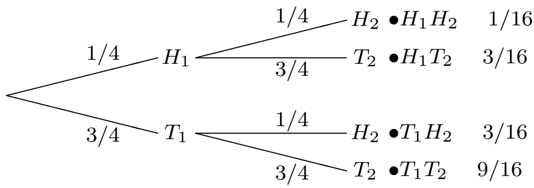
Problem Solutions – Chapter 2

Problem 2.1.1 Solution

A sequential sample space for this experiment is



(a) From the tree, we observe

$$P[H_2] = P[H_1H_2] + P[T_1H_2] = 1/4. \tag{1}$$

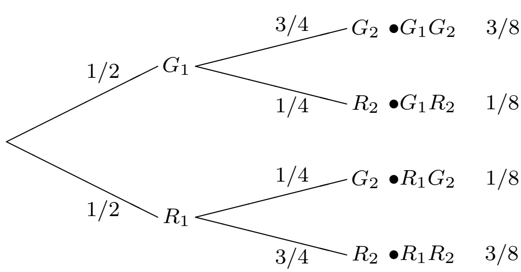
This implies

$$P[H_1|H_2] = \frac{P[H_1H_2]}{P[H_2]} = \frac{1/16}{1/4} = 1/4. \tag{2}$$

(b) The probability that the first flip is heads and the second flip is tails is $P[H_1T_2] = 3/16$.

Problem 2.1.2 Solution

The tree with adjusted probabilities is



From the tree, the probability the second light is green is

$$P[G_2] = P[G_1G_2] + P[R_1G_2] = 3/8 + 1/8 = 1/2. \tag{1}$$

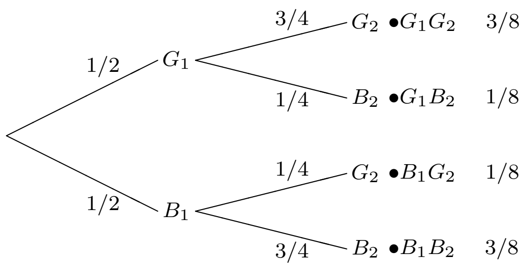
The conditional probability that the first light was green given the second light was green is

$$P[G_1|G_2] = \frac{P[G_1G_2]}{P[G_2]} = \frac{P[G_2|G_1]\,P[G_1]}{P[G_2]} = 3/4. \tag{2}$$

Finally, from the tree diagram, we can directly read that $P[G_2|G_1] = 3/4$.

Problem 2.1.3 Solution

Let G_i and B_i denote events indicating whether free throw i was good (G_i) or bad (B_i). The tree for the free throw experiment is

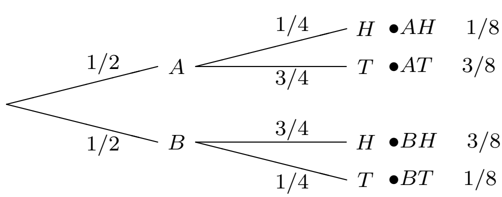


The game goes into overtime if exactly one free throw is made. This event has probability

$$P[O] = P[G_1B_2] + P[B_1G_2] = 1/8 + 1/8 = 1/4. \tag{1}$$

Problem 2.1.4 Solution

The tree for this experiment is



The probability that you guess correctly is

$$P[C] = P[AT] + P[BH] = 3/8 + 3/8 = 3/4. \tag{1}$$

Problem 2.1.5 Solution

The $P[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for the disease, is called a false-positive result and has probability $P[+|H^c]$. Since the test is correct 99% of the time,

$$P[-|H] = P[+|H^c] = 0.01. \tag{1}$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$P[H|+] = \frac{P[+,H]}{P[+]} = \frac{P[+,H]}{P[+,H] + P[+,H^c]}. \tag{2}$$

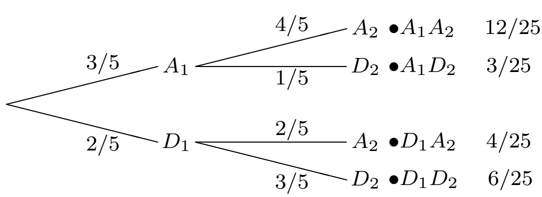
We can use Bayes' formula to evaluate these joint probabilities.

$$\begin{aligned} P[H|+] &= \frac{P[+|H]\,P[H]}{P[+|H]\,P[H] + P[+|H^c]\,P[H^c]} \\ &= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} \\ &= 0.0194. \end{aligned} \tag{3}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 2.1.6 Solution

Let A_i and D_i indicate whether the ith photodetector is acceptable or defective.



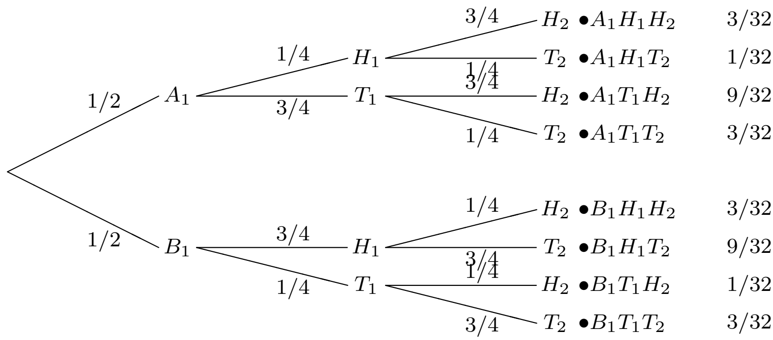
(a) We wish to find the probability $P[E_1]$ that exactly one photodetector is acceptable. From the tree, we have

$$P[E_1] = P[A_1 D_2] + P[D_1 A_2] = 3/25 + 4/25 = 7/25. \tag{1}$$

(b) The probability that both photodetectors are defective is $P[D_1 D_2] = 6/25$.

Problem 2.1.7 Solution

The tree for this experiment is



The event $H_1 H_2$ that heads occurs on both flips has probability

$$P[H_1 H_2] = P[A_1 H_1 H_2] + P[B_1 H_1 H_2] = 6/32. \tag{1}$$

The probability of H_1 is

$$\begin{aligned} P[H_1] &= P[A_1 H_1 H_2] + P[A_1 H_1 T_2] + P[B_1 H_1 H_2] + P[B_1 H_1 T_2] \\ &= 1/2. \end{aligned} \tag{2}$$

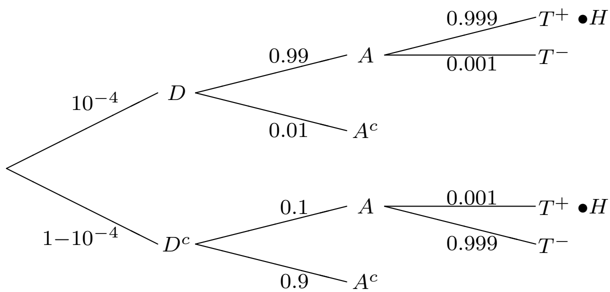
Similarly,

$$\begin{aligned} P[H_2] &= P[A_1 H_1 H_2] + P[A_1 T_1 H_2] + P[B_1 H_1 H_2] + P[B_1 T_1 H_2] \\ &= 1/2. \end{aligned} \tag{3}$$

Thus $P[H_1 H_2] \neq P[H_1] P[H_2]$, implying H_1 and H_2 are not independent. This result should not be surprising since if the first flip is heads, it is likely that coin B was picked first. In this case, the second flip is less likely to be heads since it becomes more likely that the second coin flipped was coin A.

