INSTRUCTOR'S MANUAL FOR

FUNDAMENTALS OF QUEUEING THEORY

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FIFTH EDITION

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JOHN WILEY & SONS, INC.

This edition first published 2018

Registered Office
John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, USA

Editorial Office
111 River Street, Hoboken, NJ 07030, USA

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Library of Congress Cataloging-in-Publication Data
ISBN: 9781118943526

10 9 8 7 6 5 4 3 2 1

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CHAPTER 1

Introduction

1.1

Prob.	Calling Units	Service Function	Discipline	Capacity	No. Servers	No. Stages
(a)	Airplanes	Landing runways	FCFS (PRI in emergency)	Stack ($\approx \infty$)	No. runways	1-landing only; 2-landing and taxiing
(b)	Filled Grocery carts	Checker(and bagger)	FCFS (with jockeying)	($\approx \infty$)	With jockeying and channel choice acts like a c-server model	1
(c)	People	Clerks	same as (b)	same as (b)	same as (b)	1
(d)	Cars	Paying toll (toll booth)	FCFS	∞	1 or more (in fog, acts like indep. single channels no choice or jockeying)	1
(e)	Cars	Gas filling	FCFS	Finite	No. of pump islands (similar to (b) although jockeying difficult)	1

(f)	Cars	Car-wash building	FCFS	Finite	Generally 1	Many, with no storage between stages
(g)	Calls	Lines in switchboard	FCFS	Finite	No. of lines	1
(h)	Patients	Doctor (could be batch service)	Fixed as to appoint-ments	Finite seating capacity and waiting room	1, unless a clinic	Usually 1 but could be several
(i)	Tourists	Tour group	FCFS	($\approx \infty$)	1 or more	Multiple
(j)	Components	Operations and inspection batch service	FCFS	Finite	1 or more	4
(k)	Programs	Processing Programs	FCFS (or PRI)	same as (b)	1	1

1.2 One could give a variety of illustrations, e.g., people calling into a bank to find their account status. The customers are the calls, it is generally a multi-stage process, where first an automated message of which button to press depending on what's desired is received, and then, after pressing the appropriate button, getting the desired information automatically or asking for a customer representative. We would have finite capacity - if all lines are tied up, a busy signal results and the call must be replaced. It is multi-stage and would usually be a multi-server queue, with a FCFS discipline. Another example might be a bakery, where upon entering, the customer takes a number, so that we have a true, FCFS, multi-server queue with a single waiting line (the queue being the numbers). It would be a single-stage process, since a given server serves only one customer at a time. The capacity would be finite, although there is usually enough space so that it is essentially infinite. As a final example, consider a blood donor center. We have a multi-stage process (check-in, filling out information, blood pressure and clotting-time checks, and finally giving the blood). Some stages have a single server and others have multiple servers. It is generally an appointment system, but if it is a drop-in center, customers can arrive completely randomly and we would have a FCFS discipline. There is a finite capacity in that if the waiting room is completely filled, donors might be asked to come back at another time.

1.3 The parameters are $\lambda = 40/\text{h}$ and $1/\mu = 5.5\,\text{min}$. Using units of hours, $\mu = 60/5.5 \doteq 10.91/\text{h}$. The utilization should be less than 1, so $\lambda/c\mu \doteq 40/(10.91c)$, which implies that $c > 40/10.91 \doteq 3.67$. At least 4 are required to achieve steady state.

1.4 $\text{Lq} = \lambda\text{Wq} = (3/\,\text{min})([75/60]\,\text{min}) = 3.75$ or, say, 4. The 3.75 number is, of course, the average number in the queue. We may wish to provide 5 or 6 slots to guarantee that most callers get into the queue.

1.5 **(a)** The fraction of time that a server is busy is $p_b = 1 - .01 = .99$. Now, $p_b = \lambda/c\mu = r/c$. Thus, $r = c \cdot p_b = 2 \cdot 0.99 = 1.98$. With 3 servers, $p_b = r/c = 1.98/3 = .66$, so each server is idle 34% of the time, which is more than enough time for breaks.

