PROBLEM 2.1

FIND: Define the term *signal* as it relates to measurement systems and provide examples of static and dynamic input signals to measurement systems.

SOLUTION: A *signal* is information in motion from one place to another, such as between stages of a measurement system. Signals have a variety of forms, including electrical and mechanical.

Examples of static input signals are:

1. weight, such as weighing merchandise, etc.
2. body temperature, at the moment of interest
3. length or height, such as the length of a board or a person's height

Examples of dynamic input signals:

1. input signal to an automobile speed control
2. input signal to a music amplifier from a component such as a CD player or portable personal device
3. input signal to a printer from a computer
4. wind speed on a gusty day as input signal to an anemometer

PROBLEM 2.2

FIND: List the important characteristics of input and output digital signals and define each.

SOLUTION:

1. *Magnitude* - generally refers to the maximum value of a signal
2. *Range* - difference between maximum and minimum values of a signal. For a digital signal is represented as the maximum number of bits
3. *Amplitude* - indicative of signal fluctuations relative to the mean
4. *Frequency* - describes the time variation of a signal.
5. *Sampling frequency* – For a digital signal indicates how often a digital record is output or input.
5. *Bit resolution or quantization error (see Chapter 7)* - smallest change that can be recorded by a digital system

COMMENT: The process of converting an analog signal to digital form is described in detail in Chapter 7.

PROBLEM 2.3

KNOWN: Need to transmit voice data in digital form

FIND: The importance of multiplexing and data compression in voice transmission

SOLUTION: Multiplexing is a term that represents the idea of transmitting several signals over the same medium at the same time. Historically, the need to transmit several conversations over the same telephone wires spurred the development of techniques for multiplexing, with the first applications occurring early in the 20th century.

When considering digital signals, there are several techniques available for multiplexing. The simplest to understand is termed time-division multiplexing where a time period is allocated to each of several signals being transmitted. Each receiver (or person hearing a conversation) will not "notice" that some of the time was allocated to another conversation. Implementation of this technique is made easier by the fact that there is much "dead" time in a conversation between two people that can be detected and used to advantage!

Voice data compression takes several forms. A simple voice compression scheme removes all of the frequency content that is not necessary for intelligibility. Frequency content outside of the range from 400 to 3000 Hz is generally not needed to understand speech. However, some loss of emotional content occurs as compression increases.

PROBLEM 2.4

KNOWN: Need to transmit and store digital image files

FIND: Compression schemes for image files

SOLUTION: Compression allows reducing the size of digital files for storage or transmission. The file and image resulting after compression may or may not contain all of the data present in the original file. For digital photography, a "raw" image from a 12-bit CCD allows 4096 brightness levels for each pixel. If compressed to a JPEG image, this output is only 8-bit where each pixel can have 256 brightness levels. The savings in required storage is dramatic.

Another possible scheme for compression can be termed "region of interest" compression. For example, images containing faces with a background can be reduced in size by storing less information about the background.

You may wish to research how color images are recorded and compressed to JPEG format.

PROBLEM 2.5

KNOWN:

$$y(t) = 25 + 10\sin 6\pi t$$

FIND: $\bar{y}$ and y_{rms} for the time periods t_1 to t_2 listed below

a) 0 to 0.1 sec
b) 0.4 to 0.5 sec
c) 0 to 1/3 sec
d) 0 to 20 sec

SOLUTION:

For the continuous function $y(t)$, the average may be expressed

$$\bar{y} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} y(t)\,dt$$

And the rms as

$$y_{rms} = \sqrt{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} [y(t)]^2\,dt}$$

For $y(t) = 25 + 10\sin 6\pi t$, the average is given by

$$\begin{aligned}\bar{y} &= \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \left(25 + 10\sin 6\pi t\right) dt \\ &= \frac{1}{t_2 - t_1}\left[25t - \frac{10}{6\pi}\cos 6\pi t\Big|_{t_1}^{t_2}\right] \\ &= \frac{1}{t_2 - t_1}\left[25\left(t_2 - t_1\right) - \frac{10}{6\pi}\left(\cos 6\pi t_2 - \cos 6\pi t_1\right)\right]\end{aligned}$$

And the rms as

$$\begin{aligned}y_{rms} &= \left\{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left(25 + 10\sin 6\pi t\right)^2 dt\right\}^{1/2} \\ &= \left\{\frac{1}{t_2 - t_1}\left[625t - \frac{500}{6\pi}\cos 6\pi t + 100\left(\frac{-1}{12\pi}\sin 6\pi t\cos 6\pi t + \frac{1}{2}t\right)\right]\Bigg|_{t_1}^{t_2}\right\}^{1/2}\end{aligned}$$

The resulting numerical values are

a) $\bar{y} = 25.69 \qquad y_{rms} = 31.85$

b) $\bar{y} = 25.69 \qquad y_{rms} = 31.85$

c) $\bar{y} = 25 \qquad y_{rms} = 25.98$

d) $\bar{y} = 25 \qquad y_{rms} = 25.98$

COMMENT: The average and rms values for the time period 0 to 20 seconds represents the long-term average behavior of the signal. The result in parts a) and b) are accurate over the specified time periods and for a measured signal may have specific significance. The period 0 to 1/3 represents one complete cycle of the simple periodic signal and results in average and rms values which accurately represent the long-term behavior of the signal.

PROBLEM 2.6

KNOWN: Discrete sampled data, corresponding to measurement every 0.4 seconds, as shown below.

<TBCH>t	$y_1(t)$	$y_2(t)$	t	$y_1(t)$	$y_2(t)$
<TB>0	0	0			
0.4	11.76	15.29	2.4	−11.76	−15.29
0.8	19.02	24.73	2.8	−19.02	−24.73
1.2	19.02	24.73	3.2	−19.02	−24.73
1.6	11.76	15.29	3.6	−11.76	−15.29
2.0	0	0	4.0	0	0

FIND: The mean and rms values of the measured data.

SOLUTION:

For a discrete signal the mean and rms are given by

$$\bar{y} = \frac{1}{N}\sum_{t=0}^{N-1} y_i \qquad\qquad y_{\text{rms}} = \sqrt{\frac{1}{N}\sum_{i=0}^{N-1} y_i^2}$$

The mean value for y_1 is 0 and for y_2 is also 0.

However, the rms value of y_1 is 13.49 and for y_2 is 17.53.

COMMENT: The mean value contains no information concerning the time varying nature of a signal; both these signals have an average value of 0. But the differences in the signals are made apparent when the rms value is examined.

PROBLEM 2.7

KNOWN: The effect of a moving average signal processing technique is to be determined for the signal in Figure 2.22 and $y(t) = \sin 5t + \cos 11t$

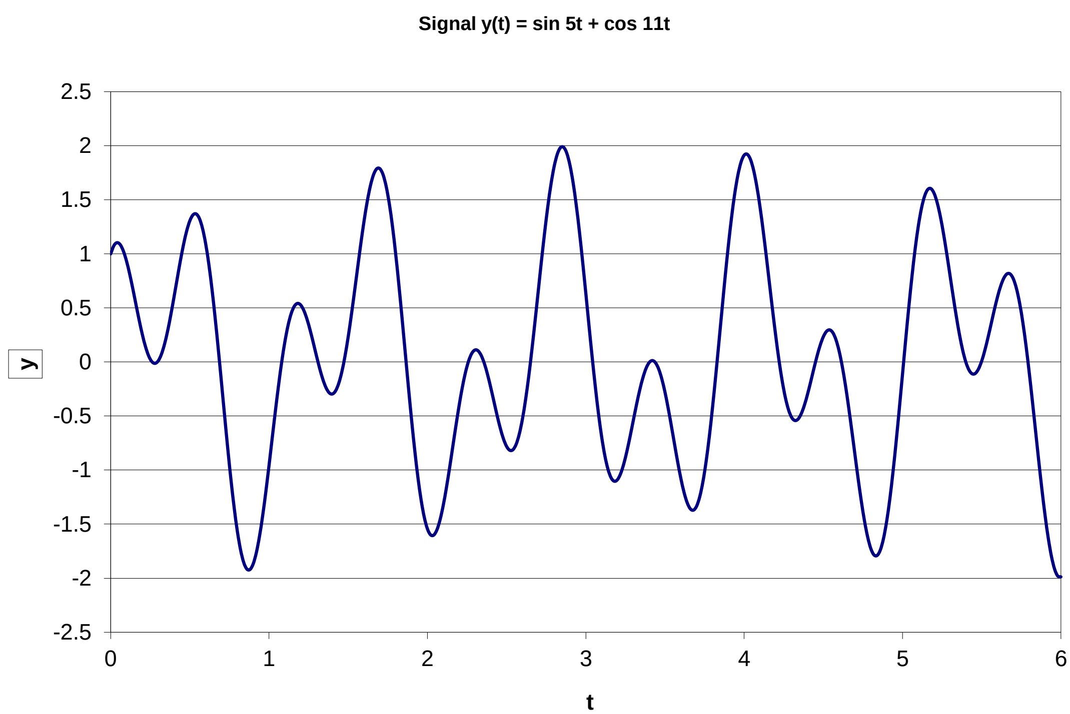
FIND: Discuss Figure 2.23 and plot the signal resulting from applying a moving average to $y(t)$.

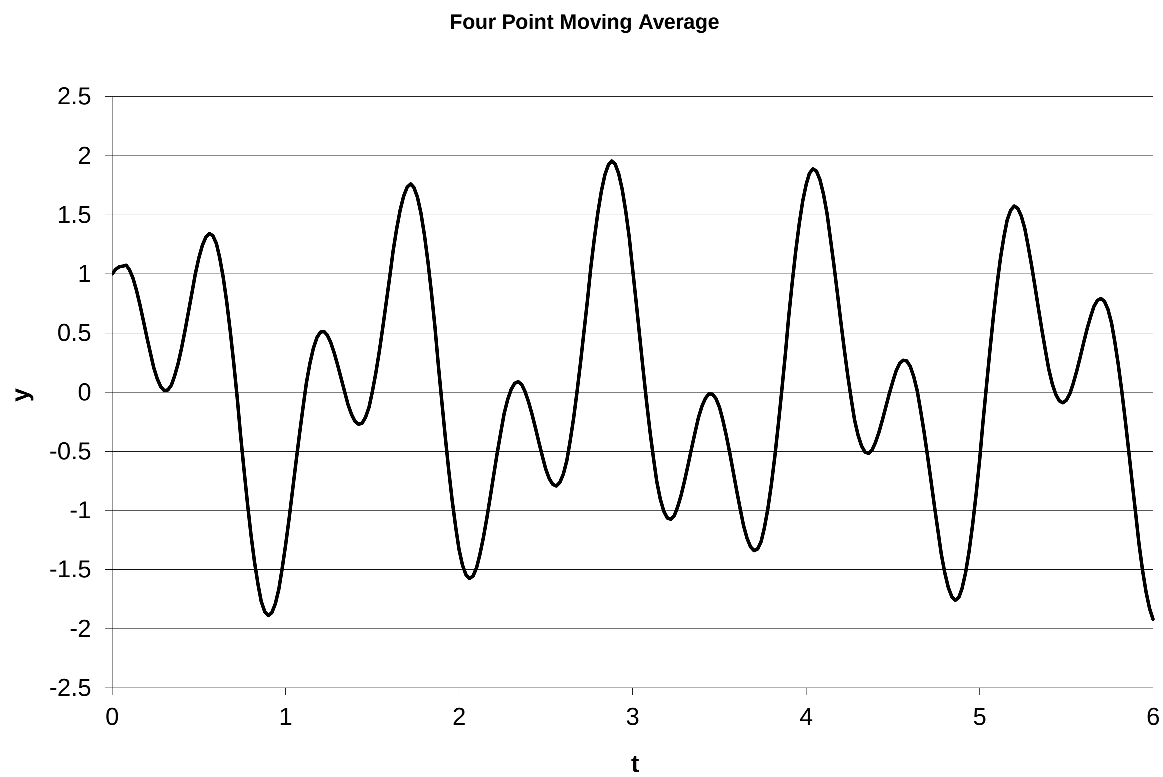
ASSUMPTIONS: The signal $y(t)$ may be represented by making a discrete representation with $\delta t = 0.05$.