Problem 2.1.8 Solution

We start with a tree diagram:



(a) Here we are asked to calculate the conditional probability $P[D|A]$. In this part, its simpler to ignore the last branches of the tree that indicate the lab test result. This yields

$$P[D|A] = \frac{P[DA]}{P[A]} = \frac{P[AD]}{P[DA] + P[D^c A]}$$
$$= \frac{(10^{-4})(0.99)}{(10^{-4})(0.99) + (0.1)(1 - 10^{-4})}$$
$$= 9.89 \times 10^{-4}. \tag{1}$$

The probability of the defect D given the arrhythmia A is still quite low because the probability of the defect is so small.

(b) Since the heart surgery occurs if and only if the event T^+ occurs, H and T^+ are the same event and (from the previous part)

$$P[H|D] = P\left[T^+|D\right] = \frac{P[DT^+]}{P[D]}$$
$$= \frac{10^{-4}(0.99)(0.999)}{10^{-4}} = (0.99)(0.999). \tag{2}$$

(c) Since the heart surgery occurs if and only if the event T^+ occurs, H and T^+ are the same event and (from the previous part)

$$P[H|D^c] = P\left[T^+|D^c\right] = \frac{P[D^c T^+]}{P[D^c]} = \frac{(1-10^{-4})(0.1)(0.001)}{1-10^{-4}} = 10^{-4}. \tag{3}$$

(d) Heart surgery occurs with probability

$$\begin{aligned} P[H] &= P[H|D]\,P[D] + P[H|D^c]\,P[D^c] \\ &= (0.99)(0.999)(10^{-4}) + (10^{-4})(1-10^{-4}) \\ &= 1.99 \times 10^{-4}. \end{aligned} \tag{4}$$

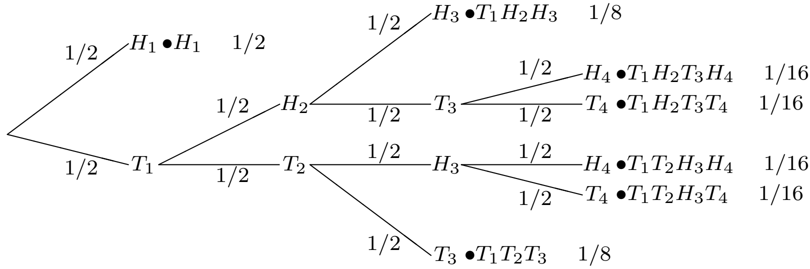
(e) Given that heart surgery was performed, the probability the child had no defect is

$$\begin{aligned} P[D^c|H] &= \frac{P[D^c H]}{P[H]} \\ &= \frac{(1-10^{-4})(0.1)(0.001)}{(0.99)(0.999)(10^{-4}) + (10^{-4})(1-10^{-4})} \\ &= \frac{1-10^{-4}}{2-10^{-2}-10^{-3}+10^{-4}} = 0.5027. \end{aligned} \tag{5}$$

Because the arrythmia is fairly common and the lab test is not fully reliable, roughly half of all the heart surgeries are performed on healthy infants.

Problem 2.1.9 Solution

(a) The primary difficulty in this problem is translating the words into the correct tree diagram. The tree for this problem is shown below.



(b) From the tree,

$$\begin{aligned} P[H_3] &= P[T_1H_2H_3] + P[T_1T_2H_3H_4] + P[T_1T_2H_3H_4] \\ &= 1/8 + 1/16 + 1/16 = 1/4. \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned} P[T_3] &= P[T_1H_2T_3H_4] + P[T_1H_2T_3T_4] + P[T_1T_2T_3] \\ &= 1/8 + 1/16 + 1/16 = 1/4. \end{aligned} \tag{2}$$

(c) The event that Dagwood must diet is

$$D = (T_1H_2T_3T_4) \cup (T_1T_2H_3T_4) \cup (T_1T_2T_3). \tag{3}$$

The probability that Dagwood must diet is

$$\begin{aligned} P[D] &= P[T_1H_2T_3T_4] + P[T_1T_2H_3T_4] + P[T_1T_2T_3] \\ &= 1/16 + 1/16 + 1/8 = 1/4. \end{aligned} \tag{4}$$

The conditional probability of heads on flip 1 given that Dagwood must diet is

$$P[H_1|D] = \frac{P[H_1D]}{P[D]} = 0. \tag{5}$$

Remember, if there was heads on flip 1, then Dagwood always postpones his diet.

(d) From part (b), we found that $\mathrm{P}[H_3] = 1/4$. To check independence, we calculate

$$\begin{aligned}\mathrm{P}[H_2] &= \mathrm{P}[T_1H_2H_3] + \mathrm{P}[T_1H_2T_3] + \mathrm{P}[T_1H_2T_4T_4] = 1/4 \\ \mathrm{P}[H_2H_3] &= \mathrm{P}[T_1H_2H_3] = 1/8.\end{aligned} \tag{6}$$

Now we find that

$$\mathrm{P}[H_2H_3] = 1/8 \neq \mathrm{P}[H_2]\,\mathrm{P}[H_3]. \tag{7}$$

Hence, H_2 and H_3 are dependent events. In fact, $\mathrm{P}[H_3|H_2] = 1/2$ while $\mathrm{P}[H_3] = 1/4$. The reason for the dependence is that given H_2 occurred, then we know there will be a third flip which may result in H_3. That is, knowledge of H_2 tells us that the experiment didn't end after the first flip.

Problem 2.1.10 Solution

(a) We wish to know what the probability that we find no good photodiodes in n pairs of diodes. Testing each pair of diodes is an independent trial such that with probability p, both diodes of a pair are bad. From Problem 2.1.6, we can easily calculate p.

$$p = \mathrm{P}[\text{both diodes are defective}] = \mathrm{P}[D_1D_2] = 6/25. \tag{1}$$

The probability of Z_n, the probability of zero acceptable diodes out of n pairs of diodes is p^n because on each test of a pair of diodes, both must be defective.

$$\mathrm{P}[Z_n] = \prod_{i=1}^{n} p = p^n = \left(\frac{6}{25}\right)^n \tag{2}$$

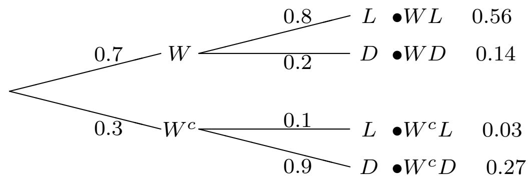
(b) Another way to phrase this question is to ask how many pairs must we test until $\mathrm{P}[Z_n] \leq 0.01$. Since $\mathrm{P}[Z_n] = (6/25)^n$, we require

$$\left(\frac{6}{25}\right)^n \leq 0.01 \quad \Rightarrow \quad n \geq \frac{\ln 0.01}{\ln 6/25} = 3.23. \tag{3}$$

Since n must be an integer, $n = 4$ pairs must be tested.

