 2ka\cos 2ka - 4\beta^4\sin^2 2ka \end{aligned}$$

$$\boxed{T = \frac{1}{1 + 4\beta^2(\cos 2ka - \beta\sin 2ka)^2}}$$

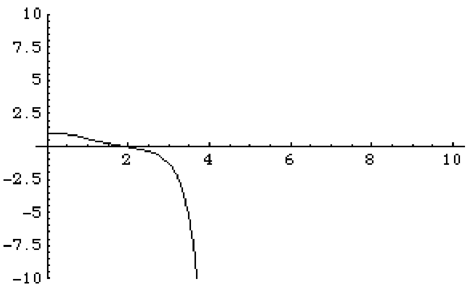
Problem 2.55

I'll just show the first two graphs, and the last two. Evidently K lies between 0.9999 and 1.0001

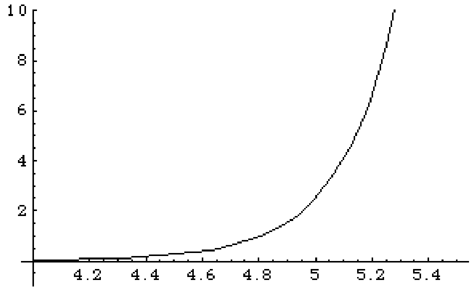
```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 0.9)*u[x] == 0, u[0] == 1,
      u'[0] == 0}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 10},
  PlotRange -> {-10, 10}];
```



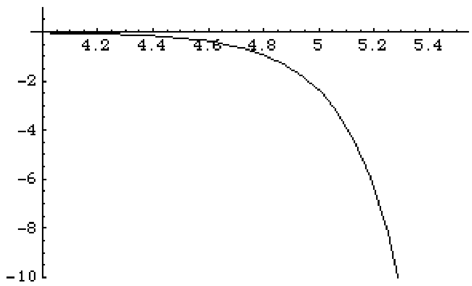
```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 1.1)*u[x] == 0, u[0] == 1,
      u'[0] == 0}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 10},
  PlotRange -> {-10, 10}];
```



```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 0.9999)*u[x] == 0, u[0] == 1,
      u'[0] == 0}, u[x], {x, 10^-8, 10}, MaxSteps -> 10000]],
  {x, 4, 5.5}, PlotRange -> {-1, 10}];
```



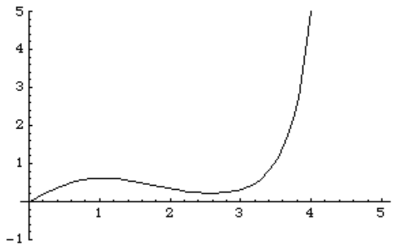
```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 1.0001)*u[x] == 0, u[0] == 1,
      u'[0] == 0}, u[x], {x, 10^-8, 10}, MaxSteps -> 10000]],
  {x, 4, 5.5}, PlotRange -> {-10, 1}];
```



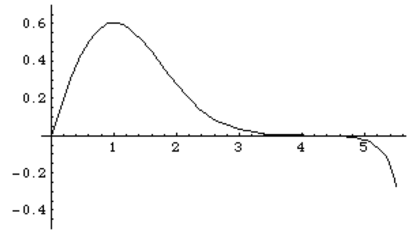
Problem 2.56

The *correct* values (in Eq. 2.73) are $K = 2n + 1$ (corresponding to $E_n = (n + \frac{1}{2})\hbar\omega$). I'll start by "guessing" 2.9, 4.9, and 6.9, and tweaking the number until I've got 5 reliable significant digits. The results (see below) are $\boxed{3.0000,\ 5.0000,\ 7.0000.}$ (The actual *energies* are these numbers multiplied by $\frac{1}{2}\hbar\omega$.)

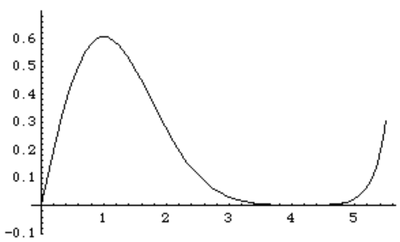
```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 2.9) * u[x] == 0, u[0] == 0,
      u'[0] == 1}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 5},
  PlotRange -> {-1, 5}];
```



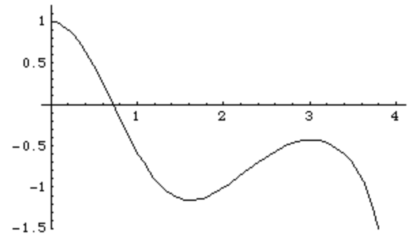

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 3.00001) * u[x] == 0,
      u[0] == 0, u'[0] == 1}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 5.5},
  PlotRange -> {-.5, .7}];
```



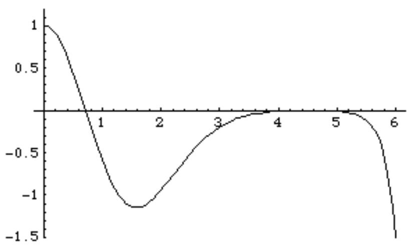

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 2.99999) * u[x] == 0,
      u[0] == 0, u'[0] == 1}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 5.5},
  PlotRange -> {-.1, .7}];
```



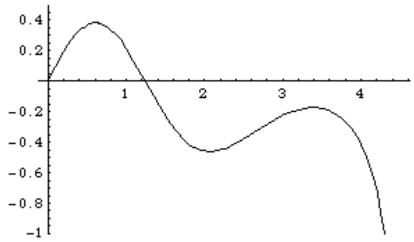

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 4.9) * u[x] == 0, u[0] == 1,
      u'[0] == 0}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 4},
  PlotRange -> {-1.5, 1.2}];
```



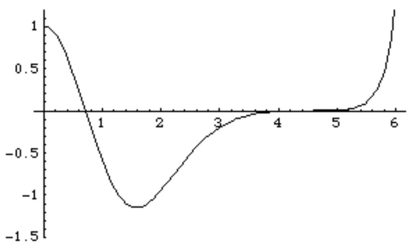

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 4.99999) * u[x] == 0,
      u[0] == 1, u'[0] == 0}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 6},
  PlotRange -> {-1.5, 1.2}];
```



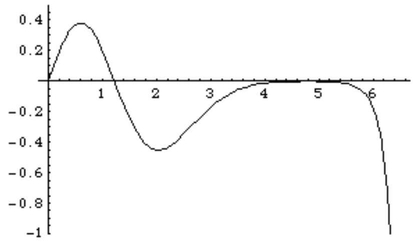

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 6.9) * u[x] == 0, u[0] == 0,
      u'[0] == 1}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 4.5},
  PlotRange -> {-1, .5}];
```



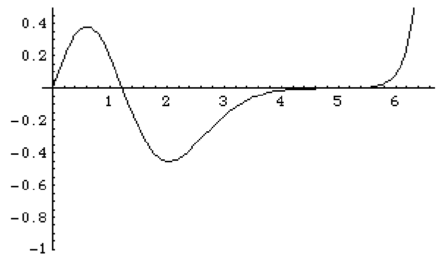

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 5.00001) * u[x] == 0,
      u[0] == 1, u'[0] == 0}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 6},
  PlotRange -> {-1.5, 1.2}];
```




```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 6.99999) * u[x] == 0,
      u[0] == 0, u'[0] == 1}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 6.5},
  PlotRange -> {-1, .5}];
```



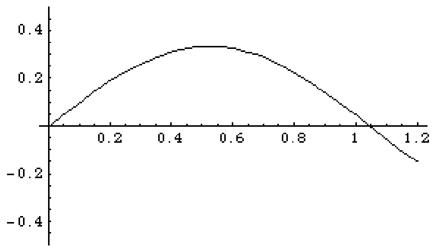
```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] - (x^2 - 7.00001) * u[x] == 0,
      u[0] == 0, u'[0] == 1}, u[x], {x, 10^-8, 10},
     MaxSteps -> 10000]], {x, 0, 6.5},
  PlotRange -> {-1, .5}];
```



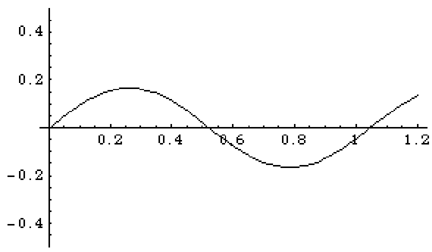
Problem 2.57

The Schrödinger equation says $-\frac{\hbar^2}{2m}\psi'' = E\psi$, or, with the *correct* energies (Eq. 2.30) and $a = 1$, $\psi'' + (n\pi)^2\psi = 0$. I'll start with a "guess" using 9 in place of π^2 (that is, I'll use 9 for the ground state, 36 for the first excited state, 81 for the next, and finally 144). Then I'll tweak the parameter until the graph crosses the axis right at $x = 1$. The results (see below) are, to five significant digits: $\boxed{9.8696,\ 39.478,\ 88.826,\ 157.91.}$ (The actual *energies* are these numbers multiplied by $\hbar^2/2ma^2$.)

