Chapter 2

Signal and Linear System Analysis

2.1 Problem Solutions

Problem 2.1

a. For the single-sided spectra, write the signal as

$$\begin{aligned} x_1(t) &= 10\cos(4\pi t + \pi/8) + 6\sin(8\pi t + 3\pi/4) \\ &= 10\cos(4\pi t + \pi/8) + 6\cos(8\pi t + 3\pi/4 - \pi/2) \\ &= 10\cos(4\pi t + \pi/8) + 6\cos(8\pi t + \pi/4) \\ &= \mathrm{Re}\left[10e^{j(4\pi t + \pi/8)} + 6e^{j(8\pi t + \pi/4)}\right] \end{aligned}$$

For the double-sided spectra, write the signal in terms of complex exponentials using Euler's theorem:

$$\begin{aligned} x_1(t) &= 5\exp[j(4\pi t + \pi/8)] + 5\exp[-j(4\pi t + \pi/8)] \\ &\quad +3\exp[j(8\pi t + 3\pi/4)] + 3\exp[-j(8\pi t + 3\pi/4)] \end{aligned}$$

The spectra are plotted in Fig. 2.1.

b. Write the given signal as

$$x_2(t) = \mathrm{Re}\left[8e^{j(2\pi t + \pi/3)} + 4e^{j(6\pi t + \pi/4)}\right]$$

to plot the single-sided spectra. For the double-side spectra, write it as

$$x_2(t) = 4e^{j(2\pi t + \pi/3)} + 4e^{-j(2\pi t + \pi/3)} + 2e^{j(6\pi t + \pi/4)} + 2e^{-j(6\pi t + \pi/4)}$$

The spectra are plotted in Fig. 2.2.

c. Change the sines to cosines by subtracting $\pi/2$ from their arguments to get

$$\begin{aligned} x_3(t) &= 2\cos(4\pi t + \pi/8 - \pi/2) + 12\cos(10\pi t - \pi/2) \\ &= 2\cos(4\pi t - 3\pi/8) + 12\cos(10\pi t - \pi/2) \\ &= \text{Re}\left[2e^{j(4\pi t - 3\pi/8)} + 12e^{j(10\pi t - \pi/2)}\right] \\ &= e^{j(4\pi t - 3\pi/8)} + e^{-j(4\pi t - 3\pi/8)} + 6e^{j(10\pi t - \pi/2)} + 6e^{-j(10\pi t - \pi/2)} \end{aligned}$$

Spectral plots are given in Fig. 2.3.

d. Use a trig identity to write

$$3\sin(18\pi t + \pi/2) = 3\cos(18\pi t)$$

and get

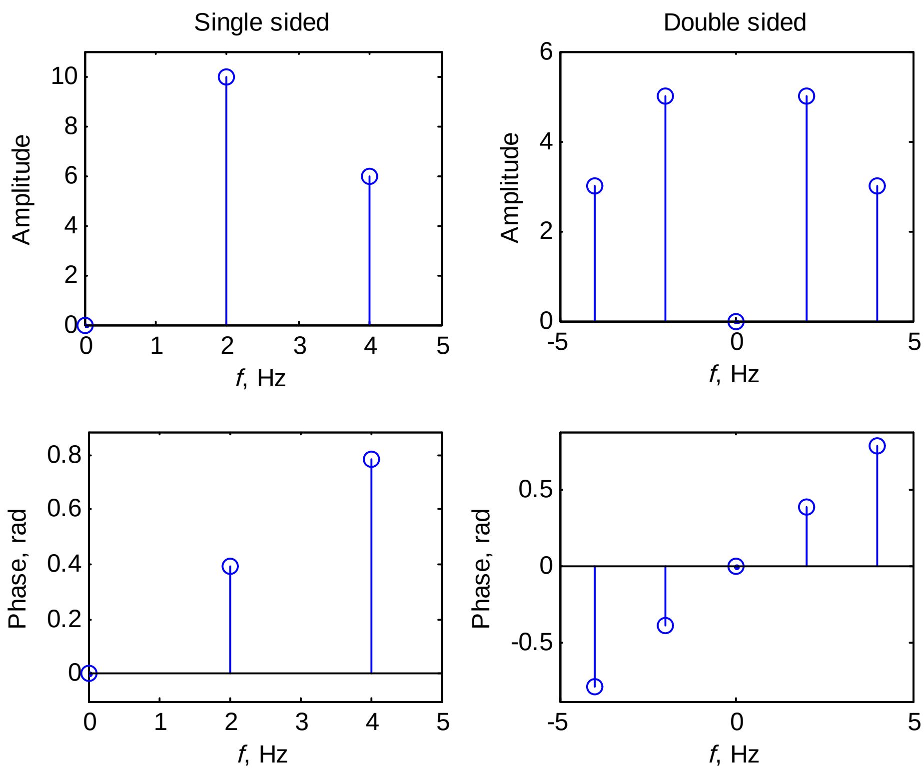
$$\begin{aligned} x_4(t) &= 2\cos(7\pi t + \pi/4) + 3\cos(18\pi t) \\ &= \text{Re}\left[2e^{j(7\pi t + \pi/4)} + 3e^{j18\pi t}\right] \\ &= e^{j(7\pi t + \pi/4)} + e^{-j(7\pi t + \pi/4)} + 1.5e^{j18\pi t} + 1.5e^{-j18\pi t} \end{aligned}$$

From this it is seen that the singe-sided amplitude spectrum consists of lines of amplitudes 2 and 3 at frequencies of 3.5 and 9 Hz, respectively, and the phase spectrum consists of a line of height $\pi/4$ at 3.5 Hz. The double-sided amplitude spectrum consists of lines of amplitudes 1, 1, 1.5, and 1.5 at frequencies of 3.5, -3.5, 9, and -9 Hz, respectively. The double-sided phase spectrum consists of lines of heights $\pi/4$ and $-\pi/4$ at frequencies 3.5 Hz and -3.5 Hz, respectively.

e. Use $\sin(2\pi t) = \cos(2\pi t - \pi/2)$ to write

$$\begin{aligned} x_5(t) &= 5\cos(2\pi t - \pi/2) + 4\cos(5\pi t + \pi/4) \\ &= \text{Re}\left[5e^{j(2\pi t - \pi/2)} + 4e^{j(5\pi t + \pi/4)}\right] \\ &= 2.5e^{j(2\pi t - \pi/2)} + 2.5e^{-j(2\pi t - \pi/2)} + 2e^{j(5\pi t + \pi/4)} + 2e^{-j(5\pi t + \pi/4)} \end{aligned}$$

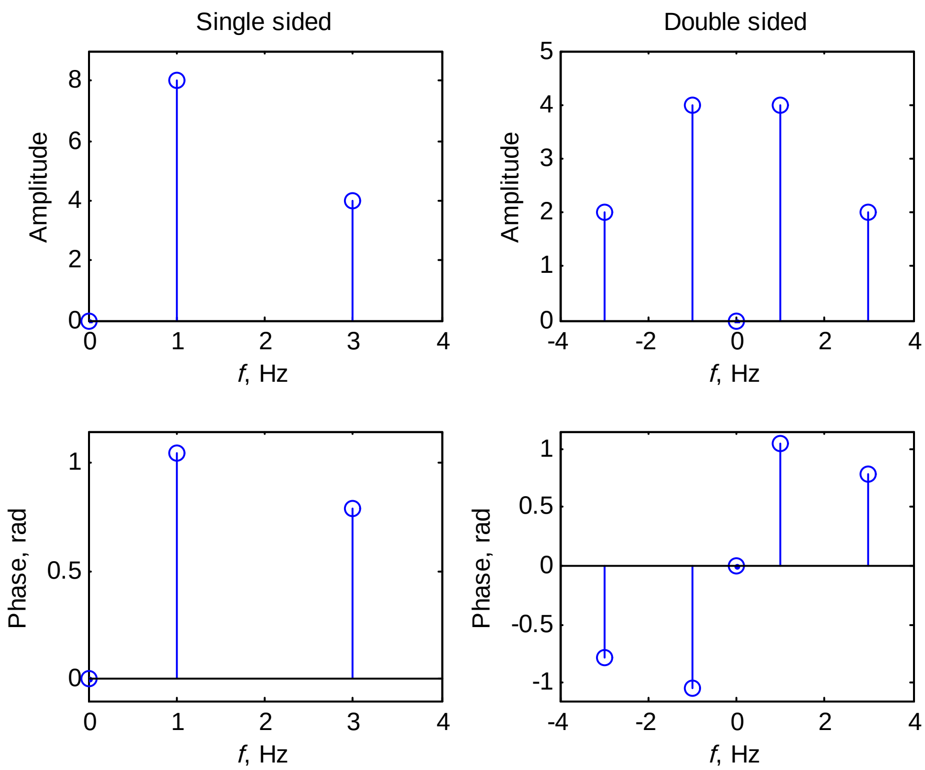
From this it is seen that the singe-sided amplitude spectrum consists of lines of amplitudes 5 and 4 at frequencies of 1 and 2.5 Hz, respectively, and the phase spectrum consists of lines of heights $-\pi/2$ and $\pi/4$ at 1 and 2.5 Hz, respectively. The double-sided amplitude spectrum consists of lines of amplitudes 2.5, 2.5, 2, and 2 at frequencies of 1, -1, 2.5, and -2.5 Hz, respectively. The double-sided phase spectrum consists of lines of heights $-\pi/2$, $\pi/2$, $\pi/4$, and $-\pi/4$ at frequencies of 1, -1, 2.5, and -2.5 Hz, respectively.

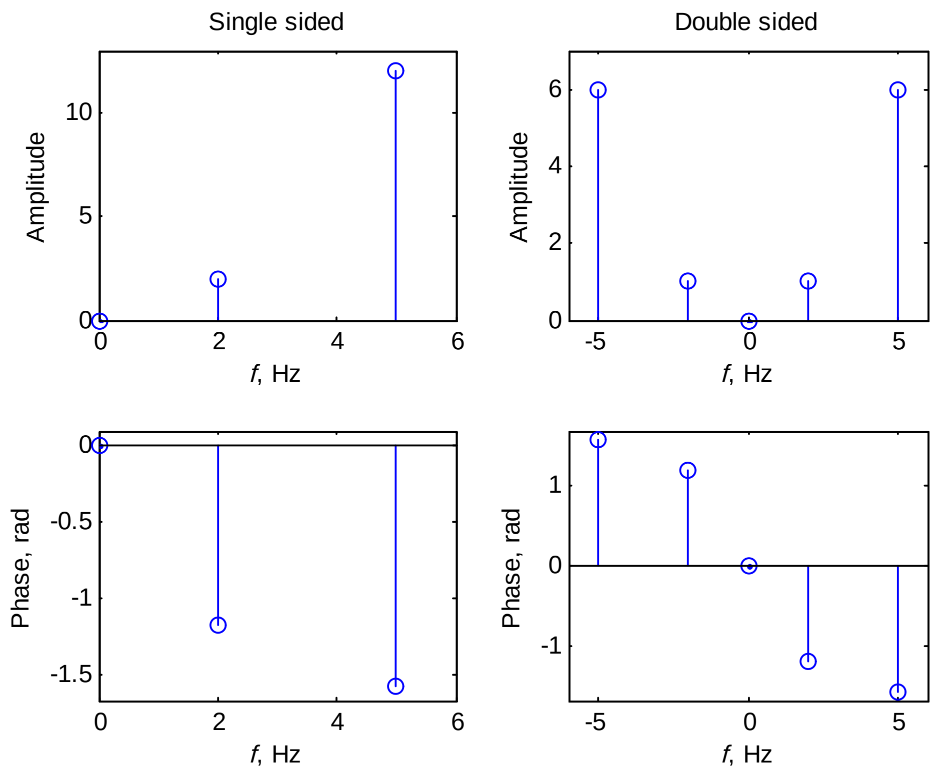


f. Use $\sin(10\pi t + \pi/6) = \cos(10\pi t + \pi/6 - \pi/2) = \cos(10\pi t - \pi/3)$ to write

$$
\begin{aligned}
x_6(t) &= 3\cos(4\pi t + \pi/8) + 4\cos(10\pi t - \pi/3) \\
&= \text{Re}\left[3e^{j(4\pi t + \pi/8)} + 4e^{j(10\pi t - \pi/3)}\right] \\
&= 1.5e^{j(4\pi t + \pi/8)} + 1.5e^{-j(4\pi t + \pi/8)} + 2e^{j10\pi t - \pi/3)} + 2e^{-j(10\pi t - \pi/3)}
\end{aligned}
$$

From this it is seen that the singe-sided amplitude spectrum consists of lines of amplitudes 3 and 4 at frequencies of 2 and 5 Hz, respectively, and the phase spectrum consists of lines of heights $\pi/8$ and $-\pi/3$ at 2 and 5 Hz, respectively. The double-sided amplitude spectrum consists of lines of amplitudes 1.5, 1.5, 2, and 2 at frequencies of 2, -2, 5, and -5 Hz, respectively. The double-sided phase spectrum consists of lines of heights $\pi/8$, $-\pi/8$, $-\pi/3$, and $\pi/3$ at frequencies of 2, -2, 5, and -5 Hz, respectively.


Single sided
Double sided
Amplitude
Amplitude
f, Hz
f, Hz
Phase, rad
Phase, rad
f, Hz
f, Hz


Single sided
Double sided
Amplitude
Phase, rad
f, Hz
0
2
4
6
10
5
-5
1
-1
-0.5
-1.5

Problem 2.2
By noting the amplitudes and phases of the various frequency components from the plots, the result is

$$
\begin{aligned}
x(t) &= 4e^{j(8\pi t+\pi/2)} + 4e^{-j(8\pi t+\pi/2)} + 2e^{j(4\pi t-\pi/4)} + 2e^{-j(4\pi t-\pi/4)} \\
&= 8\cos(8\pi t+\pi/2) + 4\cos(4\pi t-\pi/4) \\
&= -8\sin(8\pi t) + 4\cos(4\pi t-\pi/4)
\end{aligned}
$$

Problem 2.3

a. Not periodic because $f_1 = 1/\pi$ Hz and $f_2 = 3$ Hz are not commensurable.

b. Periodic. To find the period, note that

$$\frac{6\pi}{2\pi} = 3 = n_1 f_0 \text{ and } \frac{30\pi}{2\pi} = 15 = n_2 f_0$$

Therefore

$$\frac{15}{3} = \frac{n_2}{n_1}$$

Hence, take $n_1 = 1$, $n_2 = 5$, and $f_0 = 3$ Hz (we want the largest possible value for f_0 with n_1 and n_2 integer-valued).

c. Periodic. Using a similar procedure as used in (b), we find that $n_1 = 4$, $n_2 = 21$, and $f_0 = 0.5$ Hz.

d. Periodic. Using a similar procedure as used in (b), we find that $n_1 = 4$, $n_2 = 7$, $n_3 = 11$, and $f_0 = 0.5$ Hz.

e. Periodic. We find that $n_1 = 17$, $n_2 = 18$, and $f_0 = 0.5$ Hz.

f. Periodic. We find that $n_1 = 2$, $n_2 = 3$, and $f_0 = 0.5$ Hz.

g. Periodic. We find that $n_1 = 7$, $n_2 = 11$, and $f_0 = 0.5$ Hz.

h. Not periodic. The frequencies of the separate terms are incommensurable.

i. Periodic. We find that $n_1 = 19$, $n_2 = 21$, and $f_0 = 0.5$ Hz.

j. Periodic. We find that $n_1 = 6$, $n_2 = 7$, and $f_0 = 0.5$ Hz.

Problem 2.4

a. The single-sided amplitude spectrum consists of a single line of amplitude 5 at 6 Hz and the phase spectrum consists of a single line of height $-\pi/6$ rad at 6 Hz. The double-sided amplitude spectrum consists of lines of amplitude 2.5 at frequencies ± 6 Hz. The double -sided phase spectrum consists of a line of height $\pi/6$ at -6 Hz and a line of height $-\pi/6$ at 6 Hz.

b. Write the signal as

$$x_2(t) = 3\cos(12\pi t - \pi/2) + 4\cos(16\pi t)$$

From this it is seen that the single-sided amplitude spectrum consists of lines of heights 3 and 4 at frequencies 6 and 8 Hz, respectively, and the single-sided phase spectrum consists of a line of height $-\pi/2$ radians at frequency 6 Hz (the phase at 8 Hz is 0). The double-sided amplitude spectrum consists of lines of height 1.5 and 2 at frequencies of 6 and 8 Hz, respectively, and lines of height 1.5 and 2 at frequencies -6 and -8 Hz, respectively. The double-sided phase spectrum consists of a line of height $-\pi/2$ radians at frequency 6 Hz and a line of height $\pi/2$ radians at frequency -6 Hz.

c. Use the trig identity $\cos x \cos y = 0.5\cos(x+y) + 0.5\cos(x-y)$ to write

$$x_3(t) = 2\cos 20\pi t + 2\cos 4\pi t$$

From this we see that the single-sided amplitude spectrum consists of lines of height 2 at 2 and 10 Hz, and the single-sided phase spectrum is 0 at these frequencies. The double-sided amplitude spectrum consists of lines of height 1 at frequencies of -10, -2, 2, and 10 Hz. The double-sided phase spectrum is 0.

d. Use trig identies to get

$$\begin{aligned}
x_4(t) &= 4\sin(2\pi t)\left[1+\cos(10\pi t)\right] \\
&= 4\sin(2\pi t) - 2\sin(8\pi t + \pi) + 2\sin(12\pi t) \\
&= 4\cos(2\pi t - \pi/2) + 2\cos(8\pi t + \pi/2) + 2\cos(12\pi t - \pi/2) \\
&= \mathrm{Re}\left[4e^{j(2\pi t - \pi/2)} + 2e^{j(8\pi t + \pi/2)} + 2e^{j(12\pi t - \pi/2)}\right] \\
&= 2e^{j(2\pi t - \pi/2)} + 2e^{-j(2\pi t - \pi/2)} + e^{j(8\pi t + \pi/2)} + e^{-j(8\pi t + \pi/2)} + e^{j(12\pi t - \pi/2)} + e^{-j(12\pi t - \pi/2)}
\end{aligned}$$

From this we see that the single-sided amplitude spectrum consists of lines of heights 4, 2, and 2 at frequencies 1, 4, and 6 Hz, respectively and the single-sided phase spectrum is $-\pi/2$ radians at 1 and 6 Hz and $\pi/2$ radians at 4 Hz. The double-sided amplitude spectrum consists of lines of height 2 at frequencies of 1 and -1 Hz and of height 1 at frequencies of 4, -4, 6, and -6 Hz. The double-sided phase spectrum is $\pi/2$ radians at -1, 4, and -6 Hz and $-\pi/2$ radians at 1, -4, and 6 Hz.

e. Clearly, from the form of the cosine sum, the single-sided amplitude spectrum has lines of heights 1 and 7 at frequencies of 3 and 15 Hz, respectively. The single-sided phase spectrum is zero. The double-sided amplitude spectrum has lines of heights 0.5, 0.5, 3.5, and 3.5 at frequencies of 3, -3, 15, and -15 Hz, respectfully. The double-sided phase spectrum is zero.

f. The single-sided amplitude spectrum has lines of heights 1 and 9 at frequencies of 2 and 10.5 Hz, respectively. The single-sided phase spectrum is $-\pi/2$ radians at 10.5 Hz and 0 otherwise. The double-sided amplitude spectrum has lines of heights 0.5, 0.5, 4.5, and 4.5 at frequencies of 2, -2, 10.5, and -10.5 Hz, respectfully. The double-sided phase spectrum is $\pi/2$ radians at -10.5 Hz and $-\pi/2$ radians at 10.5 Hz and 0 otherwise.

g. Convert the sine to a cosine by subtracting $\pi/2$ from its argument. It then follows that the single-sided amplitude spectrum is 2, 1, and 6 at frequencies of 2, 3, and 8.5 Hz and 0 otherwise. The single-sided phase spectrum is $-\pi/2$ radians at 8.5 Hz and 0 otherwise. The double-sided amplitude spectrum is 1, 1, 0.5, 0.5, 3, and 3 at frequencies of -2, 2, -3, 3 -8.5, and 8.5 Hz, respectively, and 0 otherwise. The double-sided phase spectrum is $\pi/2$ radians at a frequency of -8.5 Hz and $-\pi/2$ radians at a frequency of 8.5 Hz. It is 0 otherwise.

Problem 2.5

a. This function has area

$$\begin{aligned} \text{Area} &= \int_{-\infty}^{\infty} \epsilon^{-1} \left[\frac{\sin(\pi t/\epsilon)}{(\pi t/\epsilon)}\right]^2 dt \\ &= \int_{-\infty}^{\infty} \left[\frac{\sin(\pi u)}{(\pi u)}\right]^2 du = 1 \end{aligned}$$

where a tabulated integral has been used for $\text{sinc}^2 u$. A sketch shows that no matter how small ϵ is, the area is still 1. With $\epsilon \to 0$, the central lobe of the function becomes narrower and higher. Thus, in the limit, it approximates a delta function.

b. The area for the function is

$$\text{Area} = \int_{-\infty}^{\infty} \frac{1}{\epsilon} \exp(-t/\epsilon) u(t)\, dt = \int_{0}^{\infty} \exp(-u) du = 1$$

A sketch shows that no matter how small ϵ is, the area is still 1. With $\epsilon \to 0$, the function becomes narrower and higher. Thus, in the limit, it approximates a delta function.

c. Area $= \int_{-\epsilon}^{\epsilon} \frac{1}{\epsilon} (1 - |t| / \epsilon)\, dt = \int_{-1}^{1} \Lambda(t)\, dt = 1$. As $\epsilon \to 0$, the function becomes narrower and higher, so it approximates a delta function in the limit.