Problem 2.1.11 Solution

The starting point is to draw a tree of the experiment. We define the events W that the plant is watered, L that the plant lives, and D that the plant dies. The tree diagram is



It follows that

(a) $P[L] = P[WL] + P[W^cL] = 0.56 + 0.03 = 0.59$.

(b)

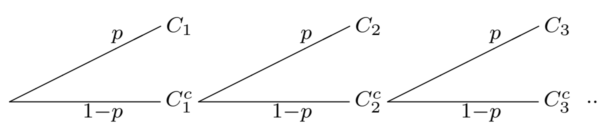
$$P[W^c|D] = \frac{P[W^cD]}{P[D]} = \frac{0.27}{0.14 + 0.27} = \frac{27}{41}. \quad (1)$$

(c) $P[D|W^c] = 0.9$.

In informal conversation, it can be confusing to distinguish between $P[D|W^c]$ and $P[W^c|D]$; however, they are simple once you draw the tree.

Problem 2.1.12 Solution

The experiment ends as soon as a fish is caught. The tree resembles



From the tree, $P[C_1] = p$ and $P[C_2] = (1 - p)p$. Finally, a fish is caught on the nth cast if no fish were caught on the previous $n - 1$ casts. Thus,

$$P[C_n] = (1 - p)^{n-1}p. \quad (1)$$

Problem 2.2.1 Solution

Technically, a gumball machine has a finite number of gumballs, but the problem description models the drawing of gumballs as sampling from the machine without replacement. This is a reasonable model when the machine has a very large gumball capacity and we have no knowledge beforehand of how many gumballs of each color are in the machine. Under this model, the requested probability is given by the multinomial probability

$$\begin{aligned} P\left[R_2 Y_2 G_2 B_2\right] &= \frac{8!}{2!\,2!\,2!\,2!}\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)^2 \\ &= \frac{8!}{4^{10}} \approx 0.0385. \end{aligned} \quad (1)$$

Problem 2.2.2 Solution

In this model of a starburst package, the pieces in a package are collected by sampling without replacement from a giant collection of starburst pieces.

(a) Each piece is "berry or cherry" with probability $p = 1/2$. The probability of only berry or cherry pieces is $p^{12} = 1/4096$.

(b) Each piece is "not cherry" with probability $3/4$. The probability all 12 pieces are "not pink" is $(3/4)^{12} = 0.0317$.

(c) For $i = 1, 2, \ldots, 6$, let C_i denote the event that all 12 pieces are flavor i. Since each piece is flavor i with probability $1/4$, $P[C_i] = (1/4)^{12}$. Since C_i and C_j are mutually exclusive,

$$P[F_1] = P[C_1 \cup C_2 \cup \cdots \cup C_4] = \sum_{i=1}^{4} P[C_i] = 4\,P[C_1] = (1/4)^{11}.$$

Problem 2.2.3 Solution

(a) Let B_i, L_i, O_i and C_i denote the events that the ith piece is Berry, Lemon, Orange, and Cherry respectively. Let F_1 denote the event that all three pieces draw are the same flavor. Thus,

$$F_1 = \{S_1S_2S_3, L_1L_2L_3, O_1O_2O_3, C_1C_2C_3\} \tag{1}$$

$$\text{P}[F_1] = \text{P}[S_1S_2S_3] + \text{P}[L_1L_2L_3] + \text{P}[O_1O_2O_3] + \text{P}[C_1C_2C_3] \tag{2}$$

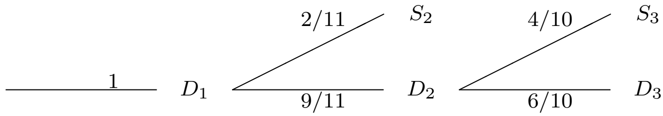
Note that

$$\text{P}[L_1L_2L_3] = \frac{3}{12} \cdot \frac{2}{11} \cdot \frac{1}{10} = \frac{1}{220} \tag{3}$$

and by symmetry,

$$\text{P}[F_1] = 4\,\text{P}[L_1L_2L_3] = \frac{1}{55}. \tag{4}$$

(b) Let D_i denote the event that the ith piece is a different flavor from all the prior pieces. Let S_i denote the event that piece i is the same flavor as a previous piece. A tree for this experiment is



Note that:

- $\text{P}[D_1] = 1$ because the first piece is "different" since there haven't been any prior pieces.
- The second piece is the same as the first piece with probability $2/11$ because in the remaining 11 pieces there are 2 pieces that are the same as the first piece. Alternatively, out of 11 pieces left, there are 3 colors each with 3 pieces (that is, 9 pieces out of 11) that are different from the first piece.
- Given the first two pieces are different, there are 2 colors, each with 3 pieces (6 pieces) out of 10 remaining pieces that are a different flavor from the first two pieces. Thus $\text{P}[D_3|D_2D_1] = 6/10$.

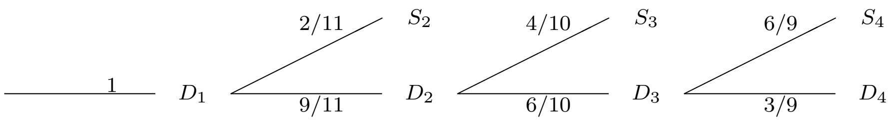
It follows that the three pieces are different with probability

$$\text{P}[D_1D_2D_3] = 1\left(\frac{9}{11}\right)\left(\frac{6}{10}\right) = \frac{27}{55}. \tag{5}$$

Problem 2.2.4 Solution

(a) Since there are only three pieces of each flavor, we cannot draw four pieces of all the same flavor. Hence $P[F_1] = 0$.

(b) Let D_i denote the event that the ith piece is a different flavor from all the prior pieces. Let S_i denote the event that piece i is the same flavor as a previous piece. A tree for this experiment is relatively simple because we stop the experiment as soon as we draw a piece that is the same as a previous piece. The tree is



Note that:

- $P[D_1] = 1$ because the first piece is "different" since there haven't been any prior pieces.
- For the second piece, there are 11 pieces left and 9 of those pieces are different from the first piece drawn.
- Given the first two pieces are different, there are 2 colors, each with 3 pieces (6 pieces) out of 10 remaining pieces that are a different flavor from the first two pieces. Thus $P[D_3|D_2D_1] = 6/10$.
- Finally, given the first three pieces are different flavors, there are 3 pieces remaining that are a different flavor from the pieces previously picked.