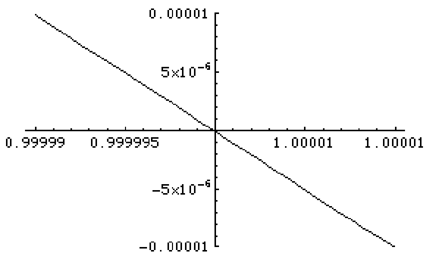
```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] + (9) * u[x] == 0, u[0] == 0, u'[0] == 1},
     u[x], {x, 10^-8, 1.5}, MaxSteps -> 10000]],
  {x, 0, 1.2}, PlotRange -> {-.5, .5}];
```



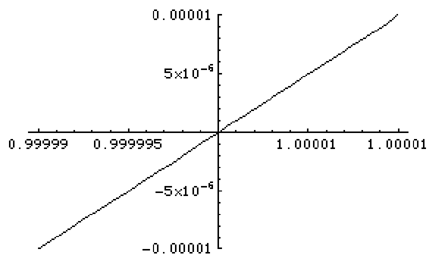

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] + (36) * u[x] == 0, u[0] == 0, u'[0] == 1},
     u[x], {x, 10^-8, 1.5}, MaxSteps -> 10000]],
  {x, 0, 1.2}, PlotRange -> {-.5, .5}];
```



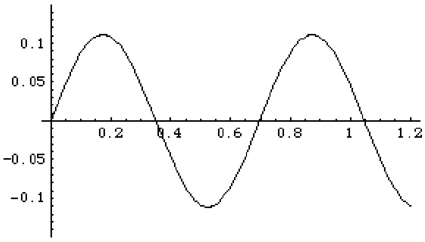

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] + (9.86959) * u[x] == 0, u[0] == 0,
      u'[0] == 1}, u[x], {x, 10^-8, 1.005},
     MaxSteps -> 10000]], {x, 0.99999, 1.00001},
  PlotRange -> {-.00001, .00001}];
```



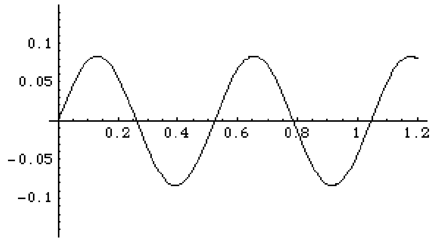

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] + (39.47803) * u[x] == 0, u[0] == 0,
      u'[0] == 1}, u[x], {x, 10^-8, 1.005},
     MaxSteps -> 10000]], {x, 0.99999, 1.00001},
  PlotRange -> {-.00001, .00001}];
```



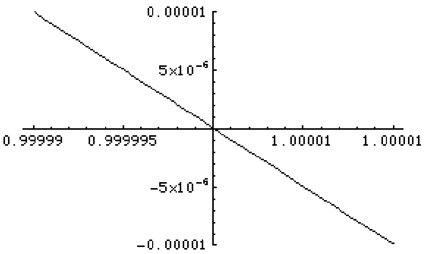
```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] + (81) *u[x] == 0, u[0] == 0, u'[0] == 1},
     u[x], {x, 10^-8, 1.5}, MaxSteps -> 10000]],
  {x, 0, 1.2}, PlotRange -> {-.15, .15}];
```



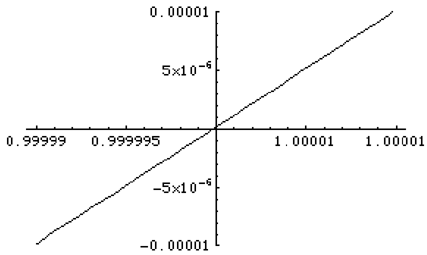

```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] + (144) *u[x] == 0, u[0] == 0, u'[0] == 1},
     u[x], {x, 10^-8, 1.5}, MaxSteps -> 10000]],
  {x, 0, 1.2}, PlotRange -> {-.1, .1}];
```




```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] + (88.82630) *u[x] == 0, u[0] == 0,
      u'[0] == 1}, u[x], {x, 10^-8, 1.005},
     MaxSteps -> 10000]], {x, 0.99999, 1.00001},
  PlotRange -> {-.00001, .00001}];
```




```
Plot[Evaluate[u[x] /.
    NDSolve[{u''[x] + (157.9129) *u[x] == 0, u[0] == 0,
      u'[0] == 1}, u[x], {x, 10^-8, 1.005},
     MaxSteps -> 10000]], {x, 0.99999, 1.00001},
  PlotRange -> {-.00001, .00001}];
```



Problem 2.58

(a) The total energy is simply N times the ground state energy of the infinite square well:

$$\boxed{E_a = N\frac{\pi^2\hbar^2}{2ma^2}.}$$

(b) Filling the lowest N energy levels of the infinite square well (with width Na) gives

$$E_b = \sum_{n=1} N\frac{n^2\pi^2\hbar^2}{2m(NA)^2} = \frac{\pi^2\hbar^2}{2mN^2a^2}\sum_{n=1}^{N} n^2.$$

The sum is $N(N+1)(2N+1)/6$, so

$$\boxed{E_b = \left(\frac{N}{3}+\frac{1}{2}+\frac{1}{6N}\right)\frac{\pi^2\hbar^2}{2ma^2}.}$$

(c)

$$\frac{\Delta E}{N} = \frac{E_a - E_b}{N} \approx \left(\frac{N-(N/3)}{N}\right)\frac{\pi^2\hbar^2}{2ma^2} = \boxed{\frac{\pi^2\hbar^2}{3ma^2}.}$$

(d) The binding energy of hydrogen (13.6 eV) is $\hbar^2/2ma_B^2$, where $a_B = 0.529$ Å is the Bohr radius, so

$$\frac{\Delta E}{N} = \frac{2}{3}\pi^2\left(\frac{a_B}{a}\right)^2 E_{\text{binding}} = \frac{2}{3}\left(\frac{0.529\,\pi}{4}\right)^2 (13.6)\,\text{eV} = \boxed{1.6\,\text{eV}.}$$

Problem 2.59

(a)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + mgx\,\psi = E\,\psi \qquad (x \geq 0; \quad \psi = 0 \text{ for } x \leq 0).$$

$$\frac{d\psi}{dx} = \frac{d\psi}{dz}\frac{dz}{dx} = a\frac{d\psi}{dz}; \quad \frac{d^2\psi}{dx^2} = a\frac{d^2\psi}{dz^2}\frac{dz}{dx} = a^2\frac{d^2\psi}{dz^2}.$$

$$-\frac{\hbar^2}{2m}a^2\frac{d^2\psi}{dz^2} + mg\frac{z}{a}\psi = E\psi \;\Rightarrow\; -\frac{\hbar^2}{2m}a^2\sqrt{a}\,y'' + mg\frac{z}{a}\sqrt{a}\,y = E\sqrt{a}\,y \;\Rightarrow\; -y'' + \frac{2m}{\hbar^2 a^2}mg\frac{z}{a}\,y = \frac{2m}{\hbar^2 a^2}E\,y.$$

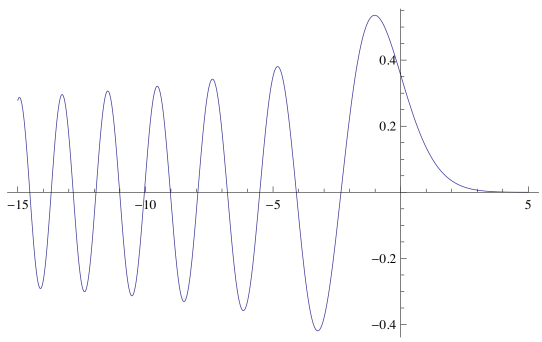
Let

$$\frac{2m}{\hbar^2 a^2}mg\frac{1}{a} = 1, \text{ or } \boxed{a \equiv \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}} \quad \text{and} \quad \epsilon \equiv \frac{2m}{\hbar^2 a^2}E = \frac{2m}{\hbar^2}\left(\frac{\hbar^2}{2m^2 g}\right)^{2/3} E = \boxed{\left(\frac{2}{m\hbar^2 g^2}\right)^{1/3} E.}$$

Then $-y'' + zy = \epsilon y$. ✓

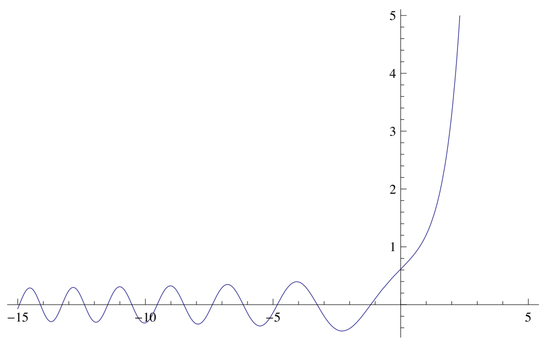
```
In[1]:= DSolve[-y''[x] + x y[x] == s y[x], y[x], x]
Out[1]= {{y[x] → AiryAi[-s + x] C[1] + AiryBi[-s + x] C[2]}}
In[2]:= Plot[AiryAi[x], {x, -15, 5}]
```

Out[2]=



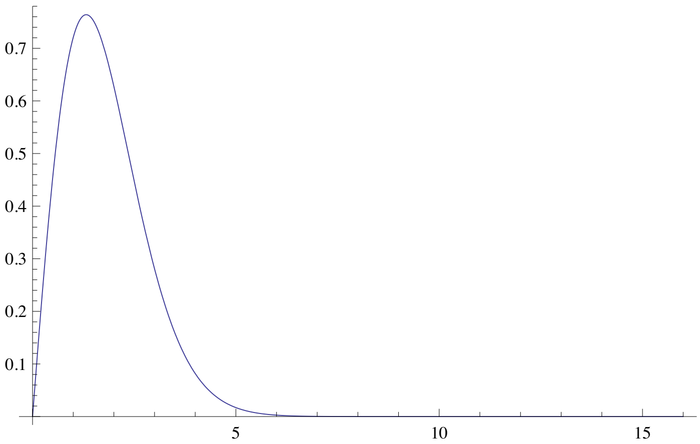