(b) The service rate is reduced to 0.8μ, so $p_b = \lambda/(3 \cdot 0.8\mu) = r/2.4 = 1.98/2.4 = 0.825$. This still gives an idle percentage for each server of 17.5%, again more than enough time for breaks.

(c) The average service *time* is reduced from $1/\mu$ to $0.8/\mu$. Thus, the new service rate is $\mu/0.8 = 1.25\mu$, so $p_b = \lambda/(2 \cdot 1.25\mu) = r/2.5 = 1.98/2.5 = 0.792$. This gives an idle percentage for each server of 20.8%. This is a cheaper solution giving each server enough time for breaks.

1.6 Let T be the total waiting time. If, when you arrive, the person in service is just about finished, then you wait on average eight service times (yours and the seven ahead of you) or $\mathrm{E}[T] = 8(2.5\,\mathrm{min}) = 20\,\mathrm{min}$. If, when you arrive, the person in service is just beginning, then you wait on average nine service times or $\mathrm{E}[T] = 9(2.5\,\mathrm{min}) = 22.5\,\mathrm{min}$. The average wait is somewhere in between.

Assuming the latter case, T is the sum of 9 IID normal random variables each with mean 2.5 and standard deviation 0.5. So T is a normal random variable with mean 22.5 and standard deviation $\sqrt{(9 \cdot 0.5^2)} = 1.5$. Then $\Pr\{T > 30\,\mathrm{min}\} = \Pr\{Z > (30 - 22.5)/1.5\} = \Pr\{Z > 5\}$, where Z is a standard normal random variable. From standard normal tables, $\Pr\{\mathrm{Z} > 5\} \doteq 0$.

1.7 **(a)** Apply Little's law to the system of active players in the league. The average number of active players in the league is represented by L, where $L = 32 \cdot 67 = 2,144$. The average rate that players enter the league is represented by λ, where $\lambda = 32 \cdot 7 = 224$ per year. The average time spent in the league is represented by W. By Little's law, $W = L/\lambda = 2144/224 = 9.57$ years.

(b) Here, it is given that $W = 3.5$ years. As before $L = 2,144$ (the number of active players in the league). The average rate that players enter the league is $\lambda = L/W = 2,144/3.5 \approx 613$ per year. Since 224 players are drafted each year, an average of $613 - 224 = 389$ players enter the league without being drafted. (This analysis assumes that a player who leaves the league never returns.)

1.8 Consider the university as a system where students enter by enrolling at the university. The average undergraduate enrollment is an estimate for L (so $L = 16,800$). The average number of new students per year (the sum of the middle two columns) is an estimate for λ (so $\lambda = 4,052$ per year). W is an estimate for the average time an undergraduate spends at the university. By Little's formula, $W = L/\lambda \approx 4.1$ years. (The main assumption here is that the system is operating in steady-state. This may not be a valid assumption, for example, if enrollment were growing. However, this particular example does not indicate a noticeable growth trend.)

1.9 Apply Little's law to the set of homes on the market. The average number of homes on the market is estimated as $L = 50$. The rate that homes enter the market is estimated as $\lambda = 5$ per week. By Little's law, a home is on the market for an average of $W = L/\lambda = 10$ weeks before it is sold. This assumes that the observed numbers are representative of long-term averages. Furthermore, it is assumed that you have no additional information that might change your estimate. For example, if you price your home at a very low price, you will probably sell it more quickly than the average.

1.10

$$\lambda_{\text{eff}} = \lambda(1 - p_K) = .9; W = L/\lambda_{\text{eff}} = 5/.9 = 50/9;$$
$$W_q = W - 1/\mu = 50/9 - 1 = 41/9;$$
$$\rho_{\text{eff}} = \lambda_{\text{eff}}/\mu = .9 \text{ and } p_0 = 1 - \rho_{\text{eff}} = .1.$$

1.11 **(a)** Use Little's law where the "system" is the set of available doses. L is the average number of available doses at a given time, and W is the length of time a dose is available from the time of its creation until the time its shelf life ends. From Little's law, $\lambda = L/W = 300,000,000/(90/365) = 1,216,667$ per year, which is the yearly rate that doses need to be made. So the yearly cost is $\$3 \cdot \lambda$ or \$3.65 billion per year.