Thus $P[D_4|D_3D_2D_1] = 3/9$. It follows that the three pieces are different with probability

$$P[D_1D_2D_3D_4] = 1\left(\frac{9}{11}\right)\left(\frac{6}{10}\right)\frac{3}{9} = \frac{9}{55}. \tag{1}$$

An alternate approach to this problem is to assume that each piece is distinguishable, say by numbering the pieces 1, 2, 3 in each flavor. In addition, we define the outcome of the experiment to be a 4-permutation of the 12 distinguishable pieces. Under this model, there are $(12)_4 = \frac{12!}{8!}$ equally likely outcomes in the sample space. The number of outcomes in which all four pieces are different is $n_4 = 12 \cdot 9 \cdot 6 \cdot 3$ since there are 12 choices for the first piece drawn, 9 choices for the second piece from the three remaining flavors, 6 choices for the third piece and three choices for the last piece. Since all outcomes are equally likely,

$$P[F_4] = \frac{n_4}{(12)_4} = \frac{12 \cdot 9 \cdot 6 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9} = \frac{9}{55} \tag{2}$$

(c) The second model of distinguishable starburst pieces makes it easier to solve this last question. In this case, let the outcome of the experiment be the $\binom{12}{4} = 495$ combinations or pieces. In this case, we are ignoring the order in which the pieces were selected. Now we count the number of combinations in which we have two pieces of each of two flavors. We can do this with the following steps:

1. Choose two of the four flavors.
2. Choose 2 out of 3 pieces of one of the two chosen flavors.
3. Choose 2 out of 3 pieces of the other of the two chosen flavors.

Let n_i equal the number of ways to execute step i. We see that

$$n_1 = \binom{4}{2} = 6, \qquad n_2 = \binom{3}{2} = 3, \qquad n_3 = \binom{3}{2} = 3. \tag{3}$$

There are $n_1 n_2 n_3 = 54$ possible ways to execute this sequence of steps. Since all combinations are equally likely,

$$P[F_2] = \frac{n_1 n_2 n_3}{\binom{12}{4}} = \frac{54}{495} = \frac{6}{55}. \tag{4}$$

Problem 2.2.5 Solution

Since there are $H = \binom{52}{7}$ equiprobable seven-card hands, each probability is just the number of hands of each type divided by H.

(a) Since there are 26 red cards, there are $\binom{26}{7}$ seven-card hands with all red cards. The probability of a seven-card hand of all red cards is

$$P[R_7] = \frac{\binom{26}{7}}{\binom{52}{7}} = \frac{26!\,45!}{52!\,19!} = 0.0049. \tag{1}$$

(b) There are 12 face cards in a 52 card deck and there are $\binom{12}{7}$ seven card hands with all face cards. The probability of drawing only face cards is

$$P[F] = \frac{\binom{12}{7}}{\binom{52}{7}} = \frac{12!\,45!}{5!\,52!} = 5.92 \times 10^{-6}. \tag{2}$$

(c) There are 6 red face cards (J, Q, K of diamonds and hearts) in a 52 card deck. Thus it is impossible to get a seven-card hand of red face cards: $P[R_7 F] = 0$.

Problem 2.2.6 Solution

There are $H_5 = \binom{52}{5}$ equally likely five-card hands. Dividing the number of hands of a particular type by H will yield the probability of a hand of that type.

(a) There are $\binom{26}{5}$ five-card hands of all red cards. Thus the probability getting a five-card hand of all red cards is

$$P[R_5] = \frac{\binom{26}{5}}{\binom{52}{5}} = \frac{26!\,47!}{21!\,52!} = 0.0253. \tag{1}$$

Note that this can be rewritten as

$$P[R_5] = \frac{26}{52}\frac{25}{51}\frac{24}{50}\frac{23}{49}\frac{22}{48},$$

which shows the successive probabilities of receiving a red card.

(b) The following sequence of subexperiments will generate all possible "full house"

1. Choose a kind for three-of-a-kind.
2. Choose a kind for two-of-a-kind.

3. Choose three of the four cards of the three-of-a-kind kind.
4. Choose two of the four cards of the two-of-a-kind kind.

The number of ways of performing subexperiment i is

$$n_1 = \binom{13}{1}, \qquad n_2 = \binom{12}{1}, \qquad n_3 = \binom{4}{3}, \qquad n_4 = \binom{4}{2}. \tag{2}$$

Note that $n_2 = \binom{12}{1}$ because after choosing a three-of-a-kind, there are twelove kinds left from which to choose two-of-a-kind. is

The probability of a full house is

$$\text{P}\,[\text{full house}] = \frac{n_1 n_2 n_3 n_4}{\binom{52}{5}} = \frac{3,744}{2,598,960} = 0.0014. \tag{3}$$

Problem 2.2.7 Solution

There are $2^5 = 32$ different binary codes with 5 bits. The number of codes with exactly 3 zeros equals the number of ways of choosing the bits in which those zeros occur. Therefore there are $\binom{5}{3} = 10$ codes with exactly 3 zeros.

Problem 2.2.8 Solution

Since each letter can take on any one of the 4 possible letters in the alphabet, the number of 3 letter words that can be formed is $4^3 = 64$. If we allow each letter to appear only once then we have 4 choices for the first letter and 3 choices for the second and two choices for the third letter. Therefore, there are a total of $4 \cdot 3 \cdot 2 = 24$ possible codes.

Problem 2.2.9 Solution

We can break down the experiment of choosing a starting lineup into a sequence of subexperiments:

1. Choose 1 of the 10 pitchers. There are $N_1 = \binom{10}{1} = 10$ ways to do this.
2. Choose 1 of the 15 field players to be the designated hitter (DH). There are $N_2 = \binom{15}{1} = 15$ ways to do this.

3. Of the remaining 14 field players, choose 8 for the remaining field positions. There are $N_3 = \binom{14}{8}$ to do this.

4. For the 9 batters (consisting of the 8 field players and the designated hitter), choose a batting lineup. There are $N_4 = 9!$ ways to do this.

So the total number of different starting lineups when the DH is selected among the field players is

$$N = N_1 N_2 N_3 N_4 = (10)(15)\binom{14}{8} 9! = 163{,}459{,}296{,}000. \tag{1}$$

Note that this overestimates the number of combinations the manager must really consider because most field players can play only one or two positions. Although these constraints on the manager reduce the number of possible lineups, it typically makes the manager's job more difficult. As for the counting, we note that our count did not need to specify the positions played by the field players. Although this is an important consideration for the manager, it is not part of our counting of different lineups. In fact, the 8 nonpitching field players are allowed to switch positions at any time in the field. For example, the shortstop and second baseman could trade positions in the middle of an inning. Although the DH can go play the field, there are some coomplicated rules about this. Here is an excerpt from Major League Baseball Rule 6.10:

> The Designated Hitter may be used defensively, continuing to bat in the same position in the batting order, but the pitcher must then bat in the place of the substituted defensive player, unless more than one substitution is made, and the manager then must designate their spots in the batting order.

If you're curious, you can find the complete rule on the web.