```
In[3]:= Plot[AiryBi[x], {x, -15, 5}]
```

Out[3]=



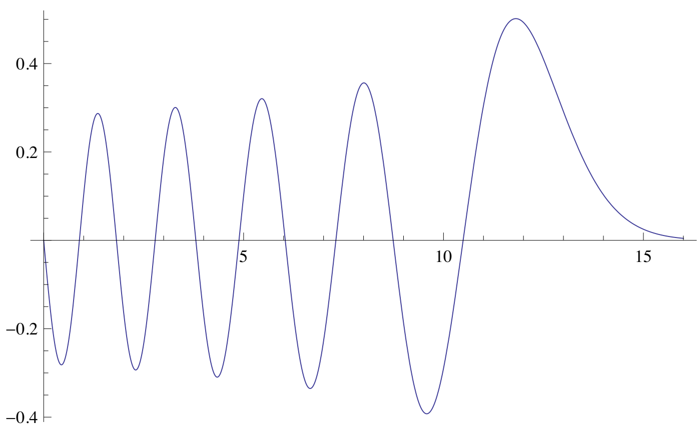
```
In[12]:= FindRoot[AiryAi[x] == 0, {x, -2}]

Out[12]= {x → -2.33811}

In[13]:= FindRoot[AiryAi[x] == 0, {x, -12.8}]

Out[13]= {x → -12.8288}

In[14]:= NIntegrate[(AiryAi[x])^2, {x, -2.338107410459767`, ∞}]

Out[14]= 0.491697

In[15]:= NIntegrate[(AiryAi[x])^2, {x, -12.828776752865757`, ∞}]

Out[15]= 1.14018

In[16]:= Pone[x_] := (0.4916966179009774`)^(-1 / 2) * AiryAi[x - 2.338107410459767`]

In[17]:= Plot[Pone[x], {x, 0, 16}]
```

Out[17]=




```
In[18]:= Pten[x_] := (1.1401837256117164`)^(-1 / 2) * AiryAi[x - 12.828776752865757`]

In[19]:= Plot[Pten[x], {x, 0, 16}]
```

Out[19]=



```
In[20]:= NIntegrate[Pone[x] * Pten[x], {x, 0, ∞}]

NIntegrate::ncvb :
  NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in x near {x} = {1.72352}. NIntegrate
    obtained -9.19403×10^-17 and 1.5861121045064646`*^-16 for the integral and error estimates. ≫

Out[20]= -9.19403 × 10^-17
```

(b)

For ψ_1 : $\sigma_x = 0.697089$, $\sigma_p = 0.882819\,\hbar$, $\sigma_x\sigma_p = 0.615403\,\hbar > 0.5\hbar$ ✓.
For ψ_{10} : $\sigma_x = 3.8248$, $\sigma_p = 2.06791\,\hbar$, $\sigma_x\sigma_p = 7.90935\,\hbar > 0.5\hbar$ ✓.

(See print-out.)

```
NIntegrate[x (Pone[x])^2, {x, 0, ∞}]
1.55874

NIntegrate[x^2 (Pone[x])^2, {x, 0, ∞}]
2.9156

√(2.9155980068599967` - (1.558738273638599`)^2)
0.697089

NIntegrate[x (Pten[x])^2, {x, 0, ∞}]
8.55252

NIntegrate[x^2 (Pten[x])^2, {x, 0, ∞}]
87.7747

√(87.77467357595424` - (8.552517834822023`)^2)
3.8248

NIntegrate[-i Pone[x] (Pone'[x]), {x, 0, ∞}]
NIntegrate::ncvb :
  NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in x near {x} = {1.2858}. NIntegrate
    obtained 0. + 6.07153×10^-18 i and 7.766131095614155`*^-17 for the integral and error estimates. ≫
0. + 6.07153 × 10^-18 i