(b) The answer is the same as before, since Little's law is stated in terms of averages, which is unchanged.

(c) The value for L remains the same (300 million). The shelf life x of the vaccine is W. Thus, $\lambda = L/x$ is the rate that vaccines must be made (per day). The daily cost to make the vaccines is therefore $(a + bx^2)(L/x)$. To minimize, take the derivative and set equal to 0:

$$L\left(\frac{-a}{x^2} + b\right) = 0.$$

This implies that $a/x^2 = b$ or $x = \sqrt{a/b} \doteq 223.6$ days

1.12 **(a)** On average, there are 50 customers in the system. The arrival rate to the system is 100 per hour. By Little's law, the average time in the system is $W = L/\lambda = 50/100 = 0.5$ hour (or 30 minutes).

(b) The arrival rate to the specialist queue is 20 per hour. On average, there are 10 customers being served or waiting to be served by a specialist. By Little's law, the average time at the specialist is $W = L/\lambda = 10/20 = 0.5$ hour.

The arrival rate to the regular queue is 100 per hour. On average, there are 40 customers being served or waiting to be served by a regular representative. By Little's law, the average time at the regular representative is $W = L/\lambda = 40/100 = 0.4$ hour.

Thus, the average time in the system for a customer who needs to see a specialist is 0.9 hour.

1.13 **(a)** The number of years an individual survives past 65 is a geometric random variable with mean $1/.05 = 20$ years. On average, a person receives benefits for 20 years. (The geometric model is somewhat unrealistic since the death rate is assumed to be the same every year, regardless of age.)

(b) Apply Little's Law to the population of people over 65. The rate of people entering this population group is $\lambda = 3$ million per year. The average time in this population group is $W = 20$ years. Thus, $L = \lambda W = 60$ million people. Thus, the average yearly payout is \$2.4 trillion.

1.14 **(a)** A path from A to C is 80 miles. A path from A to B and A to D is $40\sqrt{2}$ miles. Since the results are symmetric for every entry point, the average path length is:

$$\frac{1}{3}80 + \frac{2}{3}40\sqrt{2} = \frac{80 + 80\sqrt{2}}{3} \doteq 64.4 \text{ miles.}$$

(b) The average arrival rate to the sector is $\lambda = 20$ per hour. The average time in sector is $W = 64.4$ miles / 400 mph $\doteq .161$ hours. By Little's law, the average number in the sector is:

$$L = \lambda W = (20)(.161) \doteq 3.2.$$

(c) Avoidance maneuvers increase the path length which increases W which increases L, so the answer in part (b) would go up.

1.15 **(a)** Using Little's Law, $W = 5$ years and $L = 150$ million. Thus,

$$\lambda = \frac{L}{W} = \frac{150,000,000}{5} = 30,000,000 \text{ per year.}$$

The fact that the distribution is Erlang-3 is irrelevant.

(b) Let L_{new} and L_{used} be the average number of cars in the system that were purchased new and used, respectively. By assumption, every new car becomes a used car and then it is destroyed. Thus, the overall rate that new cars are purchased (λ) is the same rate that used cars are purchased. So,

$$150,000,000 = L_{new} + L_{used} = \lambda W_{new} + \lambda W_{used} = \lambda(5 + 7).$$

$$\lambda = \frac{150,000,000}{12} = 12,500,000 \text{ per year.}$$

1.16 Intuitive answer: The average spacing between aircraft is 6 nm. The sector is 50 nm long. Thus, the average number of aircraft in the sector is $50/6 \doteq 8.33$.

Answer using Little's Law: The average spacing between aircraft in distance is 6 nm. Since distance = velocity x time, the expected separation between aircraft in time is 6 nm / 400 knots = 3/200 hours. Thus, the arrival rate is $\lambda = 200/3$ per hour. The time in the sector (W) is 50 nm / 400 knots = 1/8 hours. By Little's Law, the average number of aircraft in the sector is: $L = \lambda W = (200/3)(50/400) \doteq 8.33$.