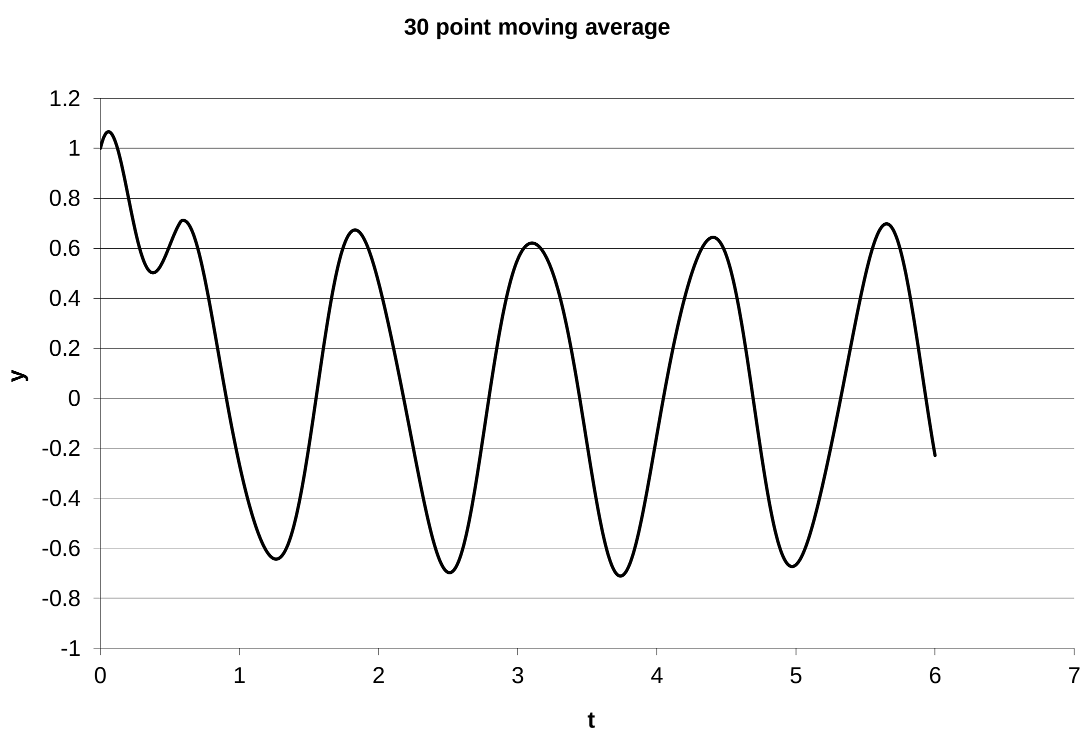
SOLUTION:

a) The signal in Figure 2.23 clearly has a reduced level of high frequency content. In essence, this emphasizes longer-term variations while removing short-term fluctuations. It is clear that the peak-to-peak value in the original signal is significantly higher than in the signal that has been averaged.

b) The figures below show in the effect of applying a moving average to $y(t) = \sin 5t + \cos 11t$.




Four Point Moving Average
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
y
0
1
2
3
4
5
6
t


30 point moving average
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y
0
1
2
3
4
5
6
7
t

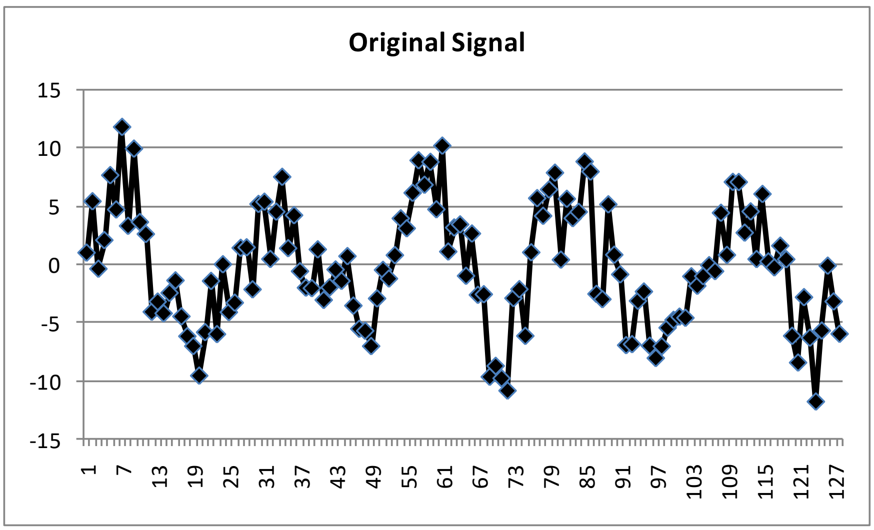
PROBLEM 2.8

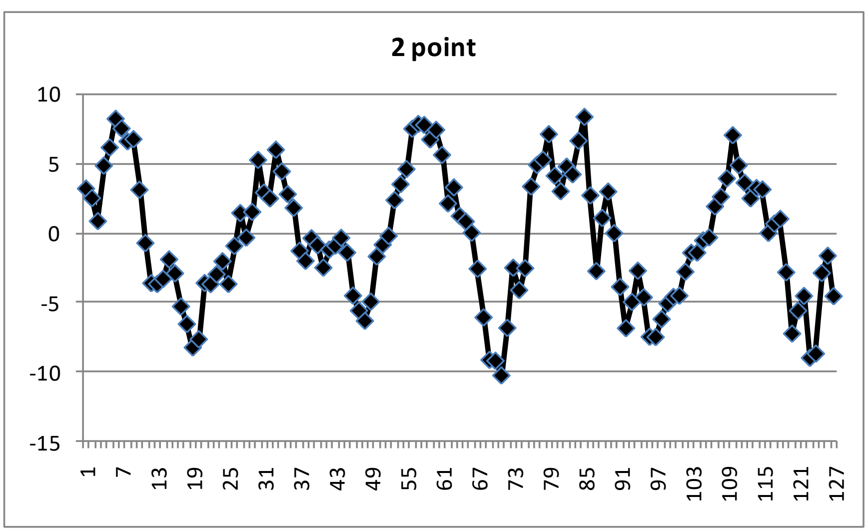
KNOWN: A signal is known to contain random noise (*noisy.txt*).
FIND: Examine the effect of 2, 3, and 4 point moving averages on the noisy signal.

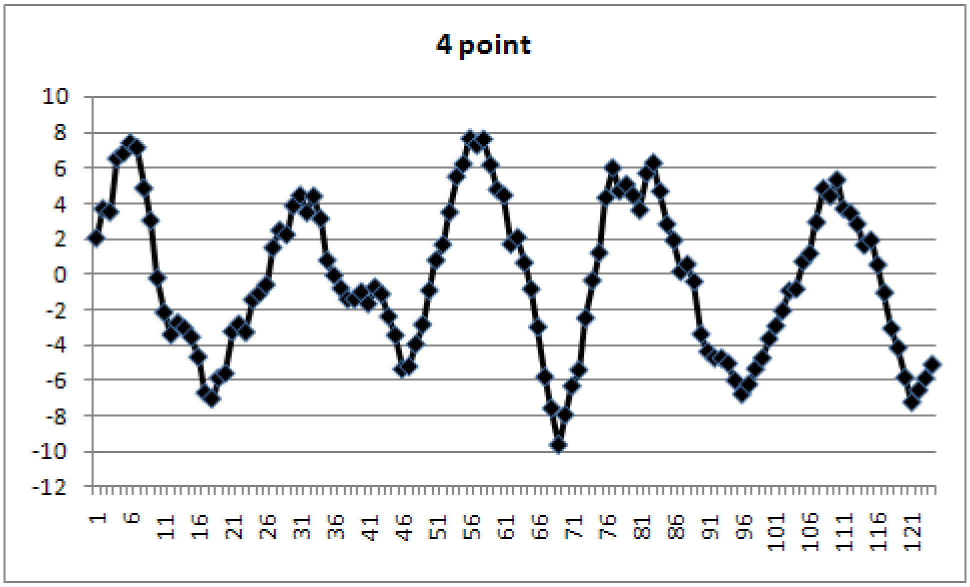
ASSUMPTIONS: The time between each data point is the same.

SOLUTION:

The figures below show in the effect of applying a moving average to the data in the file *noisy.txt*.






4 point
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
1
6
11
16
21
26
31
36
41
46
51
56
61
66
71
76
81
86
91
96
101
106
111
116
121

PROBLEM 2.9

KNOWN: A spring-mass system, with

$$m = 1 \text{ kg}$$
$$T = 2 \text{ s}$$

FIND: Spring constant, k, and natural frequency ω

SOLUTION:

Since

$$\omega = \sqrt{\frac{k}{m}}$$

(as shown in association with equation 2.7)

And

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2 \text{ s}$$
$$\omega = \pi \text{ rad/s}$$

The natural frequency is then found as $\omega = \pi = 3.14$ rad/s

And

$$\omega = 3.14 = \sqrt{\frac{k}{1 \text{ kg}}}$$
$$k = 9.87 \text{ N/m (kg/sec}^2\text{)}$$

PROBLEM 2.10

KNOWN: A spring-mass system having

$$m = 1 \text{ kg}$$

$$k = 5000 \text{ N/cm}$$

FIND: The natural frequency in rad/sec (ω) and Hz (f).

SOLUTION:

The natural frequency may be determined,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5000 \ \frac{N}{cm} 100 \frac{cm}{m}}{1 \ kg}} = 707.1 \ rad/s$$

and

$$f \ \ \frac{\omega}{2\pi} = 112.5 \ Hz$$

PROBLEM 2.11

KNOWN: Functions:

a) $\sin \dfrac{10\pi t}{5}$

b) $8\cos 8t$

c) $\sin 5n\pi t$ for n=1 to ∞

FIND: The period, frequency in Hz, and circular frequency in rad/s are found from

$$\omega = 2\pi f = \frac{2\pi}{T}$$

SOLUTION:

a) $\omega = 2\pi$ rad/s	$f = 1$ Hz	$T = 1$ s
b) $\omega = 8$ rad/s	$f = 4/\pi$ Hz	$T = \pi/4$ s
c) $\omega = 5n\pi$ rad/s	$f = 5n/2$ Hz	$T = 2/(5n)$ s

PROBLEM 2.12

KNOWN: $y(t) = 5\sin 4t + 3\cos 4t$

FIND: Equivalent expression containing a) a cosine term only, and b) a sine term only

SOLUTION:

a) From Equations 2.10 and 2.11

$$y = C\cos(\omega t - \phi) \qquad \phi = \tan^{-1}\frac{B}{A}$$

And with

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\,\cos(\omega t - \phi)$$

We find

$$C = \sqrt{A^2 + B^2} = \sqrt{5^2 + 3^2} = 5.83$$

$$\phi = \tan^{-1}\frac{5}{3} = 1.03 \text{ rad}$$

And

$$y(t) = 5.83\cos(4t - 1.03)$$

b) From Equations 2.10 and 2.11

$$y = C\sin\left(\omega t + \phi^*\right) \qquad \phi^* = \tan^{-1}\frac{A}{B}$$

And with

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\,\sin\left(\omega t + \phi^*\right)$$

We find

$$C = \sqrt{A^2 + B^2} = \sqrt{5^2 + 3^2} = 5.83$$

$$\phi = \tan^{-1}\frac{3}{5} = 0.54 \text{ rad}$$

And

$$y(t) = 5.83\sin(4t + 0.54)$$

PROBLEM 2.13

KNOWN: $y(t) = 4\sin 2\pi t + 15\cos 2\pi t$

FIND:

a) Equivalent expression containing a cosine term only

b) Equivalent expression containing a sine term only

SOLUTION: From Equations 2.10 and 2.11

$$y = C\cos(\omega t - \phi) \qquad \phi = \tan^{-1}\frac{B}{A}$$

$$y = C\sin(\omega t + \phi^*) \qquad \phi^* = \tan^{-1}\frac{A}{B}$$

And with

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\cos(\omega t - \phi)$$

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\sin(\omega t + \phi^*)$$

We find

$$C = \sqrt{A^2 + B^2} = \sqrt{15^2 + 4^2} = 15.52$$

$$\phi = \tan^{-1}\frac{4}{15} = 0.26 \text{ rad}$$

$$\phi^* = \tan^{-1}\frac{15}{4} = 1.31 \text{ rad}$$

And

$$y = 15.52\cos(2\pi t - 0.26) \quad \text{Answer (a)}$$

$$y = 15.52\sin(2\pi t + 1.31) \quad \text{Answer (b)}$$

PROBLEM 2.14

KNOWN: $y(t) = \sum_{n=1}^{\infty} \frac{2\pi n}{6} \sin n\pi t + \frac{4\pi n}{6} \cos n\pi t$

FIND:

a) Equivalent expression containing cosine terms only

SOLUTION: From Equations 2.15 and 2.17

$$y = \sum_{n=1}^{\infty} C_n \cos(\omega t - \phi_n) \qquad \phi_n = \tan^{-1} \frac{B_n}{A_n}$$

with

$$C_n = \sqrt{A_n^2 + B_n^2}$$

And with

$$C_n = \sqrt{\left(\frac{2\pi n}{6}\right)^2 + \left(\frac{4\pi n}{6}\right)^2} = \sqrt{\frac{5}{9}}\pi n \qquad \text{and } \phi_\mathrm{n} = \tan^{-1} \frac{\left(2\pi n/6\right)}{\left(4\pi n/6\right)} = \tan^{-1} 0.5 = 0.46 \text{ rad}$$

We find

$$y(t) = \sum_{n=1}^{\infty} \sqrt{\frac{5}{9}}\pi n \cos(n\pi t - 0.46)$$

PROBLEM 2.15

KNOWN: $y(t) = \sum_{n=0}^{\infty} \frac{20}{\pi} \left[\frac{1}{2n+1} \sin\left(2n+1\right)t \right]$

FIND:

a) Equivalent expression containing cosine terms only

SOLUTION: From Equations 2.15 and 2.17

$$y(t) = \sum_{n=0}^{\infty} \frac{20}{\pi} \left\{ \frac{1}{2n+1} \cos\left[\left(2n+1\right)t - \pi/2\right] \right\}$$

PROBLEM 2.16

KNOWN:

$$y(t) = \sum_{n=1}^{\infty} \frac{3n}{2} \sin nt + \frac{5n}{3} \cos nt$$

FIND: a) fundamental frequency and period
b) express this series in as cosine terms only

SOLUTION:

a) The fundamental frequency corresponds to n = 1, so ω = 1 rad/s; $T = 2\pi$

b) From equation 2.15 and 2.17

$$y(t) = A_o + \sum_{n=1}^{3} C_n \cos\left(\frac{2n\pi t}{T} - \phi_n\right)$$

$$C_n = \sqrt{A_n^2 + B_n^2} \qquad \tan\phi_n = \frac{B_n}{A_n}$$

For this Fourier series

$$C_n = \sqrt{\left(\frac{3n}{2}\right)^2 + \left(\frac{5n}{3}\right)^2} = \sqrt{\frac{181}{36} n^2}$$

$$\phi_n = \tan^{-1}\left(\frac{9}{10}\right) \Rightarrow \phi = 0.7328$$

Thus the third partial sum may be written

$$y(t) = \sum_{n=1}^{3} \sqrt{\frac{181}{36} n^2} \cos\left(nt - 0.7328\right)$$

$$y(t) = 2.24\cos\left(t - 0.7328\right) + 4.48\cos\left(2t - 0.7328\right) + 6.73\cos\left(3t - 0.7328\right)$$