Problem 2.6

a. Make use of the formula $\delta(at) = \frac{1}{|a|} \delta(t)$ to write $\delta(2t - 5) = \delta[2(t - 5/2)] = \frac{1}{2} \delta\left(t - \frac{5}{2}\right)$ and use the sifting property of the δ-function to get

$$I_a = \frac{1}{2} \left(\frac{5}{2}\right)^2 + \frac{1}{2} \exp\left[-2\left(\frac{5}{2}\right)\right] = \frac{25}{8} + \frac{1}{2} \exp[-5] = 3.1284$$

b. Impulses at -10, -5, 0, 5, 10 are included in the integral. Use the sifting property after writing the expression as the sum of five integrals to get

$$I_b = (-10)^2 + 1 + (-5)^2 + 1 + 0^2 + 1 + 5^2 + 1 + 10^2 + 1 = 255$$

c. Matching coeffi cients of like derivatives ofδ-functions on either side of the equation gives $A = 5$, $B = 10$, and $C = 3$.

d. Use $\delta(at) = \frac{1}{|a|} \delta(t)$ to write $\delta(4t + 3) = \frac{1}{4} \delta\left(t + \frac{3}{4}\right)$. The integral then becomes $I = \frac{1}{4} \left[e^{-4\pi(-3/4)} + \tan\left(10\pi \times \left(-\frac{3}{4}\right)\right)\right] = \frac{1}{4} \left[e^{3\pi} + \tan(-7.5\pi)\right] = -9.277 \times 10^{13}$.

e. Use property 5 of the unit impulse function to get

$$\begin{aligned} I_e &= (-1)^2 \frac{d^2}{dt^2} \left[\cos 5\pi t + e^{-3t}\right]_{t=2} = \frac{d}{dt} \left[-5\pi \sin 5\pi t - 3e^{-3t}\right]_{t=2} \\ &= \left[-(5\pi)^2 \cos 5\pi t + 9e^{-3t}\right]_{t=2} = -(5\pi)^2 \cos 10\pi + 9e^{-6} = -246.73 \end{aligned}$$

Problem 2.7

(a), (c), and (e) are periodic. Their periods are 2 s (fundamental frequency of 0.5 Hz), 2 s, and 3 s, respectively. The waveform of part (c) is a periodic train of triangles, each 2 units wide, extending from -∞ to ∞ spaced by 2 s ((b) is similar except that it is zero for $t < -1$ thus making it aperiodic). Waveform (d) is aperiodic because the frequencies of its two components are incommensurable. The waveform of part (e) is a doubly-infinite train of square pulses, each of which is one unit high and one unit wide, centered at $\cdots$, -6, -3, 0, 3, 6, $\cdots$. Waveform (f) is identical to (e) for $t \geq -1/2$ but 0 for $t < -1/2$ thereby making it aperiodic.

Problem 2.8

a. The result is

$$x(t) = \cos(6\pi t) + 2\cos(10\pi t - \pi/2) = \mathrm{Re}\left(e^{j6\pi t}\right) + \mathrm{Re}\left(2e^{j(10\pi t - \pi/2)}\right) = \mathrm{Re}\left[e^{j6\pi t} + 2e^{j(10\pi t - \pi/2)}\right]$$

b. The result is

$$x(t) = e^{-j(10\pi t - \pi/2)} + \frac{1}{2}e^{-j6\pi t} + \frac{1}{2}e^{j6\pi t} + e^{j(10\pi t - \pi/2)}$$

c. The single-sided amplitude spectrum consists of lines of height 1 and 2 at frequencies of 3 and 5 Hz, respectively. The single-sided phase spectrum consists of a line of height $-\pi/2$ at frequency 5 Hz. The double-sided amplitude spectrum consists of lines of height 1, 1/2, 1/2, and 1 at frequencies of -5, -3, 3, and 5 Hz, respectively. The double-sided phase spectrum consists of lines of height $\pi/2$ and $-\pi/2$ at frequencies of -5 and 5 Hz, respectively.

Problem 2.9

a. Power. Since it is a periodic signal, we obtain

$$P_1 = \frac{1}{T_0}\int_0^{T_0} 4\cos^2(4\pi t + 2\pi/3)\,dt = \frac{1}{T_0}\int_0^{T_0} 2\left[1 + \cos(8\pi t + 4\pi/3)\right]dt = 2 \text{ W}$$

where $T_0 = 1/2$ s is the period. The cosine in the above integral integrates to zero because the interval of integratation is two periods.

b. Energy. The energy is

$$E_2 = \int_{-\infty}^{\infty} e^{-2\alpha t}u^2(t)dt = \int_0^{\infty} e^{-2\alpha t}dt = \frac{1}{2\alpha} \text{ J}$$

c. Energy. The energy is

$$E_3 = \int_{-\infty}^{\infty} e^{2\alpha t} u^2(-t) dt = \int_{-\infty}^{0} e^{2\alpha t} dt = \frac{1}{2\alpha} \text{ J}$$

d. Energy. The energy is

$$\begin{aligned} E_4 &= \lim_{T\to\infty} \int_{-T}^{T} \frac{dt}{(\alpha^2 + t^2)} = \lim_{T\to\infty} \frac{1}{\alpha^2} \int_{-T}^{T} \frac{dt}{\left(1 + (t/\alpha)^2\right)} \\ &= \lim_{T\to\infty} \frac{1}{\alpha} \tan^{-1} \left[\frac{t}{\alpha}\right]_{-T}^{T} = \lim_{T\to\infty} \frac{1}{\alpha} \left[\tan^{-1}(T/\alpha) - \tan^{-1}(-T/\alpha)\right] \\ &= \frac{1}{\alpha} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{\alpha} \text{ J} \end{aligned}$$

e. Energy. Since it is the sum of $x_2(t)$ and $x_3(t)$, its energy is the sum of the energies of these two signals, or $E_5 = 1/\alpha$ J.

f. Energy. The energy is

$$\begin{aligned} E_6 &= \lim_{T\to\infty} \int_{-T}^{T} \left[e^{-\alpha t} u(t) - e^{-\alpha(t-1)} u(t-1)\right]^2 dt \\ &= \lim_{T\to\infty} \int_{-T}^{T} \left[e^{-2\alpha t} u^2(t) - e^{-\alpha t} e^{-\alpha(t-1)} u(t) u(t-1) + e^{-2\alpha(t-1)} u^2(t-1)\right] dt \\ &= \lim_{T\to\infty} \left\{ \int_0^T e^{-2\alpha t} dt - e^{-\alpha} \int_1^T e^{-2\alpha(t-1)} dt + \int_1^T e^{-2\alpha(t-1)} dt \right\} \\ &= \lim_{T\to\infty} \left\{ \int_0^T e^{-2\alpha t} dt - e^{-\alpha} \int_0^{T-1} e^{-2\alpha t'} dt' + \int_0^{T-1} e^{-2\alpha t'} dt' \right\} \\ &= \lim_{T\to\infty} \left\{ -\frac{e^{-2\alpha t}}{2\alpha}\bigg|_0^T + e^{-\alpha} \frac{e^{-2\alpha t'}}{2\alpha}\bigg|_0^{T-1} - \frac{e^{-2\alpha t'}}{2\alpha}\bigg|_0^{T-1} \right\} \\ &= \frac{1}{2\alpha} - \frac{e^{-\alpha}}{2\alpha} + \frac{1}{2\alpha} = \frac{1}{\alpha}\left(1 - \frac{1}{2} e^{-\alpha}\right) \text{ J} \end{aligned}$$

Problem 2.10

a. Power. Since the signal is periodic with period $2\pi/\omega$, we have

$$P = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} A^2 \left|\sin\left(\omega t + \theta\right)\right|^2 dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{A^2}{2}\left\{1 - \cos\left[2\left(\omega t + \theta\right)\right]\right\} dt = \frac{A^2}{2} \text{ W}$$

b. Neither. The energy calculation gives

$$E = \lim_{T\to\infty}\int_{-T}^{T} \frac{\left(A\tau\right)^2 dt}{\sqrt{\tau + jt}\sqrt{\tau - jt}} dt = \lim_{T\to\infty}\int_{-T}^{T} \frac{\left(A\tau\right)^2 dt}{\sqrt{\tau^2 + t^2}} dt \to \infty$$

The power calculation gives

$$P = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \frac{\left(A\tau\right)^2 dt}{\sqrt{\tau^2 + t^2}} dt = \lim_{T\to\infty}\frac{\left(A\tau\right)^2}{2T}\ln\left(\frac{1 + \sqrt{1 + T^2/\tau^2}}{-1 + \sqrt{1 + T^2/\tau^2}}\right) = 0 \text{ W}$$

c. Energy:

$$E = \int_0^{\infty} A^2 t^2 \exp\left(-2t/\tau\right) dt = \frac{1}{8}A^2\tau\sqrt{\frac{\pi\tau}{2}} \text{ W (use a table of integrals)}$$

d. Energy: This is a "top hat" pulse which is height 2 for $|t| \leq \tau/2$, height 1 for $\tau/2 < |t| \leq \tau$, and 0 everywhere else. Making use of the even symmetry about $t = 0$, the energy is

$$E = 2\left(\int_0^{\tau/2} 2^2 dt + \int_{\tau/2}^{\tau} 1^2 dt\right) = 5\tau \text{ J}$$

e. Energy. The signal is a "house" two units wide and one unit up to the eves with a equilateral triangle for a roof. Because of symmetry, the energy calculation need be carried out for positive t and doubled. The calculation is

$$E = 2\int_0^1 \left(2 - t\right)^2 dt = -\frac{2}{3}\left(2 - t\right)^3\Big|_0^1 = -\frac{2}{3} + \frac{2 \times 8}{3} = \frac{14}{3} \text{ J}$$

f. Power. Since the two terms are harmonically related, we may add their respective powers and get

$$P = \frac{A^2}{2} + \frac{B^2}{2} \text{ W}$$

Problem 2.11

a. Using the fact that the power contained in a sinusoid is its amplitude squared divided by 2, we get

$$P = \frac{2^2}{2} = 2 \text{ W}$$

b. This is a periodic train of "box cars" 3 units high, 2 units wide, and occurring every 4 units (period of 4 seconds). The power calculation is

$$P = \frac{1}{4}\int_{-1}^{1} 3^2 dt = \frac{3^2 \times 2}{4} = 4.5 \text{ W}$$

c. This is a train of triangles 1 unit high, 4 s wide, and occuring every 6 s. Using the waveform period centered at 0, the power calculation is

$$P = \frac{1}{6}\int_{-2}^{2}\left(1 - \frac{t}{2}\right)^2 dt = -\frac{1}{6}\frac{2}{3}\left(1 - \frac{t}{2}\right)^3\Bigg|_0^2 = \frac{2}{9} \text{ W}$$

d. This is a train of "houses" each of which is 2 s wide, 1 unit high to the eves, with an isoceles triangle on top for the roof. They are separated by 4 s (the period). Using the even symmetry of each house, the power calculation is

$$P_d = \frac{2}{4}\int_0^1 (2-t)^2\, dt = -\frac{1}{2}\frac{(2-t)^3}{3}\Bigg|_0^1 = -\frac{1}{2}\left(\frac{1}{3} - \frac{2^3}{3}\right) = \frac{7}{6} \text{ W}$$

Problem 2.12

a. The energy is

$$\begin{aligned} E &= \int_0^\infty \left|6e^{(-3+j4\pi)t}\right|^2 dt = 36\int_0^\infty e^{(-3+j4\pi)t}e^{(-3-j4\pi)t}dt \\ &= 36\int_0^\infty e^{-6t}dt = -36\,\frac{e^{-6t}}{6}\Bigg|_0^\infty = 6 \text{ J} \end{aligned}$$

The power is 0 W.

b. This signal is a "top hat" pulse which is 2 for $2 \le t \le 4$, 1 for $0 \le t < 2$ and $4 < t \le 6$, and 0 everywhere else. It is clearly an energy signal with energy

$$E = 2 \times 1^2 + 2 \times 2^2 + 2 \times 1^2 = 12 \text{ J}$$

Its power is 0 W.

c. This is a power signal with power

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 49e^{j6\pi t} e^{-j6\pi t} u(t)\, dt = \lim_{T\to\infty} \frac{49}{2T} \int_{0}^{T} dt = \frac{49}{2} = 24.5 \text{ W}$$

Its energy is infinite.

d. This is a periodic signal with power $P = \frac{2^2}{2} = 2$ W. Its energy is infinite.

e. This is neither an energy nor a power signal. Its energy is infinite and its power is

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} t^2 dt = \lim_{T\to\infty} \frac{1}{2T} \left. \frac{t^3}{3} \right|_{-T}^{T} = \lim_{T\to\infty} \frac{1}{2T} \frac{2T^3}{3} \to \infty$$

f. This is neither an energy nor a power signal. Its energy is

$$E = \int_{1}^{\infty} t^{-1} dt = \ln(t)|_1^\infty \to \infty$$

and its power is

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{1}^{T} t^{-1} dt = \lim_{T\to\infty} \frac{1}{2T} \ln(t)|_1^\infty = 0$$

Problem 2.13

a. This is a cosine burst from $t = -6$ to $t = 6$ seconds. The energy is $E_1 = \int_{-6}^{6} \cos^2(6\pi t)\, dt = 2\int_0^6 \left[\frac{1}{2} + \frac{1}{2}\cos(12\pi t)\right] dt = 6$ J

b. The energy is

$$\begin{aligned} E_2 &= \int_{-\infty}^{\infty} \left[e^{-|t|/3}\right]^2 dt = 2\int_0^\infty e^{-2t/3} dt \text{ (by even symmetry)} \\ &= -2 \left. \frac{e^{-2t/3}}{2/3} \right|_0^\infty = 3 \text{ J} \end{aligned}$$

Since the result is finite, this is an energy signal.

c. The energy is

$$E_3 = \int_{-\infty}^{\infty} \{2[u(t) - u(t-8)]\}^2 dt = \int_0^8 4dt = 32 \text{ J}$$

Since the result is finite, this is an energy signal.

d. Note that

$$r(t) \triangleq \int_{-\infty}^{t} u(\lambda)\, d\lambda = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$

which is called the unit ramp. Thus the given signal is a triangle between 0 and 20. The energy is

$$E_4 = \int_{-\infty}^{\infty} \left[r(t) - 2r(t-10) + r(t-20)\right]^2 dt = 2\int_0^{10} t^2 dt = \frac{2}{3}\, t^3\Big|_0^{10} = \frac{2000}{3} \text{ J}$$

where the last integral follows because the integrand is a symmetrical triangle about $t = 10$. Since the result is finite, this is an energy signal.

Problem 2.14

a. This is a cosine burst nonzero between 0 and 2 seconds. Its power is 0. Its energy is

$$E_1 = \int_0^2 \cos^2(10\pi t)\, dt = \frac{1}{2}\int_0^2 \left[1 + \cos(20\pi t)\right] dt = 1 \text{ J}$$

b. This is a periodic sequence of triangles of period 3 s. Its energy is infinite. Its power is

$$P_2 = \frac{2}{3}\int_0^2 (1 - t/2)^2\, dt = \frac{4}{9} \text{ J}$$

c. This is an energy signal. Its power is 0. Using evenness of the integrand, its energy is

$$\begin{aligned} E_3 &= 2\int_0^{\infty} e^{-2t}\cos^2(2\pi t)\, dt = \int_0^{\infty} e^{-2t} dt + \int_0^{\infty} e^{-2t}\cos(4\pi t)\, dt \\ &= \frac{1}{2} + \frac{2}{4 + 16\pi^2} \text{ J} \end{aligned}$$

.

d. This is an energy signal. Its energy is

$$E_4 = 2\int_0^1 (2-t)^2\, dt = -\frac{2}{3}(2-t)^3\Big|_0^1 = \frac{14}{3} \text{ J}$$

Problem 2.15

a. Use the exponential representation of the sine to get the Fourier coeffi cients as (note that the period $= \frac{1}{2}\frac{1}{f_0}$).

$$x_1(t) = \left(\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\right)^2 = -\frac{1}{4}\left(e^{-j4\pi f_0 t} - 2 + e^{j4\pi f_0 t}\right)$$

from which we find that

$$X_{-1} = X_1 = -\frac{1}{4};\ X_0 = \frac{1}{2}$$

All other coeffi cients are zero.

b. Use the exponential representations of the sine and cosine to get

$$x_2(t) = \frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t} + \frac{1}{2j}e^{j4\pi f_0 t} - \frac{1}{2j}e^{-j4\pi f_0 t}$$

Therefore, the Fourier coeffi cients for this case are

$$X_{-1} = X_1 = \frac{1}{2} \text{ and } X_2 = X_{-2}^* = \frac{1}{2j}$$

All other coeffi cients are zero.

c. Use a trig identity to write this signal as

$$x_3(t) = \frac{1}{2}\sin 8\pi f_0 t = \frac{1}{4j}e^{j8\pi f_0 t} - \frac{1}{4j}e^{-j8\pi f_0 t}$$

The fundamental frequency is $4f_0$ Hz. From this it follows that the Fourier coeffi cients are

$$X_1 = X_{-1}^* = \frac{1}{4j}$$

All other coeffi cients are zero.

d. Use trig identities and the exponential forms of cosine to write this signal as

$$\begin{aligned} x_4(t) &= \frac{3}{4}\cos 2\pi f_0 t + \frac{1}{4}\cos 6\pi f_0 t \\ &= \frac{3}{8}e^{j2\pi f_0 t} + \frac{3}{8}e^{-j2\pi f_0 t} + \frac{1}{8}e^{j6\pi f_0 t} + \frac{1}{8}e^{-j6\pi f_0 t} \end{aligned}$$

The fundamental frequency is f_0 Hz. It follows that the Fourier coeffi cients for this case are

$$X_{-1} = X_1 = \frac{3}{8};\ X_{-3} = X_3 = \frac{1}{8}$$

All other Fourier coeffi cients are zero.

e. Use trig identies to write

$$\begin{aligned} x_5(t) &= \frac{1}{2}\sin(2\pi f_0 t) - \frac{1}{4}\sin(6\pi f_0 t) + \frac{1}{4}\sin(10\pi f_0 t) \\ &= \frac{1}{4j}e^{j2\pi f_0 t} - \frac{1}{4j}e^{-j2\pi f_0 t} - \frac{1}{8j}e^{j6\pi f_0 t} + \frac{1}{8j}e^{-j6\pi f_0 t} + \frac{1}{8j}e^{j10\pi f_0 t} - \frac{1}{8j}e^{-j10\pi f_0 t} \end{aligned}$$

The fundamental frequency is f_0 Hz. It follows that the Fourier coeffi cients for this case are

$$X_{-1}^* = X_1 = -\frac{j}{4};\ X_{-3}^* = X_3 = \frac{j}{8};\ X_{-5}^* = X_5 = -\frac{j}{8}$$

All other Fourier coeffi cients are zero.

f. Use trig identities to write

$$\begin{aligned} x_6(t) &= \frac{1}{2}\cos(6\pi f_0 t) - \frac{1}{4}\cos(\pi f_0 t) - \frac{1}{4}\cos(11\pi f_0 t) \\ &= \frac{1}{4}e^{j6\pi f_0 t} + \frac{1}{4}e^{-j6\pi f_0 t} - \frac{1}{8}e^{j\pi f_0 t} - \frac{1}{8}e^{-j\pi f_0 t} - \frac{1}{8}e^{j11\pi f_0 t} - \frac{1}{8}e^{-j11\pi f_0 t} \end{aligned}$$

The fundamental frequency is $f_0/2$ Hz. It follows that the Fourier coeffi cients for this case are

$$X_{-1}^* = X_1 = -\frac{1}{8};\ X_{-12}^* = X_{12} = \frac{1}{4};\ X_{-22}^* = X_{22} = -\frac{1}{8}$$

All other Fourier coeffi cients are zero.

Problem 2.16

The expansion interval is $T_0 = 4$ so that $f_0 = 1/4$ Hz. The Fourier coeffi cients are

$$\begin{aligned} X_n &= \frac{1}{4}\int_{-2}^{2} 2t^2 e^{-jn(\pi/2)t} dt = \frac{2}{4}\int_{-2}^{2} t^2 (\cos n\pi t/2 - j\sin n\pi t/2)\ dt = \\ &= \frac{2}{4}\int_{0}^{2} 2t^2 \cos\left(\frac{n\pi t}{2}\right) dt \end{aligned}$$

which follows by the oddness of the second integrand and the eveness of the first integrand. Let $u = n\pi t/2$ to obtain the form

$$X_n = \left(\frac{2}{n\pi}\right)^3 \int_0^{n\pi} u^2 \cos u\, du = \frac{16}{(n\pi)^2}(-1)^n \quad n \neq 0 \quad \text{(use a table of integrals)}$$

If $n = 0$, the integral for the coeffi cients is

$$X_0 = \frac{1}{4}\int_{-2}^{2} 2t^2 dt = \frac{8}{3}$$

The Fourier series is therefore

$$x(t) = \frac{8}{3} + \sum_{n=-\infty,\, n\neq 0}^{\infty} (-1)^n \frac{16}{(n\pi)^2} e^{jn(\pi/2)t}$$

Problem 2.17

Parts (a) through (c) were discussed in the text. For (d), break the integral for X_n up into a part for $t < 0$ and a part for $t > 0$. Then use the odd half-wave symmetry condition.