1.17 We use the Delay Analysis for Sample Single-Server Queue model in the Basic Model category in QtsPlus:

DELAY ANALYSIS FOR SAMPLE SINGLE-SERVER QUEUE

This is a basic line waiting-time analysis for a sample G/G/1 queue constructed from an input sequence of interarrival and service times.

Output:

Clear Old Data

Number of Observations	20
Total time horizon	147
Mean interarrival time	7.35
Arrival rate (λ)	0.136054422
Mean service time	6.2
Service rate (μ)	0.161290323
Empirical traffic intensity (ρ)	84.35%
Average line delay (Wq)	3.95
Average system wait (W)	10.15

Put data below into two columns of equal length.
Enter data and then press "Solve" button.

Solve

Customer n	Line Delays Wq(n)	System Waits W(n)	Service Time S(n)	Inter-arrival Time T(n)
0	*N/A*	*N/A*	*N/A*	1.
1	0.0	3.0	3.	9.
2	0.0	7.0	7.	6.
3	1.0	10.0	9.	4.
4	6.0	15.0	9.	7.
5	8.0	18.0	10.	9.
6	9.0	13.0	4.	5.
7	8.0	16.0	8.	8.
8	8.0	13.0	5.	4.
9	9.0	14.0	5.	10.
10	4.0	7.0	3.	6.
11	1.0	7.0	6.	12.
12	0.0	3.0	3.	6.
13	0.0	5.0	5.	8.
14	0.0	4.0	4.	9.
15	0.0	9.0	9.	5.
16	4.0	13.0	9.	7.
17	6.0	14.0	8.	8.
18	6.0	12.0	6.	8.
19	4.0	12.0	8.	7.
20	5.0	8.0	3.	

1.18 Using QtsPlus Delay Analysis for Sample Single-Server Queue model in the Basic Model category:

DELAY ANALYSIS FOR SAMPLE SINGLE-SERVER QUEUE

This is a basic line waiting-time analysis for a sample G/G/1 queue constructed from an input sequence of interarrival and service times.

Output:

Number of Observations	10
Total time horizon	60
Mean interarrival time	6
Arrival rate (λ)	0.166666667
Mean service time	4.6
Service rate (μ)	0.217391304
Empirical traffic intensity (ρ)	76.67%
Average line delay (Wq)	1.7
Average system wait (W)	6.3

Clear Old Data

Put data below into two columns of equal length.
Enter data and then press "Solve" button.

Solve

Customer n	Line Delays Wq(n)	System Waits W(n)	Service Time S(n)	Inter-arrival Time T(n)
0	*N/A*	*N/A*	*N/A*	5.
1	0.0	2.0	2.	5.
2	0.0	7.0	7.	5.
3	2.0	8.0	6.	5.
4	3.0	9.0	6.	5.
5	4.0	10.0	6.	5.
6	5.0	8.0	3.	5.
7	3.0	4.0	1.	5.
8	0.0	4.0	4.	5.
9	0.0	1.0	1.	5.
10	0.0	10.0	10.	

1.19 The following table lists various statistics associated with each customer. "# in System" and "# in Queue" refer to the number of customers in the system and queue as seen by the arriving customer.

Customer # / Arrival Time	Service Start Time	Exit Time	Time in Queue	# in System	# in Queue
1	1.00	3.22	0.00	0	0
2	3.22	4.98	1.22	1	0
3	4.98	7.11	1.98	2	1
4	7.11	7.25	3.11	2	1
5	7.25	8.01	2.25	2	1
6	8.01	8.71	2.01	3	2
7	8.71	9.18	1.71	4	3
8	9.18	9.40	1.18	3	2
9	9.40	9.58	0.40	2	1
10	10.00	12.41	0.00	0	0
11	12.41	12.82	1.41	1	0
12	12.82	13.28	0.82	2	1
13	13.28	14.65	0.28	1	0
14	14.65	14.92	0.65	1	0
15	15.00	15.27	0.00	0	0

The values in the table are computed as follows:

- The exit time is the service-start time plus the service duration.