NIntegrate[- Pone[x] (Pone''[x]), {x, 0, ∞}]
0.779369

NIntegrate[- Pten[x] (Pten''[x]), {x, 0, ∞}]
4.27626

√(0.7793691368188985`)
0.882819

√(4.276258918045443`)
2.06791

0.6970889478066314` * 0.8828188584409027`
0.615403

3.8248022512288737` * 2.067911728784728`
7.90935
```

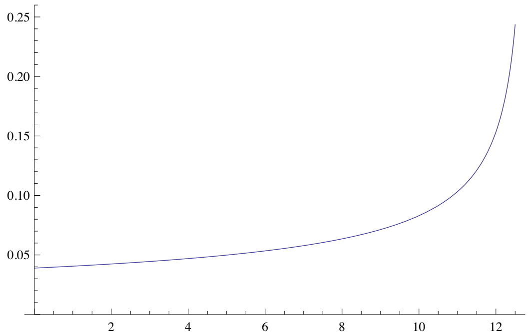
(c) $\rho_C(x) = \dfrac{1}{Tv(x)}$ (Equation 1.43). Here

$$E = \frac{1}{2}mv^2 + mgx \Rightarrow v = \sqrt{\frac{2}{m}(E - mgx)} \quad \text{and} \quad \frac{1}{2}gT^2 = h = \frac{E}{mg} \Rightarrow T = \sqrt{\frac{2E}{mg^2}}, \quad \text{so}$$

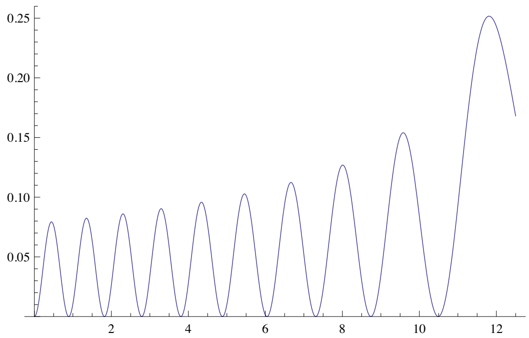
$$\rho_C(x) = \frac{mg}{2\sqrt{E(E-mgx)}} = \frac{mg}{2\sqrt{\frac{\hbar^2a^2}{2m}\epsilon\left(\frac{\hbar^2a^2}{2m}\epsilon - mgx\right)}} = \frac{1}{2}\frac{(2m^2g/\hbar^2a^2)}{\sqrt{\epsilon(\epsilon - (2m^2g/\hbar^2a^2)}} = \frac{1}{2}\frac{a}{\sqrt{\epsilon(\epsilon - a)}} \to \frac{1}{2\sqrt{\epsilon(\epsilon-1)}}.$$

For ψ_{10}, $\epsilon = 12.82877$ (Out[13] on page 64). The graphs are

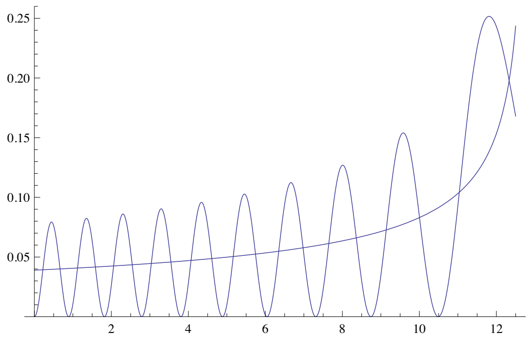
```
Plot[(4 * 12.828776752865757` (12.828776752865757` - x))^(-1/2),
 {x, 0, 12.5}, PlotRange → {0, .26}]
```



```
Plot[(Pten[x])^2, {x, 0, 12.5}, PlotRange → {0, .26}]
```



```
Show[%74, %75]
```



Comment: Well, they agree, in a kind of averaged sense.

Problem 2.60

(a)

$$E_0 = (\hbar)^n (m)^p (\alpha)^q = \left(\frac{\text{kg m}^2}{\text{s}}\right)^n (\text{kg})^p \left(\frac{\text{kg m}^4}{\text{s}^2}\right)^q = (\text{kg})^{n+p+q} (\text{m}^2)^{n+2q} (\text{s})^{-(n+2q)} = \frac{\text{kg m}^2}{\text{s}^2}.$$

So $n + p + q = 1, \quad n + 2q = 1, \quad n + 2q = 2$. The last two are incompatible. Evidently there is, on purely dimensional grounds, no possible formula for E_0.

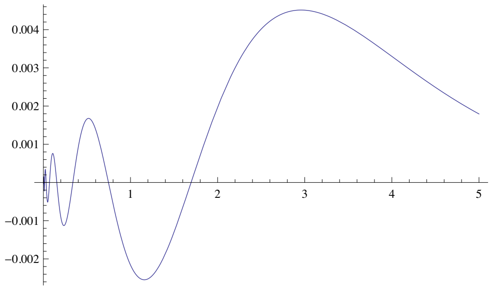
(b) Let $\psi_\lambda(x) \equiv \psi(y)$, where $y \equiv \lambda x$. Then

$$\frac{d^2\psi_\lambda(x)}{dx^2} = \frac{d^2\psi(y)}{d}y^2 \left(\frac{dy}{dx}\right)^2 = \lambda^2 \frac{d^2\psi(y)}{dy^2} = \lambda^2 \left(-\frac{\beta}{y^2}\psi(y) + \kappa^2\psi(y)\right) = -\lambda^2 \frac{\beta}{\lambda^2 x^2}\psi_\lambda(x) + \lambda^2\kappa^2\psi_\lambda(x),$$

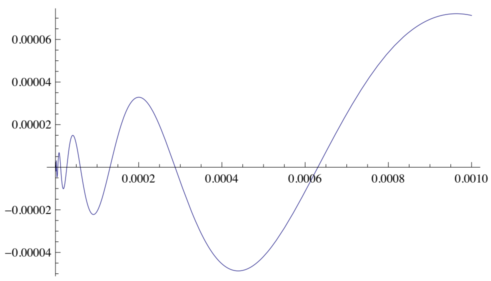
or $\dfrac{d^2\psi_\lambda(x)}{dx^2} + \dfrac{\beta}{x^2}\psi_\lambda(x) = (\lambda\kappa)^2\psi_\lambda(x) = (\kappa')^2\psi_\lambda(x)$. So $\psi_\lambda(x)$ is a solution to Equation 2.190, with $\kappa' \equiv \lambda\kappa$. The corresponding energy is given by $\left(\dfrac{-2mE'}{\hbar^2}\right) = (\kappa')^2 = \lambda^2\kappa^2 = \lambda^2\left(\dfrac{-2mE}{\hbar^2}\right) \Rightarrow E' = \lambda^2 E$.

(c)

```
f[x_] := A √x BesselK[ⅈ √(b - (1/4)), k x]

FullSimplify[f''[x] + b/x^2 f[x] - k^2 f[x] == 0]

True

h[x_] := √x BesselK[4 ⅈ, x]

Plot[h[x], {x, 0, 5}]
```




```
Plot[h[x], {x, 0, .001}]
```

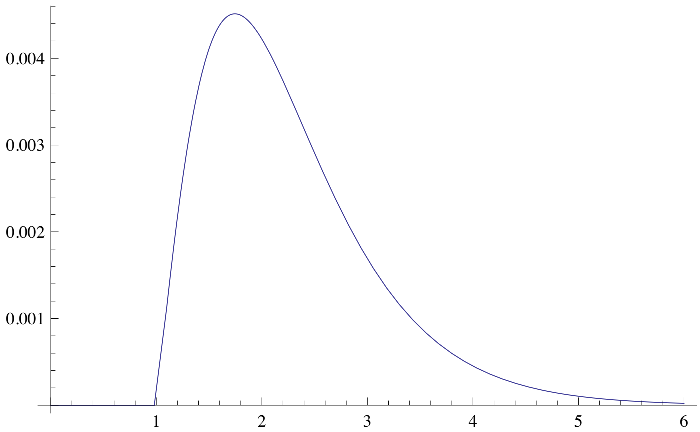



```
∫_0^∞ (f[x])^2 ⅆx

ConditionalExpression[
 (A^2 √(-1 + 4 b) π Csch[√(-1/4 + b) π]) / (4 k^2), Re[k] > 0 && -2 < Im[√(-1 + 4 b)] < 2]
```

So $\boxed{A = \kappa\sqrt{\frac{2\sinh(\pi g)}{\pi g}}.}$

(d) From the first plot on page 67 we see that the highest zero crossing occurs at $\kappa x \approx 1.7$; to find the exact value, use **FindRoot** (see print-out below). We want this to occur at $x = \epsilon = 1$, so $\kappa = 1.69541$.