The development follows:

$$\begin{aligned}
X_n &= \frac{1}{T_0}\left[\int_0^{T_0/2} x(t)\, e^{-j2\pi n f_0 t} dt + \int_{-T_0/2}^{0} x(t)\, e^{-j2\pi n f_0 t} dt\right] \\
&= \frac{1}{T_0}\left[\int_0^{T_0/2} x(t)\, e^{-j2\pi n f_0 t} dt + \int_0^{T_0/2} x\left(t' + T_0/2\right) e^{-j2\pi n f_0 (t'+T_0/2)} dt\right],\ t' = t + T_0/2 \\
&= \frac{1}{T_0}\left[\int_0^{T_0/2} x(t)\, e^{-j2\pi n f_0 t} dt - \int_0^{T_0/2} x\left(t'\right) e^{-j2\pi n f_0 t' - jn\pi} dt\right],\ f_0 = 1/T_0 \\
&= \frac{1}{T_0}\left[\int_0^{T_0/2} x(t)\, e^{-j2\pi n f_0 t} dt - (-1)^n \int_0^{T_0/2} x\left(t'\right) e^{-j2\pi n f_0 t'} dt\right] \\
&= \begin{cases} 0,\ n \text{ even} \\ \frac{2}{T_0}\int_0^{T_0/2} x(t)\, e^{-j2\pi n f_0 t} dt,\ n \text{ odd} \end{cases}
\end{aligned}$$

Problem 2.18

This is a matter of integration. Only the solution for part (b) will be given here. The integral for the Fourier coeffi cients is

$$\begin{aligned}
X_n &= \frac{A}{T_0}\int_0^{T_0/2}\sin(\omega_0 t)\, e^{-jn\omega_0 t}dt = \frac{A}{2jT_0}\int_0^{T_0/2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)e^{-jn\omega_0 t}dt \\
&= \frac{A}{2jT_0}\left[\int_0^{T_0/2} e^{j(1-n)\omega_0 t}dt - \int_0^{T_0/2} e^{-j(1+n)\omega_0 t}dt\right] \\
&= \frac{A}{2jT_0}\left[\left.\frac{e^{j(1-n)\omega_0 t}}{j(1-n)\omega_0}\right|_0^{T_0/2} - \left.\frac{e^{-j(1+n)\omega_0 t}}{-j(1+n)\omega_0}\right|_0^{T_0/2}\right] \\
&= \frac{A}{-4\pi}\left[\frac{e^{j(1-n)\pi}-1}{1-n} + \frac{e^{-j(1+n)\pi}-1}{1+n}\right],\ n \neq \pm 1\ \ (\omega_0 T_0/2 = \pi) \\
&= \frac{A}{4\pi}\left[\frac{(-1)^n+1}{1-n} + \frac{(-1)^n+1}{1+n}\right],\ n \neq \pm 1 \\
&= \begin{cases} 0,\ n \text{ odd and } n \neq \pm 1 \\ \frac{A}{\pi(1-n^2)},\ n \text{ even} \end{cases}
\end{aligned}$$

For $n = 1$, the integral is

$$\begin{aligned}
X_1 &= \frac{A}{2jT_0}\int_0^{T_0/2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)e^{-j\omega_0 t}dt \\
&= \frac{A}{2jT_0}\int_0^{T_0/2}\left(1 - e^{-j2\omega_0 t}\right)dt = -\frac{jA}{4} = X_{-1}^*
\end{aligned}$$

This is the same result as given in Table 2.1.

Problem 2.19

a. Use Parseval's theorem to get

$$P_{|nf_0|\,\leq\,1/\tau} = \sum_{n=-N}^{N}|X_n|^2 = \sum_{n=-N}^{N}\left(\frac{A\tau}{T_0}\right)^2\text{sinc}^2(nf_0\tau)$$

where N is an appropriately chosen limit on the sum. We are given that only frequences for which $|nf_0| \leq 1/\tau$ are to be included. This is the same as requiring that $|n| \leq 1/(\tau f_0) =$

$T_0/\tau = 2$. Also, for a pulse train, $P_{\text{total}} = A^2\tau/T_0$ and, in this case, $P_{\text{total}} = A^2/2$. Thus

$$\begin{aligned}\frac{P_{|nf_0| \leq 1/\tau}}{P_{\text{total}}} &= \frac{2}{A^2}\sum_{n=-2}^{2}\left(\frac{A}{2}\right)^2 \operatorname{sinc}^2(nf_0\tau) \\ &= \frac{1}{2}\sum_{n=-2}^{2}\operatorname{sinc}^2(nf_0\tau) \\ &= \frac{1}{2}\left[1 + 2\left(\operatorname{sinc}^2(1/2) + \operatorname{sinc}^2(1)\right)\right] \\ &= \frac{1}{2}\left[1 + 2\left(\frac{2}{\pi}\right)^2\right] = 0.9053\end{aligned}$$

b. In this case, $|n| \leq 5$, $P_{\text{total}} = A^2/5$, and

$$\begin{aligned}\frac{P_{|nf_0| \leq 1/\tau}}{P_{\text{total}}} &= \frac{1}{5}\sum_{n=-5}^{5}\operatorname{sinc}^2(n/5) \\ &= \frac{1}{5}\left\{1 + 2\left[(0.9355)^2 + (0.7568)^2 + (0.5046)^2 + (0.2339)^2\right]\right\} \\ &= 0.9029\end{aligned}$$

c. In this case, $|n| \leq 10$, $P_{\text{total}} = A^2/10$, and

$$\begin{aligned}\frac{P_{|nf_0| \leq 1/\tau}}{P_{\text{total}}} &= \frac{1}{10}\sum_{n=-10}^{10}\operatorname{sinc}^2(n/10) \\ &= \frac{1}{10}\left\{1 + 2\left[\begin{array}{c}(0.9836)^2 + (0.9355)^2 + (0.8584)^2 + (0.7568)^2 + (0.6366)^2 \\ + (0.5046)^2 + (0.3679)^2 + (0.2339)^2 + (0.1093)^2\end{array}\right]\right\} \\ &= 0.9028\end{aligned}$$

d. In this case, $|n| \leq 20$, $P_{\text{total}} = A^2/20$, and

$$\begin{aligned}\frac{P_{|nf_0| \leq 1/\tau}}{P_{\text{total}}} &= \frac{1}{20}\sum_{n=-20}^{20}\operatorname{sinc}^2(n/20) \\ &= \frac{1}{20}\left\{1 + 2\sum_{n=1}^{20}\operatorname{sinc}^2(n/20)\right\} \\ &= 0.9028\end{aligned}$$

Problem 2.20

a. The integral for Y_n is

$$Y_n = \frac{1}{T_0}\int_{T_0} y(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0}\int_0^{T_0} x(t - t_0)\, e^{-jn\omega_0 t} dt,\ \omega_0 = 2\pi f_0$$

Let $t' = t - t_0$, which results in

$$Y_n = \left[\frac{1}{T_0}\int_{-t_0}^{T_0 - t_0} x\left(t'\right) e^{-jn\omega_0 t'} dt'\right] e^{-jn\omega_0 t_0} = X_n e^{-j2\pi n f_0 t_0}$$

b. Note that

$$y(t) = A\cos\omega_0 t = A\sin(\omega_0 t + \pi/2) = A\sin[\omega_0 (t + \pi/2\omega_0)]$$

Thus, t_0 in the theorem proved in part (a) here is $-\pi/2\omega_0$. By Euler's theorem, a sine wave can be expressed as

$$\sin(\omega_0 t) = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

Its Fourier coeffi cients are therefore$X_1 = \frac{1}{2j}$ and $X_{-1} = -\frac{1}{2j}$. According to the theorem proved in part (a), we multiply these by the factor

$$e^{-jn\omega_0 t_0} = e^{-jn\omega_0(-\pi/2\omega_0)} = e^{jn\pi/2}$$

For $n = 1$, we obtain

$$Y_1 = \frac{1}{2j}e^{j\pi/2} = \frac{1}{2}$$

For $n = -1$, we obtain

$$Y_{-1} = -\frac{1}{2j}e^{-j\pi/2} = \frac{1}{2}$$

which gives the Fourier series representation of a cosine wave as

$$y(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} = \cos\omega_0 t$$

We could have written down this Fourier representation directly by using Euler's theorem.

Problem 2.21

a. Use the Fourier series of a square wave (specialize the Fourier series of a pulse train) with $A = 1$ and $t = 0$ to obtain the series

$$1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

Multiply both sides by $\frac{\pi}{4}$ to get the series in the problem statement. Hence, the sum is $\frac{\pi}{4}$.

b. Use the Fourier series of a triangular wave as given in Table 2.1 with $A = 1$ and $t = 0$ to obtain the series

$$1 = \cdots + \frac{4}{25\pi^2} + \frac{4}{9\pi^2} + \frac{4}{\pi^2} + \frac{4}{\pi^2} + \frac{4}{9\pi^2} + \frac{4}{25\pi^2} + \cdots$$

Multiply both sides by $\frac{\pi^2}{8}$ to get the series in given in the problem. Hence, its sum is $\frac{\pi^2}{8}$.

Problem 2.22

a. In the expression for the Fourier series of a pulse train (Table 2.1), let $t_0 = -T_0/8$ and $\tau = T_0/4$ to get

$$X_n = \frac{A}{4}\text{sinc}\left(\frac{n}{4}\right)\exp\left(j\frac{\pi n f_0}{4}\right)$$

The spectra are shown in Fig. 2.4.

b. The amplitude spectrum is the same as for part (a) except that $X_0 = \frac{3A}{4}$. Note that this can be viewed as having a sinc-function envelope with zeros at multiples of $\frac{4}{3T_0}$. The phase spectrum can be obtained from that of part (a) by subtracting a phase shift of π for negative frequencies and adding π for postitive frequencies (or vice versa). The Fourier coeffi cients are given by

$$X_n = \frac{3A}{4}\text{sinc}\left(\frac{3n}{4}\right)\exp\left(-j\frac{3\pi n f_0}{4}\right)$$

See Fig. 2.4 for amplitude and phase plots.

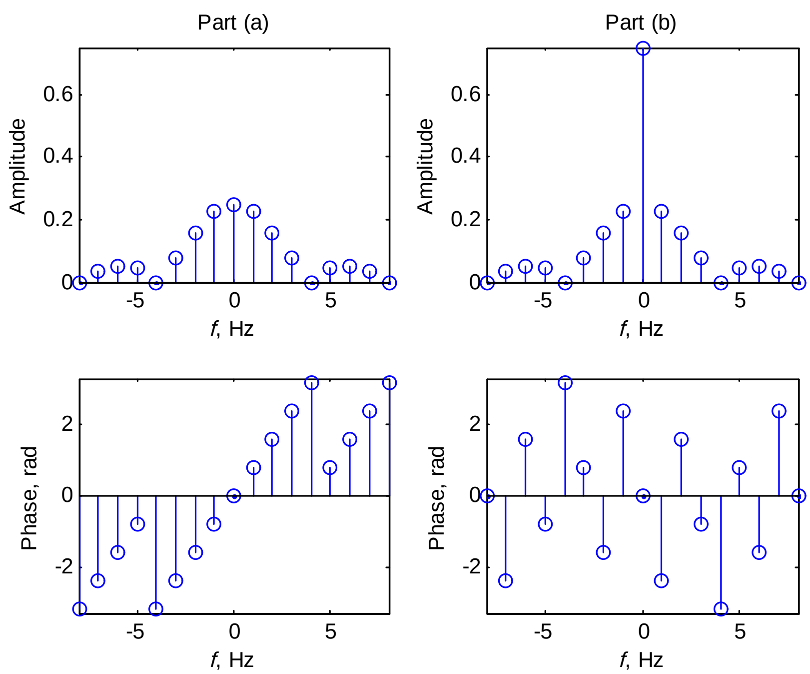
Problem 2.23

a. Use the rectangular pulse waveform of Table 2.1 specialized to

$$x_a(t) = 2A\Pi\left(\frac{t - T_0/4}{T_0/2}\right) - A, |t| < T_0/2$$

and periodically extended. Hence, from Table 2.1, we have

$$\begin{aligned} X_n^A &= \frac{2AT_0/2}{T_0}\text{sinc}\left(\frac{nT_0/2}{T_0}\right)\exp\left(-j\frac{2\pi n T_0/4}{T_0}\right) \\ &= A\text{sinc}(n/2)\exp(-j\pi n/2),\ n \neq 0 \\ &= A\frac{\sin(n\pi/2)}{n\pi/2}\exp(-j\pi n/2),\ n \neq 0 \end{aligned}$$


Part (a)
Part (b)
Amplitude
0.6
0.4
0.2
0
-5
0
5
f, Hz
Phase, rad
2
0
-2

where the superscript A refers to $x_A(t)$. The dc component is 0 so $X_0^A = 0$. The Fourier coeffi cients are therefore

$$\begin{aligned}
X_0^A &= 0 \\
X_1^A &= -j2A/\pi;\ X_{-1}^A = j2A/\pi \\
X_2^A &= 0 = X_{-2}^A \\
X_3^A &= j2A/3\pi;\ X_{-3}^A = -j2A/3\pi \\
&\cdots
\end{aligned}$$

b. Note that

$$x_A(t) = \frac{dx_B(t)}{dt}$$

where $A = \frac{2B}{T_0/2} = \frac{4B}{T_0}$ obtained from matching the amplitude of $x_A(t)$ with the slope of $x_B(t)$ or $B = \frac{T_0}{4}A$. The relationship between spectral components is therefore

$$X_n^A = (jn\omega_0)\, X_n^B = j2\pi n f_0 X_n^B$$

or

$$X_n^B = \frac{X_n^A}{j2\pi n f_0} = \frac{T_0 X_n^A}{j2\pi n}$$

where the superscript A refers to $x_A(t)$ and B refers to $x_B(t)$. For example,

$$X_1^B = -\frac{j2A/\pi}{j2\pi f_0} = -\frac{2B}{\pi^2} = X_{-1}^B$$

Problem 2.24

a. This is a decaying exponential starting at $t = 0$ and its Fourier transform is

$$\begin{aligned}
X_1(f) &= A\int_0^\infty e^{-t/\tau}e^{-j2\pi ft}dt = A\int_0^\infty e^{-(1/\tau+j2\pi f)t}dt \\
&= \left.\frac{Ae^{-(1/\tau+j2\pi f)t}}{1/\tau+j2\pi f}\right|_0^\infty = \frac{A}{1/\tau+j2\pi f} \\
&= \frac{A\tau}{1+j2\pi f\tau}
\end{aligned}$$

b. Since $x_2(t) = x_1(-t)$ we have, by the time reversal theorem, that

$$\begin{aligned} X_2(f) &= X_1^*(f) = X_1(-f) \\ &= \frac{A\tau}{1 - j2\pi f\tau} \end{aligned}$$

c. Since $x_3(t) = x_1(t) - x_2(t)$ we have, after some simplification, that

$$\begin{aligned} X_3(f) &= X_1(f) - X_2(f) \\ &= \frac{A\tau}{1 + j2\pi f\tau} - \frac{A\tau}{1 - j2\pi f\tau} \\ &= \frac{-j4A\pi f\tau}{1 + (2\pi f\tau)^2} \end{aligned}$$

d. Since $x_4(t) = x_1(t) + x_2(t)$ we have, after some simplification, that

$$\begin{aligned} X_4(f) &= X_1(f) + X_2(f) \\ &= \frac{A\tau}{1 + j2\pi f\tau} + \frac{A\tau}{1 - j2\pi f\tau} \\ &= \frac{2A\tau}{1 + (2\pi f\tau)^2} \end{aligned}$$

This is the expected result since $x_4(t)$ is really a double-sided decaying exponential.

e. By part a and the delay theorem

$$X_5(f) = X_1(f)\, e^{-j10\pi f} = \frac{A\tau e^{-j10\pi f}}{1 + j2\pi f\tau}$$

f. By parts a and e and superposition

$$X_6(f) = X_1(f)\left[1 - e^{-j10\pi f}\right] = \frac{A\tau\left[1 - e^{-j10\pi f}\right]}{1 + j2\pi f\tau}$$

Problem 2.25

a. Using a table of Fourier transforms and the time reversal theorem, the Fourier transform of the given signal is

$$X(f) = \frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f}$$

Note that $x(t) \to \text{sgn}(t)$ in the limit as $\alpha \to 0$. Taking the limit of the above Fourier transform as $\alpha \to 0$, we deduce that

$$F[\text{sgn}(t)] = \frac{1}{j2\pi f} - \frac{1}{-j2\pi f} = \frac{1}{j\pi f}$$

b. Using the given relationship between the unit step and the signum function and the linearity property of the Fourier transform, we obtain

$$\begin{aligned} F[u(t)] &= \frac{1}{2}F[\text{sgn}(t)] + \frac{1}{2}F[1] \\ &= \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \end{aligned}$$

c. The same result as obtained in part (b) is obtained.

Problem 2.26

a. One differentiation gives

$$\frac{dx_a(t)}{dt} = \Pi(t-0.5) - \Pi(t-2.5)$$

Two differentiations give

$$\frac{d^2x_a(t)}{dt^2} = \delta(t) - \delta(t-1) - \delta(t-2) + \delta(t+3)$$

Application of the differentiation theorem of Fourier transforms gives

$$\begin{aligned} (j2\pi f)^2 X_a(f) &= 1 - 1 \cdot e^{-j2\pi f} - 1 \cdot e^{-j4\pi f} + 1 \cdot e^{-j6\pi f} \\ &= \left(e^{j3\pi f} - e^{j\pi f} - e^{-j\pi f} + e^{-j3\pi f}\right) e^{-j3\pi f} \\ &= 2(\cos 3\pi f - \cos \pi f) e^{-j3\pi f} \end{aligned}$$

where the time delay theorem and the Fourier transform of a unit impulse have been used. Dividing both sides by $(j2\pi f)^2$, we obtain

$$X_a(f) = \frac{2(\cos 3\pi f - \cos \pi f) e^{-j3\pi f}}{(j2\pi f)^2} = \frac{\cos \pi f - \cos 3\pi f}{2\pi^2 f^2} e^{-j3\pi f}$$

Use the trig identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ or $\cos 2x = 1 - 2\sin^2 x$ to rewrite this result as

$$\begin{aligned} X_a(f) &= \frac{1 - 2\sin^2(0.5\pi f) - 1 + 2\sin^2(1.5\pi f)}{2\pi^2 f^2} e^{-j3\pi f} \\ &= \frac{-\sin^2(0.5\pi f) + \sin^2(1.5\pi f)}{\pi^2 f^2} e^{-j3\pi f} \\ &= \frac{\sin^2(1.5\pi f)}{\pi^2 f^2} e^{-j3\pi f} - \frac{\sin^2(0.5\pi f)}{\pi^2 f^2} e^{-j3\pi f} \\ &= \left[1.5^2 \left(\frac{\sin 1.5\pi f}{1.5\pi f}\right)^2 - 0.5^2 \left(\frac{\sin 0.5\pi f}{0.5\pi f}\right)^2\right] e^{-j3\pi f} \\ &= \left[1.5^2 \mathrm{sinc}^2(1.5f) - 0.5^2 \mathrm{sinc}^2(0.5f)\right] e^{-j3\pi f} \end{aligned}$$

This is the same result as would have been obtained by writing

$$x_a(t) = 1.5\Lambda\left(\frac{t-1.5}{1.5}\right) - 0.5\Lambda\left(\frac{t-1.5}{0.5}\right)$$

and using the Fourier transform of the triangular pulse along with the superposition and time delay theorems.

b. Two differentiations give (sketch $dx_b(t)/dt$ to see this)

$$\frac{d^2 x_b(t)}{dt^2} = \delta(t) - 2\delta(t-1) + 2\delta(t-3) - \delta(t-4)$$

Application of the differentiation theorem gives

$$(j2\pi f)^2 X_b(f) = 1 - 2e^{-j2\pi f} + 2e^{-j6\pi f} - e^{-j8\pi f}$$

Dividing both sides by $(j2\pi f)^2$, we obtain

$$X_b(f) = \frac{1 - 2e^{-j2\pi f} + 2e^{-j6\pi f} - e^{-j8\pi f}}{-4\pi^2 f^2}$$

Further manipulation may be applied to this result to convert it to

$$\begin{aligned} X_b(f) &= \mathrm{sinc}^2(f)\left[e^{-j2\pi f} - e^{-j6\pi f}\right] \\ &= \mathrm{sinc}^2(f)\left[e^{j2\pi f} - e^{-j2\pi f}\right] e^{-j4\pi f} \\ &= 2j\sin(2\pi f)\,\mathrm{sinc}^2(f)\, e^{-j4\pi f} \end{aligned}$$

which would have resulted from Fourier transforming the waveform written as

$$x_b(t) = \Lambda(t-1) - \Lambda(t-3)$$

c. Two differentiations give (sketch $dx_c(t)/dt$ to see this)

$$\frac{d^2x_c(t)}{dt^2} = \delta(t) - 2\delta(t-1) + 2\delta(t-2) - 2\delta(t-3) + \delta(t-4)$$

Application of the differentiation theorem gives

$$(j2\pi f)^2 X_c(f) = 1 - 2e^{-j2\pi f} + 2e^{-j4\pi f} - 2e^{-j6\pi f} + e^{-j8\pi f}$$

Dividing both sides by $(j2\pi f)^2$, we obtain

$$X_c(f) = \frac{1 - 2e^{-j2\pi f} + 2e^{-j4\pi f} - 2e^{-j6\pi f} + e^{-j8\pi f}}{(j2\pi f)^2}$$

This result may be further arranged to give

$$\begin{aligned} X_c(f) &= \operatorname{sinc}^2(f)\left[e^{-j2\pi f} + e^{-j6\pi f}\right] \\ &= 2\cos(2\pi f)\operatorname{sinc}^2(f)\, e^{-j4\pi f} \end{aligned}$$

which would have resulted from Fourier transforming the waveform written as

$$x_c(t) = \Lambda(t-1) + \Lambda(t-3)$$

d. Two differentiations give (sketch $dx_d(t)/dt$ to see this)

$$\frac{d^2x_d(t)}{dt^2} = \delta(t) - 2\delta(t-1) + \delta(t-1.5) + \delta(t-2.5) - 2\delta(t-3) + \delta(t-4)$$

Application of the differentiation theorem gives

$$(j2\pi f)^2 X_d(f) = 1 - 2e^{-j2\pi f} + e^{-j3\pi f} + e^{-j5\pi f} - 2e^{-j6\pi f} + e^{-j8\pi f}$$

Dividing both sides by $(j2\pi f)^2$, we obtain

$$X_d(f) = \frac{1 - 2e^{-j2\pi f} + e^{-j3\pi f} + e^{-j5\pi f} - 2e^{-j6\pi f} + e^{-j8\pi f}}{(j2\pi f)^2}$$

This result may be further arranged to give

$$X_d(f) = \operatorname{sinc}^2(f)\left[e^{-j2\pi f} + 0.5e^{-j4\pi f} + e^{-j6\pi f}\right]$$

which would have resulted from Fourier transforming the waveform written as

$$x_d(t) = \Lambda(t-1) + 0.5\Lambda(t-2) + \Lambda(t-3)$$

Problem 2.27
See the solutions to Problem 2.26.