Problem 2.2.10 Solution

When the DH can be chosen among all the players, including the pitchers, there are two cases:

- The DH is a field player. In this case, the number of possible lineups, N_F, is given in Problem 2.2.9. In this case, the designated hitter must be chosen from the 15 field players. We repeat the solution of Problem 2.2.9 here: We can break down the experiment of choosing a starting lineup into a sequence of subexperiments:

1. Choose 1 of the 10 pitchers. There are $N_1 = \binom{10}{1} = 10$ ways to do this.
2. Choose 1 of the 15 field players to be the designated hitter (DH). There are $N_2 = \binom{15}{1} = 15$ ways to do this.
3. Of the remaining 14 field players, choose 8 for the remaining field positions. There are $N_3 = \binom{14}{8}$ to do this.
4. For the 9 batters (consisting of the 8 field players and the designated hitter), choose a batting lineup. There are $N_4 = 9!$ ways to do this.

So the total number of different starting lineups when the DH is selected among the field players is

$$N = N_1 N_2 N_3 N_4 = (10)(15)\binom{14}{8} 9! = 163{,}459{,}296{,}000. \tag{1}$$

- The DH is a pitcher. In this case, there are 10 choices for the pitcher, 10 choices for the DH among the pitchers (including the pitcher batting for himself), $\binom{15}{8}$ choices for the field players, and 9! ways of ordering the batters into a lineup. The number of possible lineups is

$$N' = (10)(10)\binom{15}{8} 9! = 233,513,280,000. \tag{2}$$

The total number of ways of choosing a lineup is $N + N' = 396{,}972{,}576{,}000$.

Problem 2.2.11 Solution

(a) This is just the multinomial probability

$$\begin{aligned} \mathrm{P}[A] &= \binom{40}{19, 19, 2} \left(\frac{19}{40}\right)^{19} \left(\frac{19}{40}\right)^{19} \left(\frac{2}{40}\right)^{2} \\ &= \frac{40!}{19!19!2!} \left(\frac{19}{40}\right)^{19} \left(\frac{19}{40}\right)^{19} \left(\frac{2}{40}\right)^{2}. \end{aligned} \tag{1}$$

(b) Each spin is either green (with probability 19/40) or not (with probability 21/40). If we call landing on greeen a success, then G_{19} is the probability of 19 successes in 40 trials. Thus

$$\mathrm{P}\left[G_{19}\right]=\binom{40}{19}\left(\frac{19}{40}\right)^{19}\left(\frac{21}{40}\right)^{21}. \tag{2}$$

(c) If you bet on red, the probability you win is 19/40. If you bet green, the probability that you win is 19/40. If you first make a random choice to bet red or green, (say by flipping a coin), the probability you win is still $p = 19/40$.

Problem 2.2.12 Solution

(a) We can find the number of valid starting lineups by noticing that the swingman presents three situations: (1) the swingman plays guard, (2) the swingman plays forward, and (3) the swingman doesn't play. The first situation is when the swingman can be chosen to play the guard position, and the second where the swingman can only be chosen to play the forward position. Let N_i denote the number of lineups corresponding to case i. Then we can write the total number of lineups as $N_1 + N_2 + N_3$. In the first situation, we have to choose 1 out of 3 centers, 2 out of 4 forwards, and 1 out of 4 guards so that

$$N_1=\binom{3}{1}\binom{4}{2}\binom{4}{1}=72. \tag{1}$$

In the second case, we need to choose 1 out of 3 centers, 1 out of 4 forwards and 2 out of 4 guards, yielding

$$N_2=\binom{3}{1}\binom{4}{1}\binom{4}{2}=72. \tag{2}$$

Finally, with the swingman on the bench, we choose 1 out of 3 centers, 2 out of 4 forward, and 2 out of four guards. This implies

$$N_3=\binom{3}{1}\binom{4}{2}\binom{4}{2}=108, \tag{3}$$

and the total number of lineups is $N_1 + N_2 + N_3 = 252$.

Problem 2.2.13 Solution

What our design must specify is the number of boxes on the ticket, and the number of specially marked boxes. Suppose each ticket has n boxes and $5+k$ specially marked boxes. Note that when $k>0$, a winning ticket will still have k unscratched boxes with the special mark. A ticket is a winner if each time a box is scratched off, the box has the special mark. Assuming the boxes are scratched off randomly, the first box scratched off has the mark with probability $(5+k)/n$ since there are $5+k$ marked boxes out of n boxes. Moreover, if the first scratched box has the mark, then there are $4+k$ marked boxes out of $n-1$ remaining boxes. Continuing this argument, the probability that a ticket is a winner is

$$p = \frac{5+k}{n}\frac{4+k}{n-1}\frac{3+k}{n-2}\frac{2+k}{n-3}\frac{1+k}{n-4} = \frac{(k+5)!(n-5)!}{k!n!}. \quad (1)$$

By careful choice of n and k, we can choose p close to 0.01. For example,

n	9	11	14	17
k	0	1	2	3
p	0.0079	0.012	0.0105	0.0090

(2)

A gamecard with $N=14$ boxes and $5+k=7$ shaded boxes would be quite reasonable.

Problem 2.3.1 Solution

(a) Since the probability of a zero is 0.8, we can express the probability of the code word 00111 as 2 occurrences of a 0 and three occurrences of a 1. Therefore

$$P[00111] = (0.8)^2(0.2)^3 = 0.00512. \quad (1)$$

(b) The probability that a code word has exactly three 1's is

$$P[\text{three 1's}] = \binom{5}{3}(0.8)^2(0.2)^3 = 0.0512. \quad (2)$$

Problem 2.3.2 Solution

Given that the probability that the Celtics win a single championship in any given year is 0.32, we can find the probability that they win 8 straight NBA championships.

$$P[8 \text{ straight championships}] = (0.32)^8 = 0.00011. \tag{1}$$

The probability that they win 10 titles in 11 years is

$$P[10 \text{ titles in } 11 \text{ years}] = \binom{11}{10}(.32)^{10}(.68) = 0.00084. \tag{2}$$

The probability of each of these events is less than 1 in 1000! Given that these events took place in the relatively short fifty year history of the NBA, it should seem that these probabilities should be much higher. What the model overlooks is that the sequence of 10 titles in 11 years started when Bill Russell joined the Celtics. In the years with Russell (and a strong supporting cast) the probability of a championship was much higher.