- The service-start time is the maximum of the exit time of the previous customer and the arrival time of the customer in question. (The first customer starts service immediately upon arrival.)
- The time in queue is the service-start time minus the arrival time.
- The number in system is the number of previously arriving customers whose exit time is after the arrival time of the customer in question.
- The number in queue is the number in system minus one, with a minimum value of zero.

$L_q^{(A)}$ is the average of the last column. $L_q^{(A)} = 12/15 = 0.8$. L_q is the total person minutes spent in the queue (the sum of the "Time in Queue" column) divided by the total time interval. $L_q = 17.02/15.27 = 1.1146$. Note that $L_q \neq L_q^{(A)}$.

CHAPTER 2

Review of Stochastic Processes

2.1 The CDF and CCDF are evaluated as follows:

$$F(x) = \Pr\{X \leq x\} = \int_0^x \lambda e^{-\lambda u}\, du = -e^{-\lambda u}\Big|_{u=0}^{u=x} = 1 - e^{-\lambda x},$$
$$F^c(x) = 1 - F(x) = e^{-\lambda x}.$$

To compute $\mathrm{E}[X]$, use integration by parts ($u = x$, $du = dx$, $dv = \lambda e^{-\lambda x}dx$, $v = -e^{-\lambda x}$):

$$\mathrm{E}[X] = \int_0^\infty x\lambda e^{-\lambda x}\, dx = -xe^{-\lambda x}\Big|_{x=0}^{x=\infty} + \int_0^\infty e^{-\lambda x} dx = 0 + \frac{-e^{-\lambda x}}{\lambda}\Bigg|_{x=0}^{x=\infty}$$
$$= \frac{1}{\lambda}$$

To compute $\mathrm{E}[X^2]$, use integration by parts ($u = x^2$, $du = 2x\, dx$, $dv = \lambda e^{-\lambda x}dx$, $v = -e^{-\lambda x}$):

$$\mathrm{E}[X^2] = \int_0^\infty x^2\lambda e^{-\lambda x}\, dx = -x^2 e^{-\lambda x}\Big|_{x=0}^{x=\infty} + \int_0^\infty 2xe^{-\lambda x} dx$$
$$= 0 + \frac{2}{\lambda}\int_0^\infty x\lambda e^{-\lambda x} dx = \frac{2}{\lambda}\mathrm{E}[X] = \frac{2}{\lambda^2},$$
$$\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

2.2 From differential equation theory, the solution to $\frac{dy(t)}{dt} + \phi(t)y(t) = \psi(t)$ is $y(t) = ce^{-\int \phi(t)dt} + e^{-\int \phi(t)dt}\int e^{-\int \phi(t)dt\psi(t)dt}$. So $\frac{dp_0(t)}{dt} + \lambda p_0(t) = 0; p_0(0) = 1$.

Set $\phi(t) = \lambda$ and $\psi(t) = 0 \Rightarrow p_0(t) = ce^{-\lambda t}$.

From the boundary condition: $1 = ce^0 \Rightarrow c = 1$. Therefore, $p_0(t) = e^{-\lambda t}$.

$\frac{dp_1(t)}{dt} + \lambda p_1(t) = \lambda p_0(t) = \lambda e^{-\lambda t}; p_1(0) = 0$.

Set $\phi(t) = \lambda$ and $\psi(t) = \lambda e^{-\lambda t} \Rightarrow p_1(t) = ce^{-\lambda t} + \lambda te^{-\lambda t}$.

From the boundary condition: $0 = ce^0 + 0 \Rightarrow c = 0 \Rightarrow p_1(t) = \lambda te^{-\lambda t}$.

$\frac{dp_2(t)}{dt} + \lambda p_2(t) = \lambda p_1(t) = \lambda^2 te^{-\lambda t}; p_2(0) = 0.$

Set $\phi(t) = \lambda$ and $\psi(t) = \lambda^2 te^{-\lambda t} \Rightarrow p_2(t) = ce^{-\lambda t} + \frac{(\lambda t)^2}{2}e^{-\lambda t}$.