Problem 2.28

a. The steps in finding the Fourier transform for (i) are as follows:

$$\begin{aligned}\Pi(t) &\longleftrightarrow \operatorname{sinc}(f)\\ \Pi(t)\exp[j4\pi t] &\longleftrightarrow \operatorname{sinc}(f-2)\\ \Pi(t-1)\exp[j4\pi(t-1)] &\longleftrightarrow \operatorname{sinc}(f-2)\exp(-j2\pi f)\end{aligned}$$

The steps in finding the Fourier transform for (ii) are as follows:

$$\begin{aligned}\Pi(t) &\longleftrightarrow \operatorname{sinc}(f)\\ \Pi(t)\exp[j4\pi t] &\longleftrightarrow \operatorname{sinc}(f-2)\\ \Pi(t+1)\exp[j4\pi(t+1)] &\longleftrightarrow \operatorname{sinc}(f-2)\exp(j2\pi f)\end{aligned}$$

b. The steps in finding the Fourier transform for (i) are as follows:

$$\begin{aligned}\Pi(t) &\longleftrightarrow \operatorname{sinc}(f)\\ \Pi(t-1) &\longleftrightarrow \operatorname{sinc}(f)\exp(-j2\pi f)\\ \Pi(t-1)\exp[j4\pi(t-1)] &= \Pi(t-1)\exp(j4\pi t)\\ &\longleftrightarrow \operatorname{sinc}(f-2)\exp[-j2\pi(f-2)] = \operatorname{sinc}(f-2)\exp(-j2\pi f)\end{aligned}$$

which follows because $\exp(\pm jn2\pi) = 1$ where n is an integer. The steps in finding the Fourier transform for (ii) are as follows:

$$\begin{aligned}\Pi(t) &\longleftrightarrow \operatorname{sinc}(f)\\ \Pi(t+1) &\longleftrightarrow \operatorname{sinc}(f)\exp(j2\pi f)\\ \Pi(t+1)\exp[j4\pi(t+1)] &= \Pi(t+1)\exp(j4\pi t)\\ &\longleftrightarrow \operatorname{sinc}(f-2)\exp[j2\pi(f-2)] = \operatorname{sinc}(f-2)\exp(j2\pi f)\end{aligned}$$

Problem 2.29

a. The time reversal theorem states that $x(-t) \longleftrightarrow X(-f) \neq X^*(f)$ if $x(t)$ is complex, so

$$\begin{aligned}
x_a(t) &\longleftrightarrow \frac{1}{2}X_1(f) + \frac{1}{2}X_1(-f) = \frac{1}{2}\text{sinc}(f-2)\exp(-j2\pi f) + \frac{1}{2}\text{sinc}(-f-2)\exp(j2\pi f) \\
&= \frac{1}{2}\text{sinc}(f-2)\exp(-j2\pi f) + \frac{1}{2}\text{sinc}(f+2)\exp(j2\pi f)
\end{aligned}$$

Note that

$$\begin{aligned}
x_a(t) &= \frac{1}{2}\Pi(t-1)\exp[j4\pi(t-1)] + \frac{1}{2}\Pi(-t-1)\exp[j4\pi(-t-1)] \\
&= \frac{1}{2}\Pi(t-1)\exp(j4\pi t) + \frac{1}{2}\Pi(t+1)\exp(-j4\pi t) \ \text{(by the eveness of } \Pi(u)\text{)}
\end{aligned}$$

b. Similarly to (a), we obtain

$$x_b(t) \longleftrightarrow \frac{1}{2}X_2(f) + \frac{1}{2}X_2(-f) = \frac{1}{2}\text{sinc}(f-2)\exp(j2\pi f) + \frac{1}{2}\text{sinc}(f+2)\exp(-j2\pi f)$$

Problem 2.30

a. The result is

$$X_1(f) = 2\text{sinc}(2f)\exp(-j2\pi f)$$

b. The result is

$$X_2(f) = 2 \times \frac{1}{2}\Pi\left(\frac{f}{2}\right)\exp(-j2\pi f)$$

c. The result is

$$X_3(f) = 8\text{sinc}^2(8f)\exp(-j4\pi f)$$

d. The result is

$$X_4(f) = 4\Lambda(4f)\exp(-j6\pi f)$$

e. The result is

$$\begin{aligned} X_5(f) &= 5 \times \frac{1}{2}\Pi\left(\frac{f}{2}\right)\exp(-j2\pi f) + 5 \times \frac{1}{2}\Pi\left(\frac{f}{2}\right)\exp(j2\pi f) \\ &= 5\Pi\left(\frac{f}{2}\right)\cos(2\pi f) \end{aligned}$$

f. The result is

$$\begin{aligned} X_6(f) &= 16\text{sinc}^2(8f)\exp(-j4\pi f) + 16\text{sinc}^2(8f)\exp(j4\pi f) \\ &= 32\text{sinc}^2(8f)\cos(4\pi f) \end{aligned}$$

Problem 2.31

a. This is an odd signal, so its Fourier transform is odd and purely imaginary.

b. This is an even signal, so its Fourier transform is even and purely real.

c. This is an odd signal, so its Fourier transform is odd and purely imaginary.

d. This signal is neither even nor odd, so its Fourier transform is complex.

e. This is an even signal, so its Fourier transform is even and purely real.

f. This signal is even, so its Fourier transform is real and even.

Problem 2.32

In the Poisson sum formula, we identify $p(t) = \Pi(t/2)$ which has Fourier transform $P(f) = 2\text{sinc}(2f)$. Thus, for this case, the Poisson sum formula becomes

$$\sum_{m=-\infty}^{\infty} p(t - mT_s) = \sum_{m=-\infty}^{\infty} \Pi\left(\frac{t-4m}{2}\right) = \frac{1}{4}\sum_{n=-\infty}^{\infty} 2\text{sinc}\left(\frac{2n}{4}\right) e^{j2\pi(n/4)t} = f_s \sum_{n=-\infty}^{\infty} P(nf_s)\, e^{j2\pi n f_s t}$$

or

$$\sum_{m=-\infty}^{\infty} \Pi\left(\frac{t-4m}{2}\right) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\text{sinc}\left(\frac{n}{2}\right) e^{j\pi(n/2)t}$$

The fundamental frequency is 0.25 Hz and the Fourier coeffi cients are$X_0 = 1/2$, $X_1 = X_{-1} = \frac{1}{2}\text{sinc}\left(\frac{1}{2}\right) = \frac{1}{\pi}$, $X_2 = X_{-2} = 0$, $X_3 = X_{-3} = \frac{1}{2}\text{sinc}\left(\frac{3}{2}\right) = -\frac{1}{3\pi}$, etc.

Problem 2.33

a. The Fourier transform of this signal is

$$X_1(f) = \frac{10}{5 + j2\pi f} = \frac{2}{1 + j2\pi f/5}$$

Thus, the energy spectral density is

$$G_1(f) = \frac{4}{1 + (2\pi f/5)^2}$$

b. The Fourier transform of this signal is

$$X_2(f) = 5\Pi\left(\frac{f}{2}\right)$$

Thus, the energy spectral density is

$$X_2(f) = 25\Pi^2\left(\frac{f}{2}\right) = 25\Pi\left(\frac{f}{2}\right)$$

c. The Fourier transform of this signal is

$$X_3(f) = \frac{3}{2}\text{sinc}\left(\frac{f}{2}\right)$$

so the energy spectral density is $G_3(f) = \frac{9}{4}\text{sinc}^2\left(\frac{f}{2}\right)$

d. The Fourier transform of this signal is

$$X_4(f) = \frac{3}{4}\left[\text{sinc}\left(\frac{f-5}{2}\right) + \text{sinc}\left(\frac{f+5}{2}\right)\right]$$

so the energy spectral density is

$$G_4(f) = \frac{9}{16}\left[\text{sinc}\left(\frac{f-5}{2}\right) + \text{sinc}\left(\frac{f+5}{2}\right)\right]^2$$

Problem 2.34

a. Use the transform pair

$$x_1(t) = e^{-\alpha t}u(t) \longleftrightarrow \frac{1}{\alpha + j2\pi f}$$

Using Rayleigh's energy theorem, we obtain the integral relationship

$$\int_{-\infty}^{\infty} |X_1(f)|^2\, df = \int_{-\infty}^{\infty} \frac{df}{\alpha^2 + (2\pi f)^2} df = \int_{-\infty}^{\infty} |x_1(t)|^2\, dt = \int_0^{\infty} e^{-2\alpha t} dt = \frac{1}{2\alpha}$$

b. Use the transform pair

$$x_2(t) = \frac{1}{\tau}\Pi\left(\frac{t}{\tau}\right) \longleftrightarrow \text{sinc}(\tau f) = X_2(f)$$

Rayleigh's energy theorem gives

$$\begin{aligned}\int_{-\infty}^{\infty} |X_2(f)|^2\, df &= \int_{-\infty}^{\infty} \text{sinc}^2(\tau f)\, df = \int_{-\infty}^{\infty} |x_2(t)|^2\, dt \\ &= \int_{-\infty}^{\infty} \frac{1}{\tau^2}\Pi^2\left(\frac{t}{\tau}\right) dt = \int_{-\tau/2}^{\tau/2} \frac{dt}{\tau^2} = \frac{1}{\tau}\end{aligned}$$

c. Use the transform pair

$$e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \text{ or } \frac{e^{-\alpha|t|}}{2\alpha} \longleftrightarrow \frac{1}{\alpha^2 + (2\pi f)^2}$$

The desired integral, by Rayleigh's energy theorem, is

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} \left[\frac{1}{\alpha^2 + (2\pi f)^2}\right]^2 df = \int_{-\infty}^{\infty} \frac{e^{-2\alpha|t|}}{4\alpha^2} dt \\ &= \frac{2}{4\alpha^2} \int_0^{\infty} e^{-2\alpha t} dt = \frac{1}{2\alpha^2} \frac{e^{-2\alpha t}}{-2\alpha}\Big|_0^{\infty} = \frac{1}{4\alpha^3} \end{aligned}$$

d. Use the transform pair

$$\frac{1}{\tau}\Lambda\left(\frac{t}{\tau}\right) \longleftrightarrow \text{sinc}^2(\tau f)$$

The desired integral, by Rayleigh's energy theorem, is

$$\begin{aligned} I_4 &= \int_{-\infty}^{\infty} |X_4(f)|^2 df = \int_{-\infty}^{\infty} \text{sinc}^4(\tau f)\, df \\ &= \frac{1}{\tau^2} \int_{-\infty}^{\infty} \Lambda^2(t/\tau)\, dt = \frac{2}{\tau^2} \int_0^{\tau} [1 - (t/\tau)]^2 dt \\ &= \frac{2}{\tau} \int_0^1 (1-u)^2 du = \frac{2}{\tau}\left[-\frac{1}{3}(1-u^2)\right]_0^1 = \frac{2}{3\tau} \end{aligned}$$

Problem 2.35

a. The convolution operation gives

$$y_1(t) = \begin{cases} 0, & t \le \tau - 1/2 \\ \frac{1}{\alpha}\left[1 - e^{-\alpha(t-\tau+1/2)}\right], & \tau - 1/2 < t \le \tau + 1/2 \\ \frac{1}{\alpha}\left[e^{-\alpha(t-\tau-1/2)} - e^{-\alpha(t-\tau+1/2)}\right], & t > \tau + 1/2 \end{cases}$$

b. The convolution of these two signals gives

$$y_2(t) = \Lambda(t) + \text{trap}(t)$$

where trap(t) is a trapezoidal function given by

$$\text{trap}(t) = \begin{cases} 0, & t < -3/2 \text{ or } t > 3/2 \\ 1, & -1/2 \le t \le 1/2 \\ 3/2 + t, & -3/2 \le t < -1/2 \\ 3/2 - t, & 1/2 \le t < 3/2 \end{cases}$$

c. The convolution results in

$$y_3(t) = \int_{-\infty}^{\infty} e^{-\alpha|\lambda|}\Pi(\lambda - t)\, d\lambda = \int_{t-1/2}^{t+1/2} e^{-\alpha|\lambda|} d\lambda$$

Sketches of the integrand for various values of t give the following cases:

$$y_3(t) = \begin{cases} \int_{t-1/2}^{t+1/2} e^{\alpha\lambda} d\lambda, & t \le -1/2 \\ \int_{t-1/2}^{0} e^{\alpha\lambda} d\lambda + \int_0^{t+1/2} e^{-\alpha\lambda} d\lambda, & -1/2 < t \le 1/2 \\ \int_{t-1/2}^{t+1/2} e^{-\alpha\lambda} d\lambda, & t > 1/2 \end{cases}$$

Integration of these three cases gives

$$y_3(t) = \begin{cases} \frac{1}{\alpha}\left[e^{\alpha(t+1/2)} - e^{\alpha(t-1/2)}\right], & t \le -1/2 \\ \frac{1}{\alpha}\left[e^{-\alpha(t-1/2)} - e^{-\alpha(t+1/2)}\right], & -1/2 < t \le 1/2 \\ \frac{1}{\alpha}\left[e^{-\alpha(t-1/2)} - e^{-\alpha(t+1/2)}\right], & t > 1/2 \end{cases}$$

d. The convolution gives

$$y_4(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$$

Problem 2.36

a. The inverse FT of $\Pi(f)$ is sinc(t). By the time delay theorem, the inverse Fourier transform of $\Pi(f)\exp(-j4\pi f)$ is sinc$(t-2)$. The product of this and $2\cos(2\pi f)$ in the frequency domain has an inverse Fourier transform which is the convolution of their respective Fourier transforms. Thus

$$\begin{aligned} x_1(t) &= \operatorname{sinc}(t-2) * [\delta(t-1) + \delta(t+1)] \\ &= \operatorname{sinc}(t-3) + \operatorname{sinc}(t-1) \end{aligned}$$

b. The inverse Fourier transform of $\Lambda(f/2)$ is $2\text{sinc}^2(2t)$. By the time delay theorem

$$x_2(t) = 2\text{sinc}^2[2(t-2.5)]$$

c. The inverse Fourier transform of $\Pi(f/2)$ is 2sinc$(2t)$. By the modulation theorem, the inverse Fourier transform of $\Pi\left(\frac{f+4}{2}\right) + \Pi\left(\frac{f-4}{2}\right)$ is sinc$(2t)\cos(8\pi t)$. By the time delay theorem

$$x_3(t) = \operatorname{sinc}[2(t-4)]\cos[2\pi(t-4)]$$

Problem 2.37

a. From before, the total energy is $E_{1,\text{ total}} = \frac{1}{2\alpha}$. The Fourier transform of the given signal is

$$X_1(f) = \frac{1}{\alpha + j2\pi f}$$

so that the energy spectral density is

$$G_1(f) = |X_1(f)|^2 = \frac{1}{\alpha^2 + (2\pi f)^2}$$

By Rayleigh's energy theorem, the normalized inband energy is

$$\frac{E_1(|f| \le W)}{E_{1,\text{ total}}} = 2\alpha \int_{-W}^{W} \frac{df}{\alpha^2 + (2\pi f)^2} = \frac{2}{\pi} \tan^{-1}\left(\frac{2\pi W}{\alpha}\right)$$

b. The total energy is $E_{2,\text{ total}} = \tau$. The Fourier transform of the given signal and its energy spectral density are, respectively,

$$X_2(f) = \tau \text{sinc}(f\tau) \text{ and } G_2(f) = |X_2(f)|^2 = \tau^2 \text{sinc}^2(f\tau)$$

By Rayleigh's energy theorem, the normalized inband energy is

$$\frac{E_2(|f| \le W)}{E_{2,\text{ total}}} = \frac{1}{\tau} \int_{-W}^{W} \tau^2 \text{sinc}^2(f\tau)\, df = 2 \int_0^{\tau W} \text{sinc}^2(u)\, du$$

The integration must be carried out numerically.

c. The total energy is

$$\begin{aligned} E_{3,\text{ total}} &= \int_0^\infty \left[e^{-\alpha t} - e^{-\beta t}\right]^2 dt = \int_0^\infty \left[e^{-2\alpha t} - 2e^{-(\alpha+\beta)t} + e^{-2\beta t}\right]^2 dt \\ &= \frac{1}{2\alpha} - \frac{2}{\alpha + \beta} + \frac{1}{2\beta} = \frac{\beta(\alpha+\beta) - 4\alpha\beta + \alpha(\alpha+\beta)}{2\alpha\beta(\alpha+\beta)} = \frac{(\beta-\alpha)^2}{2\alpha\beta(\alpha+\beta)} \end{aligned}$$

The Fourier transform of the given signal and its energy spectral density are, respectively,

$$X_3(f) = \frac{1}{\alpha + j2\pi f} - \frac{1}{\beta + j2\pi f}$$

and

$$\begin{aligned}
G_3(f) &= |X_3(f)|^2 = \left| \frac{1}{\alpha + j2\pi f} - \frac{1}{\beta + j2\pi f} \right|^2 \\
&= \frac{1}{\alpha^2 + (2\pi f)^2} - 2\,\mathrm{Re}\left[\frac{1}{\alpha + j2\pi f} \frac{1}{\beta - j2\pi f} \right] + \frac{1}{\beta^2 + (2\pi f)^2} \\
&= \frac{1}{\alpha^2 + (2\pi f)^2} - 2\,\mathrm{Re}\left[\frac{\alpha - j2\pi f}{\alpha^2 + (2\pi f)^2} \frac{\beta + j2\pi f}{\beta^2 + (2\pi f)^2} \right] + \frac{1}{\beta^2 + (2\pi f)^2} \\
&= \frac{1}{\alpha^2 + (2\pi f)^2} - 2\,\mathrm{Re}\left[\frac{\alpha\beta - (\alpha - \beta)\, j2\pi f + (2\pi f)^2}{\left(\alpha^2 + (2\pi f)^2\right)\left(\beta^2 + (2\pi f)^2\right)} \right] + \frac{1}{\beta^2 + (2\pi f)^2} \\
&= \frac{1}{\alpha^2 + (2\pi f)^2} - 2 \frac{\alpha\beta + (2\pi f)^2}{\left(\alpha^2 + (2\pi f)^2\right)\left(\beta^2 + (2\pi f)^2\right)} + \frac{1}{\beta^2 + (2\pi f)^2}
\end{aligned}$$

The normalized inband energy is

$$\frac{E_3(|f| \le W)}{E_{3,\ \text{total}}} = \frac{1}{E_{3,\ \text{total}}} \int_{-W}^{W} G_3(f)\, df$$