```
h[x_] := √x BesselK[4 i, x]

FindRoot[h[x] == 0, {x, 1.7}]

{x → 1.69541 + 0. i}

j[x_] := If[x >= 1, h[1.6954066933505485` x], 0]

Plot[j[x], {x, 0, 6}]
```



The parameter $\beta \equiv 2m\alpha/\hbar^2$ is dimensionless, so we may as well eliminate $\hbar$ (in favor of β, m, and α) in the dimensional analysis. This leaves $E_0 = m^p\,\alpha^q\,\epsilon^r = (\text{kg})^p\left(\frac{\text{kg m}^4}{\text{s}^2}\right)^q(\text{m})^r = (\text{kg})^{p+q}(\text{m})^{4q+r}(\text{s})^{-2q} = \frac{\text{kg m}^2}{\text{s}^2}$, so $p+q = 1, 4q+r = 2,\ q = 1, \Rightarrow p = 0, r = -2$. On dimensional grounds, therefore, the expression for E_0 has to be of the form $E_0 = -\alpha/\epsilon^2$ times some function of β. As $\epsilon \to 0$ (for fixed values of m and α), $E_0 \to -\infty$, indicating once again that there is no ground state for this system.

Problem 2.61

(a) From Equation 2.197:

$$N = 1:\ j = 1:\ -\lambda\cancel{\psi_2} + (2\lambda)\psi_1 - \lambda\cancel{\psi_0} = E\psi_1 \quad \Rightarrow \quad \boxed{\mathsf{H} = (2\lambda).}$$

$$N = 2:\ \begin{cases} j = 1:\ -\lambda\psi_2 + (2\lambda)\psi_1 - \lambda\cancel{\psi_0} = E\psi_1 \\ j = 2:\ -\lambda\cancel{\psi_3} + (2\lambda)\psi_2 - \lambda\psi_1 = E\psi_2 \end{cases} \quad \Rightarrow \quad \boxed{\mathsf{H} = \begin{pmatrix} 2\lambda & -\lambda \\ -\lambda & 2\lambda \end{pmatrix}.}$$

$$N = 3:\ \begin{cases} j = 1:\ -\lambda\psi_2 + (2\lambda)\psi_1 - \lambda\cancel{\psi_0} = E\psi_1 \\ j = 2:\ -\lambda\psi_3 + (2\lambda)\psi_2 - \lambda\psi_1 = E\psi_2 \\ j = 3:\ -\lambda\cancel{\psi_4} + (2\lambda)\psi_3 - \lambda\psi_2 = E\psi_2 \end{cases} \quad \Rightarrow \quad \boxed{\mathsf{H} = \begin{pmatrix} 2\lambda & -\lambda & 0 \\ -\lambda & 2\lambda & -\lambda \\ 0 & -\lambda & 2\lambda \end{pmatrix}.}$$

(b) Denote the eigenvalues by $\tilde{E}_n$:

$N=1:$ $\boxed{\tilde{E}_1 = 2\lambda} = \dfrac{2\hbar^2}{2m(a/2)^2} = \dfrac{8\hbar^2}{2ma^2}$. The exact energies are $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$, so $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$; the agreement is not too bad: $8 \approx \pi^2 = 9.87$.

$N=2:$

$$\det\begin{pmatrix} 2\lambda-\tilde{E} & -\lambda \\ -\lambda & 2\lambda-\tilde{E}\end{pmatrix} = 0 \Rightarrow (2\lambda-\tilde{E})^2 - \lambda^2 = 0 \Rightarrow 2\lambda - \tilde{E} = \pm\lambda.$$

$\tilde{E}_1 = 2\lambda - \lambda = \boxed{\lambda} = \dfrac{\hbar^2}{2m(a/3)^2} = \dfrac{9\hbar^2}{2ma^2}$. This is better: 9 is closer to π^2 than 8 was.

$\tilde{E}_2 = 2\lambda + \lambda = \boxed{3\lambda} = \dfrac{3\hbar^2}{2m(a/3)^2} = \dfrac{27\hbar^2}{2ma^2}$. The exact answer has $4\pi^2 = 39.5$ instead of 27.

$N=3:$

$$\det\begin{pmatrix} 2\lambda-\tilde{E} & -\lambda & 0 \\ -\lambda & 2\lambda-\tilde{E} & -\lambda \\ 0 & -\lambda & 2\lambda-\tilde{E}\end{pmatrix} = 0 \Rightarrow (2\lambda-\tilde{E})^3 - 2\lambda^2(2\lambda-\tilde{E}) = 0 \Rightarrow 2\lambda-\tilde{E} = 0 \text{ or } 2\lambda - \tilde{E} = \pm\sqrt{2}\,\lambda.$$

$\tilde{E}_1 = 2\lambda - \sqrt{2}\,\lambda = \boxed{\lambda(2-\sqrt{2})} = \dfrac{(2-\sqrt{2})\hbar^2}{2m(a/4)^2} = \dfrac{16(2-\sqrt{2})\hbar^2}{2ma^2}$. This is better yet: $16(2-\sqrt{2}) = 9.37$.

$\tilde{E}_2 = \boxed{2\lambda} = \dfrac{2\hbar^2}{2m(a/4)^2} = \dfrac{32\hbar^2}{2ma^2}$. Improving: the exact answer is $4\pi^2 = 39.5$ instead of 32.

$\tilde{E}_3 = 2\lambda + \sqrt{2}\,\lambda = \boxed{\lambda(2+\sqrt{2})} = \dfrac{(2+\sqrt{2})\hbar^2}{2m(a/4)^2} = \dfrac{16(2+\sqrt{2})\hbar^2}{2ma^2}$; $16(2+\sqrt{2}) = 54.6 \approx 9\pi^2 = 88.8$.

(c)

```
h = Table[If[i == j, 2 λ, 0], {i, 10}, {j, 10}]

k = Table[If[i == j + 1, -λ, 0], {i, 10}, {j, 10}]

m = Table[If[i == j - 1, -λ, 0], {i, 10}, {j, 10}]

p = Table[h[[i, j]] + k[[i, j]] + m[[i, j]], {i, 10}, {j, 10}]

P = MatrixForm[%]
```

$$\begin{pmatrix}
2\lambda & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\lambda & 2\lambda & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\lambda & 2\lambda & -\lambda & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\lambda & 2\lambda & -\lambda & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\lambda & 2\lambda & -\lambda & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\lambda & 2\lambda & -\lambda & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\lambda & 2\lambda & -\lambda & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 2\lambda & -\lambda & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 2\lambda & -\lambda\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda & 2\lambda
\end{pmatrix}$$

```
λ = 1
1

EIG = Eigenvalues[N[p]]
{3.91899, 3.68251, 3.30972, 2.83083, 2.28463,
 1.71537, 1.16917, 0.690279, 0.317493, 0.0810141}
```

To get the eigenvalues, multiply by $\lambda = \dfrac{\hbar^2}{2m(a/11)^2} = \left(\dfrac{121}{\pi^2}\right) E_1$:

```
121 * EIG / (π^2)
{48.0462, 45.147, 40.5767, 34.7056,
 28.0092, 21.0302, 14.3339, 8.46272, 3.89242, 0.993221}
```

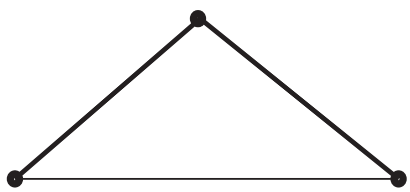
Thus the lowest five eigenvalues, in units of E_1, are $\boxed{0.9932, 3.8924, 8.4627, 14.3339, 21.0302,}$ as compared to the exact values $1, 4, 9, 16, 25$. Doing the same for $N = 100$:

```
h = Table[If[i == j, 2 λ, 0], {i, 100}, {j, 100}]

k = Table[If[i == j + 1, -λ, 0], {i, 100}, {j, 100}]

m = Table[If[i == j - 1, -λ, 0], {i, 100}, {j, 100}]

p = Table[h[[i, j]] + k[[i, j]] + m[[i, j]], {i, 100}, {j, 100}]

λ = 1

1

EIG = Eigenvalues[N[p]]

10 201 * EIG / (π^2)