PROBLEM 2.17

KNOWN:

$$y(t) = 4 + \sum_{n=1}^{\infty} \frac{2n\pi}{10} \cos \frac{n\pi}{4} t + \frac{120n\pi}{30} \sin \frac{n\pi}{4} t$$

FIND: a) f_1 and ω_1

b) T_1

c) $$y(t) = A_o + \sum_{n=1}^{\infty} C_n \sin\left(\frac{2n\pi t}{T} + \phi^* \right)$$

SOLUTION:

a) When n = 1, $\omega_1 = \pi/4, \quad f_1 = 1/8$

b) $T_1 = 8$ sec

c) From Equations 2.16 and 2.17

$$C_n = \sqrt{A_n^2 + B_n^2} \quad \text{and} \quad \tan\phi^* = \frac{A_n}{B_n}$$

$$C_n = \sqrt{\left(\frac{2n\pi}{10}\right)^2 + \left(\frac{120n\pi}{30}\right)^2} = 4n\pi$$

$$\phi_n^* = \tan^{-1}\frac{(2n\pi/10)}{(120n\pi/30)} = \tan^{-1}\left(\frac{1}{20}\right) = 0.05 \text{ rad}$$

And the resulting Fourier sine series is

$$y(t) = 4 + \sum_{n=1}^{\infty} 4n\pi \sin\left(\frac{n\pi t}{4} + 0.05 \right)$$

PROBLEM 2.18

KNOWN:

$$y(t) = \sin t$$

FIND: Fourier series that represents the function $y(t) = \sin t$

SOLUTION: The function $y(t) = \sin t$ has a period of 2π so that the Fourier coefficients may be found from Equation 2.18 as

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(t)$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\cos ntdt$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\sin ntdt$$

The term A_o is zero as seen from

$$A_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin tdt = 0$$

And A_n is given by

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nt \cos ntdt = 0$$

And B_1 is given by

$$B_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 t = \frac{\pi}{\pi} = 1$$

And all other B_n are given by

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin t \sin nt = 0$$

The result of the Fourier coefficient equations is consistent with the expected result.

PROBLEM 2.19

KNOWN: $y(t) = t^2$ for $-\pi \le t \le \pi$; $y(t+2\pi) = y(t)$

FIND: Fourier series for the function $y(t)$.

SOLUTION:

Since the function y(t) is an even function, the Fourier series will contain only cosine terms,

$$y(t) = A_o + \sum_{n=1}^{\infty} A_n \cos\frac{2n\pi t}{T} = A_o + \sum_{n=1}^{\infty} A_n \cos n\omega t$$

The coefficients are found as

$$A_o = \frac{1}{T}\int_{-T/2}^{T/2} y(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos\frac{2n\pi t}{2\pi}\,dt$$

$$= \frac{1}{\pi}\left[\frac{2t\cos nt}{n^2} + \frac{n^2t^2 - 2}{n^3}\sin nt\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2}\cos(n\pi) + \frac{2\pi}{n^2}\cos(-n\pi)\right]$$

For n even $A_n = 4/n^2$ for n odd $A_n = -4/n^2$ and the resulting Fourier series is

$$y(t) = \frac{\pi^2}{3} - 4\left[\cos t - \frac{1}{4}\cos 2t + \frac{1}{9}\cos 3t - + \ldots\right]$$

A series approximation for π is

$$y(\pi) = \pi^2 = \frac{\pi^2}{3} - 4\left[\cos\pi - \frac{1}{4}\cos 2\pi + \frac{1}{9}\cos 3\pi - + \ldots\right]$$

or $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \ldots$

PROBLEM 2.20

KNOWN:

$$y(t) = 0 \ \text{ for } -\pi \le t \le 0$$
$$y(t) = -1 \ \text{ for } 0 \le t \le \pi/2$$
$$y(t) = 1 \ \text{ for } \pi/2 \le t \le \pi$$

FIND: Fourier series for $y(t)$ assuming that the function has a period of 2π

SOLUTION: Since the function is neither even nor odd, the Fourier series will contain both sine and cosine terms. The coefficients are found as

$$A_o = \frac{1}{T}\int_{-T/2}^{-T/2} y(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(t)dt$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0dt + \int_{0}^{\pi/2} -1dt + \int_{\pi/2}^{\pi} 1dt\right]$$
$$= \frac{1}{2\pi}\left[\left(-\pi/2 - 0\right) + \left(\pi - \pi/2\right)\right] = 0 \quad \therefore \quad A_o = 0$$

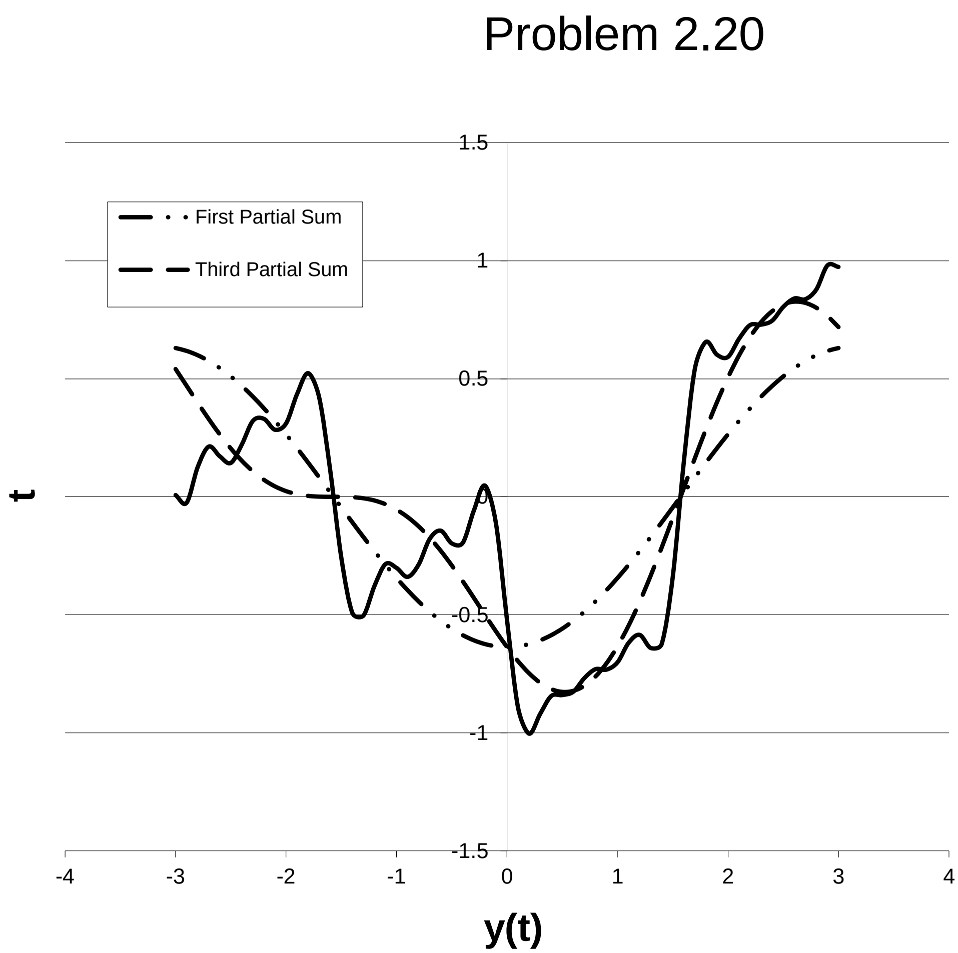
Note: Since the contribution from $-\pi$ to 0 is identically zero, it will be omitted.

$$A_n = \frac{2}{2\pi}\left[\int_0^{\pi/2} -1\cos\frac{2n\pi t}{T} + \int_{\pi/2}^{\pi} 1\cos\frac{2n\pi t}{T}\right]dt$$
$$= \frac{1}{\pi}\left\{\left[\frac{-1}{n}\sin nt\right]_0^{\pi/2} + \left[\frac{1}{n}\sin nt\right]_{\pi/2}^{\pi}\right\}$$
$$= \frac{-1}{\pi n}2\sin\left(\frac{n\pi}{2}\right)$$

$$B_n = \frac{2}{2\pi}\left[\int_0^{\pi/2} -1\sin\frac{2\pi nt}{T}dt + \int_{\pi/2}^{\pi} 1\sin\frac{2\pi nt}{T}dt\right]$$
$$= \frac{1}{n\pi}\left\{\left[\cos nt\right]_0^{\pi/2} + \left[-\cos nt\right]_{\pi/2}^{\pi}\right\} = \frac{1}{n\pi}\left[-\cos(0) - \cos(n\pi)\right]$$

Noting that A_n is zero for n even, and B_n is zero for n odd, the resulting Fourier series is

$$y(t) = \frac{2}{\pi}\left[-\cos t - \frac{1}{2}\sin 2t + \frac{1}{3}\cos 3t - \frac{1}{4}\sin 4t - \frac{1}{5}\cos 5t - \frac{1}{6}\sin 6t + \frac{1}{7}\cos 7t - ...\right]$$



Problem 2.20
First Partial Sum
Third Partial Sum
1.5
1
0.5
0
-0.5
-1
-1.5
t
-4
-3
-2
-1
0
1
2
3
4
y(t)

PROBLEM 2.21

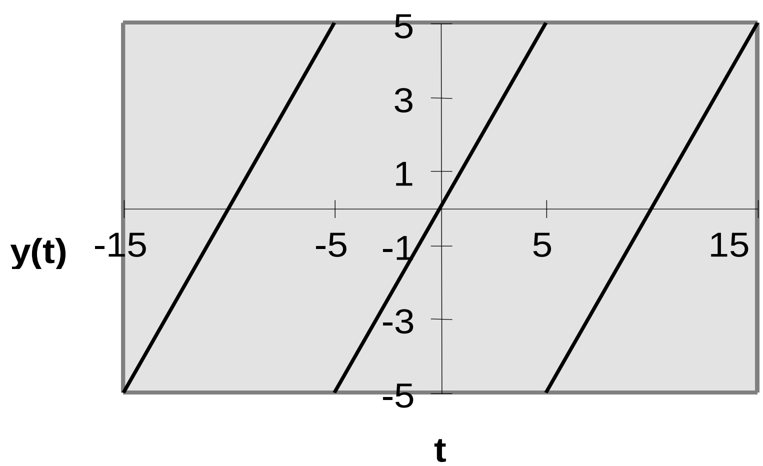
KNOWN: $y(t) = t$ for $-5 < t < 5$

FIND: Fourier series for the function $y(t)$.

ASSUMPTIONS: An odd periodic extension is assumed.

SOLUTION:

The function is approximated as shown below



Since the function is odd, the Fourier series will contain only sine terms

$$y(t) = \sum_{n=1}^{\infty} B_n \sin\frac{2n\pi t}{T}$$

Where, from (2.13)

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin\frac{2n\pi t}{T}\,dt$$

Thus

$$B_n = \frac{2}{10}\int_{-5}^{5} t\sin\frac{2n\pi t}{10}\,dt$$

Which is of the form $x\sin ax$, and

$$B_n = \frac{2}{10}\left[\left(\frac{5}{n\pi}\right)^2 \sin\frac{2n\pi t}{10} - \frac{10t}{2n\pi}\cos\frac{2n\pi t}{10}\right]_{-5}^{5}$$

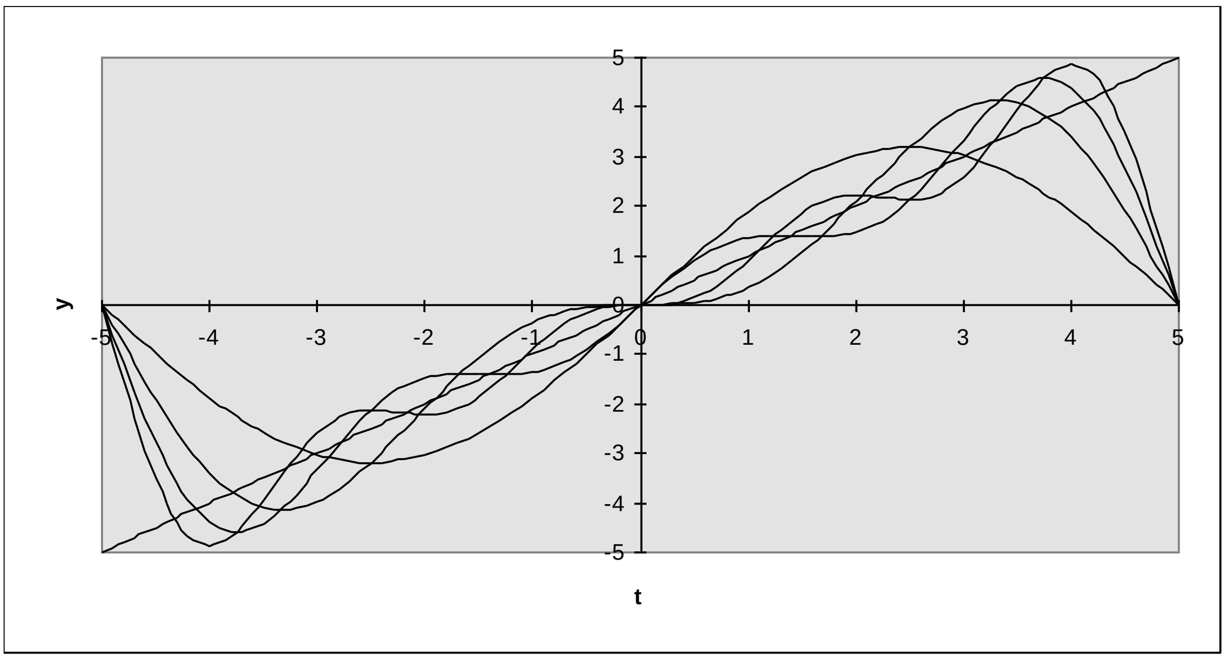
$$= \frac{2}{10}\left[\left(\frac{5}{n\pi}\right)^2 \{\sin(n\pi) - \sin(-n\pi)\} - \left(\frac{50}{2n\pi}\right)\{\cos(n\pi) + \cos(-n\pi)\}\right]$$