Problem 2.3.3 Solution

We know that the probability of a green and red light is 7/16, and that of a yellow light is 1/8. Since there are always 5 lights, G, Y, and R obey the multinomial probability law:

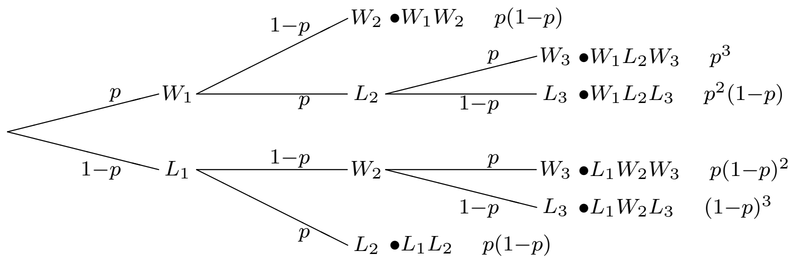
$$P[G = 2, Y = 1, R = 2] = \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right)\left(\frac{7}{16}\right)^2. \tag{1}$$

The probability that the number of green lights equals the number of red lights

$$\begin{aligned} P[G = R] &= P[G = 1, R = 1, Y = 3] + P[G = 2, R = 2, Y = 1] \\ &\qquad + P[G = 0, R = 0, Y = 5] \\ &= \frac{5!}{1!1!3!}\left(\frac{7}{16}\right)\left(\frac{7}{16}\right)\left(\frac{1}{8}\right)^3 + \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right) \\ &\qquad + \frac{5!}{0!0!5!}\left(\frac{1}{8}\right)^5 \\ &\approx 0.1449. \end{aligned} \tag{2}$$

Problem 2.3.4 Solution

For the team with the homecourt advantage, let W_i and L_i denote whether game i was a win or a loss. Because games 1 and 3 are home games and game 2 is an away game, the tree is



The probability that the team with the home court advantage wins is

$$\begin{aligned} P[H] &= P[W_1W_2] + P[W_1L_2W_3] + P[L_1W_2W_3] \\ &= p(1-p) + p^3 + p(1-p)^2. \end{aligned} \tag{1}$$

Note that $P[H] \le p$ for $1/2 \le p \le 1$. Since the team with the home court advantage would win a 1 game playoff with probability p, the home court team is less likely to win a three game series than a 1 game playoff!

Problem 2.3.5 Solution

(a) There are 3 group 1 kickers and 6 group 2 kickers. Using G_i to denote that a group i kicker was chosen, we have

$$P[G_1] = 1/3, \qquad P[G_2] = 2/3. \tag{1}$$

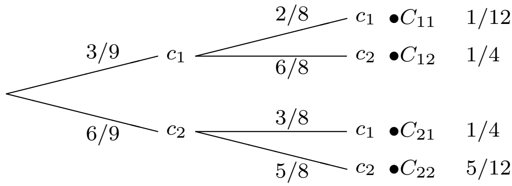
In addition, the problem statement tells us that

$$P[K|G_1] = 1/2, \qquad P[K|G_2] = 1/3. \tag{2}$$

Combining these facts using the Law of Total Probability yields

$$\begin{aligned} P[K] &= P[K|G_1]P[G_1] + P[K|G_2]P[G_2] \\ &= (1/2)(1/3) + (1/3)(2/3) = 7/18. \end{aligned} \tag{3}$$

(b) To solve this part, we need to identify the groups from which the first and second kicker were chosen. Let c_i indicate whether a kicker was chosen from group i and let C_{ij} indicate that the first kicker was chosen from group i and the second kicker from group j. The experiment to choose the kickers is described by the sample tree:



Since a kicker from group 1 makes a kick with probability 1/2 while a kicker from group 2 makes a kick with probability 1/3,

$$P[K_1K_2|C_{11}] = (1/2)^2, \qquad P[K_1K_2|C_{12}] = (1/2)(1/3), \tag{4}$$

$$P[K_1K_2|C_{21}] = (1/3)(1/2), \qquad P[K_1K_2|C_{22}] = (1/3)^2. \tag{5}$$

By the law of total probability,

$$\begin{aligned} P[K_1K_2] &= P[K_1K_2|C_{11}]\,P[C_{11}] + P[K_1K_2|C_{12}]\,P[C_{12}] \\ &\quad + P[K_1K_2|C_{21}]\,P[C_{21}] + P[K_1K_2|C_{22}]\,P[C_{22}] \\ &= \frac{1}{4}\frac{1}{12} + \frac{1}{6}\frac{1}{4} + \frac{1}{6}\frac{1}{4} + \frac{1}{9}\frac{5}{12} = 15/96. \end{aligned} \tag{6}$$

It should be apparent that $P[K_1] = P[K]$ from part (a). Symmetry should also make it clear that $P[K_1] = P[K_2]$ since for any ordering of two kickers, the reverse ordering is equally likely. If this is not clear, we derive this result by calculating $P[K_2|C_{ij}]$ and using the law of total probability to calculate $P[K_2]$.

$$P[K_2|C_{11}] = 1/2, \qquad P[K_2|C_{12}] = 1/3, \tag{7}$$

$$P[K_2|C_{21}] = 1/2, \qquad P[K_2|C_{22}] = 1/3. \tag{8}$$

By the law of total probability,

$$\begin{aligned}
\mathrm{P}[K_2] &= \mathrm{P}[K_2|C_{11}]\,\mathrm{P}[C_{11}] + \mathrm{P}[K_2|C_{12}]\,\mathrm{P}[C_{12}] \\
&\quad + \mathrm{P}[K_2|C_{21}]\,\mathrm{P}[C_{21}] + \mathrm{P}[K_2|C_{22}]\,\mathrm{P}[C_{22}] \\
&= \frac{1}{2}\frac{1}{12} + \frac{1}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{4} + \frac{1}{3}\frac{5}{12} = \frac{7}{18}.
\end{aligned} \tag{9}$$

We observe that K_1 and K_2 are not independent since

$$\mathrm{P}[K_1K_2] = \frac{15}{96} \neq \left(\frac{7}{18}\right)^2 = \mathrm{P}[K_1]\,\mathrm{P}[K_2]\,. \tag{10}$$

Note that $15/96$ and $(7/18)^2$ are close but not exactly the same. The reason K_1 and K_2 are dependent is that if the first kicker is successful, then it is more likely that kicker is from group 1. This makes it more likely that the second kicker is from group 2 and is thus more likely to miss.

(c) Once a kicker is chosen, each of the 10 field goals is an independent trial. If the kicker is from group 1, then the success probability is $1/2$. If the kicker is from group 2, the success probability is $1/3$. Out of 10 kicks, there are 5 misses iff there are 5 successful kicks. Given the type of kicker chosen, the probability of 5 misses is

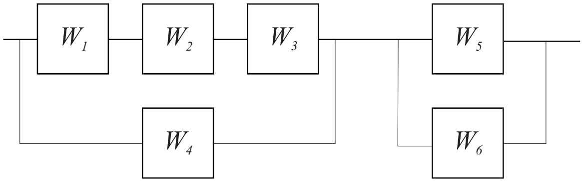
$$\mathrm{P}[M|G_1] = \binom{10}{5}(1/2)^5(1/2)^5, \qquad \mathrm{P}[M|G_2] = \binom{10}{5}(1/3)^5(2/3)^5. \tag{11}$$

We use the Law of Total Probability to find

$$\begin{aligned}
\mathrm{P}[M] &= \mathrm{P}[M|G_1]\,\mathrm{P}[G_1] + \mathrm{P}[M|G_2]\,\mathrm{P}[G_2] \\
&= \binom{10}{5}\left((1/3)(1/2)^{10} + (2/3)(1/3)^5(2/3)^5\right).
\end{aligned} \tag{12}$$

Problem 2.4.1 Solution

From the problem statement, we can conclude that the device components are configured in the following way.