{4133.31, 4130.31, 4125.32, 4118.33, 4109.36, 4098.41, 4085.5, 4070.64, 4053.84,
 4035.11, 4014.49, 3991.97, 3967.6, 3941.39, 3913.36, 3883.55, 3851.98,
 3818.69, 3783.7, 3747.04, 3708.77, 3668.9, 3627.49, 3584.57, 3540.18, 3494.36,
 3447.16, 3398.63, 3348.81, 3297.75, 3245.5, 3192.11, 3137.63, 3082.12,
 3025.62, 2968.2, 2909.9, 2850.79, 2790.92, 2730.35, 2669.14, 2607.35, 2545.03,
 2482.25, 2419.07, 2355.55, 2291.76, 2227.74, 2163.57, 2099.3, 2035.01,
 1970.74, 1906.57, 1842.55, 1778.75, 1715.24, 1652.06, 1589.28, 1526.96,
 1465.17, 1403.96, 1343.39, 1283.52, 1224.41, 1166.11, 1108.69, 1052.19,
 996.679, 942.201, 888.81, 836.559, 785.499, 735.679, 687.147, 639.95, 594.133,
 549.742, 506.819, 465.405, 425.541, 387.265, 350.614, 315.624, 282.328,
 250.76, 220.948, 192.922, 166.71, 142.336, 119.824, 99.1963, 80.4724,
 63.6704, 48.8067, 35.8956, 24.9496, 15.9794, 8.99347, 3.99871, 0.999919}
```

This time the lowest five eigenvalues are $\boxed{0.9999, 3.9987, 8.9934, 15.9794, 24.9496.}$

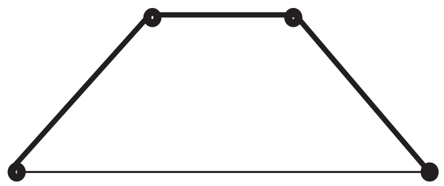
(d) Always, $\psi_0 = \psi_{N+1} = 0$; might as well set $\lambda = 1$ for this part.

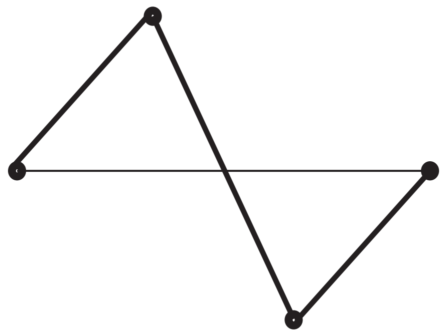
$N = 1:\ 2\psi_1 = \tilde{E}\psi_1,\ \tilde{E}_1 = 2$. Up to normalization:



$N = 2: \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \tilde{E} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \Rightarrow 2\psi_1 - \psi_2 = \tilde{E}\psi_1$ and $-\psi_1 + 2\psi_2 = \tilde{E}\psi_2$.

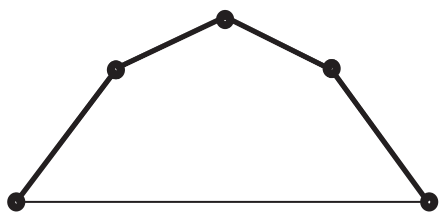
$$n = 1: \ \tilde{E}_1 = 1 \Rightarrow 2\psi_1 - \psi_2 = \psi_1 \Rightarrow \psi_2 = \psi_1$$
$$n = 2: \ \tilde{E}_2 = 3 \Rightarrow 2\psi_1 - \psi_2 = 3\psi_1 \Rightarrow \psi_2 = -\psi_1$$

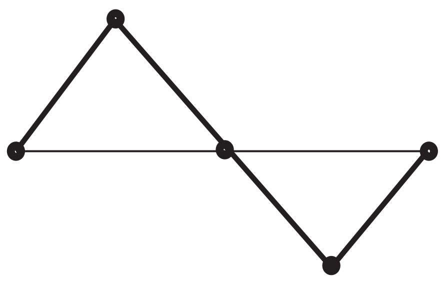


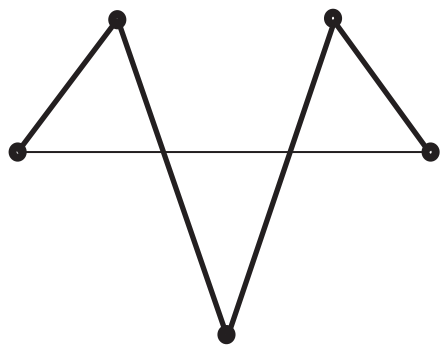


$$N = 3: \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \tilde{E} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \Rightarrow 2\psi_1 - \psi_2 = \tilde{E}\psi_1, \ -\psi_1 + 2\psi_2 - \psi_3 = \tilde{E}\psi_2, \ -\psi_2 + 2\psi_3 = \tilde{E}\psi_3.$$

$$n = 1: \ \tilde{E}_1 = 2 - \sqrt{2} \Rightarrow 2\psi_1 - \psi_2 = (2 - \sqrt{2})\psi_1 \Rightarrow \psi_2 = \sqrt{2}\psi_1;$$
$$-\psi_1 + 2\psi_2 - \psi_3 = (2 - \sqrt{2})\psi_2 \Rightarrow \psi_1 + \psi_3 = \sqrt{2}\psi_2 = 2\psi_1 \Rightarrow \psi_3 = \psi_1$$
$$n = 2: \ \tilde{E}_2 = 2 \Rightarrow 2\psi_1 - \psi_2 = 2\psi_1 \Rightarrow \psi_2 = 0; \ -\psi_1 + 2\psi_2 - \psi_3 = 2\psi_2 \Rightarrow \psi_3 = -\psi_1$$
$$n = 3: \ \tilde{E}_3 = 2 + \sqrt{2} \Rightarrow 2\psi_1 - \psi_2 = (2 + \sqrt{2})\psi_1 \Rightarrow \psi_2 = -\sqrt{2}\psi_1;$$
$$-\psi_1 + 2\psi_2 - \psi_3 = (2 + \sqrt{2})\psi_2 \Rightarrow \psi_1 + \psi_3 = -\sqrt{2}\psi_2 = 2\psi_1 \Rightarrow \psi_3 = \psi_1$$



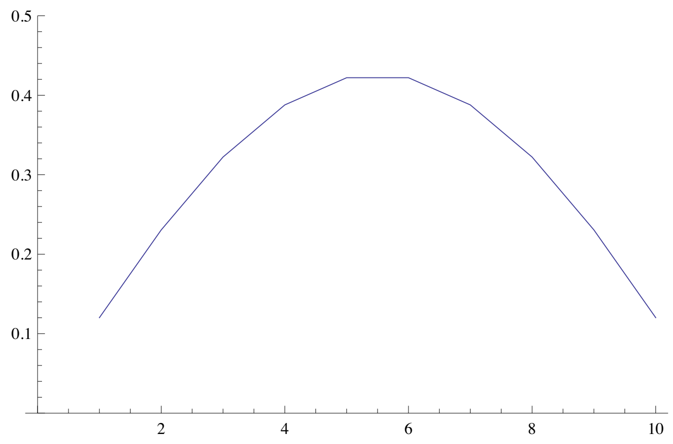




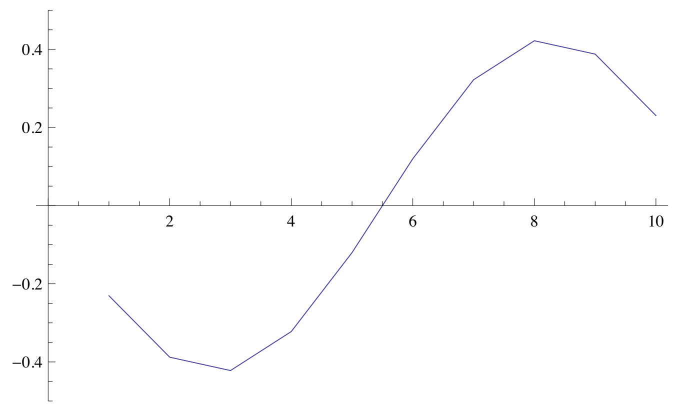
For $N = 10$ we get

```
EVE = Eigenvectors[N[p]]
```

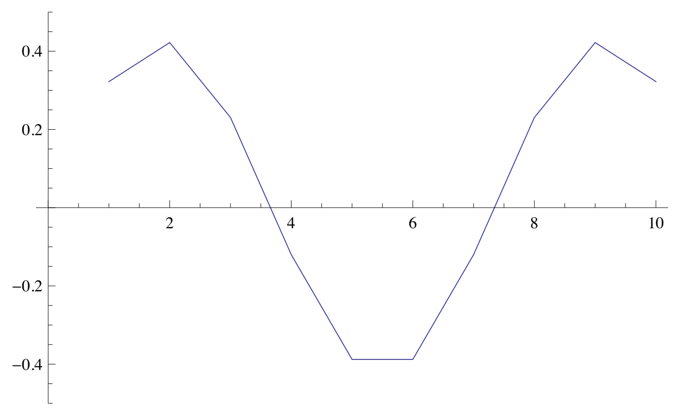
```
ListLinePlot[EVE[[10]], PlotRange → {0, 0.5}]
```