For n even $B_n = \frac{-10}{n\pi}$

For n odd $B_n = \frac{10}{n\pi}$

The resulting Fourier series is

$$y(t) = \frac{10}{\pi}\sin\frac{2\pi t}{10} - \frac{10}{2\pi}\sin\frac{4\pi t}{10} + \frac{10}{3\pi}\sin\frac{6\pi t}{10} - \frac{10}{4\pi}\sin\frac{8\pi t}{10} +\text{-} \ldots$$



PROBLEM 2.22

KNOWN:

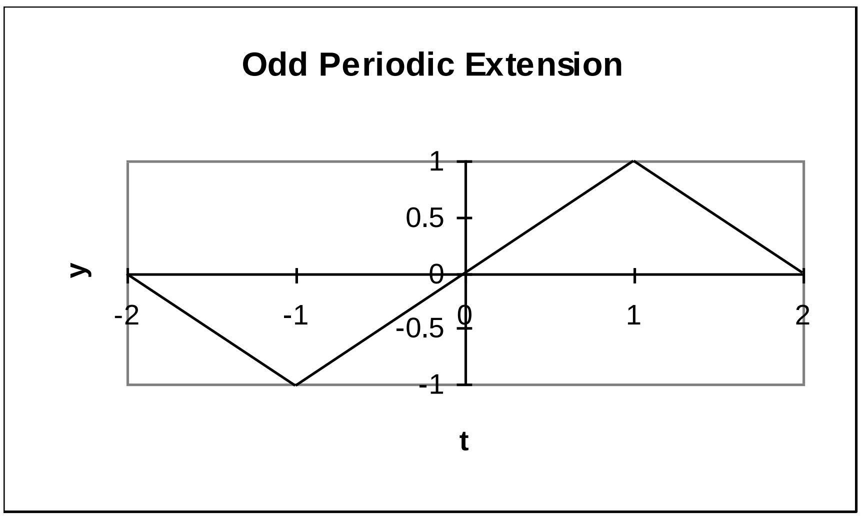
$$y(t) = \begin{Bmatrix} t \ \text{ for } 0 < t < 1 \\ 2 - t \ \text{ for } 1 < \text{t} < 2 \end{Bmatrix}$$

FIND: Fourier series representation of $y(t)$

ASSUMPTION: Utilize an odd periodic extension of $y(t)$

SOLUTION:

The function is extended as shown below with a period of 4.



The Fourier series for an odd function contains only sine terms and can be written

$$y(t) = \sum_{n=1}^{\infty} B_n \sin \frac{2n\pi t}{T} = \sum_{n=1}^{\infty} B_n \sin n\omega t$$

Where

$$B_n = \int_{-T/2}^{T/2} y(t) \sin \frac{2n\pi t}{T} \, dt$$

For the odd periodic extension of the function $y(t)$ shown above, this integral can be expressed as the sum of three integrals

$$B_n = \int_{-2}^{-1} -(2+t)\sin\frac{2n\pi t}{4}\,dt + \int_{-1}^{1} t\sin\frac{2n\pi t}{4}\,dt + \int_{1}^{2} (2-t)\sin\frac{2n\pi t}{4}\,dt$$

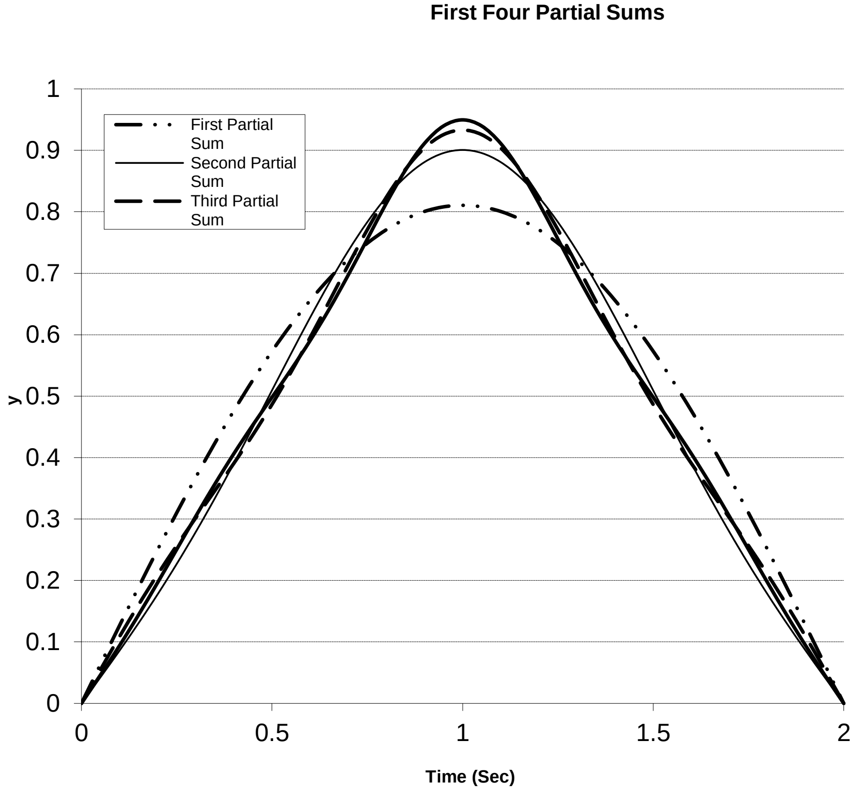
These integrals can be evaluated and simplified to yield the following expression for B_n

$$B_n = 4\frac{-\sin(n\pi) + 2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

Since $\sin(n\pi)$ is identically zero, and $\sin(n\pi/2)$ is zero for n even, the Fourier series can be written

$$y(t) = \frac{8}{\pi^2}\left[\sin\frac{\pi t}{2} - \frac{1}{9}\sin\frac{3\pi t}{2} + \frac{1}{25}\sin\frac{5\pi t}{2} - \frac{1}{49}\sin\frac{7\pi t}{2} + - \ldots\right]$$

The first four partial sums of this series are shown below



PROBLEM 2.23

KNOWN:

$$y(t) = \begin{cases} \dfrac{-2}{\pi}t - 2 \ \text{ for } -\pi < t < \dfrac{-\pi}{2} \\ \dfrac{2}{\pi}t \ \text{ for } \dfrac{-\pi}{2} < t < \dfrac{\pi}{2} \\ \dfrac{-2}{\pi}t + 2 \ \text{ for } \dfrac{\pi}{2} < t < \pi \end{cases}$$

FIND: Fourier series representation of $y(t)$

ASSUMPTION: $y(t)$ is an odd function with a period 2π

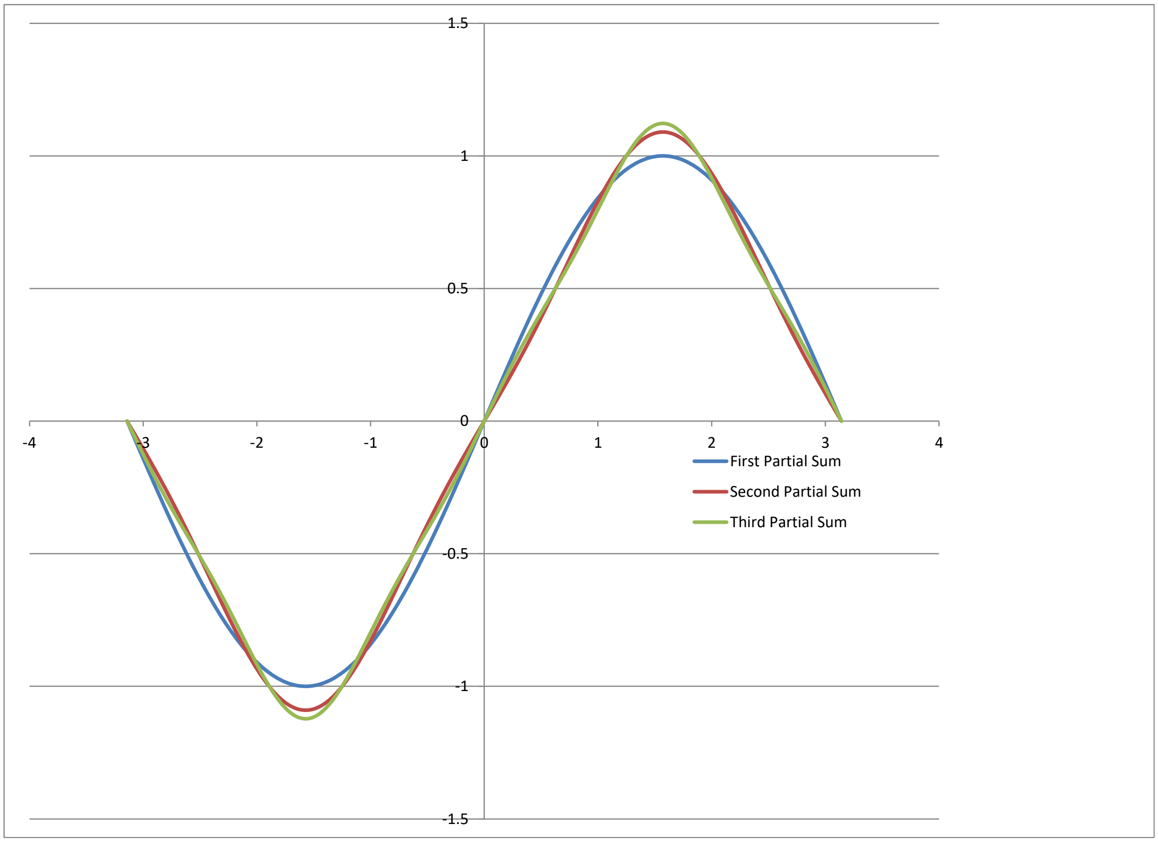
SOLUTION:

Because the function is odd
$A_o = 0$ and $A_n = 0$
And

$$B_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}\left(-\frac{2}{\pi}t - 2\right)\sin ntdt + \int_{-\pi/2}^{\pi/2}\left(\frac{2}{\pi}t\right)\sin ntdt + \int_{\pi/2}^{\pi}\left(-\frac{2}{\pi}t + 2\right)\sin ntdt\right]$$

The integration yields the Fourier series

$$y(t) = \frac{8}{\pi^2}\sum_{n=1,3,5...}^{\infty}\frac{(-1)^{(n-1)/2}}{n^2}\sin\left(\frac{n\pi t}{\pi}\right)$$



1.5
1
0.5
0
-0.5
-1
-1.5
-4
-3
-2
-1
0
1
2
3
4
First Partial Sum
Second Partial Sum
Third Partial Sum

PROBLEM 2.24

KNOWN:

a) sin 10t V
b) $5+2\cos 2t$ m
c) $5t$ s
d) 2 V

FIND: Classification of signals

SOLUTION:

a) Dynamic, deterministic, simple periodic waveform
b) Dynamic, deterministic periodic with a zero offset
c) Dynamic, deterministic, unbounded as $t \to \infty$
d) Static, deterministic

PROBLEM 2.25

KNOWN: At time zero ($t = 0$)

$$x = 0$$

$$\frac{dx}{dt} = 5 \text{ cm / s} \quad f = 1 \text{ Hz}$$

FIND:

a) period, T
b) amplitude, A
c) displacement as a function of time, $x(t)$
d) maximum speed

SOLUTION:

The position of the particle as a function of time may be expressed

$$x(t) = A \sin 2\pi t$$

So that

$$\frac{dx}{dt} = 2A\pi \cos 2\pi t$$

Thus, at t = 0 $\frac{dx}{dt} = 5$

From these expressions we find

a) $T = 1$ s

b) amplitude, $A = 5/2\pi$

c) $x(t) = (5/2\pi)\sin 2\pi t$

d) maximum speed = 5 cm/s

PROBLEM 2.26

KNOWN:

a) Frequency content
b) Amplitude
c) Magnitude
d) Period

FIND:

Define the terms listed above

SOLUTION:

a) Frequency content - for a complex periodic waveform, refers to the relative amplitude of the terms which comprise the Fourier series for the signal, or the result of a Fourier transform.

b) Amplitude - the range of variation of a particular frequency component in a complex periodic waveform

c) Magnitude - the value of a signal, which may be a function of time

d) Period - the time for a signal to repeat, or the time associated with a particular frequency component in a complex periodic waveform.