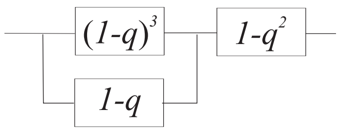


To find the probability that the device works, we replace series devices 1, 2, and 3, and parallel devices 5 and 6 each with a single device labeled with the probability that it works. In particular,

$$\mathrm{P}\left[W_1W_2W_3\right] = (1-q)^3, \tag{1}$$

$$\mathrm{P}\left[W_5 \cup W_6\right] = 1 - \mathrm{P}\left[W_5^cW_6^c\right] = 1 - q^2. \tag{2}$$

This yields a composite device of the form



The probability $\mathrm{P}[W']$ that the two devices in parallel work is 1 minus the probability that neither works:

$$\mathrm{P}\left[W'\right] = 1 - q(1-(1-q)^3). \tag{3}$$

Finally, for the device to work, both composite device in series must work. Thus, the probability the device works is

$$\mathrm{P}\left[W\right] = [1 - q(1-(1-q)^3)][1-q^2]. \tag{4}$$

Problem 2.4.2 Solution

Suppose that the transmitted bit was a 1. We can view each repeated transmission as an independent trial. We call each repeated bit the receiver decodes as 1 a success. Using $S_{k,5}$ to denote the event of k successes in the five trials, then the probability k 1's are decoded at the receiver is

$$\mathrm{P}\left[S_{k,5}\right] = \binom{5}{k} p^k (1-p)^{5-k}, \quad k = 0, 1, \ldots, 5. \tag{1}$$

The probability a bit is decoded correctly is

$$P[C] = P[S_{5,5}] + P[S_{4,5}] = p^5 + 5p^4(1-p) = 0.91854. \tag{2}$$

The probability a deletion occurs is

$$P[D] = P[S_{3,5}] + P[S_{2,5}] = 10p^3(1-p)^2 + 10p^2(1-p)^3 = 0.081. \tag{3}$$

The probability of an error is

$$P[E] = P[S_{1,5}] + P[S_{0,5}] = 5p(1-p)^4 + (1-p)^5 = 0.00046. \tag{4}$$

Note that if a 0 is transmitted, then 0 is sent five times and we call decoding a 0 a success. You should convince yourself that this a symmetric situation with the same deletion and error probabilities. Introducing deletions reduces the probability of an error by roughly a factor of 20. However, the probability of successfull decoding is also reduced.

Problem 2.4.3 Solution

Note that each digit 0 through 9 is mapped to the 4 bit binary representation of the digit. That is, 0 corresponds to 0000, 1 to 0001, up to 9 which corresponds to 1001. Of course, the 4 bit binary numbers corresponding to numbers 10 through 15 go unused, however this is unimportant to our problem. the 10 digit number results in the transmission of 40 bits. For each bit, an independent trial determines whether the bit was correct, a deletion, or an error. In Problem 2.4.2, we found the probabilities of these events to be

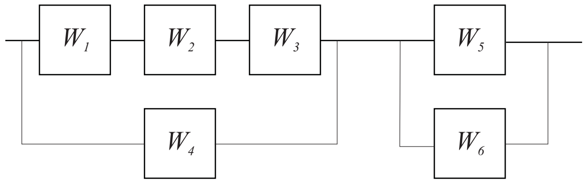
$$P[C] = \gamma = 0.91854, \qquad P[D] = \delta = 0.081, \qquad P[E] = \epsilon = 0.00046. \tag{1}$$

Since each of the 40 bit transmissions is an independent trial, the joint probability of c correct bits, d deletions, and e erasures has the multinomial probability

$$P[C = c, D = d, E = d] = \begin{cases} \frac{40!}{c!d!e!}\gamma^c\delta^d\epsilon^e & c + d + e = 40; c, d, e \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 2.4.4 Solution

From the statement of Problem 2.4.1, the configuration of device components is



By symmetry, note that the reliability of the system is the same whether we replace component 1, component 2, or component 3. Similarly, the reliability is the same whether we replace component 5 or component 6. Thus we consider the following cases:

I **Replace component 1** In this case

$$P[W_1W_2W_3] = (1 - \frac{q}{2})(1-q)^2, \tag{1}$$
$$P[W_4] = 1 - q, \tag{2}$$
$$P[W_5 \cup W_6] = 1 - q^2. \tag{3}$$

This implies

$$\begin{aligned} P[W_1W_2W_3 \cup W_4] &= 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) \\ &= 1 - \frac{q^2}{2}(5 - 4q + q^2). \end{aligned} \tag{4}$$

In this case, the probability the system works is

$$\begin{aligned} P[W_I] &= P[W_1W_2W_3 \cup W_4]\, P[W_5 \cup W_6] \\ &= \left[1 - \frac{q^2}{2}(5 - 4q + q^2)\right](1 - q^2). \end{aligned} \tag{5}$$

II **Replace component 4** In this case,

$$P[W_1W_2W_3] = (1-q)^3, \tag{6}$$
$$P[W_4] = 1 - \frac{q}{2}, \tag{7}$$
$$P[W_5 \cup W_6] = 1 - q^2. \tag{8}$$

This implies

$$\begin{aligned} P[W_1W_2W_3 \cup W_4] &= 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) \\ &= 1 - \frac{q}{2} + \frac{q}{2}(1-q)^3. \end{aligned} \tag{9}$$

In this case, the probability the system works is

$$\begin{aligned} P[W_{II}] &= P[W_1W_2W_3 \cup W_4]\, P[W_5 \cup W_6] \\ &= \left[1 - \frac{q}{2} + \frac{q}{2}(1-q)^3\right](1-q^2). \end{aligned} \tag{10}$$

III **Replace component 5** In this case,

$$P[W_1W_2W_3] = (1-q)^3, \tag{11}$$

$$P[W_4] = 1 - q, \tag{12}$$

$$P[W_5 \cup W_6] = 1 - \frac{q^2}{2}. \tag{13}$$

This implies

$$\begin{aligned} P[W_1W_2W_3 \cup W_4] &= 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) \\ &= (1-q)\left[1 + q(1-q)^2\right]. \end{aligned} \tag{14}$$

In this case, the probability the system works is

$$\begin{aligned} P[W_{III}] &= P[W_1W_2W_3 \cup W_4]\, P[W_5 \cup W_6] \\ &= (1-q)\left(1 - \frac{q^2}{2}\right)\left[1 + q(1-q)^2\right]. \end{aligned} \tag{15}$$

From these expressions, its hard to tell which substitution creates the most reliable circuit. First, we observe that $P[W_{II}] > P[W_I]$ if and only if

$$1 - \frac{q}{2} + \frac{q}{2}(1-q)^3 > 1 - \frac{q^2}{2}(5 - 4q + q^2). \tag{16}$$

Some algebra will show that $P[W_{II}] > P[W_I]$ if and only if $q^2 < 2$, which occurs for all nontrivial (i.e., nonzero) values of q. Similar algebra will show that $P[W_{II}] > P[W_{III}]$ for all values of $0 \le q \le 1$. Thus the best policy is to replace component 4.