```
ListLinePlot[EVE[[9]], PlotRange → {-0.5, 0.5}]
```



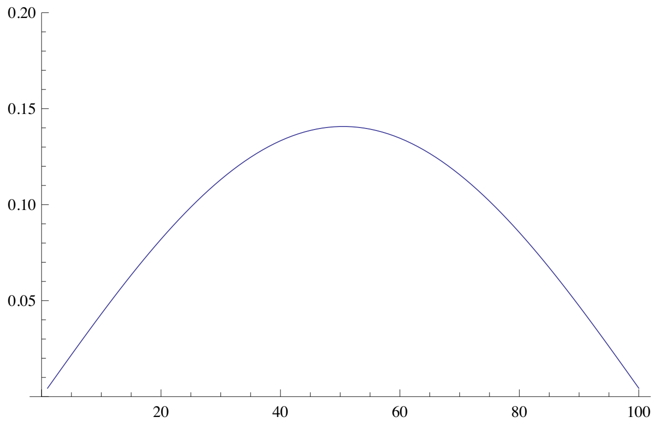

```
ListLinePlot[EVE[[8]], PlotRange → {-0.5, 0.5}]
```



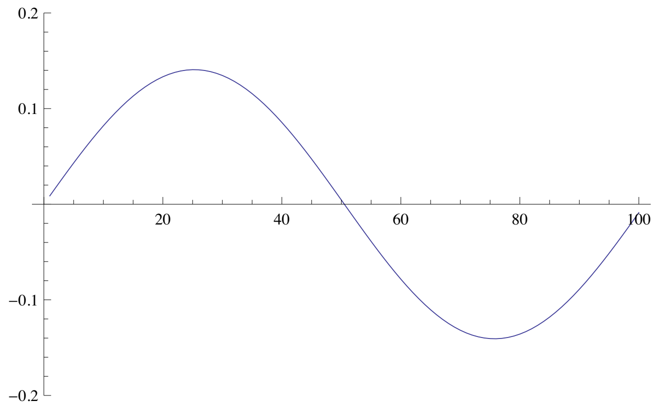
and for $N = 100$

```
EVE = Eigenvectors[N[p]]
```

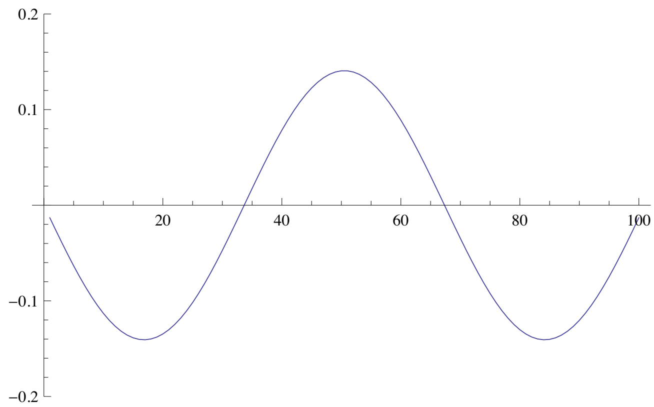
```
ListLinePlot[EVE[[100]], PlotRange -> {0, 0.2}]
```




```
ListLinePlot[EVE[[99]], PlotRange -> {-0.2, 0.2}]
```




```
ListLinePlot[EVE[[98]], PlotRange -> {-0.2, 0.2}]
```



Problem 2.62

$$\lambda = \frac{\hbar^2}{2ma^2}(N+1)^2 = (N+1)^2 V_0 \quad \text{(here } N = 100\text{);}$$
$$V_j = bV_0 \sin\left(\frac{\pi x_j}{a}\right) = bV_0 \sin\left(\frac{\pi j \Delta x}{a}\right) = bV_0 \sin\left(\frac{\pi j}{N+1}\right) \quad \text{(here } b = 500\text{);}$$
$$v_j = \frac{V_j}{\lambda} = \frac{b}{(N+1)^2} \sin\left(\frac{\pi j}{N+1}\right).$$

Factoring out λ, the diagonal elements of H are

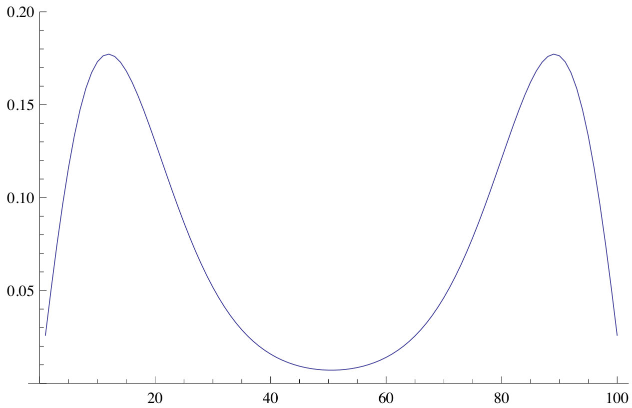
$$2 + v_j = 2 + \frac{b}{(N+1)^2} \sin\left(\frac{\pi j}{N+1}\right) = 2 + \frac{500}{10201} \sin\left(\frac{\pi j}{101}\right).$$

```
h = Table[If[i == j, 2 + (500 / 10201) Sin[π j / 101], 0], {i, 100}, {j, 100}]

k = Table[If[i == j + 1, -1, 0], {i, 100}, {j, 100}]
m = Table[If[i == j - 1, -1, 0], {i, 100}, {j, 100}]

p = Table[h[[i, j]] + k[[i, j]] + m[[i, j]], {i, 100}, {j, 100}]

EIG = Eigenvalues[N[p]]

10201 * EIG

{41255., 41158.1, 41063.4, 40968.8, 40869.4, 40758.8, 40632., 40486.7, 40322.1,
 40138.4, 39935.6, 39714.1, 39474., 39215.7, 38939.5, 38645.5, 38334.2,
 38005.7, 37660.6, 37299., 36921.3, 36528., 36119.3, 35695.8, 35257.7,
 34805.6, 34339.9, 33860.9, 33369.3, 32865.4, 32349.7, 31822.8, 31285.1,
 30737.3, 30179.7, 29613., 29037.7, 28454.3, 27863.4, 27265.6, 26661.5,
 26051.7, 25436.7, 24817.1, 24193.6, 23566.7, 22937., 22305.2, 21671.9,
 21037.6, 20403.1, 19768.8, 19135.5, 18503.7, 17874.1, 17247.2, 16623.6,
 16004., 15389., 14779.2, 14175.1, 13577.3, 12986.5, 12403.1, 11827.8,
 11261.1, 10703.5, 10155.6, 9617.98, 9091.08, 8575.44, 8071.55, 7579.91,
 7100.98, 6635.24, 6183.13, 5745.1, 5321.57, 4912.95, 4519.64, 4142.02, 3780.47,
 3435.34, 3106.97, 2795.68, 2501.8, 2225.62, 1967.43, 1727.51, 1506.15, 1303.63,
 1120.24, 956.36, 812.457, 689.191, 590.237, 499.854, 476.163, 304.8, 304.66}
```

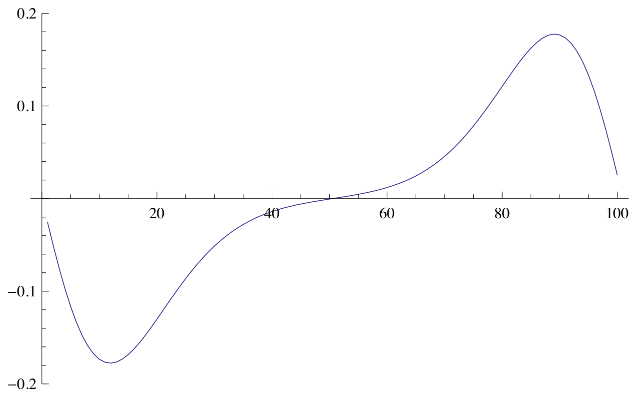
So the lowest three energies are $\boxed{304.66\ V_0,\ 304.8\ V_0, \text{ and } 476.163\, V_0.}$ Notice that the ground state is almost degenerate—essentially we have two separated wells with a huge barrier in between them, and the particle can be either in the left one or in the right one (or the even and odd linear combinations thereof).