PROBLEM 2.27

KNOWN:

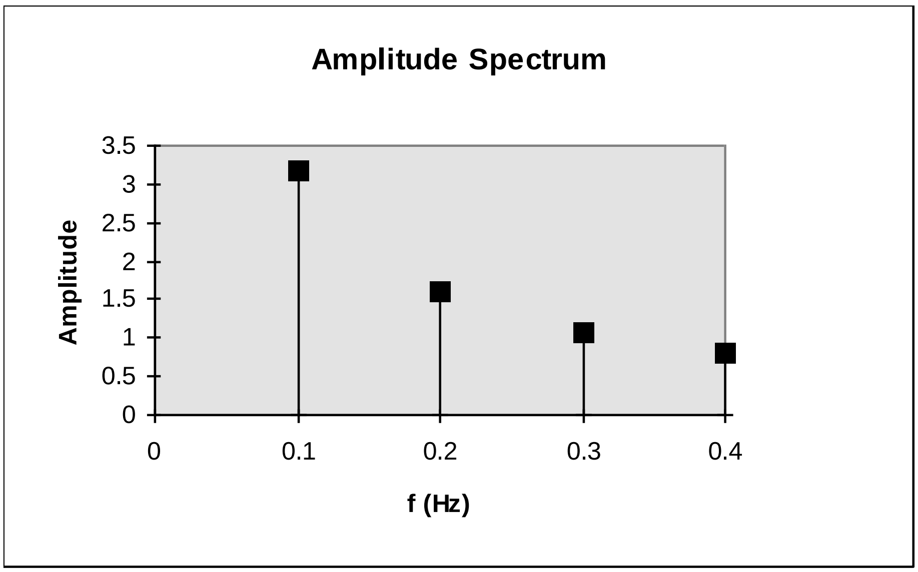
Fourier series for the function $y(t) = t$ in Problem 2.21

$$y(t) = \frac{10}{\pi}\sin\frac{2\pi t}{10} - \frac{10}{2\pi}\sin\frac{4\pi t}{10} + \frac{10}{3\pi}\sin\frac{6\pi t}{10} - \frac{10}{4\pi}\sin\frac{8\pi t}{10} + \ldots$$

FIND:

Construct an amplitude spectrum plot for this series.

SOLUTION:



PROBLEM 2.28

KNOWN: Fourier series for the function $y(t) = t^2$ in Problem 2.19

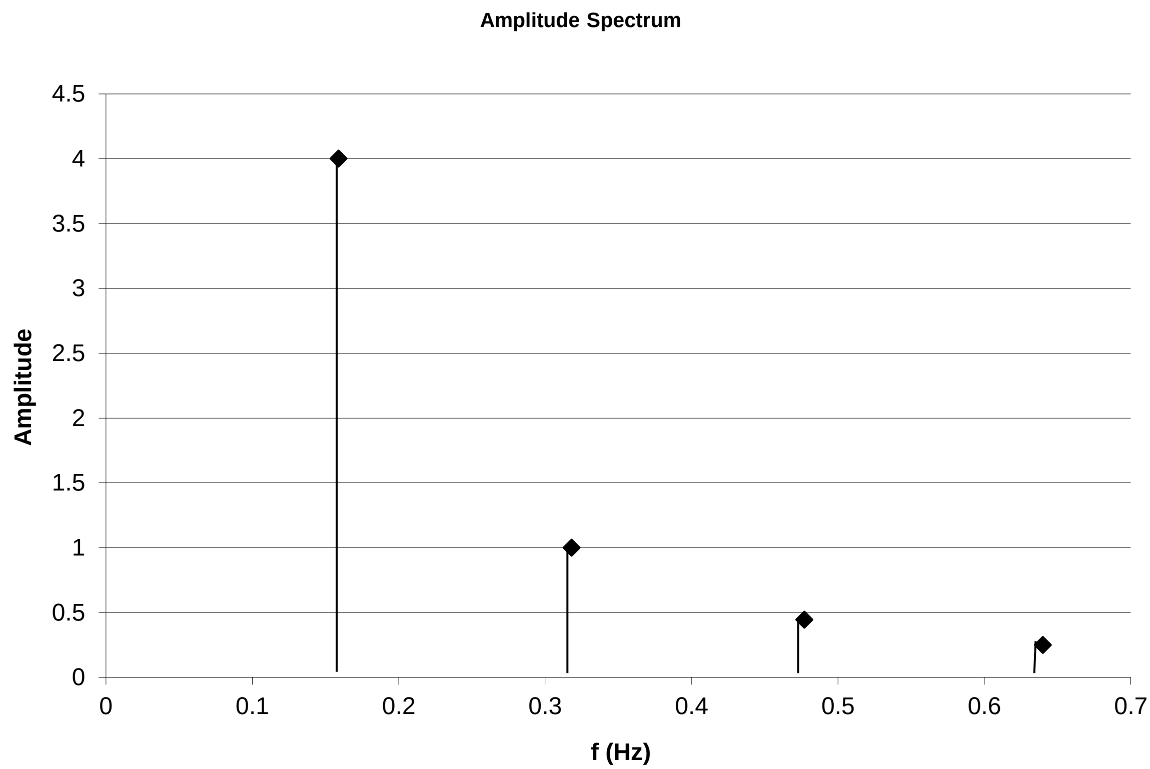
$$y(t) = \frac{\pi^2}{3} - 4\left(\cos t - \frac{1}{4}\cos 2t + \frac{1}{9}\cos 3t - + \ldots\right)$$

FIND:

Construct an amplitude spectrum plot for this series.

SOLUTION:

The corresponding frequency spectrum is shown below



COMMENT: The relative importance of the various terms in the Fourier series as discerned from the amplitude of each term would aid in specifying the required frequency response for a measurement system. For example, the term cos $4x$ has an amplitude of 1/16, which for many purposes may not influence a measurement, and would allow a measurement system to be selected to measure frequencies up to 0.6 Hz.

PROBLEM 2.29

KNOWN: Signal sources:

a) thermostat on a refrigerator

b) input to a spark plug

c) input to a cruise control

d) a pure musical tone

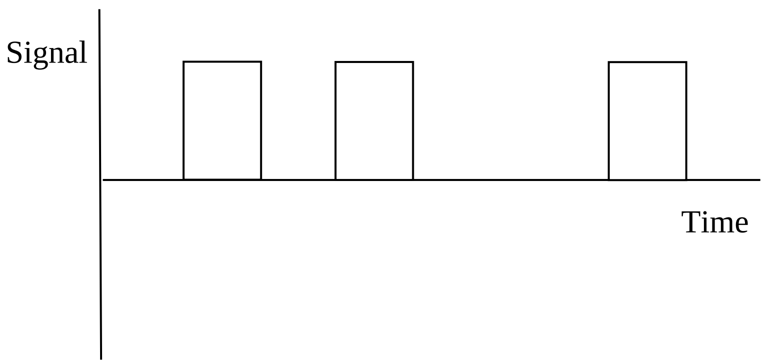
e) note produced by a guitar string

f) AM and FM radio signals

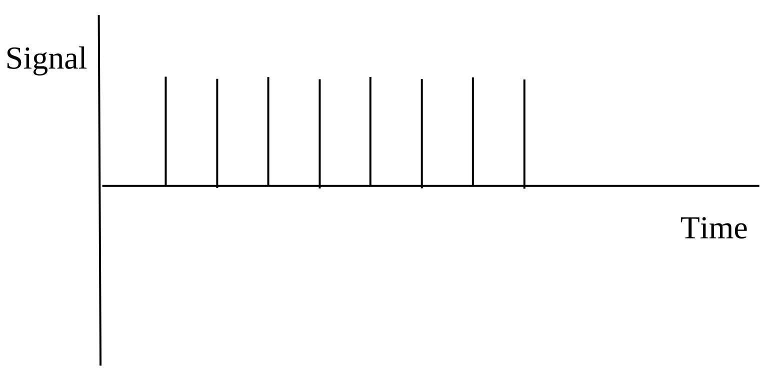
FIND: Sketch representative signal waveforms.

SOLUTION:

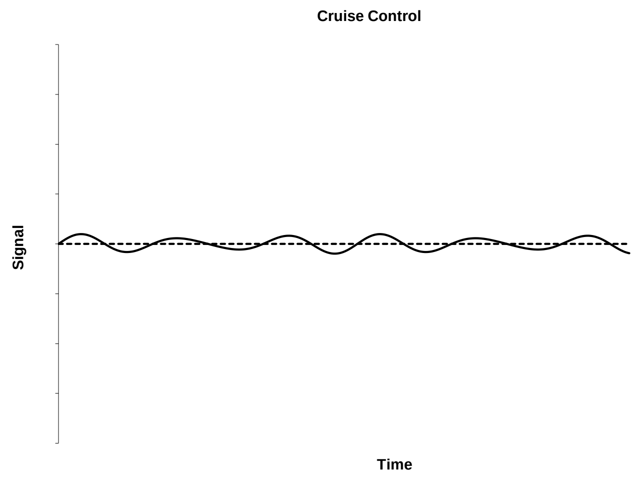
a)



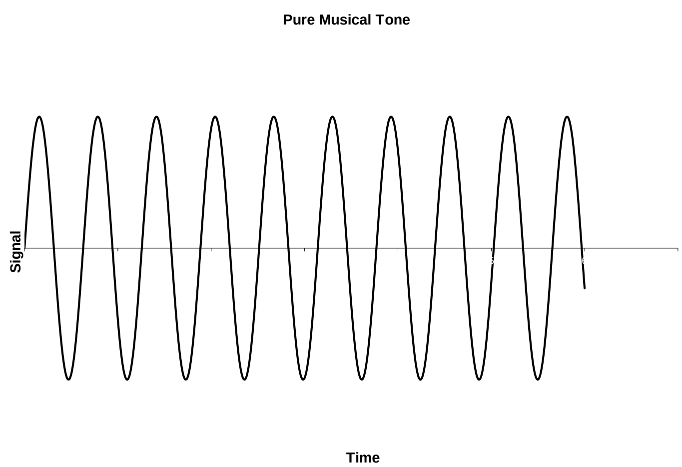
b)



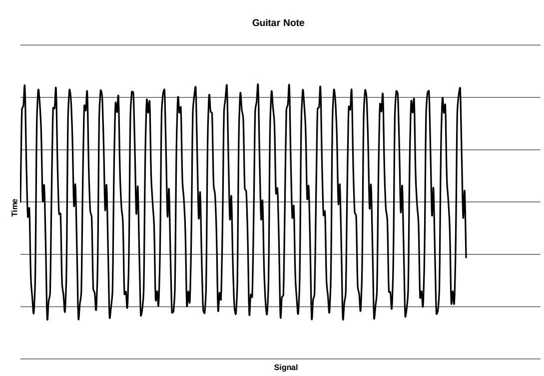
c)


Cruise Control
Signal
Time

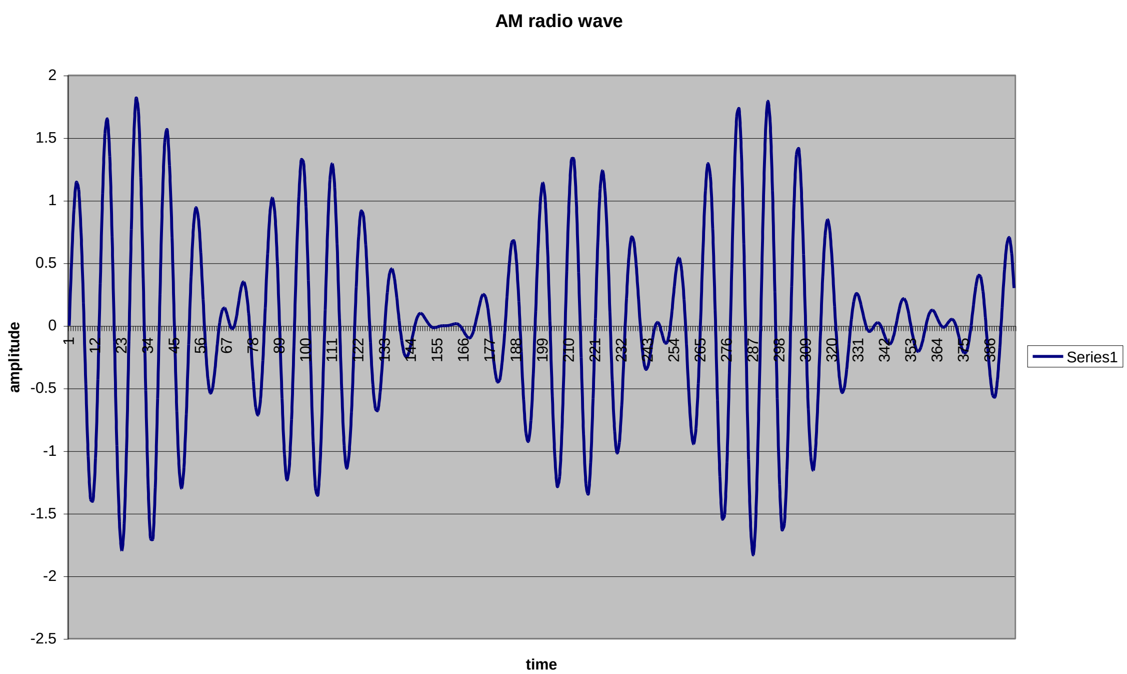
d)


Pure Musical Tone
Signal
Time

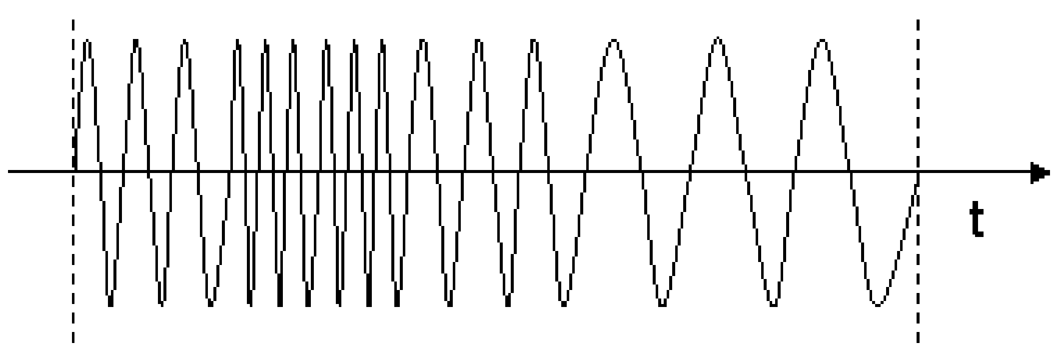
e)


Guitar Note
Time
Signal

f)



FM Radio Wave



PROBLEM 2.30

KNOWN: $e(t) = 5\sin 31.4t + 2\sin 44t$

FIND: $e(t)$ as a discrete-time series of $N = 128$ numbers separated by a time increment of δt. Find the amplitude-frequency spectrum.