The first and third terms may be integrated easily as inverse tangents. The second term may be integrated after partial fraction expansion:

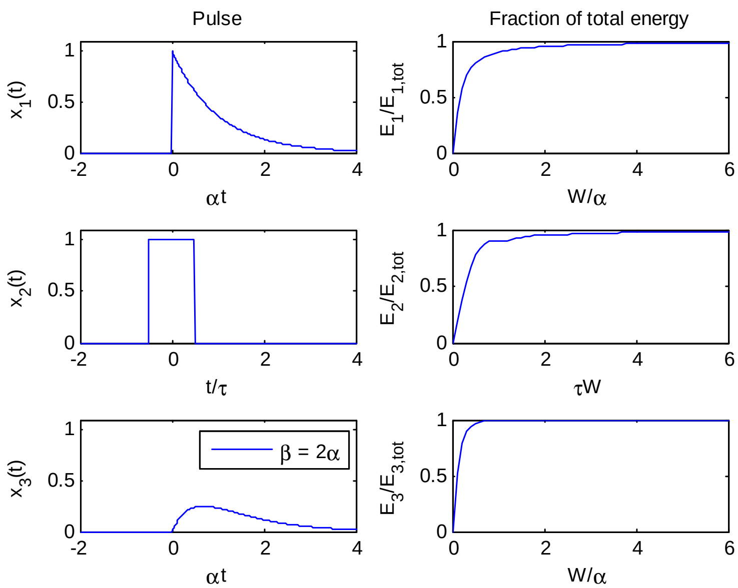
$$2 \frac{\alpha\beta + (2\pi f)^2}{\left(\alpha^2 + (2\pi f)^2\right)\left(\beta^2 + (2\pi f)^2\right)} = \frac{A}{\alpha^2 + (2\pi f)^2} + \frac{B}{\beta^2 + (2\pi f)^2}$$

where

$$A = 2\frac{\alpha^2 - \alpha\beta}{\alpha^2 - \beta^2} \text{ and } B = 2\frac{\alpha\beta - \beta^2}{\alpha^2 - \beta^2}$$

Therefore

$$\begin{aligned}
\frac{E_3(|f| \le W)}{E_{3,\ \text{total}}} &= \frac{1}{E_{3,\ \text{total}}} \int_{-W}^{W} \left[\frac{1}{\alpha^2 + (2\pi f)^2} - \frac{A}{\alpha^2 + (2\pi f)^2} - \frac{B}{\beta^2 + (2\pi f)^2} + \frac{1}{\beta^2 + (2\pi f)^2} \right] df \\
&= \frac{1}{\pi E_{3,\ \text{total}}} \left[\begin{array}{c} \frac{1}{\alpha} \tan^{-1}\left(\frac{2\pi W}{\alpha}\right) - \frac{A}{\alpha} \tan^{-1}\left(\frac{2\pi W}{\alpha}\right) \\ -\frac{B}{\beta} \tan^{-1}\left(\frac{2\pi W}{\beta}\right) + \frac{1}{\beta} \tan^{-1}\left(\frac{2\pi W}{\beta}\right) \end{array} \right] \\
&= \frac{1}{\pi E_{3,\ \text{total}}} \left[\frac{1}{\alpha}(1 - A) \tan^{-1}\left(\frac{2\pi W}{\alpha}\right) + \frac{1}{\beta}(1 - B) \tan^{-1}\left(\frac{2\pi W}{\beta}\right) \right] \\
&= \frac{1}{\pi E_{3,\ \text{total}}} \left(\frac{\beta - \alpha}{\alpha + \beta}\right) \left[\frac{1}{\alpha} \tan^{-1}\left(\frac{2\pi W}{\alpha}\right) - \frac{1}{\beta} \tan^{-1}\left(\frac{2\pi W}{\beta}\right) \right] \\
&= \frac{2}{\pi(\beta - \alpha)} \left[\beta \tan^{-1}\left(\frac{2\pi W}{\alpha}\right) - \alpha \tan^{-1}\left(\frac{2\pi W}{\beta}\right) \right]
\end{aligned}$$



Plots of the signals and inband energy for all three cases are shown in Fig. 2.5

Problem 2.38

a. By the modulation theorem

$$\begin{aligned} X(f) &= \frac{AT_0}{4}\left\{\text{sinc}\left[(f-f_0)\frac{T_0}{2}\right]+\text{sinc}\left[(f+f_0)\frac{T_0}{2}\right]\right\} \\ &= \frac{AT_0}{4}\left\{\text{sinc}\left[\frac{1}{2}\left(\frac{f}{f_0}-1\right)\right]+\text{sinc}\left[\frac{1}{2}\left(\frac{f}{f_0}+1\right)\right]\right\} \end{aligned}$$

b. Use the superposition and modulation theorems to get

$$X(f) = \frac{AT_0}{4}\left\{\text{sinc}\left(\frac{f}{2f_0}\right)+\frac{1}{2}\left[\text{sinc}\left[\frac{1}{2}\left(\frac{f}{f_0}-2\right)\right]+\text{sinc}\left[\frac{1}{2}\left(\frac{f}{f_0}+2\right)\right]\right]\right\}$$

c. In this case, $p(t) = x(t)$ and $P(f) = X(f)$ of part (a) and $T_s = T_0$. From part (a), we have

$$P(nf_0) = \frac{AT_0}{4}\left[\text{sinc}\left(\frac{n-1}{2}\right)+\text{sinc}\left(\frac{n+1}{2}\right)\right]$$

Using this in (2.149), we have the Fourier transform of the half-wave rectified cosine waveform as

$$X(f) = \sum_{n=-\infty}^{\infty}\frac{A}{4}\left[\text{sinc}\left(\frac{n-1}{2}\right)+\text{sinc}\left(\frac{n+1}{2}\right)\right]\ \delta(f-nf_0)$$

Note that $\text{sinc}(x) = 0$ for integer values of its argument and it is 1 for its argument 0. Also, use $\text{sinc}(1/2) = 2/\pi$, $\text{sinc}(3/2) = -2/3\pi$, etc. to get

$$\begin{aligned} X(f) &= \frac{A}{\pi}\delta(f)+\frac{A}{4}\left[\delta(f-f_0)+\delta(f+f_0)\right]+\frac{A}{3\pi}\left[\delta(f-2f_0)+\delta(f+2f_0)\right] \\ &\quad -\frac{A}{15\pi}\left[\delta(f-4f_0)+\delta(f+4f_0)\right]+\cdots \end{aligned}$$

Problem 2.39

Signals $x_1(t)$, $x_2(t)$, and $x_6(t)$ are real and even. Therefore their Fourier transforms are real and even. Signals $x_3(t)$, $x_4(t)$, and $x_5(t)$ are real and odd. Therefore their Fourier transforms are imaginary and odd. Using the Fourier transforms for a square pulse and a triangle along with superposition, time delay, and scaling, the Fourier transforms of these signals are the following.

a. $X_1(f) = 2\text{sinc}^2(2f) + 2\text{sinc}(2f)$

b. $X_2(f) = 2\text{sinc}(2f) - \text{sinc}^2(f)$

c. $X_3(f) = \text{sinc}(f)\, e^{j\pi f} - \text{sinc}(f)\, e^{-j\pi f} = 2j \sin(\pi f)\text{sinc}(f)$

d. $X_4(f) = \text{sinc}^2(f)\, e^{-j2\pi f} - \text{sinc}^2(f)\, e^{j2\pi f} = -2j \sin(2\pi f)\text{sinc}^2(f)$

e. By duality $\text{sgn}(t) \longleftrightarrow j/(\pi f)$ so, by the convolution theorem of Fourier transforms, $X_5(f) = \text{sinc}^2(f) * j/(\pi f)$. The convolution cannot be carried out in closed form, but it is clear that the result is imaginary and odd.

f. By the modulation theorem $X_6(f) = \frac{1}{2}\text{sinc}^2(f-1) + \frac{1}{2}\text{sinc}^2(f+1)$ which is real and even.

Problem 2.40

a. Write

$$\begin{aligned} 6\cos(20\pi t) + 3\sin(20\pi t) &= R\cos(20\pi t - \theta) \\ &= R\cos\theta\cos(20\pi t) + R\sin\theta\sin(20\pi t) \end{aligned}$$

Thus

$$\begin{aligned} R\cos\theta &= 6\cos(20\pi t) \\ R\sin\theta &= 3\sin(20\pi t) \end{aligned}$$

Square both equations, add, and take the square root to obtain

$$R = \sqrt{6^2 + 3^2} = \sqrt{45} = 6.7082$$

Divide the second equation by the first to obtain

$$\begin{aligned} \tan\theta &= 0.5 \\ \theta &= 0.4636 \text{ rad} \end{aligned}$$

So

$$x(t) = 3 + \sqrt{45}\cos(20\pi t - 0.4636)$$

Following Example 2.19, we obtain

$$R_x(\tau) = 3^2 + \frac{45}{2}\cos(20\pi\tau)$$

b. Taking the Fourier transform of $R_x(\tau)$ we obtain

$$S_x(f) = 9\delta(f) + \frac{45}{4}\left[\delta(f-10) + \delta(f+10)\right]$$

Problem 2.41

Use the facts that the power spectral density integrates to give total power, it must be even, and contains no phase information.

a. The total power of this signal is $2^2/2 = 2$ watts which is distributed equally at the frequencies ± 10 hertz. Therefore, by inspection we write

$$S_1(f) = \delta(f - 10) + \delta(f + 10) \text{ W/Hz}$$

b. The total power of this signal is $3^2/2 = 4.5$ watts which is distributed equally at the frequencies ± 15 hertz. Therefore, by inspection we write

$$S_2(f) = 2.25\delta(f - 15) + 2.25\delta(f + 15) \text{ W/Hz}$$

c. The total power of this signal is $5^2/2 = 12.5$ watts which is distributed equally at the frequencies ± 5 hertz. Therefore, by inspection we write

$$S_3(f) = 6.25\delta(f - 5) + 6.25\delta(f + 5) \text{ W/Hz}$$

d. The power of the first component of this signal is $3^2/2 = 4.5$ watts which is distributed equally at the frequencies ± 15 hertz. The power of the second component of this signal is $5^2/2 = 12.5$ watts which is distributed equally at the frequencies ± 5 hertz. Therefore, by inspection we write

$$S_4(f) = 2.25\delta(f - 15) + 2.25\delta(f + 15) + 6.25\delta(f - 5) + 6.25\delta(f + 5) \text{ W/Hz}$$

Problem 2.42

Since the autocorrelation function and power spectral density of a signal are Fourier transform pairs, we may write down the answers by inspection using the Fourier transform pair $A\cos(2\pi f_0 t) \longleftrightarrow \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0)$. The answers are the following.

a. $R_1(\tau) = 8\cos(30\pi\tau)$; Average power $= R_1(0) = 8$ W.

b. $R_2(\tau) = 18\cos(40\pi\tau)$; Average power $= R_2(0) = 18$ W.

c. $R_3(\tau) = 32\cos(10\pi\tau)$; Average power $= R_3(0) = 32$ W.

d. $R_4(\tau) = 18\cos(40\pi\tau) + 32\cos(10\pi\tau)$; Average power $= R_4(0) = 50$ W.

Problem 2.43
The autocorrelation function must be (1) even, (2) have an absolute maximum at $\tau = 0$, and (3) have a Fourier transform that is real and nonnegative.

a. Acceptable - all properties satisfied;

b. Acceptable - all properties satisfied;

c. Not acceptable - none of the properties satisfied;

d. Acceptable - all properties satisfied;

e. Not acceptable - property (3) not satisfied;

f. Not acceptable - none of the properties satisfied.

Problem 2.44
Given that the autocorrelation function of $x(t) = A\cos(2\pi f_0 t + \theta)$ is $R_x(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$ (special case of Ex. 2.19), the results are as follows.

a. $R_1(\tau) = 2\cos(10\pi\tau)$;

b. $R_2(\tau) = 2\cos(10\pi\tau)$;

c. $R_3(\tau) = 5\cos(10\pi\tau)$ (write the signal as $x_3(t) = \text{Re}\left[5\exp\left(j\tan^{-1}(4/3)\exp(j10\pi t)\right)\right]$);

d. $R_4(\tau) = \frac{2^2+2^2}{2}\cos(10\pi\tau) = 4\cos(10\pi\tau)$.

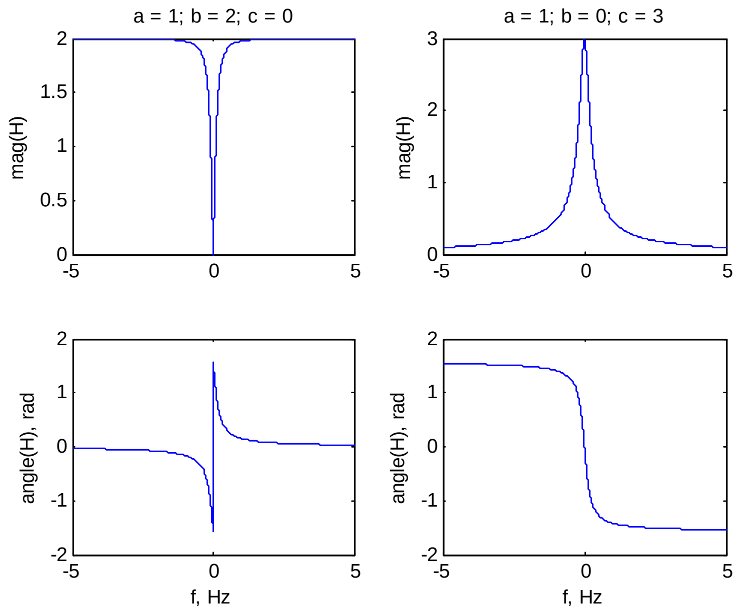
Problem 2.45
This is a matter of applying (2.151) by making the appropriate identifications with the parameters given in Example 2.20

Problem 2.46
Fourier transform both sides of the differential equation using the differentiation theorem of Fourier transforms to get

$$[j2\pi f + a]\, Y(f) = [j2\pi bf + c]\, X(f)$$

Therefore, the frequency response function is

$$H(f) = \frac{Y(f)}{X(f)} = \frac{c + j2\pi bf}{a + j2\pi f}$$



The amplitude response function is

$$|H(f)| = \frac{\sqrt{c^2 + (2\pi bf)^2}}{\sqrt{a^2 + (2\pi f)^2}}$$

and the phase response is

$$\arg[H(f)] = \tan^{-1}\left(\frac{2\pi bf}{c}\right) - \tan^{-1}\left(\frac{2\pi f}{a}\right)$$

Amplitude and phase responses for various values of the constants are plotted in Figure 2.6.

Problem 2.47

a. Use the transform pair

$$Ae^{-\alpha t}u(t) \longleftrightarrow \frac{A}{\alpha + j2\pi f}$$

to find the unit impulse response as

$$h_1(t) = e^{-7t}u(t)$$

b. Long division gives

$$H_2(f) = 1 - \frac{7}{7 + j2\pi f}$$

Use the transform pair $\delta(t) \longleftrightarrow 1$ along with superposition and the transform pair in part (a) to get

$$h_2(t) = \delta(t) - 7e^{-7t}u(t)$$

c. Use the time delay theorem along with the result of part (a) to get

$$h_3(t) = e^{-7(t-3)}u(t-3)$$

d. Use superposition and the results of parts (a) and (c) to get

$$h_4(t) = e^{-7t}u(t) - e^{-7(t-3)}u(t-3)$$

Problem 2.48

Use the transform pair for a sinc function to find that

$$Y(f) = \Pi\left(\frac{f}{2B}\right)\Pi\left(\frac{f}{2W}\right)$$

a. If $W < B$, it follows that

$$Y(f) = \Pi\left(\frac{f}{2W}\right)$$

because $\Pi\left(\frac{f}{2B}\right) = 1$ throughout the region where $\Pi\left(\frac{f}{2W}\right)$ is nonzero.

b. If $W > B$, it follows that

$$Y(f) = \Pi\left(\frac{f}{2B}\right)$$

because $\Pi\left(\frac{f}{2W}\right) = 1$ throughout the region where $\Pi\left(\frac{f}{2B}\right)$ is nonzero.

c. Part (b) gives distortion because the output spectrum differs from the input spectrum. In fact, the output is

$$y(t) = 2B\text{sinc}(2Bt)$$

which clearly differs from the input for $W > B$.

Problem 2.49

a. Replace the capacitors with $1/j\omega C$ which is their ac-equivalent impedance. Call the junction of the input resistor, feedback resistor, and capacitors 1. Call the junction at the positive input of the operational amplifier 2. Call the junction at the negative input of the operational amplifier 3. Write down the KCL equations at these three junctions. Use the constraint equation for the operational amplifier, which is $V_2 = V_3$, and the definitions for ω_0, Q, and K to get the given transfer function as $H(j\omega) = V_o(j\omega)/V_i(j\omega)$. The node equations are

$$\begin{aligned} \frac{V_1 - V_i}{R} + j\omega C V_1 + \frac{V_1 - V_o}{R} + j\omega C(V_1 - V_2) &= 0 \\ j\omega C(V_2 - V_1) + \frac{V_2}{R} &= 0 \\ \frac{V_3}{R_b} + \frac{V_3 - V_o}{R_a} &= 0 \\ \text{(constraint on op amp input)} \ V_2 &= V_3 \end{aligned}$$

b. See plot given in Figure 2.7.

c. In terms of f, the transfer function magnitude is

$$|H(f)| = \frac{K}{\sqrt{2}} \frac{(f/f_0)}{\sqrt{\left[1 - \left(\frac{f}{f_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{f}{f_0}\right)^2}}$$

It can be shown that the maximim of $|H(f)|$ is at $f = f_0$. By substitution, this maximum is $|H(f_0)| = KQ/\sqrt{2}$. To find the 3-dB bandwidth, we must find the frequencies for which $|H(f)| = |H(f_0)|/\sqrt{2}$. This results in

$$\frac{Q}{\sqrt{2}} = \frac{(f_3/f_0)}{\sqrt{\left[1 - \left(\frac{f_3}{f_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{f_3}{f_0}\right)^2}}$$

which reduces to the quadratic equation

$$\left(\frac{f_3}{f_0}\right)^4 - \left(2 + \frac{1}{Q^2}\right)\left(\frac{f_3}{f_0}\right)^2 + 1 = 0$$

Using the quadratic formula, the solutions to this equation are

$$\begin{aligned}
\left(\frac{f_{\pm 3}}{f_0}\right)^2 &= \frac{1}{2}\left(2+\frac{1}{Q^2}\right) \pm \sqrt{\left(2+\frac{1}{Q^2}\right) - 4} \\
&= \frac{1}{2}\left(2+\frac{1}{Q^2}\right) \pm \frac{1}{Q}\sqrt{1+\frac{1}{4Q^2}} \approx 1 \pm \frac{1}{Q} \\
\left(\frac{f_{\pm 3}}{f_0}\right) &\approx \sqrt{1 \pm \frac{1}{Q}} \approx 1 \pm \frac{1}{2Q}, \quad Q >> 1
\end{aligned}$$

Therefore, the 3-dB bandwidth is, for large Q, approximately

$$B = f_3 - f_{-3} \approx \left(1+\frac{1}{2Q}\right) f_0 - \left(1-\frac{1}{2Q}\right) f_0 = \frac{f_0}{Q}$$

d. Combinations of components giving

$$RC = 2.2508 \times 10^{-4} \text{ seconds}$$

and

$$\frac{R_a}{R_b} = 2.5757$$

will work.

Problem 2.42

a. By voltage division, with the inductor replaced by $j2\pi fL$, the frequency response function is

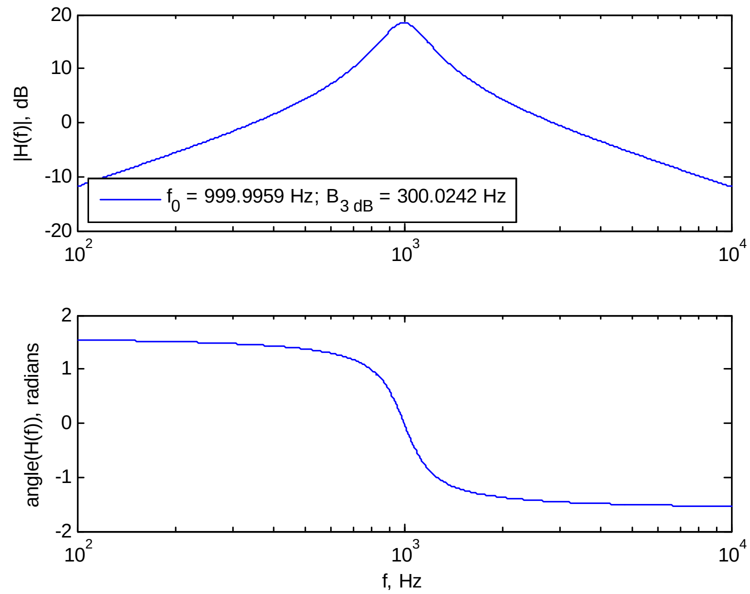
$$H_1(f) = \frac{R_2 + j2\pi fL}{R_1 + R_2 + j2\pi fL} = \frac{R_2/L + j2\pi f}{(R_1+R_2)/L + j2\pi f}$$

By long division

$$H_1(f) = 1 - \frac{R_1/L}{\frac{R_1+R_2}{L} + j2\pi f}$$

Using the transforms of a delta function and a one-sided exponential, we obtain

$$h_1(t) = \delta(t) - \frac{R_1}{L}\exp\left(-\frac{R_1+R_2}{L}t\right)u(t)$$



b. Substituting the ac-equivalent impedance for the inductor and using voltage division, the frequency response function is

$$
\begin{aligned}
H_2(f) &= \frac{R_2 || (j2\pi f L)}{R_1 + R_2 || (j2\pi f L)} \text{ where } R_2 || (j2\pi f L) = \frac{j2\pi f L R_2}{R_2 + j2\pi f L} \\
&= \frac{R_2}{R_1 + R_2} \frac{j2\pi f L}{\frac{R_1 R_2}{R_1 + R_2} + j2\pi f L} = \frac{R_2}{R_1 + R_2} \left(1 - \frac{(R_1 \parallel R_2)/L}{(R_1 \parallel R_2)/L + j2\pi f} \right)
\end{aligned}
$$

where $R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2}$. Therefore, the impulse response is

$$
h_2(t) = \frac{R_2}{R_1 + R_2} \left[\delta(t) - \frac{R_1 R_2}{(R_1 + R_2) L} \exp\left(-\frac{R_1 R_2}{(R_1 + R_2) L} t \right) u(t) \right]
$$

Both have a high pass amplitude response, with the dc gain of the first circuit being $\frac{R_2}{R_1 + R_2}$ and the second being 0; the high frequency gain of the first is 1 and that of the second is $\frac{R_2}{R_1 + R_2}$.