```
EVE = Eigenvectors[N[p]]
```

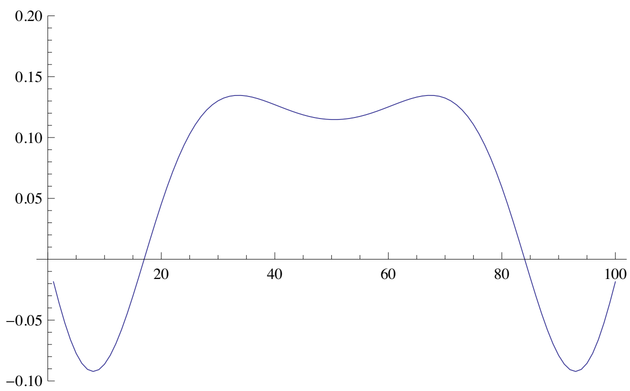
```
ListLinePlot[EVE[[100]], PlotRange → {0, 0.2}]
```




```
ListLinePlot[EVE[[99]], PlotRange → {-.2, 0.2}]
```




```
ListLinePlot[EVE[[98]], PlotRange → {-.1, 0.2}]
```



Notice that the central barrier pushes the wave function out to the wings.

Problem 2.63

(a) $-\dfrac{\partial}{\partial\beta}\ln(Z) = -\dfrac{1}{Z}\dfrac{\partial Z}{\partial\beta} = -\dfrac{1}{Z}\sum_n (-E_n)e^{-\beta E_n} = \sum_n E_n P(n).$ ✓

(b) Geometric series: $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$. Here $x = e^{-\beta\hbar\omega}$:

$$Z = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} = e^{-\beta\hbar\omega/2}\sum_{n=0}^{\infty} e^{(-\beta\hbar\omega)n} = e^{-\beta\hbar\omega/2}\frac{1}{1-e^{-\beta\hbar\omega}}. \quad ✓$$

(c)

$$\ln Z = \ln\left(e^{-\beta\hbar\omega/2}\right) - \ln\left(1 - e^{-\beta\hbar\omega}\right) = -\frac{\beta\hbar\omega}{2} - \ln\left(1 - e^{-\beta\hbar\omega}\right),$$

$$\frac{\partial}{\partial\beta}\ln Z = -\frac{\hbar\omega}{2} - \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = -\left(\frac{\hbar\omega}{2}\right)\frac{1 - e^{-\beta\hbar\omega} + 2e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \quad \Rightarrow \quad \bar{E} = \left(\frac{\hbar\omega}{2}\right)\frac{1+e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}. \quad ✓$$

(d)

$$\frac{\partial\bar{E}}{\partial T} = \frac{\partial\bar{E}}{\partial\beta}\frac{d\beta}{dT} = -\frac{1}{k_B T^2}\frac{\partial\bar{E}}{\partial\beta}.$$

$$\frac{\partial\bar{E}}{\partial\beta} = \left(\frac{\hbar\omega}{2}\right)\frac{\left(1-e^{-\beta\hbar\omega}\right)\left(-\hbar\omega e^{-\beta\hbar\omega}\right) - \left(1+e^{-\beta\hbar\omega}\right)\left(\hbar\omega e^{-\beta\hbar\omega}\right)}{\left(1-e^{-\beta\hbar\omega}\right)^2} = \left(\frac{\hbar\omega}{2}\right)\frac{-2\hbar\omega e^{-\beta\hbar\omega}}{\left(1-e^{-\beta\hbar\omega}\right)^2} = -\frac{(\hbar\omega)^2 e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}.$$

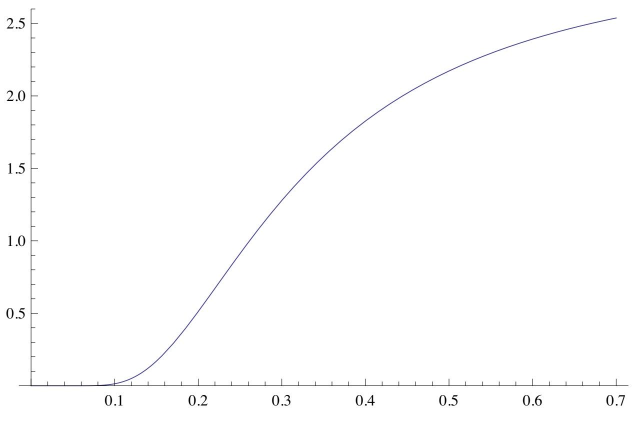
$$C = 3\left(\frac{1}{k_B T^2}\right)(\hbar\omega)^2\frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}.$$

Or, using $\beta = 1/k_B T$ and $\hbar\omega = k_B\theta_E$,

$$C = 3\left(\frac{1}{k_B T^2}\right)(k_B\theta_E)^2\frac{e^{\theta_E/T}}{\left(e^{\theta_E/T}-1\right)^2} = 3k_B\left(\frac{\theta_E}{T}\right)^2\frac{e^{\theta_E/T}}{\left(e^{\theta_E/T}-1\right)^2}. \quad ✓$$

(e)

```
Plot[3 * (x^(-2)) * Exp[1 / x] / (Exp[1 / x] - 1)^2, {x, 0, .7}, PlotRange → {0, 2.6}]
```



Incidentally, comparing the graphs suggests that $x = 0.7$ corresponds to $T = 1000\,\text{K}$, so $\theta_E = 1000/0.7\,\text{K} = 1400\,\text{K}$. Then $\hbar\omega = (1400k_B)\text{K} = (1400)(8.6\times10^{-5})\,\text{eV} = 0.12\,\text{eV}$.

Problem 2.64

(a) Plugging into the differential equation we have

$$\sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^{n-2} - \sum_{n=2}^{\infty} a_n\, n\,(n-1)\, x^n - 2\sum_{n=1}^{\infty} a_n\, n\, x^n + \ell\,(\ell+1)\sum_{n=0}^{\infty} a_n\, x^n = 0\,.$$

Reindexing the sums so that all the powers of x match we have

$$\sum_{p=0}^{\infty} a_{p+2}\,(p+2)\,(p+1)\, x^p - \sum_{p=2}^{\infty} a_p\, p\,(p-1)\, x^p - 2\sum_{p=1}^{\infty} a_p\, p\, x^p + \ell\,(\ell+1)\sum_{p=0}^{\infty} a_p\, x^p = 0\,.$$

We can then combine the sums (extending the second and third sums to begin at $p = 0$) to get

$$\sum_{p=0}^{\infty} \left\{(p+2)\,(p+1)\; a_{p+2} - \left[p\,(p-1) + 2\,p - \ell\,(\ell+1)\right]\, a_p\right\}\, x^p = 0\,.$$

From this we read off the recursion relation

$$\boxed{a_{p+2} = \frac{p\,(p+1) - \ell\,(\ell+1)}{(p+2)\,(p+1)}\, a_p\,.}$$

(b) For large values of p,

$$a_{p+2} \approx \frac{p}{p+2}\, a_p\,.$$

with the (approximate) solution

$$a_p \approx \frac{C}{p}\,.$$

and this gives the behavior

$$f(x) \approx C\sum \frac{1}{p}\, x^p \approx \log\left(\frac{1}{1-x}\right)$$

which diverges at $x = 1$. (There will be finite corrections coming from the low values of p, but these cannot fix the divergence.)

(c) For $\ell = 0$ we need the even function ($a_1 = 0$) and the recursion relation gives $a_2 = 0$ so that

$$P_0(x) = a_0\,.$$

For $\ell = 1$ we need the odd function ($a_0 = 0$) and the recursion relation gives $a_3 = 0$ so that

$$P_1(x) = a_1\, x\,.$$

For $\ell = 2$ we need the even function again and the recursion relation gives $a_2 = -3\,a_0$ and $a_4 = 0$ so that

$$P_2(x) = a_0\left(1 - 3\,x^2\right)\,.$$

For $\ell = 3$ we need the odd function again and the recursion relation gives $a_3 = -5/3\,a_1$ and $a_5 = 0$ so that

$$P_3(x) = a_1\left(x - \frac{5}{3}\,x^3\right)\,.$$