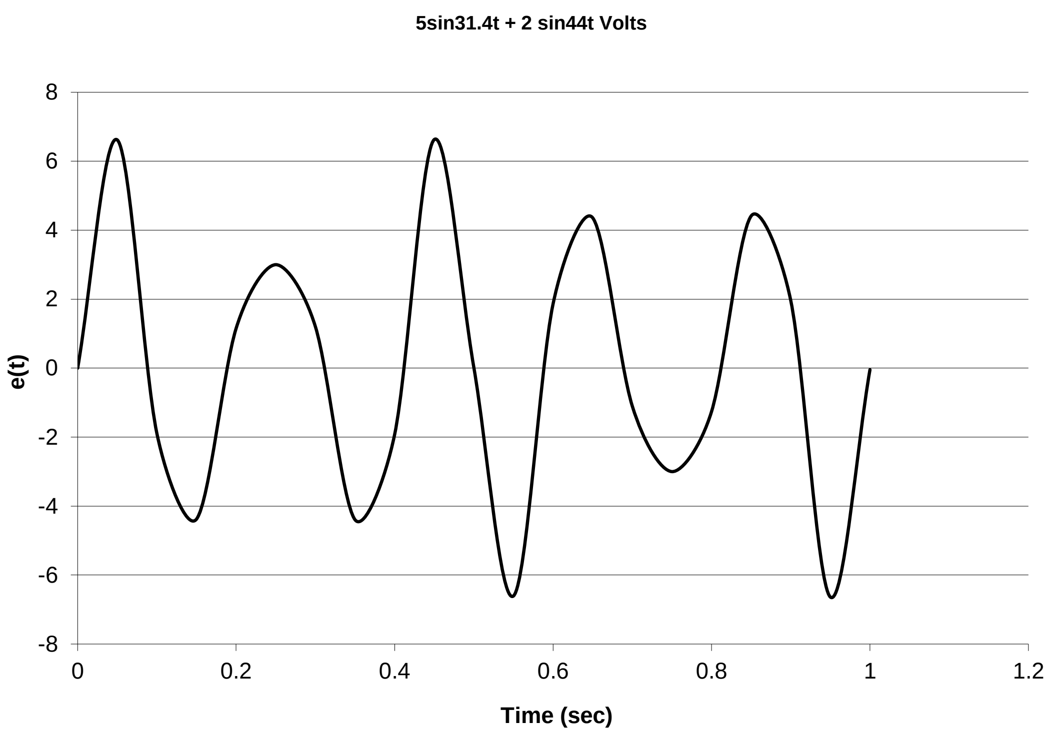
SOLUTION:

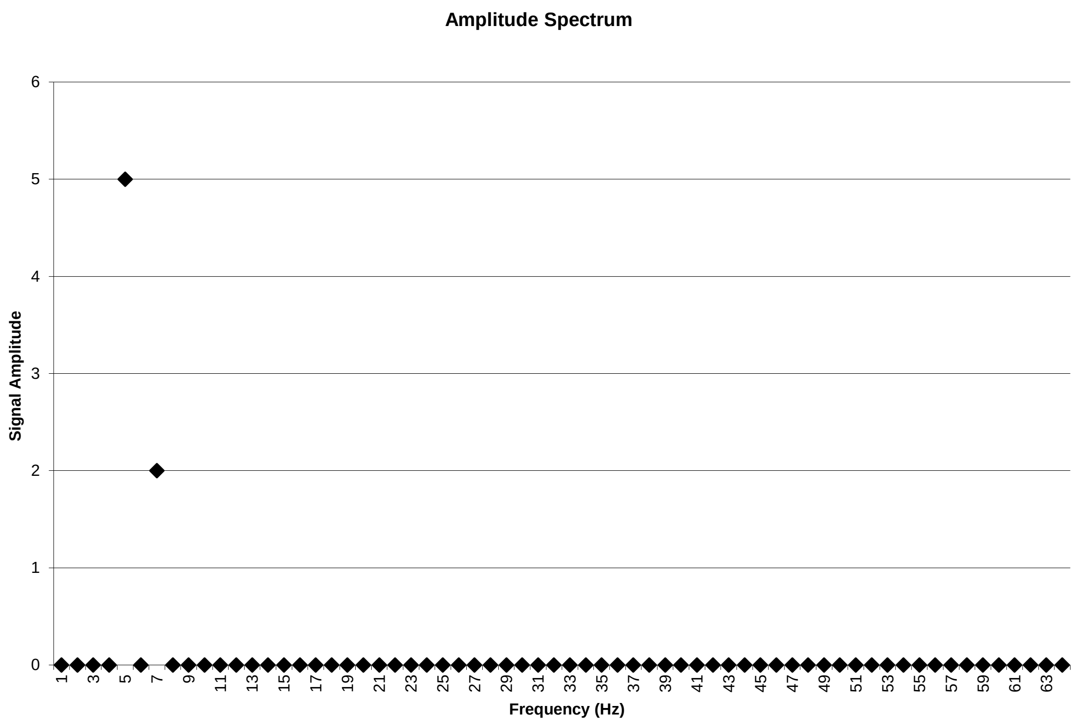
With $N = 128$ and $\delta t = 1/N$, the discrete-time series will represent a total time (or series length) of $N\,\delta t = 1$ sec. The signal to be represented contains two fundamental frequencies,

$$f_1 = 31.4/2\pi = 5 \text{ Hz} \quad \text{and} \quad f_2 = 44/2\pi = 7 \text{ Hz}$$

We see that the total time length of the series will represent more than one period of the signal $e(t)$ and, in fact, will represent 5 periods of the f_1 component and 7 periods of the f_2 component of this signal. This is important because if we represent the signal by a discrete-time series that has an exact integer number of the periods of the fundamental frequencies, then the discrete Fourier series will be exact.

Any DFT or FFT program can be used to solve this problem. Using the companion software disk, issues associated with sampling continuous signals to create discrete-time series can be explored. The time series and the amplitude spectrum are plotted below.



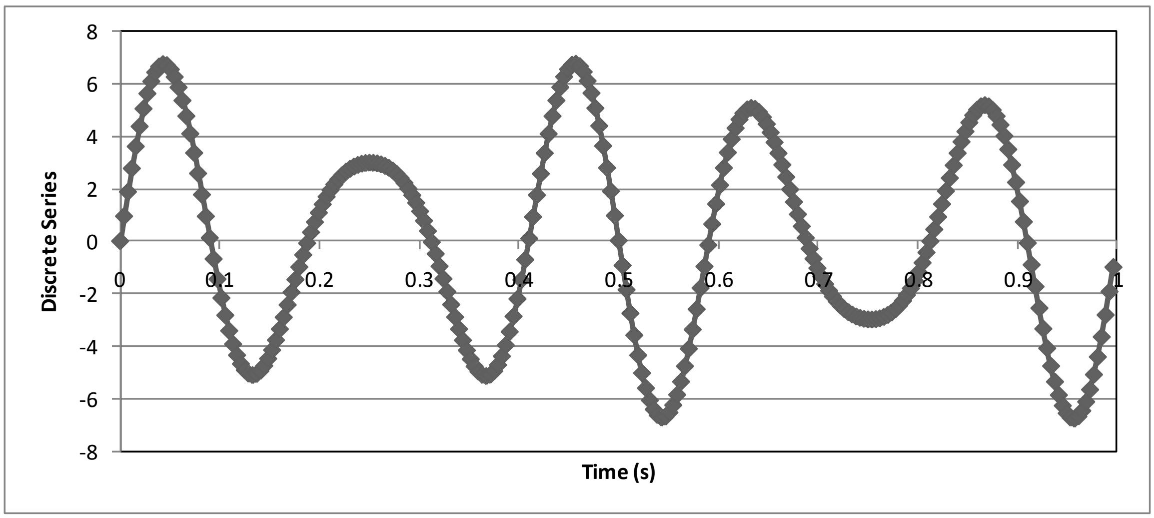

Amplitude Spectrum
Signal Amplitude
Frequency (Hz)
0
1
2
3
4
5
6
1
3
5
7
9
11
13
15
17
19
21
23
25
27
29
31
33
35
37
39
41
43
45
47
49
51
53
55
57
59
61
63

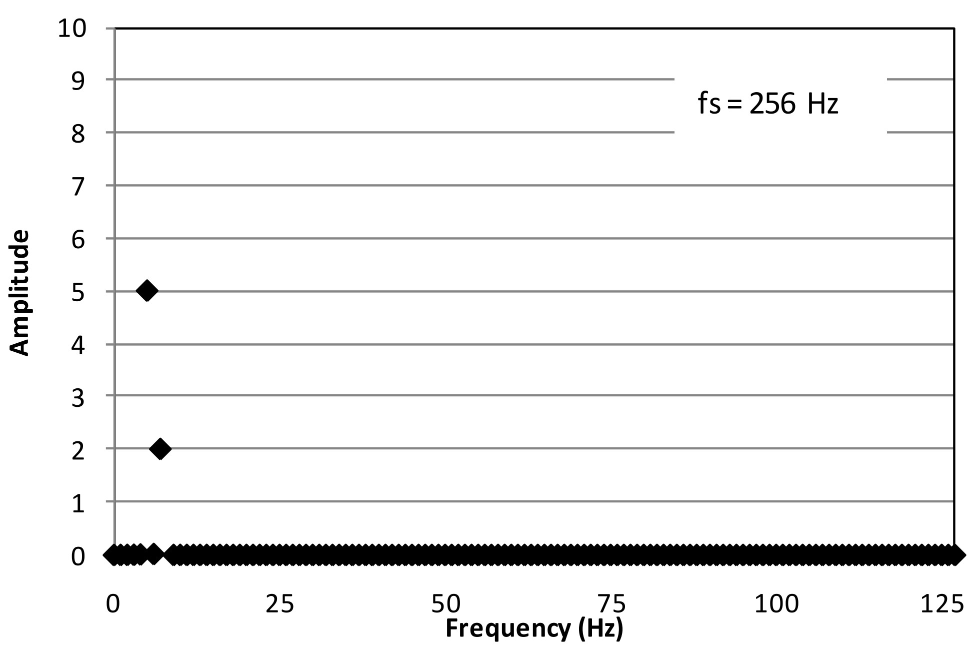
PROBLEM 2.31

KNOWN: $e(t) = 5\sin 31.4t + 2\sin 44t$ Volts

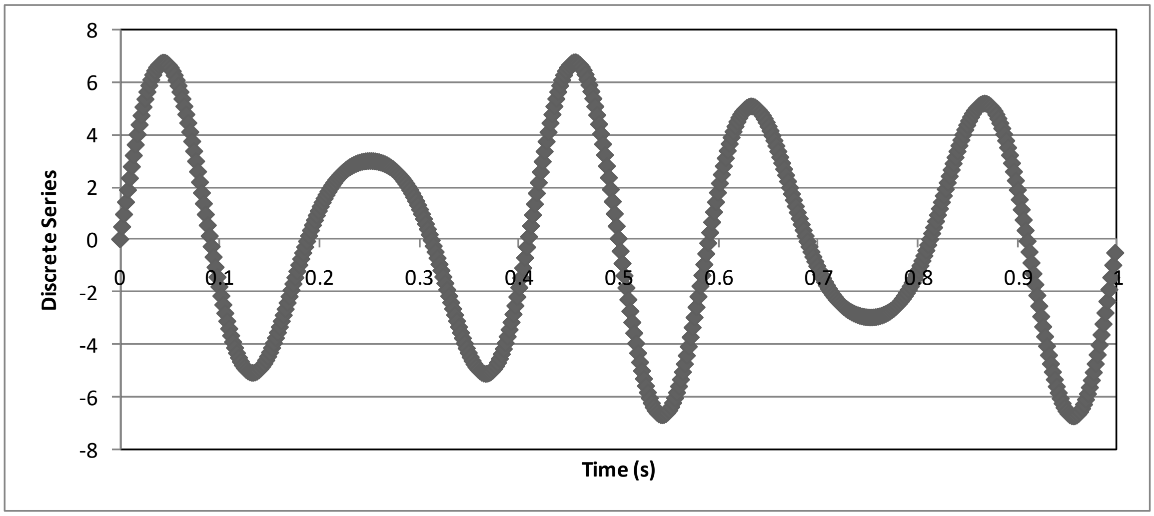
FIND: Repeat Problem 2.30 using N = 256 and δt = 1/256 s and δt = 1/512 s

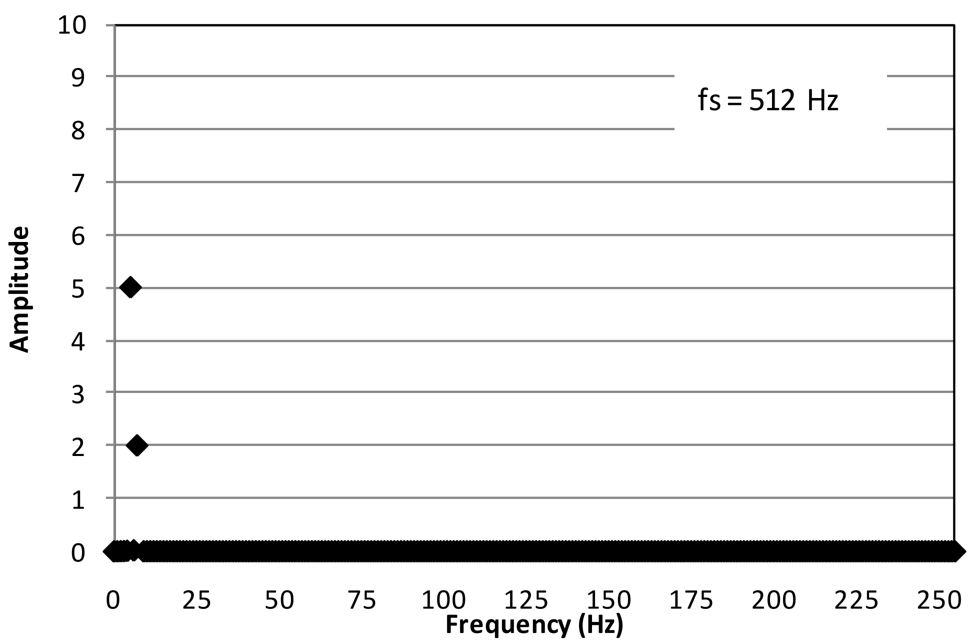
The resolution and number of data points changes but not the signal content or the spectrum quality. In Chapter 7, we study the relation between sample rate and discrete-series quality.