Problem 2.51

Application of the Payley-Wiener criterion gives the integral

$$
I = \int_{-\infty}^{\infty} \frac{\beta f^2}{1 + f^2} df
$$

which does not converge. Hence, the given function is not suitable as the frequency response function of a causal LTI system.

Problem 2.52

a. The condition for stability is

$$\begin{aligned}\int_{-\infty}^{\infty} |h_1(t)|\, dt &= \int_{-\infty}^{\infty} |\exp(-\alpha|t|)\cos(2\pi f_0 t)|\, dt \\ &= 2\int_0^{\infty} \exp(-\alpha t)\,|\cos(2\pi f_0 t)|\, dt < 2\int_0^{\infty} \exp(-\alpha t)\, dt = \frac{2}{\alpha} < \infty\end{aligned}$$

which follows because $|\cos(2\pi f_0 t)| \leq 1$. Hence this system is BIBO stable.

b. The condition for stability is

$$\begin{aligned}\int_{-\infty}^{\infty} |h_2(t)|\, dt &= \int_{-\infty}^{\infty} |\cos(2\pi f_0 t)\, u(t)|\, dt \\ &= \int_0^{\infty} |\cos(2\pi f_0 t)|\, dt \to \infty\end{aligned}$$

which follows by integrating one period of $|\cos(2\pi f_0 t)|$ and noting that the total integral is the limit of one period of area times N as $N \to \infty$. This system is not BIBO stable.

c. The condition for stability is

$$\begin{aligned}\int_{-\infty}^{\infty} |h_3(t)|\, dt &= \int_{-\infty}^{\infty} \frac{1}{|t|} u(t-1)\, dt \\ &= \int_1^{\infty} \frac{dt}{t} = \ln(t)|_1^{\infty} \to \infty\end{aligned}$$

This system is not BIBO stable.

d. The condition for stability is

$$\begin{aligned}\int_{-\infty}^{\infty} |h_4(t)|\, dt &= \int_{-\infty}^{\infty} |e^{-t}u(t) - e^{-((t-1))}u(t-1)|dt \\ &\leq \int_0^{\infty} \left(e^{-t} - e^{-(t-1)}\right) dt = 1 + e < \infty\end{aligned}$$

This system is BIBO stable.

e. The condition for stability is

$$\int_{-\infty}^{\infty} |h_5(t)|\, dt = \int_{1}^{\infty} t^{-2}|dt = 1 < \infty$$

This system is BIBO stable.

f. The condition for stability is

$$\int_{-\infty}^{\infty} |h_6(t)|\, dt = \int_{-\infty}^{\infty} \operatorname{sinc}(2t)\, dt = \frac{1}{2} < \infty$$

This system is BIBO stable.

Problem 2.53

The energy spectral density of the output is

$$G_y(f) = |H(f)|^2 \, |X(f)|^2$$

where

$$\begin{aligned} |H(f)|^2 &= \frac{25}{16 + (2\pi f)^2} \\ X(f) &= \frac{1}{3 + j2\pi f};\ G_x(f) = |X(f)|^2 = \frac{1}{9 + (2\pi f)^2} \end{aligned}$$

Hence

$$G_y(f) = \frac{25}{\left[9 + (2\pi f)^2\right]\left[16 + (2\pi f)^2\right]}$$

Plots of the input and output energy spectral densities are left to the student.

Problem 2.54

Using the Fourier coeffi cients of a half-rectified sine wave from Table 2.1 and noting that those of a half-rectified cosine wave are related by

$$X_{c_n} = X_{s_n} e^{-jn\pi/2}$$

The fundamental frequency is 10 Hz. The ideal rectangular filter passes all frequencies less than 31 Hz and rejects all frequencies greater than 31 Hz. Therefore

$$\begin{aligned}
y(t) &= -\frac{A}{3\pi}e^{j\pi}e^{-j40\pi t}+\frac{jA}{4}e^{j\pi/2}e^{-j20\pi t}+\frac{A}{\pi}+\frac{jA}{4}e^{-j\pi/2}e^{j20\pi t}-\frac{A}{3\pi}e^{-j\pi}e^{j40\pi t} \\
&= \frac{A}{\pi}-\frac{A}{2}\left(\frac{e^{j(20\pi t-\pi/2)}+e^{-j(20\pi t-\pi/2)}}{2j}\right)-\frac{2A}{3\pi}\left(\frac{e^{j(40\pi t-\pi)}+e^{-j(40\pi t-\pi)}}{2}\right) \\
&= \frac{A}{\pi}-\frac{A}{2}\sin(20\pi t-\pi/2)-\frac{2A}{3\pi}\cos(40\pi t-\pi) \\
&= \frac{A}{\pi}+\frac{A}{2}\cos(20\pi t)+\frac{2A}{3\pi}\cos(40\pi t)
\end{aligned}$$

Problem 2.55

a. The energy spectral density is given by

$$G_1(f) = |X_1(f)|^2 = \left|\frac{1}{\alpha + j2\pi f}\right|^2 = \frac{1}{\alpha^2 + (2\pi f)^2}$$

Since $E_{total} = \frac{1}{2\alpha}$, the 90% energy containment bandwidth is given by

$$\begin{aligned}
\frac{0.9}{2\alpha} &= 2\int_0^{B_{90}} \frac{1}{\alpha^2 + (2\pi f)^2} df = \frac{2}{\alpha^2}\int_0^{B_{90}} \frac{1}{1 + \left(\frac{2\pi f}{\alpha}\right)^2} df, \quad u = \frac{2\pi f}{\alpha} \\
&= \frac{1}{\pi\alpha}\int_0^{2\pi B_{90}/\alpha} \frac{du}{1+u^2} = \frac{1}{\pi\alpha}\tan^{-1}(2\pi B_{90}/\alpha) \\
\text{or } B_{90} &= \frac{\alpha}{2\pi}\tan(0.45\pi) = 1.0049\alpha
\end{aligned}$$

b. For this case, using $X_2(f) = \Pi(f/2W)$ and $E_{total} = 2W$, we obtain

$$\begin{aligned}
0.9(2W) &= 2\int_0^{B_{90}} df = 2B_{90} \\
\text{or } B_{90} &= 0.9W
\end{aligned}$$

c. For this case, using $X_3(f) = \tau\text{sinc}(f\tau)$ and $E_3 = \tau$, we obtain

$$\begin{aligned}
0.9\tau &= 2\int_0^{B_{90}} \tau^2\text{sinc}^2(f\tau)\, df \\
\text{or } \quad 0.45 &= \int_0^{\tau B_{90}} \text{sinc}^2(u)\, du
\end{aligned}$$

Numerical integration gives

$$B_{90} = 0.9/\tau$$

d. For this case, using $X_4(f) = \tau\text{sinc}^2(f\tau)$ and $E_4 = 2\int_0^{\tau}(1 - t/\tau)^2\, dt = 2\tau/3$, we obtain

$$\begin{aligned}
1.8\tau/3 &= 2\int_0^{B_{90}} \tau^2\text{sinc}^4(f\tau)\, df \\
\text{or } \quad 0.3 &= \int_0^{\tau B_{90}} \text{sinc}^4(u)\, du
\end{aligned}$$

Numerical integration gives

$$B_{90} = 0.35/\tau$$

e. For this case, using $X_5(f) = \frac{2\alpha}{\alpha^2+(2\pi f)^2}$ and $E_5 = 2\int_0^\infty \exp(-2\alpha t)\,dt = 1/\alpha$, we obtain

$$\begin{aligned} 0.9/\alpha &= 2\int_0^{B_{90}} \left[\frac{2\alpha}{\alpha^2+(2\pi f)^2}\right]^2 df \\ \text{or } 0.225\pi &= \int_0^{2\pi B_{90}/\alpha} \frac{du}{(1+u^2)^2},\ u = 2\pi f/\alpha \end{aligned}$$

Numerical integration gives $2\pi B_{90}/\alpha = 1.18$ or $B_{90}/\alpha = 0.188$ or $B_{90} = 0.188\alpha$.

Problem 2.56

The outputs are the inputs phase shifted by $-\pi/2$ radians for frequencies greater than 0 and $\pi/2$ for frequencies less than 0 Hz.

a. $y_1(t) = \exp[j100\pi t - j\pi/2] = -j\exp(j100\pi t)$;

b. $y_2(t) = \frac{1}{2}\exp[j100\pi t - j\pi/2] + \frac{1}{2}\exp[-j100\pi t + j\pi/2] = \sin(100\pi t)$;

c. $y_3(t) = \frac{1}{2j}\exp[j100\pi t - j\pi/2] - \frac{1}{2j}\exp[-j100\pi t + j\pi/2] = -\cos(100\pi t)$;

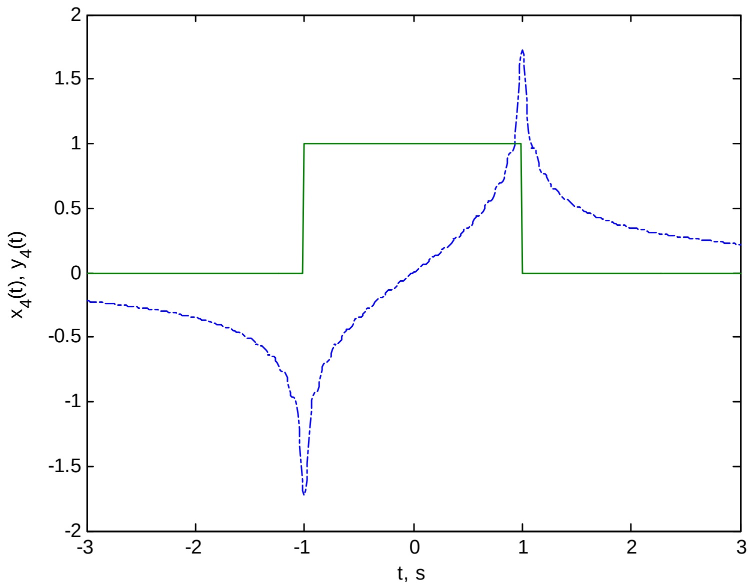
d. The spectrum of the input is $X_4(f) = 2\text{sinc}(2f)$. The spectrum of the output is

$$Y_4(f) = \begin{cases} 2\text{sinc}(2f)\exp(-j\pi/2),\ f>0 \\ 2\text{sinc}(2f)\exp(j\pi/2),\ f<0 \end{cases}$$

The time domain output signal is the inverse Fourier transform of this, which is

$$\begin{aligned} y_4(t) &= \int_0^\infty 2\text{sinc}(2f)\exp(-j\pi/2)\exp(j2\pi ft)\,df + \int_{-\infty}^0 2\text{sinc}(2f)\exp(j\pi/2)\exp(j2\pi ft)\,df \\ &= -2j\int_0^\infty \text{sinc}(2f)\exp(j2\pi ft)\,df + 2j\int_\infty^0 \text{sinc}(-2u)\exp(-j2\pi ut)(-du)\,;\quad u=-f \\ &= -2j\int_0^\infty \text{sinc}(2f)\exp(j2\pi ft)\,df + 2j\int_0^\infty \text{sinc}(2u)\exp(-j2\pi ut)\,du;\quad \text{sinc}2u \text{ is even} \\ &= 2\int_0^\infty \text{sinc}(2f)\left[-j\exp(j2\pi ft) + j\exp(-j2\pi ft)\right]df;\quad \text{rewrite 2nd int in terms of } f \\ &= 4\int_0^\infty \text{sinc}(2f)\sin(2\pi ft)\,df \end{aligned}$$

This requires numerical integration. A plot is given in Fig. 2.8.



Problem 2.57

a. Amplitude distortion; no phase distortion.

The output for (a) is

$$\begin{aligned} y_a(t) &= 4\cos\left(48\pi t - 24\frac{\pi}{150}\right) + 10\cos\left(126\pi t - 63\frac{\pi}{150}\right) \\ &= 4\cos 48\pi\left(t - \frac{1}{300}\right) + 10\cos 126\pi\left(t - \frac{1}{300}\right) \end{aligned}$$

b. No amplitude distortion; phase distortion.

The output for (b) is

$$\begin{aligned} y_b(t) &= 2\cos\left(126\pi t - 63\frac{\pi}{150}\right) + \cos(170\pi t) \\ &= 2\cos 126\pi\left(t - \frac{1}{300}\right) + \cos(170\pi t) \end{aligned}$$

c. No amplitude distortion; no phase distortion.

The output for (c) is

$$y_c(t) = 2\cos 126\pi \left(t - \frac{1}{300}\right) + 6\cos 144\pi \left(t - \frac{1}{300}\right)$$

d. No amplitude distortion; no phase distortion.

The output for (d) is

$$y_d(t) = 4\cos 10\pi \left(t - \frac{1}{300}\right) + 16\cos 50\pi \left(t - \frac{1}{300}\right)$$

Problem 2.58

a. The frequency response function corresponding to this impulse response is

$$H_1(f) = \frac{3}{5 + j2\pi f} = \frac{3}{\sqrt{25 + (2\pi f)^2}} \exp\left[-j\tan^{-1}\left(\frac{2\pi f}{5}\right)\right]$$

The group delay is

$$\begin{aligned} T_{g_1}(f) &= -\frac{1}{2\pi}\frac{d}{df}\left[-\tan^{-1}\left(\frac{2\pi f}{5}\right)\right] \\ &= \frac{1}{2\pi}\frac{1}{1 + \left(\frac{2\pi f}{5}\right)^2}\frac{2\pi}{5} \\ &= \frac{5}{25 + (2\pi f)^2} \end{aligned}$$

The phase delay is

$$T_{p1}(f) = -\frac{\theta_1(f)}{2\pi f} = \frac{\tan^{-1}\left(\frac{2\pi f}{5}\right)}{2\pi f}$$

b. The frequency response function corresponding to this impulse response is

$$\begin{aligned} H_2(f) &= \frac{5}{3 + j2\pi f} - \frac{2}{5 + j2\pi f} \\ &= \frac{5(5 + j2\pi f) - 2(3 + j2\pi f)}{(3 + j2\pi f)(5 + j2\pi f)} \\ &= \frac{19 + j6\pi f}{15 - (2\pi f)^2 + j16\pi f} \\ &= \frac{\sqrt{361 + (6\pi f)^2}\exp\left[j\tan^{-1}\left(\frac{6\pi f}{19}\right)\right]}{\sqrt{\left[15 - (2\pi f)^2\right]^2 + (16\pi f)^2}\exp\left[j\tan^{-1}\left(\frac{16\pi f}{15 - (2\pi f)^2}\right)\right]} \end{aligned}$$

Therefore

$$\theta_2(f) = \tan^{-1}\left(\frac{6\pi f}{19}\right) - \tan^{-1}\left(\frac{16\pi f}{15-(2\pi f)^2}\right)$$

The group delay is

$$\begin{aligned}
T_{g_2}(f) &= -\frac{1}{2\pi}\frac{d}{df}\left[\tan^{-1}\left(\frac{6\pi f}{19}\right) - \tan^{-1}\left(\frac{16\pi f}{15-(2\pi f)^2}\right)\right] \\
&= -\frac{1}{2\pi}\left\{ \begin{array}{c} \frac{1}{1+\left(\frac{6\pi f}{19}\right)^2}\left(\frac{6\pi}{19}\right) \\ -\frac{1}{1+\left(\frac{16\pi f}{15-(2\pi f)^2}\right)^2}\frac{16\pi\left(15-(2\pi f)^2\right)-16\pi f[-2(2\pi f)(2\pi)]}{\left[15-(2\pi f)^2\right]^2} \end{array} \right\} \\
&= -\frac{1}{2\pi}\left\{\frac{19(6\pi)}{361+(6\pi f)^2} - \frac{16\pi\left[15+(2\pi f)^2\right]}{\left[15-(2\pi f)^2\right]^2+(16\pi f)^2}\right\} \\
&= -\frac{57}{361+(6\pi f)^2} + \frac{8\left[15+(2\pi f)^2\right]}{225+34(2\pi f)^2+(2\pi f)^4}
\end{aligned}$$

The phase delay is

$$T_{p2}(f) = -\frac{\theta_2(f)}{2\pi f} = \frac{\tan^{-1}\left(\frac{16\pi f}{15-(2\pi f)^2}\right) - \tan^{-1}\left(\frac{6\pi f}{19}\right)}{2\pi f}$$

The group and phase delays for (a) and (b) are shown in Fig. 2.9.

c. The frequency response is

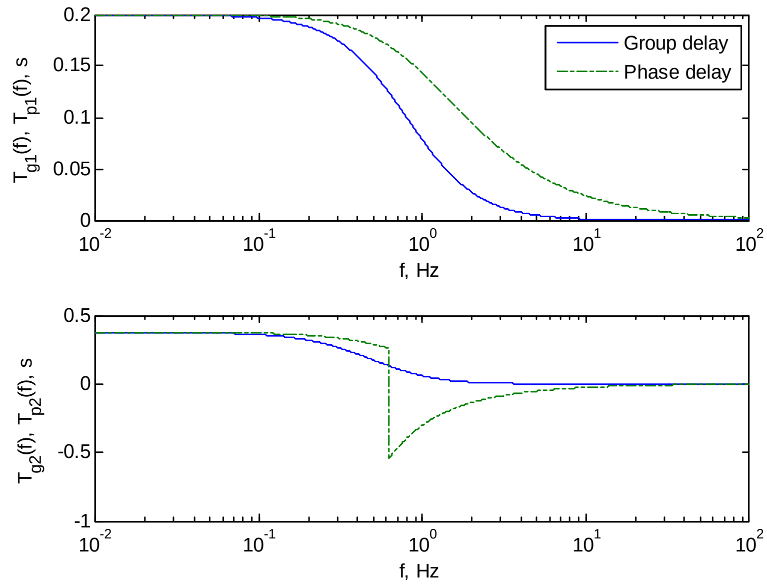
$$H(f) = \Pi\left(\frac{f}{2B}\right)\exp(-j2\pi t_0 f)$$

The group delay is

$$\begin{aligned}
T_g(f) &= -\frac{1}{2\pi}\frac{d}{df}\left[-2\pi t_0 f\right],\ -B \le f \le B \\
&= t_0,\ -B \le f \le B \text{ and 0 otherwise}
\end{aligned}$$

The phase delay is

$$T_p(f) = -\frac{(-2\pi t_0 f)}{2\pi f} = t_0,\ -B \le f \le B \text{ and 0 otherwise}$$



d. The frequency response function is

$$
\begin{aligned}
H(f) &= \frac{5}{3+j2\pi f} - \frac{2}{3+j2\pi f}\exp\left(-j2\pi t_0 f\right) \\
&= \frac{2}{3+j2\pi f}\left[2.5 - \exp\left(-j2\pi t_0 f\right)\right]
\end{aligned}
$$

The phase shift function is

$$
\theta(f) = \frac{\sin 2\pi t_0 f}{2.5 - \cos 2\pi t_0 f} - \tan^{-1}\frac{2\pi f}{3}
$$

The phase delay is

$$
T_p(f) = \frac{1}{2\pi f}\left[\tan^{-1}\frac{2\pi f}{3} = \frac{\sin 2\pi t_0 f}{2.5 - \cos 2\pi t_0 f}\right]
$$

The group delay is

$$
\begin{aligned}
T_g(f) &= \frac{1}{2\pi}\frac{d}{df}\left[\tan^{-1}\frac{2\pi f}{3} = \frac{\sin 2\pi t_0 f}{2.5 - \cos 2\pi t_0 f}\right] \\
&= \frac{3}{9+(2\pi f)^2} + \frac{1 - 2.5\cos 2\pi t_0 f}{(2.5 - \cos 2\pi t_0 f)^2}
\end{aligned}
$$

Problem 2.59

a. The amplitude response is

$$|H(f)| = \frac{2\pi|f|}{\sqrt{64 + (2\pi f)^2}\sqrt{9 + (2\pi f)^2}}$$

b. The phase response is

$$\theta(f) = \frac{\pi}{2}\operatorname{sgn}(f) - \tan^{-1}\left(\frac{\pi f}{4}\right) - \tan^{-1}\left(\frac{2\pi f}{3}\right)$$

c. The phase delay is

$$T_p(f) = \frac{1}{2\pi f}\left[\tan^{-1}\left(\frac{\pi f}{4}\right) + \tan^{-1}\left(\frac{2\pi f}{3}\right) - \frac{\pi}{2}\operatorname{sgn}(f)\right]$$

d. The group delay is

$$T_g(f) = \frac{1/8}{1 + (\pi f/4)^2} + \frac{1/3}{1 + (\pi f/3)^2} - \frac{1}{4}\delta(f)$$

Problem 2.60
In terms of the input spectrum, the output spectrum is

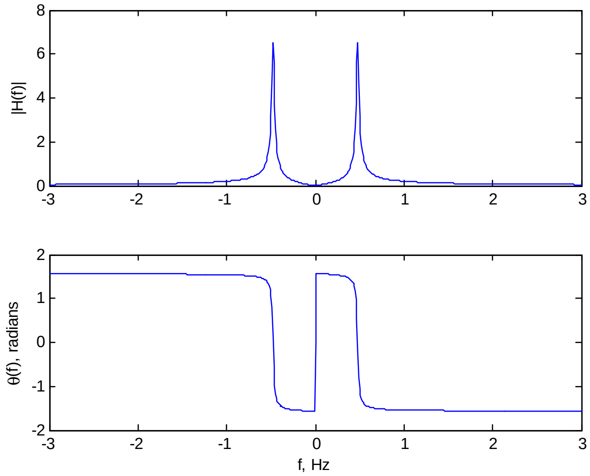
$$\begin{aligned}
Y(f) &= X(f) + 0.1X(f) * X(f) \\
&= 2\left[\Pi\left(\frac{f-10}{4}\right) + \Pi\left(\frac{f+10}{4}\right)\right] \\
&\quad +0.4\left[\Pi\left(\frac{f-10}{4}\right) + \Pi\left(\frac{f+10}{4}\right)\right] * \left[\Pi\left(\frac{f-10}{4}\right) + \Pi\left(\frac{f+10}{4}\right)\right] \\
&= 2\left[\Pi\left(\frac{f-10}{4}\right) + \Pi\left(\frac{f+10}{4}\right)\right] \\
&\quad +0.4\left[4\Lambda\left(\frac{f-20}{4}\right) + 8\Lambda\left(\frac{f}{4}\right) + 4\Lambda\left(\frac{f+20}{4}\right)\right]
\end{aligned}$$

where $\Lambda(f)$ is an isoceles triangle of unit height going from -1 to 1. The student should sketch the output spectrum given the above analytical result.