From the boundary condition: $c = 0 \Rightarrow p_2(t) = \frac{(\lambda t)^2}{2}e^{-\lambda t}$.

$\frac{dp_3(t)}{dt} + \lambda p_3(t) = \lambda p_2(t) = \frac{\lambda^3 t^2}{2}e^{-\lambda t}; p_3(0) = 0.$

Set $\phi(t) = \lambda$ and $\psi(t) = \frac{\lambda^3 t^2}{2}e^{-\lambda t} \Rightarrow p_3(t) = ce^{-\lambda t} + \frac{(\lambda t)^3}{3 \cdot 2}e^{-\lambda t}$.

The boundary condition gives $c = 0 \Rightarrow p_3(t) = \frac{(\lambda t)^3}{3!}e^{-\lambda t}$.

Now assume $p_{n-1}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!}e^{-\lambda t}$. Set $\phi(t) = \lambda$ and $\psi(t) = \frac{\lambda^n t^{n-1}}{(n-1)!}e^{-\lambda t} \Rightarrow p_n(t) = ce^{-\lambda t} + \frac{(\lambda t)^n}{n!}e^{-\lambda t}$ and boundary condition gives $c = 0 \Rightarrow p_n(t) = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$.

2.3

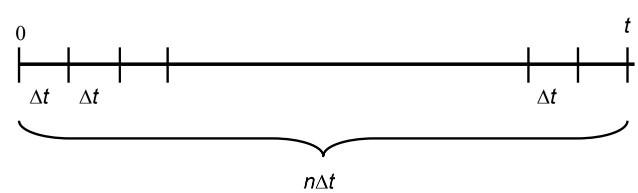
$$p_n(t) = \frac{\tau^n e^{-\tau}}{n!}, \tau = \lambda t, n = 0, 1, 2 \dots$$

$$M_{N(t)}(\theta) = E[e^{\theta N(t)}] = \sum_{n=0}^{\infty} \frac{\tau^n e^{-\tau} e^{\theta n}}{n!} = e^{-\tau} \sum_{n=0}^{\infty} \frac{(\tau e^{\theta})^n}{n!} = e^{-\tau} e^{\tau e^{\theta}} = e^{\tau(e^{\theta}-1)}$$

$$E[N(t)] = \left.\frac{dM_{N(t)}(\theta)}{d\theta}\right|_{\theta=0} = \left.\tau e^{\theta} e^{\tau(e^{\theta}-1)}\right|_{\theta=0} = \tau$$

$$E[(N(t) - E[N(t)])^2] = E[(N(t))^2] - \{E[N(t)]\}^2 = \left.\frac{d^2 M_{N(t)}(\theta)}{d\theta^2}\right|_{\theta=0} - \tau^2$$
$$= [\tau e^{\theta} e^{\tau(e^{\theta}-1)} + \tau^2 e^{2\theta} e^{\tau(e^{\theta}-1)}]_{\theta=0} - \tau^2 = \tau + \tau^2 - \tau^2 = \tau$$

2.4



Divide the interval $[0, t]$ into n subintervals of length Δt, so that $t = n\,\Delta t$. The probability of one arrival in a subinterval is

$$p \equiv \Pr\{\text{one arrival in } \Delta t\} = \lambda \Delta t + o(\Delta t) \approx \frac{\lambda t}{n}.$$

The probability of more than one arrival in a subinterval is $o(\Delta t)$, which can be made arbitrarily small. Assuming that there can be at most one arrival in a subinterval and using the assumption of independence of non-overlapping intervals, the total number of arrivals in $[0, t]$ is the sum of n Bernoulli trials. This follows a binomial distribution:

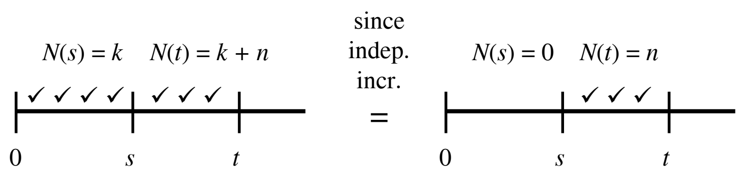
$$\begin{aligned} b(x; n, p) &= \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n \\ &= \frac{n(n-1)\cdots(n-x+1)}{x!} p^x (1-p)^n (1-p)^{-x} \\ &= \frac{1 \cdot (1-1/n)\cdots(1-\frac{x-1}{n})}{x!} (np)^x \left(1 - \frac{\lambda t}{n}\right)^n \left(1 - \frac{\lambda t}{n}\right)^{-x}. \end{aligned}$$

So,

$$\lim_{n \to \infty} b(x; n, p) = \frac{1}{x!} (\lambda t)^x e^{-\lambda t}, \quad x = 0, 1, \ldots,$$

which is the Poisson distribution.

2.5 Consider



Let $P_r\{N(t) - N(s) = n\} = q_n(t, s)$.
So $q_n(t + \Delta t, s) = q_n(t, s)[1 - \lambda\Delta t] + q_{n-1}(t, s)\lambda\Delta t + o(\Delta t) \quad n > 0$.
$q_0(t + \Delta t, s) = q_0(t, s)[1 - \lambda\Delta t] + o(\Delta t)$.
Rearranging & dividing by Δt, then taking $\lim \Delta t \to 0$ gives

$$\begin{aligned} \frac{\partial q_n(t, s)}{\partial t} &= -\lambda q_n(t, s) + \lambda q_{n-1}(t, s) \\ \frac{\partial q_0(t, s)}{\partial t} &= -\lambda q_0(t, s) \end{aligned}$$

Solve in a similar manner to (2.5) & (2.8) by the general solution to a first order linear differential equation was in Problem 1.2 solution. Here, however, the boundary conditions are

$$\begin{aligned} &q_0(s, s) = 1, q_n(s, s) = 0, n \neq 0. \\ &q_0(t, s) = ce^{-\lambda t} + e^{-\lambda t}(0) = ce^{-\lambda t} \\ &q_0(s, s) = 1 = ce^{-\lambda s} \end{aligned}$$

Therefore $c = \frac{1}{e^{-\lambda s}} = e^{\lambda s}$ and $q_0(t, s) = e^{\lambda s} e^{-\lambda t} = e^{-\lambda(t-s)}$

$$\begin{aligned} q_1(t, s) &= ce^{-\lambda t} + e^{-\lambda t} \int e^{\lambda t} \lambda e^{-\lambda(t-s)} dt = ce^{-\lambda t} + e^{-\lambda t} e^{\lambda s} \lambda \int dt \\ &= ce^{-\lambda t} + \lambda e^{-\lambda(t-s)} \cdot t = ce^{-\lambda t} + \lambda t e^{-\lambda(t-s)} \\ q_1(s, s) &= 0 = ce^{-\lambda s} + \lambda s \Rightarrow ce^{-\lambda s} = -\lambda s \Rightarrow c = -\lambda s e^{\lambda s} \end{aligned}$$

Therefore

$$q_1(t,s) = -\lambda s e^{\lambda s} e^{-\lambda t} + \lambda t e^{-\lambda(t-s)} = -\lambda s e^{-\lambda(t-s)} + \lambda t e^{-\lambda(t-s)} = \lambda(t-s)e^{-\lambda(t-s)}$$

etc...

Therefore, $q_n(t,s) = p_n(t-s)$. Similarly for $q_n(t+h, s+h)$.

2.6 Let $P_n(t) \equiv CDF$ of the arrival counting process.

Then,

$$\begin{aligned} P_n(t) &= \Pr\{(\text{sum of } n+1 \text{Erlang interarrival times}) \geq t\} \\ &= \int_t^\infty \frac{k\lambda(k\lambda x)^{(n+1)k-1}}{[(n+1)k-1]!} e^{-k\lambda x} dx \end{aligned}$$

since the sum of IID Erlang random variables is also an Erlang.