Problem 2.31 continued





PROBLEM 2.32

KNOWN: 3250 με < ε < 4150 με , *f* = 1 Hz

FIND:
a) average value
b) amplitude and frequency when the output is expressed as a simple periodic function
c) express the signal, *y*(*t*), as a Fourier series
d) construct an amplitude spectrum plot.

SOLUTION:

a) $A_o =$ average value $= (3250 + 4150)/2 = 3700\ \mu\varepsilon$

b) $C_1 = (4150 - 3250)/2 = 450\ \mu\varepsilon$
$f_1 = 1$ Hz

c) $y(t) = 3700 + 450\sin\left(2\pi t \pm \pi/2\right)\ \mu\varepsilon$

d) An amplitude spectrum would display an entry representing the mean value of 3700×10^{-6} at 0 Hz and an entry representing an amplitude of 450×10^{-6} at 1 Hz.

PROBLEM 2.33

KNOWN: A force input signal varies between 100 and 170 N ($100 \le F \le 170$) at a frequency of ω = 10 rad/s.

FIND: Signal average value, amplitude and frequency. Express the signal, $y(t)$, as a Fourier series.

SOLUTION:

The signal characteristics may be determined by writing the signal as

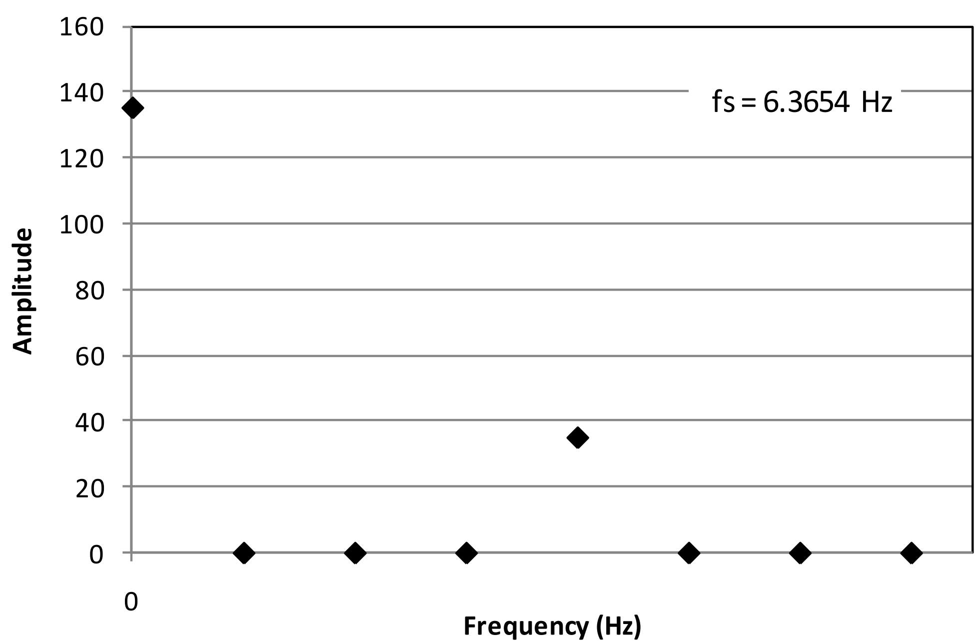
$$y(t) = 135 + 35 \sin 10t \quad [\text{N}]$$

a) A_o = Average value = (170 + 100)/2 = 135 N

b) C_1 = (170 – 100)/2 = 135 N; $f_1 = \omega/2\pi = 10/2\pi = 1.59$ Hz

c) $y(t) = 135 + 35 \sin (10t \pm \pi/2)$

A discrete time series was created using 16 points each separated by 0.1571 s. The amplitude spectrum is shown below.



PROBLEM 2.34

KNOWN: A periodic displacement varies between 2 and 5 mm ($2 \le x \le 5$ 2) at a frequency of f = 100 Hz.

FIND: Express the signal as a Fourier series and plot the signal in the time domain, and construct an amplitude spectrum plot.

SOLUTION:
Noting that

$A_o =$ Average value $= (2+5)/2 =$ 3.5 mm

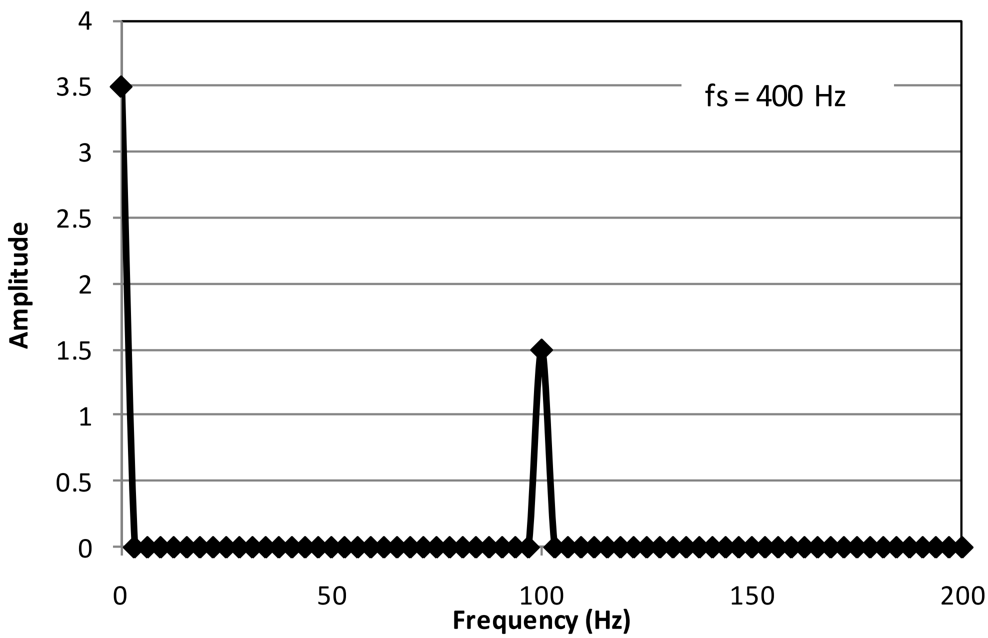
$C_1 = (5-2)/2 =$ 1.5 mm

$f_1 = 100$ Hz

The displacement may be expressed

$$y(t) = 3.5 + 1.5\sin\left(200\pi t \pm \pi/2\right) \text{ mm}$$

The resulting time domain behavior is a simple periodic function; an amplitude spectrum plot should show a value of 3.5 mm at zero frequency and a value of 1.5 mm at a frequency of 100 Hz. Below, we set N = 128 and used a sample rate of 400 Hz. The Fourier analysis returns 200 points, each separated by 3.125 Hz.



PROBLEM 2.35

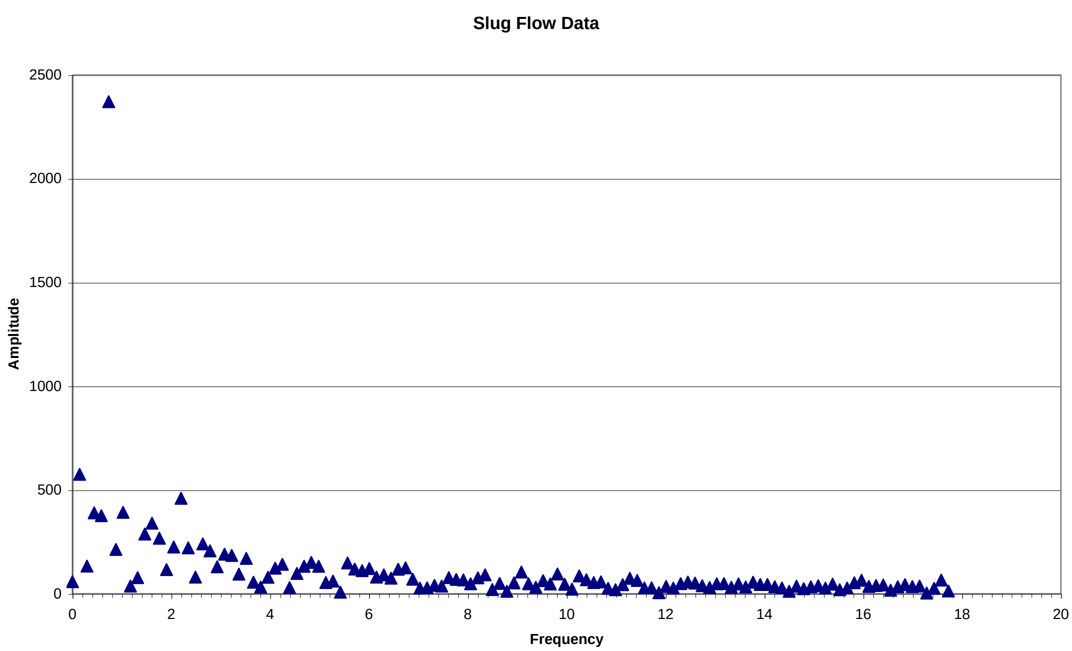
KNOWN: Wall pressure is measured in the upward flow of water and air as provided in the file *gas_ liquid_data.txt*. The flow is in the slug flow regime, with slugs of liquid and large gas bubbles alternating in the flow, as shown in the text Figure 2.27. Pressure measurements were acquired at a sample frequency of 300 Hz, and the average flow velocity is 1 m/sec.

FIND: Construct an amplitude spectrum for the signal, and determine the length of the repeating bubble/slug flow pattern.

SOLUTION:

The figure below shows the amplitude spectrum for the measured data. There is clearly a dominant frequency at 0.73 Hz. Then with an average flow velocity of 1 m/sec, the length is determined as

$$L = \frac{1\text{ m/sec}}{0.73\text{ Hz}} = 1.37\text{ m}$$

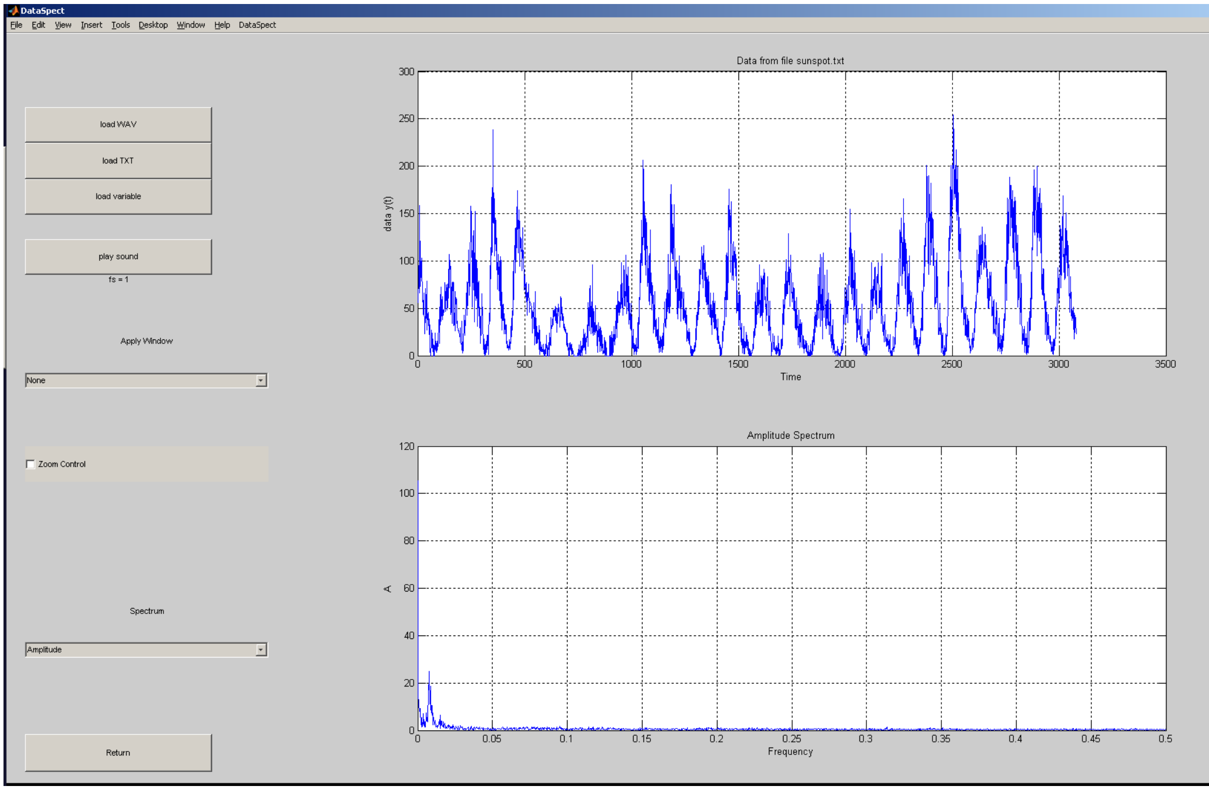


PROBLEM 2.36

KNOWN: Sunspot data for the years 1746 to 2005, from the file *sunspot.txt*.