Problem 2.61

a. The amplitude response is

$$|H(f)| = \frac{2\pi|f|}{\sqrt{(9 - 4\pi^2 f^2)^2 + (0.3\pi f)^2}}$$



b. The phase response is

$$\theta(f) = \frac{\pi}{2}\text{sgn}(f) - \tan^{-1}\left(\frac{0.3\pi f}{9-4\pi^2 f^2}\right)$$

These are shown in Fig. 2.10.

c. The phase delay is

$$T_p(f) = -\frac{1}{2\pi}\left[\frac{\pi}{2}\text{sgn}(f) - \tan^{-1}\left(\frac{0.3\pi f}{9-4\pi^2 f^2}\right)\right]$$

d. The group delay is

$$\begin{aligned} T_g(f) &= -\frac{1}{2\pi}\frac{d}{df}\left[\frac{\pi}{2}\text{sgn}(f) - \tan^{-1}\left(\frac{0.3\pi f}{9-4\pi^2 f^2}\right)\right] \\ &= -\frac{1}{2\pi}\left[\pi\delta(f) - \frac{1}{1+\left(\frac{0.3\pi f}{9-4\pi^2 f^2}\right)^2}\frac{0.3\pi\left(9-4\pi^2 f^2\right)+(0.3\pi f)\left(8\pi^2 f\right)}{\left(9-4\pi^2 f^2\right)^2}\right] \\ &= \frac{1.35+0.6\pi^2 f^2}{81-71.91\pi^2 f^2+16\pi^4 f^4} - \frac{1}{2}\delta(f) \end{aligned}$$

Problem 2.62
Let $u = 2\pi t$. We then have

$$\begin{aligned} y(t) &= [\cos(u) + \cos(3u)]^3 \\ &= \cos^3(u) + 3\cos^2(u)\cos(3u) + 3\cos(u)\cos^2(3u) + \cos^3(3u) \end{aligned}$$

Use the trig identities

$$\begin{aligned} \cos^2(z) &= \frac{1}{2}[1 + \cos(2z)] \\ \cos^3(z) &= \frac{3}{4}\cos(z) + \frac{1}{4}\cos(3z) \\ \cos(w)\cos(z) &= \frac{1}{2}\cos(w - z) + \frac{1}{2}\cos(w + z) \end{aligned}$$

to get

$$\begin{aligned} y(t) &= \frac{3}{4}\cos(u) + \frac{1}{4}\cos(3u) + \frac{3}{2}[1 + \cos(2u)]\cos(3u) \\ &\quad + \frac{3}{2}\cos(u)[1 + \cos(6u)] + \frac{3}{4}\cos(3u) + \frac{1}{4}\cos(9u) \\ &= 3\cos(u) + \frac{5}{2}\cos(3u) + \frac{3}{2}\cos(5u) + \frac{3}{4}\cos(7u) + \frac{1}{4}\cos(9u) \\ &= 3\cos(2\pi t) + \frac{5}{2}\cos(6\pi t) + \frac{3}{2}\cos(10\pi t) + \frac{3}{4}\cos(14\pi t) + \frac{1}{4}\cos(18\pi t) \end{aligned}$$

Problem 2.63
Write the transfer function as

$$H(f) = H_0 e^{-j2\pi f t_0} - H_0 \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}$$

Use the inverse Fourier transform of a constant, the delay theorem, and the inverse Fourier transform of a rectangular pulse function to get

$$h(t) = H_0 \delta(t - t_0) - 2BH_0 \text{sinc}[2B(t - t_0)]$$

Problem 2.64

a. The Fourier transform of this signal is

$$X(f) = A\sqrt{2\pi b^2} \exp\left(-2\pi^2 \tau^2 f^2\right)$$

By definition, using a table of integrals,

$$T = \frac{1}{x(0)} \int_{-\infty}^{\infty} |x(t)|\, dt = \sqrt{2\pi}\tau$$

Similarly,

$$W = \frac{1}{2X(0)} \int_{-\infty}^{\infty} |X(f)|\, df = \frac{1}{2\sqrt{2\pi}\tau}$$

Therefore,

$$2WT = \frac{2}{2\sqrt{2\pi}\tau} \sqrt{2\pi}\tau = 1$$

b. The Fourier transform of this signal is

$$X(f) = \frac{2A/\alpha}{1 + (2\pi f/\alpha)^2}$$

The pulse duration is

$$T = \frac{1}{x(0)} \int_{-\infty}^{\infty} |x(t)|\, dt = \frac{2}{\alpha}$$

The bandwidth is

$$W = \frac{1}{2X(0)} \int_{-\infty}^{\infty} |X(f)|\, df = \frac{\alpha}{4}$$

Thus,

$$2WT = 2\left(\frac{\alpha}{4}\right)\left(\frac{2}{\alpha}\right) = 1$$

Problem 2.65

a. The poles for a second order Butterworth filter are given by

$$s_1 = s_2^* = -\frac{\omega_3}{\sqrt{2}}(1 - j)$$

where ω_3 is the 3-dB cutoff frequency of the Butterworth filter. Its s-domain transfer function is

$$H(s) = \frac{\omega_3^2}{\left[s + \frac{\omega_3}{\sqrt{2}}(1-j)\right]\left[s + \frac{\omega_3}{\sqrt{2}}(1+j)\right]} = \frac{\omega_3^2}{s^2 + \sqrt{2}\omega_3 s + \omega_3^2}$$

Letting $\omega_3 = 2\pi f_3$ and $s = j\omega = j2\pi f$, we obtain

$$H(j2\pi f) = \frac{4\pi^2 f_3^2}{-4\pi^2 f^2 + \sqrt{2}(2\pi f_3)(j2\pi f) + 4\pi^2 f_3^2} = \frac{f_3^2}{-f^2 + j\sqrt{2} f_3 f + f_3^2}$$

b. If the phase response function of the filter is $\theta(f)$, the group delay is

$$T_g(f) = -\frac{1}{2\pi}\frac{d}{df}[\theta(f)]$$

For the second-order Butterworth filter considered here,

$$\theta(f) = -\tan^{-1}\left(\frac{\sqrt{2}f_3 f}{f_3^2 - f^2}\right)$$

Therefore, the group delay is

$$\begin{aligned} T_g(f) &= \frac{1}{2\pi}\frac{d}{df}\left[\tan^{-1}\left(\frac{\sqrt{2}f_3 f}{f_3^2 - f^2}\right)\right] \\ &= \frac{f_3}{\sqrt{2}\pi}\frac{f_3^2 + f^2}{f_3^4 + f^4} = \frac{1}{\sqrt{2}\pi f_3}\frac{1 + (f/f_3)^2}{1 + (f/f_3)^4} \end{aligned}$$

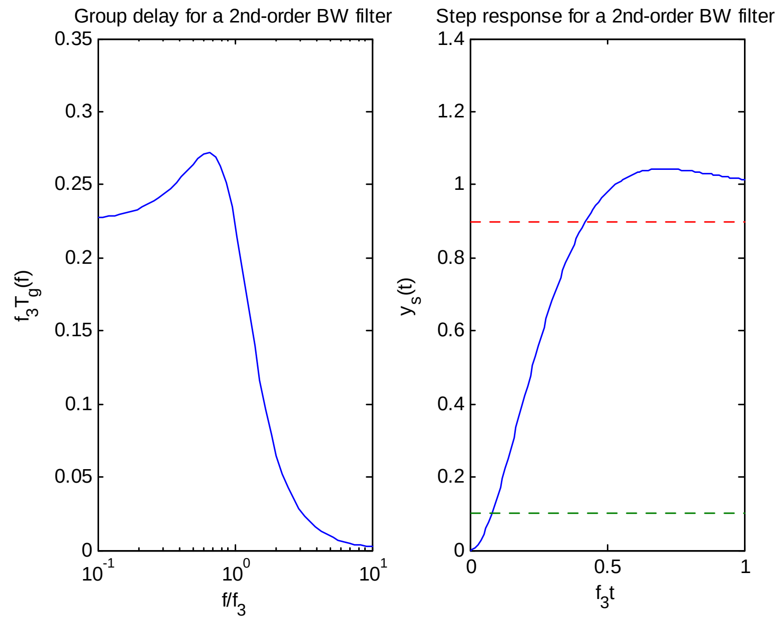
This is plotted in Fig. 2.11.

c. Use partial fraction expansion of $H(s)/s$ and then inverse Laplace transform it to get the given step response. The expansion is

$$\begin{aligned} \frac{H(s)}{s} &= \frac{\omega_3^2}{s\left(s^2 + \sqrt{2}\omega_3 s + \omega_3^2\right)} = \frac{A}{s} + \frac{Bs + C}{s^2 + \sqrt{2}\omega_3 s + \omega_3^2} \\ \text{where } A &= 1, \quad B = -1, \quad C = -\sqrt{2}\omega_3 \end{aligned}$$

This allows $H(s)/s$ to be written as

$$\begin{aligned} \frac{H(s)}{s} &= \frac{1}{s} - \frac{s + \sqrt{2}\omega_3}{s^2 + \sqrt{2}\omega_3 s + \omega_3^2} \\ &= \frac{1}{s} - \frac{s + \sqrt{2}\omega_3}{s^2 + \sqrt{2}\omega_3 s + \omega_3^2/2 - \omega_3^2/2 + \omega_3^2} \\ &= \frac{1}{s} - \frac{s + \sqrt{2}\omega_3}{\left(s + \omega_3/\sqrt{2}\right)^2 + \omega_3^2/2} \\ &= \frac{1}{s} - \frac{s + \omega_3/\sqrt{2} - \omega_3/\sqrt{2} + \sqrt{2}\omega_3}{\left(s + \omega_3/\sqrt{2}\right)^2 + \omega_3^2/2} \\ &= \frac{1}{s} - \frac{s + \omega_3/\sqrt{2}}{\left(s + \omega_3/\sqrt{2}\right)^2 + \omega_3^2/2} - \frac{\omega_3/\sqrt{2}}{\left(s + \omega_3/\sqrt{2}\right)^2 + \omega_3^2/2} \end{aligned}$$



Using the s-shift theorem of Laplace transforms, this inverse transforms to the given expression for the step response (with $\omega_3 = 2\pi f_3$). Plot it and estimate the 10% and 90% times from the plot. From the MATLAB plot of Fig. 2.10, $f_3 t_{10\%} \approx 0.08$ and $f_3 t_{90\%} \approx 0.42$ so that the 10-90 % rise time is about $0.34/f_3$ seconds.

Problem 2.66

a. Slightly less than 0.5 seconds;

b & c. Use sketches to show.

Problem 2.67

a. The leading edges of the flat-top samples follow the waveform at the sampling instants.

b. The spectrum is

$$Y(f) = X_\delta(f) H(f)$$

where

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

and

$$H(f) = \tau \text{sinc}(f\tau) \exp(-j\pi f\tau)$$

The latter represents the frequency response of a filter whose impulse response is a square pulse of width τ and implements flat top sampling. If W is the bandwidth of $X(f)$, very little distortion will result if $\tau^{-1} >> W$.

Problem 2.68

a. The sampling frequency should be large compared with the bandwidth of the signal.

b. The output spectrum of the zero-order hold circuit is

$$Y(f) = \text{sinc}(T_s f) \sum_{n=-\infty}^{\infty} X(f - nf_s) \exp(-j\pi f T_s)$$

where $f_s = T_s^{-1}$. For small distortion, we want $T_s << W^{-1}$.

Problem 2.69

Use trig identities to rewrite the signal as a sum of sinusoids:

$$\begin{aligned} x(t) &= 10\cos^2(600\pi t)\cos(2400\pi t) \\ &= 5[1 + \cos(600\pi t)]\cos(2400\pi t) \\ &= 5\cos(2400\pi t) + 2.5\cos(1800\pi t) + 2.5\cos(3000\pi t) \end{aligned}$$

The lowpass recovery filter can cut off in the range 1.5^+ kHz to 3^- kHz where the superscript $+$ means just above and the superscript $-$ means just below. The lower of these is the highest frequency of $x(t)$ and the larger is equal to the sampling frequency minus the highest frequency of $x(t)$. The minimum allowable sampling frequency is just above 3 kHz.

Problem 2.70

For bandpass sampling and recovery, all but (b) and (e) will work theoretically, although an ideal filter with bandwidth exactly equal to the unsampled signal bandwidth is necessary. For lowpass sampling and recovery, only (f) will work.

Problem 2.71

The Fourier transform is

$$\begin{aligned} Y(f) &= \frac{1}{2}X(f-f_0)+\frac{1}{2}X(f+f_0) \\ &\quad +[-j\text{sgn}(f)X(f)]*\left[\frac{1}{2}\delta(f-f_0)e^{-j\pi/2}+\frac{1}{2}\delta(f+f_0)e^{j\pi/2}\right] \\ &= \frac{1}{2}X(f-f_0)[1-\text{sgn}(f-f_0)]+\frac{1}{2}X(f+f_0)[1+\text{sgn}(f+f_0)] \end{aligned}$$

Noting that

$$\begin{aligned} \frac{1}{2}[1-\text{sgn}(f-f_0)] &= u(f_0-f) \\ \text{and}\quad \frac{1}{2}[1+\text{sgn}(f+f_0)] &= u(f+f_0) \end{aligned}$$

this may be rewritten as

$$Y(f)=X(f-f_0)\,u(f_0-f)+X(f+f_0)\,u(f+f_0)$$

Thus, if $X(f)=\Pi\left(\frac{f}{2}\right)$ (a unit-height rectangle 2 units wide centered at $f=0$) and $f_0=10$ Hz, $Y(f)$ would consist of unit-height rectangles going from -10 to -9 Hz and from 9 to 10 Hz.

Problem 2.72

a. $\widehat{x}_a(t)=\cos(\omega_0 t-\pi/2)=\sin(\omega_0 t)$, so

$$\begin{aligned} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t)\,\widehat{x}(t)\,dt &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sin(\omega_0 t)\cos(\omega_0 t)\,dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{1}{2}\sin(2\omega_0 t)\,dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\left.\frac{\cos(2\omega_0 t)}{4\omega_0}\right|_{-T}^{T}=0 \end{aligned}$$

b. Use trigonometric identities to express $x(t)$ in terms of sines and cosines. Then find the Hilbert transform of $x(t)$ by phase shifting by $-\pi/2$. Multiply $x(t)$ and $\widehat{x}(t)$ together term by term, use trigonometric identities for the product of sines and cosines, then integrate. The integrand will be a sum of terms similar to that of part (a). The limit as $T\to\infty$ will be zero term-by-term.

c. For this case $\widehat{x}(t) = A\exp(j\omega_0 t - j\pi/2) = -jA\exp(j\omega_0 t)$, take the product, integrate over time to get

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,\widehat{x}(t)\,dt &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[A\exp(j\omega_0 t)][jA\exp(j\omega_0 t)]\,dt \\ &= \lim_{T\to\infty}\frac{jA^2}{2T}\int_{-T}^{T}\exp(j2\omega_0 t)\,dt = 0\end{aligned}$$

by periodicity of the integrand

Problem 2.73

a. Note that $F[j\widehat{x}(t)] = j[-j\text{sgn}(f)]X(f)$. Hence

$$\begin{aligned}x_1(t) &= \frac{2}{3}x(t) + \frac{1}{3}j\widehat{x}(t) \to X_1(f) = \frac{2}{3}X(f) + \frac{1}{3}j[-j\text{sgn}(f)]X(f) \\ &= \left[\frac{2}{3} + \frac{1}{3}\text{sgn}(f)\right]X(f) \\ &= \begin{cases}\frac{1}{3}X(f), & f < 0 \\ X(f), & f > 0\end{cases}\end{aligned}$$

b. It follows that

$$\begin{aligned}x_2(t) &= \left[\frac{3}{4}x(t) + \frac{3}{4}j\widehat{x}(t)\right]\exp(j2\pi f_0 t) \\ &\Rightarrow X_2(f) = \frac{3}{4}[1 + \text{sgn}(f - f_0)]X(f - f_0) \\ &= \begin{cases}0, & f < f_0 \\ \frac{3}{2}X(f - f_0), & f > f_0\end{cases}\end{aligned}$$

c. This case has the same spectrum as part (a), except that it is shifted right by W Hz. That is,

$$\begin{aligned}x_3(t) &= \left[\frac{2}{3}x(t) + \frac{1}{3}j\widehat{x}(t)\right]\exp(j2\pi Wt) \\ &\to X_3(f) = \left[\frac{2}{3} + \frac{1}{3}\text{sgn}(f - W)\right]X(f - W)\end{aligned}$$

d. For this signal

$$\begin{aligned}x_4(t) &= \left[\frac{2}{3}x(t) - \frac{1}{3}j\widehat{x}(t)\right]\exp(j\pi Wt) \\ &\to X_4(f) = \left[\frac{2}{3} - \frac{1}{3}\text{sgn}(f - W/2)\right]X(f - W/2)\end{aligned}$$