Let $u = x - t$,

$$\begin{aligned} P_n(t) &= \int_0^\infty \frac{(k\lambda)^{(n+1)k}(u+t)^{(n+1)k-1}}{[(n+1)k-1]!} e^{-k\lambda u} e^{-k\lambda t} du \\ &= \int_0^\infty \frac{(k\lambda)^{(n+1)k} e^{-k\lambda u} e^{-k\lambda t}}{[(n+1)k-1]!} \sum_{i=0}^{(n+1)k-1} \frac{u^{(n+1)k-1-i} t^i}{[(n+1)k-1-i]!} \cdot \frac{[(n+1)k-1]!}{i!} du \\ &= \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda)^{(n+1)k} t^i e^{-k\lambda t}}{[(n+1)k-1-i]! i!} \cdot \int_0^\infty e^{-k\lambda u} u^{(n+1)k-1-i} du \\ &= \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda)^{(n+1)k} t^i e^{-k\lambda t}}{[(n+1)k-1-i]! i!} \cdot \frac{[(n+1)k-1-i]!}{(k\lambda)^{(n+1)k-1}} = \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} \end{aligned}$$

The probability function of the counting process is thus,

$$\begin{aligned} p_n(t) = P_n(t) - P_{n-1}(t) &= \sum_{i=0}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} - \sum_{i=0}^{nk-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} \\ &= \sum_{i=nk}^{(n+1)k-1} \frac{(k\lambda t)^i}{i!} e^{-k\lambda t} \end{aligned}$$

2.7 First, assume that n is even. Then,

$$\begin{aligned} p_n(t) &= \Pr\{N(t) = n\} \\ &= \Pr\{n \text{ singles}\} + \Pr\{(n-2) \text{ singles and 1 double}\} \\ &\quad + \Pr\{(n-4) \text{ singles and 2 doubles}\} + \cdots + \Pr\{n/2 \text{ doubles}\}. \end{aligned}$$

Then,

$$p_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}p^n + \binom{n-1}{1} e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}p^{n-2}(1-p)$$
$$+ \binom{n-2}{2} e^{-\lambda t}\frac{(\lambda t)^{n-2}}{(n-2)!}p^{n-4}(1-p)^2 + \cdots$$
$$+ \binom{n-n/2}{n/2} e^{-\lambda t}\frac{(\lambda t)^{n-n/2}}{(n-n/2)!}p^{n-2(n/2)}(1-p)^{n/2}.$$

So,

$$p_n(t) = e^{-\lambda t}\left\{\frac{(\lambda t)^n}{n!}p^n + \frac{(\lambda t)^{n-1}}{1!(n-2)!}p^{n-2}(1-p)\right.$$
$$\left. + \frac{(\lambda t)^{n-2}}{2!(n-4)!}p^{n-4}(1-p)^2 + \cdots + \frac{(\lambda t)^{n/2}}{(n/2)!}(1-p)^{n/2}\right\}$$
$$= e^{-\lambda t}\sum_{k=0}^{n/2}\frac{(\lambda t)^{n-k}}{k!(n-2k)!}p^{n-2k}(1-p)^k.$$

Similarly, if n is odd,

$$\begin{aligned} p_n(t) &= \Pr\{N(t) = n\} \\ &= \Pr\{n \text{ singles}\} + \Pr\{(n-2) \text{ singles and 1 double}\} \\ &\quad + \Pr\{(n-4) \text{ singles and 2 doubles}\} \\ &\quad + \cdots + \Pr\{1 \text{ single and } (n-1)/2 \text{ doubles}\}. \end{aligned}$$

Proceeding in the same manner gives

$$p_n(t) = e^{-\lambda t}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(\lambda t)^{n-k}}{k!(n-2k)!}p^{n-2k}(1-p)^k.$$

2.8 **(a)** Denote respective first recurrence times as T_1 and T_2. The joint PDF is $f(t_1, t_2) = \lambda_1 e^{-\lambda_1 t_1} \cdot \lambda_2 e^{-\lambda_2 t_2}$, since the processes are independent.