FIND: Plot the data and create an amplitude spectrum using the companion software program *Dataspect*.

SOLUTION:



Using the zoom control, we can estimate that the amplitude peak occurs at approximately 0.008 which corresponds to a period in months of 125 and a period in years of 10.4. Most references quote an 11 year period.

PROBLEM 2.37

KNOWN:

Signals,

a) Clock face having hands
b) Morse code
c) Musical score
d) Flashing neon sign
e) Telephone conversation
f) Fax transmission

FIND: Classify signals as completely as possible

SOLUTION:

a) Analog, time-dependent, deterministic, periodic, steady-state

b) Digital, time-dependent, nondeterministic

c) Digital, time-dependent, nondeterministic

d) Digital, time-dependent, deterministic

e) Analog or digital, time-dependent, nondeterministic

f) Digital, time-dependent, nondeterministic

PROBLEM 2.38

KNOWN: Amplitude and phase spectrum for $\{y(r\delta t)\}$ from Figure 2.28

FIND: $\{y(r\delta t)\}$, δf, δt

SOLUTION:

By inspection of Figure 2.27:

$$C_1 = 5\text{ V} \quad C_2 = 0\text{ V} \quad C_3 = 3\text{ V} \quad C_4 = 0\text{ V} \quad C_5 = 1\text{ V}$$
$$f_1 = 1\text{ Hz} \quad f_2 = 2\text{ Hz} \quad f_3 = 3\text{ Hz} \quad f_4 = 4\text{ Hz} \quad f_5 = 5\text{ Hz}$$
$$\phi_1 = 0\text{ rad} \quad \phi_2 = 0\text{ rad} \quad \phi_3 = 0.2\text{ rad} \quad \phi_4 = 0\text{ rad} \quad \phi_5 = 0.1\text{ rad}$$

And $\delta f = 1\text{ Hz}$.

The signal can be reconstructed from the above information, as

$$y(t) = 5\sin(2\pi t) + 3\sin(6\pi t + 0.2) + \sin(10\pi t + 0.1)$$

The exact phase of the signal relative to t = 0 is not known, so $y(t)$ is ambiguous within $\pm\pi/2$ in terms of its overall phase.

A DFT returns N/2 values. Therefore 5 spectral values implies that N = 10. Then

$\delta f = 1/N\delta t = 1\text{ Hz} = 1/10\delta t$ giving $\delta t = 0.1$ sec or $f_s = 10$ Hz

Alternatively, by inspection of the plots

$f_N = f_s/2 = 5$ Hz giving $f_s = 10$ Hz or $\delta t = 0.1$ sec

PROBLEM 2.39

KNOWN:

$$y(t) = (4C/T)t + C \quad -T/2 \le t \le 0$$
$$y(t) = (-4C/T)t + C \quad 0 \le t \le T/2$$

FIND: Show that the signal $y(t)$ can be represented by the Fourier series

$$y(t) = A_o + \sum_{n=1}^{\infty} \frac{4C(1-\cos n\pi)}{(\pi n)^2} \cos\frac{2n\pi t}{T}$$

SOLUTION:

a) Since the function $y(t)$ is an even function, the Fourier series will contain only cosine terms,

$$y(t) = A_o + \sum_{n=1}^{\infty} A_n \cos\frac{2n\pi t}{T} = A_o + \sum_{n=1}^{\infty} A_n \cos n\omega t$$

The value of A_o is determined from Equation (2.17)

$$A_o = \frac{1}{T}\int_{-T/2}^{T/2} y(t)dt$$

$$A_o = \int_{-T/2}^{0}\left(\frac{4Ct}{T} + C\right)dt + \int_{0}^{T/2}\left(\frac{-4Ct}{T} + C\right)dt$$

Integrating yields a value of zero for A_o

$$A_o = \frac{1}{T}\left\{\left[\frac{2Ct^2}{T} + Ct\right]_{-T/2}^{0} + \left[\frac{-2Ct^2}{T} + Ct\right]_{0}^{T/2}\right\} = 0$$

Then to determine A_n

$$A_n = \frac{2}{T}\left\{\int_{-T/2}^{0}\left(\frac{4Ct}{T} + C\right)\cos\frac{2n\pi t}{T}dt + \int_{0}^{T/2}\left(\frac{-4Ct}{T} + C\right)\cos\frac{2n\pi t}{T}dt\right\}$$

$$A_n = -2\frac{C(-2 + 2\cos(n\pi) + n\pi\sin(n\pi))}{(n\pi)^2}$$

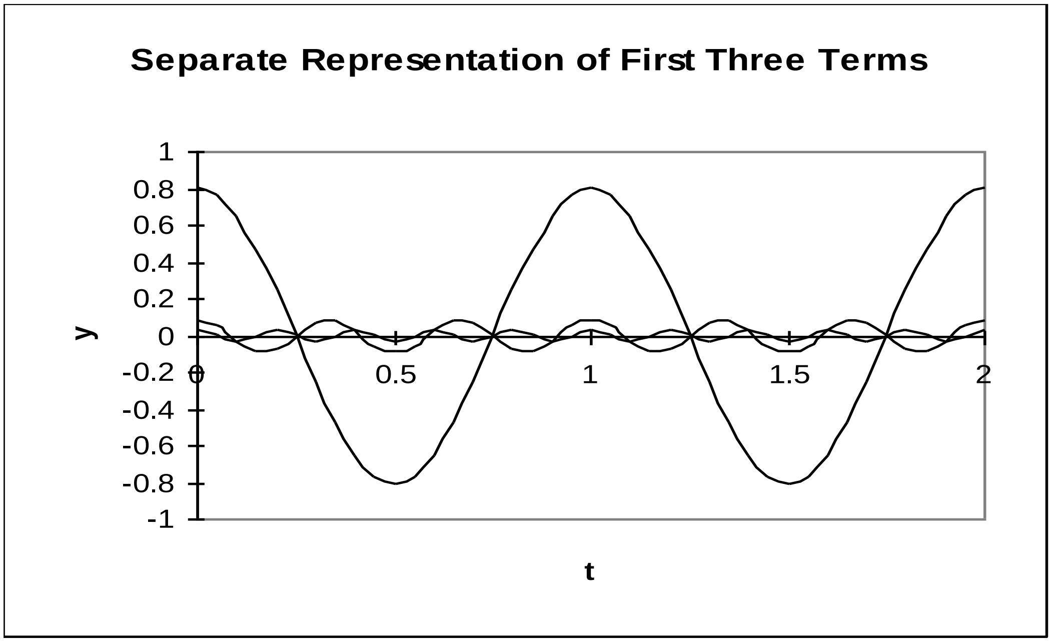
Since $\sin(n\pi) = 0$, then the Fourier series is

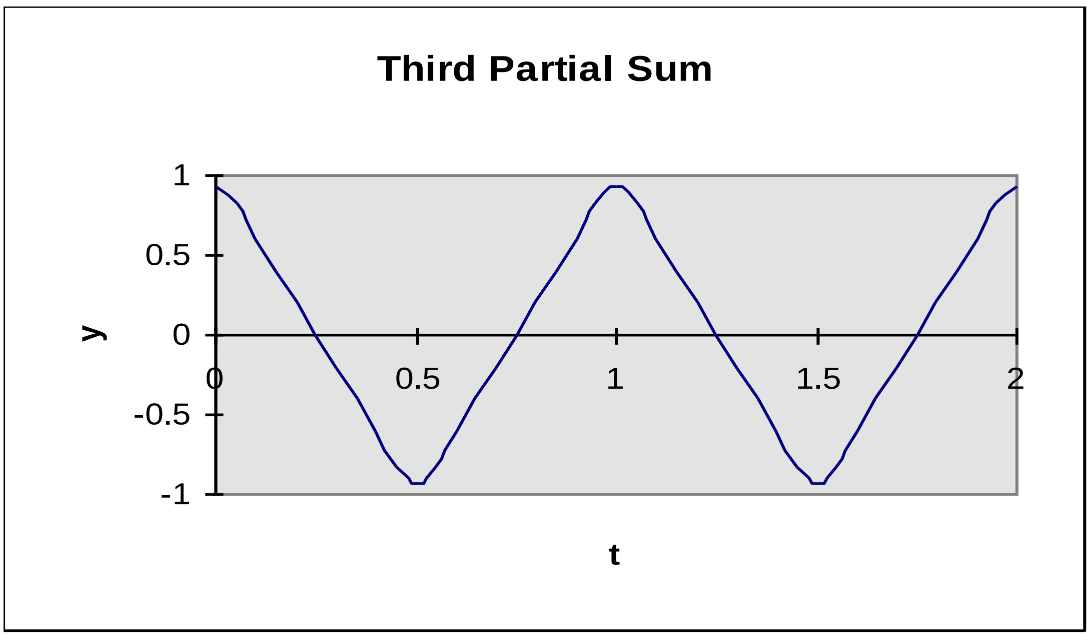
$$y(t) = \sum_{n=1}^{\infty} \frac{4C(1-\cos n\pi)}{(\pi n)^2} \cos\frac{2n\pi t}{T}$$

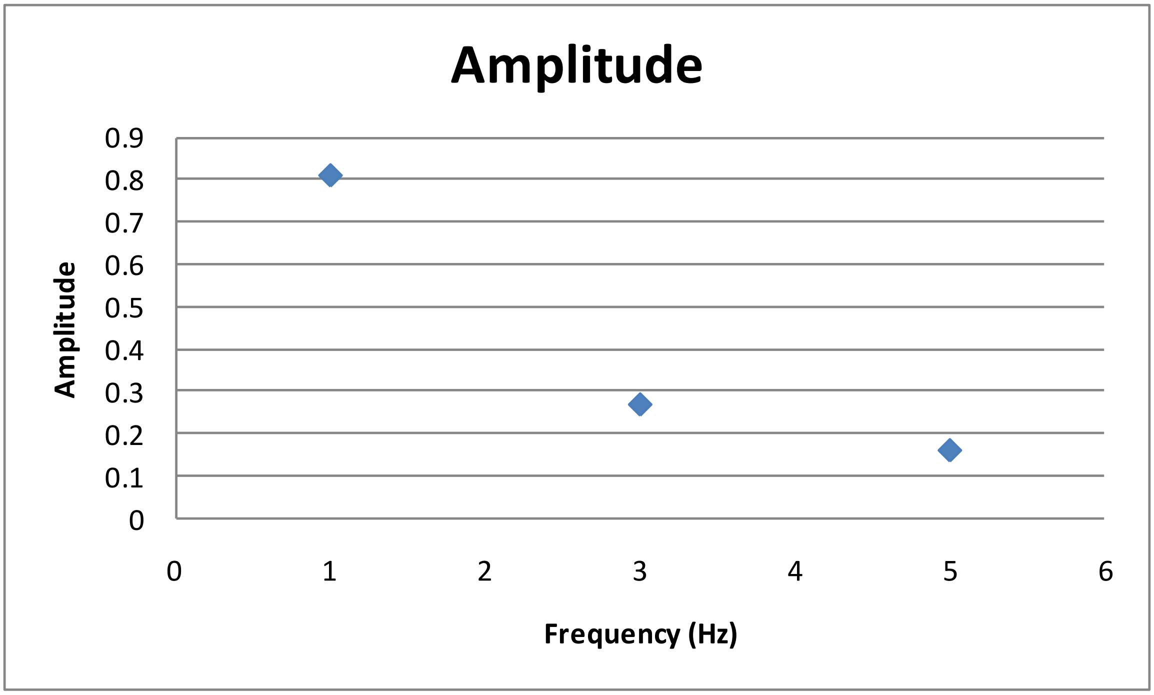
The values of A_n are zero for n even, and the first three nonzero terms of the Fourier series are

$$\frac{8C}{(\pi)^2}\cos\frac{2\pi t}{T}+\frac{8C}{(3\pi)^2}\cos\frac{6\pi t}{T}+\frac{8C}{(5\pi)^2}\cos\frac{10\pi t}{T}$$

The first term represents the fundamental frequency.






Amplitude
Amplitude
Frequency (Hz)
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6

PROBLEM 2.40

KNOWN: Figure 2.14 illustrates the nature of spectral distribution or frequency distribution on a signal.

FIND: Discuss the effects of low amplitude high frequency noise on signals.

SOLUTION:

Assume that Figure 2.14a represents a signal, and that Figures 2.14 b-d represents the effects of noise superimposed on the signal. Several aspects of the effects of noise are apparent. The waveform can be altered significantly by the presence of noise, particularly if rates of change of the signal are important for specific purposes such as control. Generally, high frequency, low amplitude noise will not influence a mean value, and most of the signal statistics are not affected when calculated for a sufficiently long signal.