Problem 2.74
The Hilbert transform of the given signal is

$$\widehat{x}(t) = 2\sin(52\pi t)$$

The signal $x_p(t)$ is

$$\begin{aligned} x_p(t) &= x(t) + j\widehat{x}(t) \\ &= 2\cos(52\pi t) + j2\sin(52\pi t) \\ &= 2e^{j52\pi t} \end{aligned}$$

a. We have, for $f_0 = 25$ Hz,

$$\tilde{x}(t) = x_p(t)\, e^{-j2\pi f_0 t} = 2e^{j52\pi t}e^{-j50\pi t} = 2e^{j2\pi t} = 2\cos(2\pi t) + j2\sin(2\pi t)$$

$$\begin{aligned} x_R(t) &= 2\cos(2\pi t) \\ x_I(t) &= 2\sin(2\pi t) \end{aligned}$$

b. For $f_0 = 27$ Hz,

$$\tilde{x}(t) = 2e^{j52\pi t}e^{-j54\pi t} = 2e^{-j2\pi t} = 2\cos(2\pi t) - j2\sin(2\pi t)$$

$$\begin{aligned} x_R(t) &= 2\cos(2\pi t) \\ x_I(t) &= -2\sin(2\pi t) \end{aligned}$$

c. For $f_0 = 10$ Hz,

$$\tilde{x}(t) = 2e^{j52\pi t}e^{-j20\pi t} = 2e^{j32\pi t} = 2\cos(32\pi t) + j2\sin(32\pi t)$$

$$\begin{aligned} x_R(t) &= 2\cos(32\pi t) \\ x_I(t) &= 2\sin(32\pi t) \end{aligned}$$

d. For $f_0 = 15$ Hz,

$$\tilde{x}(t) = 2e^{j52\pi t}e^{-j30\pi t} = 2e^{j22\pi t} = 2\cos(22\pi t) + j2\sin(22\pi t)$$

$$\begin{aligned} x_R(t) &= 2\cos(22\pi t) \\ x_I(t) &= 2\sin(22\pi t) \end{aligned}$$

e. For $f_0 = 30$ Hz,

$$\tilde{x}(t) = 2e^{j52\pi t}e^{-j60\pi t} = 2e^{-j8\pi t} = 2\cos(8\pi t) - j2\sin(8\pi t)$$

$$\begin{aligned} x_R(t) &= 2\cos(8\pi t) \\ x_I(t) &= -2\sin(8\pi t) \end{aligned}$$

f. For $f_0 = 20$ Hz,

$$\tilde{x}(t) = 2e^{j52\pi t}e^{-j40\pi t} = 2e^{j12\pi t} = 2\cos(12\pi t) + j2\sin(12\pi t)$$

$$\begin{aligned} x_R(t) &= 2\cos(12\pi t) \\ x_I(t) &= 2\sin(12\pi t) \end{aligned}$$

Problem 2.75

For $t < \tau/2$, the output is zero. For $|t| \leq \tau/2$, the result is

$$\begin{aligned} y(t) &= \frac{\alpha/2}{\sqrt{\alpha^2 + (2\pi\Delta f)^2}} \\ &\times \left\{\cos\left[2\pi(f_0 + \Delta f)t - \theta\right] - e^{-\alpha(t+\tau/2)}\cos\left[2\pi(f_0 + \Delta f)t + \theta\right]\right\} \end{aligned}$$

For $t > \tau/2$, the result is

$$\begin{aligned} y(t) &= \frac{(\alpha/2)e^{-\alpha t}}{\sqrt{\alpha^2 + (2\pi\Delta f)^2}} \\ &\times \left\{e^{\alpha\tau/2}\cos\left[2\pi(f_0 + \Delta f)t - \theta\right] - e^{-\alpha\tau/2}\cos\left[2\pi(f_0 + \Delta f)t + \theta\right]\right\} \end{aligned}$$

In the above equations, θ is given by

$$\theta = -\tan^{-1}\left(\frac{2\pi\Delta f}{\alpha}\right)$$

2.2 Computer Exercises

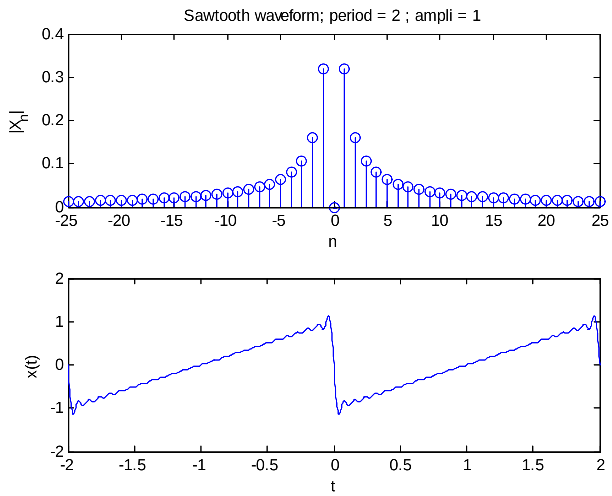
Computer Exercise 2.1

```
% ce2_1.m: Amplitude spectra and Fourier series synthesized
% for various periodic waveforms
%
% R. Ziemer & W. Tranter, Principles of Communications, 7th edition
%
clear all; clf
N = input('Number of harmonics in Fourier sum => ');
T = input('Period of periodic waveform => ');
A = input('Amplitude of waveform => ');
n = -N:1:N;
I_type = input('1 = rect pulse train; 2 = HR sinewave; 3 = FR sinewave; 4 = triang
pulse train; 5 = triangle wave; 6 = sawtooth wave => ');
X = zeros(size(n));
if I_type == 1
tau = input('Width of rectangular pulse => ');
t0 = input('Delay of rectangular pulse center => ');
d = tau/T;
X = A*d*sinc(n*d).*exp(-j*2*pi*n*t0/T);
elseif I_type == 2
I = find(n == 1 | n == -1);
II = find(rem(n, 2) == 0);
III = find(rem(abs(n), 2) == 1 & (n ~= 1 & n ~= -1));
X(I) = -0.25*j*n(I)*A;
X(II) = A./(pi*(1-n(II).*n(II)));
X(III) = 0;
elseif I_type == 3
X = 2*A./(pi*(1-4*n.*n));
elseif I_type == 4
tau = input('Half width of triangle pulse => ');
t0 = input('Delay of triangle pulse => ');
d = tau/T;
X = A*d*sinc(n*d).*sinc(n*d).*exp(-j*2*pi*n*t0/T);
elseif I_type == 5
I = find(rem(abs(n), 2) == 1);
X(I) = 4*A./(pi^2*n(I).^2);
elseif I_type == 6
```

```
I = find(n~=0);
X(I) = 2*A*(-exp(-j*2*pi*n(I))+(1 - exp(-j*2*pi*n(I)))./(j*2*pi*n(I)))./(j*2*pi*n(I));
end
subplot(2,1,1), stem(n, abs(X)),xlabel('n'), ylabel('|X_n|'), ...
if I_type == 1
title(['Rectangular pulse train; period = ', num2str(T), '; delay = ', num2str(t0), '; ampli = ', num2str(A)])
elseif I_type == 2
title(['Half-rectified sinewave; period = ', num2str(T),' ; ampli = ', num2str(A)])
elseif I_type == 3
title(['Full-rectified sinewave; period = ', num2str(T),' ; ampli = ', num2str(A)])
elseif I_type == 4
title(['Triangle pulse train; ', num2str(2*N+1), ' terms. A = ', num2str(A), ', \tau = ', num2str(tau), ', T = ', num2str(T) ' s; d = ', num2str(d), '; t_0 = ', num2str(t0), ' s'])
elseif I_type == 5
title(['Triangle waveform; period = ', num2str(T),'; ampli = ', num2str(A)])
elseif I_type == 6
title(['Sawtooth waveform; period = ', num2str(T),' ; ampli = ', num2str(A)])
end
fn = n./T;
t = -T:T/500:T;
x = real(X*exp(j*2*pi*fn'*t));
subplot(2,1,2), plot(t, x), xlabel('t'), ylabel('x(t)'), ...

>> ce2_1
Number of harmonics in Fourier sum => 25
Period of periodic waveform => 2
Amplitude of waveform => 1
1 = rect pulse train; 2 = HR sinewave; 3 = FR sinewave; 4 = triang pulse train; 5 = triangle wave; 6 = sawtooth wave => 6
>>
```

Computer Exercise 2.2

```
% ce2_2.m: Plot of line spectra for half-rectified,
% full-rectified sinewave, square wave, and triangle wave
%
% R. Ziemer & W. Tranter, Principles of Communications, 7th edition
%
clear all; clf
```



waveform = input('Enter type of waveform: 1 = HR sine; 2 = FR sine; 3 = square; 4 = triangle: ');

A = 1;

n_max = 13; % maximum harmonic plotted; odd

n = -n_max:1:n_max;

if waveform == 1

X = A./(pi*(1+eps - n.^2)); % Offset 1 slightly to avoid divide by zero

for m = 1:2:2*n_max+1

X(m) = 0; % Set odd harmonic lines to zero

X(n_max+2) = -j*A/4; % Compute lines for n = 1 and n = -1

X(n_max) = j*A/4;

end

elseif waveform == 2

X = 2*A./(pi*(1+eps - 4*n.^2));

elseif waveform == 3

X = abs(4*A./(pi*n+eps));

for m = 2:2:2*n_max+1

X(m) = 0;

end

elseif waveform == 4

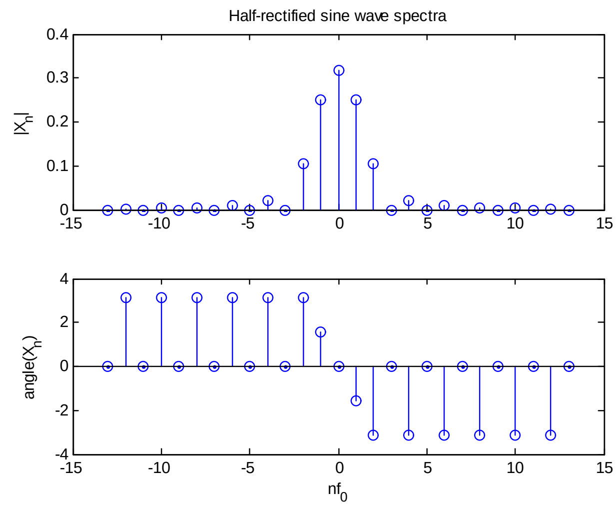
X = 4*A./(pi*n+eps).^2;

for m = 2:2:2*n_max+1

```
X(m) = 0;
end
end
[arg_X, mag_X] = cart2pol(real(X),imag(X));      % Convert to magnitude and phase
if waveform == 1
for m = n_max+3:2:2*n_max+1
arg_X(m) = arg_X(m) - 2*pi;                      % Force phase to be odd
end
elseif waveform == 2
m = find(n > 0);
arg_X(m) = arg_X(m) - 2*pi;                      % Force phase to be odd
elseif waveform == 4
arg_X = mod(arg_X, 2*pi);
end
subplot(2,1,1),stem(n, mag_X),ylabel('|X_n|')
if waveform == 1
title('Half-rectified sine wave spectra')
elseif waveform == 2
title('Full-rectified sine wave spectra')
elseif waveform == 3
title('Spectra for square wave with even symmetry ')
elseif waveform == 4
title('Spectra for triangle wave with even symmetry')
end
subplot(2,1,2),stem(n, arg_X),xlabel('nf_0'),ylabel('angle(X_n)')
>> ce2_2
Enter type of waveform: 1 = HR sine; 2 = FR sine; 3 = square; 4 = triangle: 1
>>
```

Computer Exercise 2.3

```
% ce2_3.m: FFT plotting of line spectra for half-rectified, full-rectified sinewave, square
wave,
% and triangular waveforms
%
% R. Ziemer & W. Tranter, Principles of Communications, 7th edition
%
% User defined functions used: pls_fn( ); trgl_fn( )
%
clf
```



I_wave = input('Enter type of waveform: 1 = positive squarewave; 2 = 0-dc level triangular; 3 = half-rect. sine; 4 = full-wave sine: ');

```
T = 2;
del_t = 0.001;
t = 0:del_t:T;
L = length(t);
fs = (L-1)/T;
del_f = 1/T;
n = 0:9;
if I_wave == 1
x = pls_fn(2*(t-T/4)/T);
X_th = abs(0.5*sinc(n/2));
disp(' ')
disp(' 0 - 1 level squarewave')
elseif I_wave == 2
x = 2*trgl_fn(2*(t-T/2)/T)-1;
X_th = 4./(pi^2*n.^2);
X_th(1) = 0; % Set n = even coeffi cients to zero (odd indexed because of MATLAB)
X_th(3) = 0;
X_th(5) = 0;
X_th(7) = 0;
X_th(9) = 0;
disp(' ')
```

```
disp(' 0-dc level triangular wave')
elseif I_wave == 3
x = sin(2*pi*t/T).*pls_fn(2*(t-T/4)/T);
X_th = abs(1./(pi*(1-n.^2)));
X_th(2) = 0.25;
X_th(4) = 0;        % Set n = odd coeffi cients to zero (even indexed because of MATLAB)
X_th(6) = 0;
X_th(8) = 0;
X_th(10) = 0;
disp(' ')
disp(' Half-rectified sinewave')
elseif I_wave == 4
x = abs(sin(pi*t/T));        % Period of full-rectified sinewave is T/2
X_th = abs(2./(pi*(1-4*n.^2)));
disp(' ')
disp(' Full-rectified sinewave')
end
X = 0.5*fft(x)*del_t; % Multiply by 0.5 because of 1/T_0 with T_0 = 2
f = 0:del_f:fs;
Y = abs(X(1:10));
Z = [n' Y' X_th'];
disp('Magnitude of the Fourier coeffi cients');
disp(' ')
disp(' n FFT Theory');
disp(' ____________________________')
disp(' ')
disp(Z);
subplot(2,1,1),plot(t, x), xlabel('t'), ylabel('x(t)')
subplot(2,1,2),plot(f, abs(X),'o'),axis([0 10 0 1]),...
xlabel('n'), ylabel('|X_n|')

% Unit-width pulse function
%
function y = pls_fn(t)
y = stp_fn(t+0.5)-stp_fn(t-0.5);

function y = stp_fn(t)
% Function for generating the unit step
%
y = zeros(size(t));
```

```
I = find(t >= 0);
y(I) = ones(size(I));

function y = trgl_fn(t)
%     This function generates a unit-high triangle centered
%     at zero and extending from -1 to 1
%
y = (1 - abs(t)).*pls_fn(t/2);
%
% End of script file
```

A typical run follows with a plot given in Fig. 2.13.

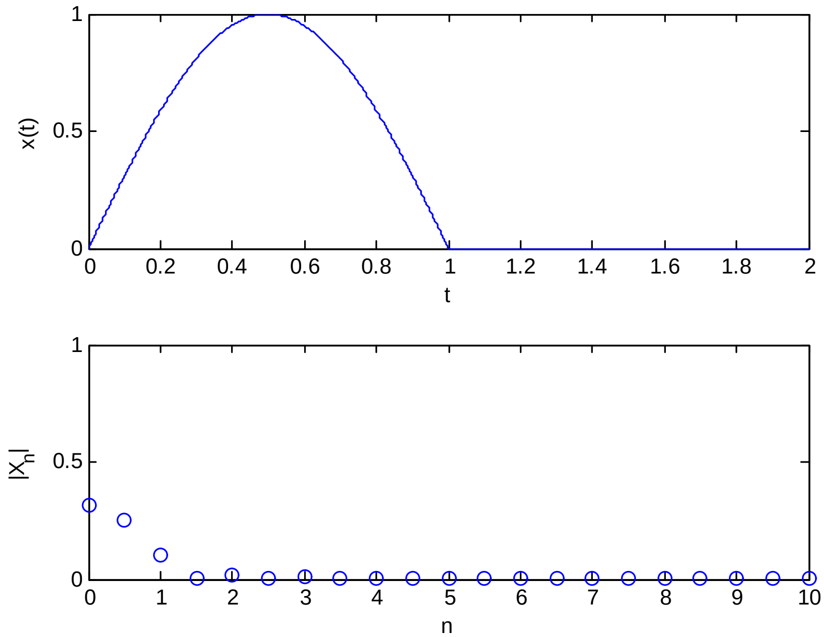
```
>> ce2_3
Enter type of waveform: 1 = positive squarewave; 2 = 0-dc level triangular; 3 = half-rect.
sine; 4 = full-wave sine: 3
    Half-rectified sinewave

Magnitude of the Fourier coeffi cients
n          FFT       Theory
____________________________
0          0.3183    0.3183
1.0000     0.2501    0.2500
2.0000     0.1062    0.1061
3.0000     0.0001    0
4.0000     0.0212    0.0212
5.0000     0.0001    0
6.0000     0.0091    0.0091
7.0000     0.0000    0
8.0000     0.0051    0.0051
9.0000     0.0000    0
```

Computer Exercise 2.4

Make the time window long compared with the pulse width.

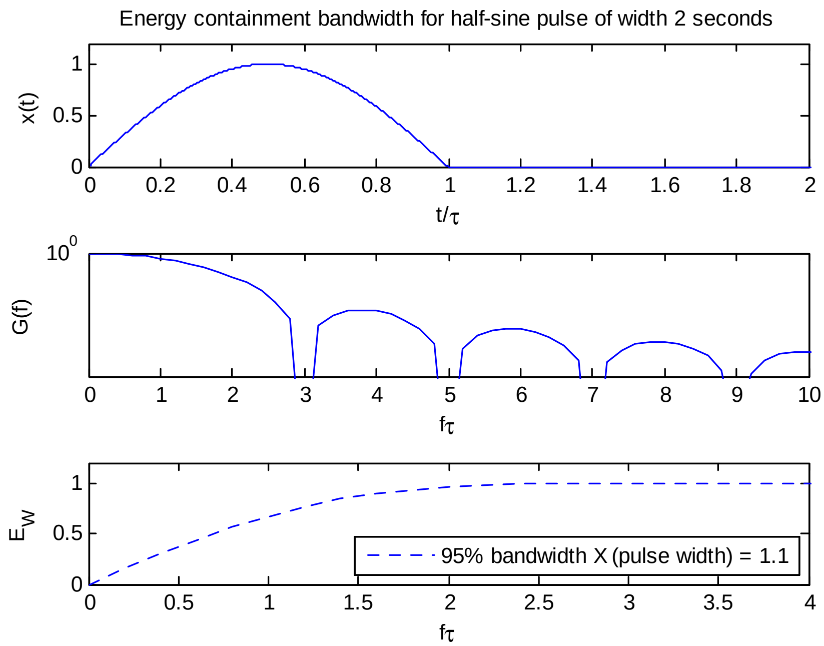
Computer Exercise 2.5

```
% ce2_5.m: Finding the energy ratio in a preset bandwidth
%
% R. Ziemer & W. Tranter, Principles of Communications, 7th edition
%
% User defined functions used: pls_fn( ); trgl_fn( )
```


x(t)
t
|X_n|
n

```
%
I_wave = input('Enter type of waveform: 1 = rectangular; 2 = triangular; 3 = half-rect. sine; 4 = raised cosine: ');
tau = input('Enter pulse width: ');
per_cent = input('Enter percent total desired energy: ');
clf
T = 20;
f = [];
G = [];
del_t = 0.001;
t = 0:del_t:T;
L = length(t);
fs = (L-1)/T;
del_f = 1/T;
% n = [0 1 2 3 4 5 6 7 8 9];
if I_wave == 1
    x = pls_fn((t-tau/2)/tau);
    disp(' ')
    disp(' Rectangular pulse')
elseif I_wave == 2
    x = trgl_fn(2*(t-tau/2)/tau);
    disp(' ')
    disp(' Triangular pulse')
elseif I_wave == 3
    x = sin(pi*t/tau).*pls_fn((t-tau/2)/tau);
    disp(' ')
    disp(' Half sinewave')
elseif I_wave == 4
    x = abs(sin(pi*t/tau).^2.*pls_fn((t-tau/2)/tau));
    disp(' ')
    disp(' Raised sinewave')
end
X = fft(x)*del_t;
f1 = 0:del_f*tau:fs*tau;
G1 = X.*conj(X);
NN = floor(length(G1)/2);
G = G1(1:NN);
ff = f1(1:NN);
f = f1(1:NN+1);
E_tot = sum(G);
```

```
E_f = cumsum(G);
E_W = [0 E_f]/E_tot;
test = E_W - per_cent/100;
L_test = length(test);
k = 1;
while test(k) <= 0
    k = k+1;
end
B = k*del_f;
if I_wave == 2
    tau1 = tau/2;
else
    tau1 = tau;
end
subplot(3,1,1),plot(t/tau, x), xlabel('t/\tau'), ylabel('x(t)'), axis([0 2 0 1.2])
if I_wave == 1
    title(['Energy containment bandwidth for rectangular pulse of width ', num2str(tau),' seconds'])
elseif I_wave == 2
    title(['Energy containment bandwidth for triangular pulse of width ', num2str(tau),' seconds'])
elseif I_wave == 3
    title(['Energy containment bandwidth for half-sine pulse of width ', num2str(tau),' seconds'])
elseif I_wave == 4
    title(['Energy containment bandwidth for raised cosine pulse of width ', num2str(tau),' seconds'])
end
subplot(3,1,2),semilogy(ff*tau1, abs(G./max(G))), xlabel('f\tau'), ylabel('G(f)'), axis([0 10 1e-5 1])
subplot(3,1,3),plot(f*tau1, E_W,'--'), xlabel('f\tau'), ylabel('E_W'), axis([0 4 0 1.2])
legend([num2str(per_cent), '% bandwidth X (pulse width) = ', num2str(B*tau)],4)

%    This function generates a unit-high triangle centered
%    at zero and extending from -1 to 1
%
function y = trgl_fn(t)
y = (1 - abs(t)).*pls_fn(t/2);

% Unit-width pulse function
```




```
%
function y = pls_fn(t)
y = stp_fn(t+0.5)-stp_fn(t-0.5);
%
% End of script file
```

A typical run follows with a plot given in Fig. 2.14.

```
>> ce2_5
Enter type of waveform: 1 = positive squarewave; 2 = triangular; 3 = half-rect. sine; 4 = raised cosine: 3
Enter pulse width: 2
Enter percent total desired energy: 95
```

Computer Exercise 2.6

The program for this exercise is similar to that for Computer Exercise 2.5, except that the waveform is used in the energy calculation.

Computer Exercise 2.7

Use Computer Example 2.2 as a pattern for the solution .