Problem 2.5.1 Solution

Rather than just solve the problem for 50 trials, we can write a function that generates vectors **C** and **H** for an arbitrary number of trials n. The code for this task is

```
function [C,H]=twocoin(n);
C=ceil(2*rand(n,1));
P=1-(C/4);
H=(rand(n,1)< P);
```

The first line produces the $n \times 1$ vector `C` such that `C(i)` indicates whether coin 1 or coin 2 is chosen for trial i. Next, we generate the vector `P` such that `P(i)=0.75` if `C(i)=1`; otherwise, if `C(i)=2`, then `P(i)=0.5`. As a result, `H(i)` is the simulated result of a coin flip with heads, corresponding to `H(i)=1`, occurring with probability `P(i)`.

Problem 2.5.2 Solution

Rather than just solve the problem for 100 trials, we can write a function that generates n packets for an arbitrary number of trials n. The code for this task is

```
function C=bit100(n);
% n is the number of 100 bit packets sent
B=floor(2*rand(n,100));
P=0.03-0.02*B;
E=(rand(n,100)< P);
C=sum((sum(E,2)<=5));
```

First, `B` is an $n \times 100$ matrix such that `B(i,j)` indicates whether bit `i` of packet `j` is zero or one. Next, we generate the $n \times 100$ matrix `P` such that `P(i,j)=0.03` if `B(i,j)=0`; otherwise, if `B(i,j)=1`, then `P(i,j)=0.01`. As a result, `E(i,j)` is the simulated error indicator for bit `i` of packet `j`. That is, `E(i,j)=1` if bit `i` of packet `j` is in error; otherwise `E(i,j)=0`. Next we sum across the rows of `E` to obtain the number of errors in each packet. Finally, we count the number of packets with 5 or more errors.

For $n = 100$ packets, the packet success probability is inconclusive. Experimentation will show that `C=97`, `C=98`, `C=99` and `C=100` correct packets are typica values that might be observed. By increasing n, more consistent results are obtained.

For example, repeated trials with $n = 100,000$ packets typically produces around $C = 98,400$ correct packets. Thus 0.984 is a reasonable estimate for the probability of a packet being transmitted correctly.

Problem 2.5.3 Solution

To test n 6-component devices, (such that each component works with probability q) we use the following function:

```
function N=reliable6(n,q);
% n is the number of 6 component devices
%N is the number of working devices
W=rand(n,6)>q;
D=(W(:,1)&W(:,2)&W(:,3))|W(:,4);
D=D&(W(:,5)|W(:,6));
N=sum(D);
```

The $n \times 6$ matrix `W` is a *logical* matrix such that `W(i,j)=1` if component `j` of device `i` works properly. Because `W` is a logical matrix, we can use the MATLAB logical operators `|` and `&` to implement the logic requirements for a working device. By applying these logical operators to the $n \times 1$ columns of `W`, we simulate the test of `n` circuits. Note that `D(i)=1` if device `i` works. Otherwise, `D(i)=0`. Lastly, we count the number `N` of working devices. The following code snippet produces ten sample runs, where each sample run tests `n=100` devices for $q = 0.2$.

```
>> for n=1:10, w(n)=reliable6(100,0.2); end
>> w
w =
    82   87   87   92   91   85   85   83   90   89
>>
```

As we see, the number of working devices is typically around 85 out of 100. Solving Problem 2.4.1, will show that the probability the device works is actually 0.8663.

Problem 2.5.4 Solution

The code

```
function n=countequal(x,y)
%Usage:  n=countequal(x,y)
%n(j)= # elements of x = y(j)
[MX,MY]=ndgrid(x,y);
%each column of MX = x
%each row of MY = y
n=(sum((MX==MY),1))';
```

for `countequal` is quite short (just two lines excluding comments) but needs some explanation. The key is in the operation

`[MX,MY]=ndgrid(x,y).`

The MATLAB built-in function `ndgrid` facilitates plotting a function $g(x,y)$ as a surface over the x,y plane.The x,y plane is represented by a grid of all pairs of points $x(i),y(j)$. When `x` has n elements, and `y` has m elements, `ndgrid(x,y)` creates a grid (an $n \times m$ array) of all possible pairs `[x(i) y(j)]`. This grid is represented by two separate $n \times m$ matrices: `MX` and `MY` which indicate the `x` and `y` values at each grid point. Mathematically,

`MX(i,j) = x(i),    MY(i,j)=y(j).`

Next, `C=(MX==MY)` is an $n \times m$ array such that `C(i,j)=1` if `x(i)=y(j)`; otherwise `C(i,j)=0`. That is, the jth column of `C` indicates indicates which elements of `x` equal `y(j)`. Lastly, we sum along each column j to count number of elements of `x` equal to `y(j)`. That is, we sum along column j to count the number of occurrences (in `x`) of `y(j)`.

Problem 2.5.5 Solution

For arbitrary number of trials n and failure probability q, the following functions evaluates replacing each of the six components by an ultrareliable device.

```
function N=ultrareliable6(n,q);
% n is the number of 6 component devices
%N is the number of working devices
for r=1:6,
    W=rand(n,6)>q;
    R=rand(n,1)>(q/2);
    W(:,r)=R;
    D=(W(:,1)&W(:,2)&W(:,3))|W(:,4);
    D=D&(W(:,5)|W(:,6));
    N(r)=sum(D);
end
```

This code is based on the code for the solution of Problem 2.5.3. The $n \times 6$ matrix `W` is a *logical* matrix such that `W(i,j)=1` if component `j` of device `i` works properly. Because `W` is a logical matrix, we can use the MATLAB logical operators `|` and `&` to implement the logic requirements for a working device. By applying these logical opeators to the $n \times 1$ columns of `W`, we simulate the test of `n` circuits. Note that `D(i)=1` if device `i` works. Otherwise, `D(i)=0`. Note that in the code, we first generate the matrix `W` such that each component has failure probability q. To simulate the replacement of the jth device by the ultrareliable version by replacing the jth column of `W` by the column vector `R` in which a device has failure probability $q/2$. Lastly, for each column replacement, we count the number `N` of working devices. A sample run for $n = 100$ trials and $q = 0.2$ yielded these results:

```
>> ultrareliable6(100,0.2)
ans =
    93    89    91    92    90    93
```

From the above, we see, for example, that replacing the third component with an ultrareliable component resulted in 91 working devices. The results are fairly inconclusive in that replacing devices 1, 2, or 3 should yield the same probability of device failure. If we experiment with $n = 10,000$ runs, the results are more definitive:

```
>> ultrareliable6(10000,0.2)
ans =
        8738        8762        8806        9135        8800        8796
>> >> ultrareliable6(10000,0.2)
ans =
        8771        8795        8806        9178        8886        8875
>>
```

In both cases, it is clear that replacing component 4 maximizes the device reliability. The somewhat complicated solution of Problem 2.4.4 will confirm this observation.