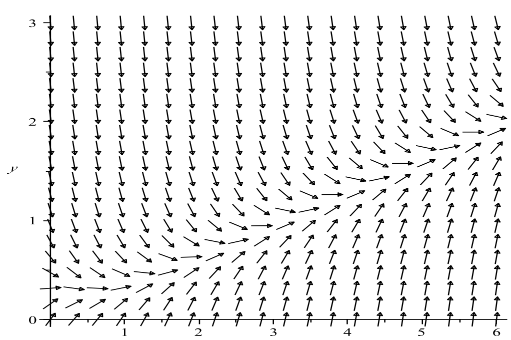
CHAPTER

2

First-Order Differential Equations

2.1

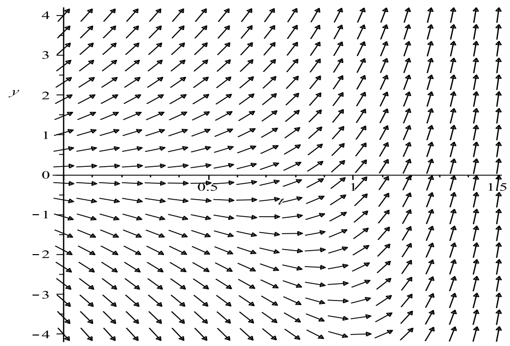
SS 1.(a)



(b) All solutions seem to approach a line in the region where the negative and positive slopes meet each other.

(c) $\mu(t) = e^{\int 3\,dt} = e^{3t}$. Thus $e^{3t}(y' + 3y) = e^{3t}(t + e^{-2t})$ or $(ye^{3t})' = te^{3t} + e^t$. Integration of both sides yields $ye^{3t} = te^{3t}/3 - e^{3t}/9 + e^t + c$, where integration by parts is used on the right side, with $u = t$ and $dv = e^{3t}dt$. Division by e^{3t} gives $y(t) = ce^{-3t} + t/3 - 1/9$, so y approaches $t/3 - 1/9$ as $t \to \infty$. This is the line identified in part (b).

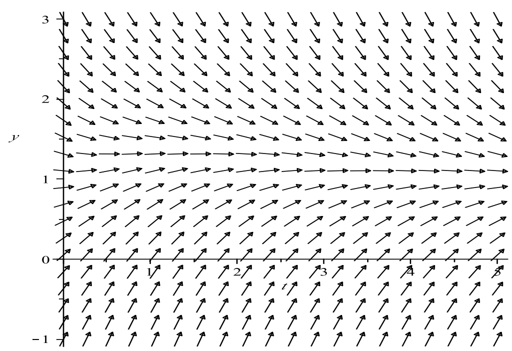
SS 2.(a)



(b) All solutions eventually have positive slopes, and hence increase without bound.

(c) The integrating factor is $\mu(t) = e^{-2t}$, and hence $y(t) = t^3 e^{2t}/3 + c\,e^{2t}$. It is evident that all solutions increase at an exponential rate.

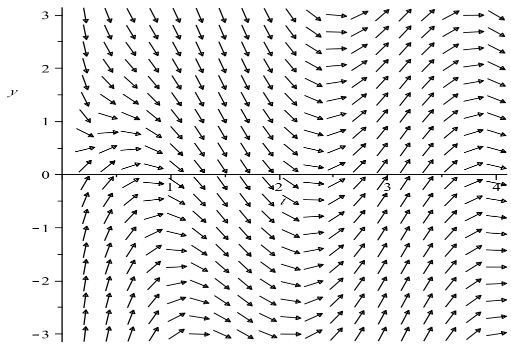
SS 3.(a)



(b) All solutions seem to converge to the function $y_0(t) = 1$.

(c) The integrating factor is $\mu(t) = e^{t}$, and hence $y(t) = t^2 e^{-t}/2 + 1 + c\,e^{-t}$. It is clear that all solutions converge to the specific solution $y_0(t) = 1$.

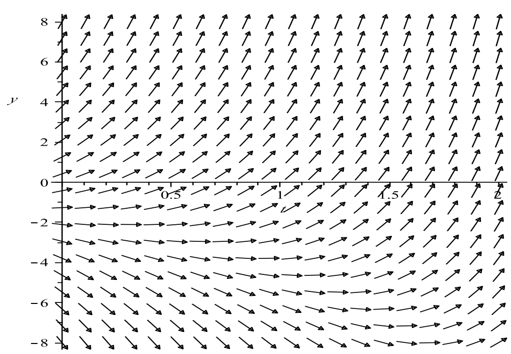
SS 4.(a)



(b) Based on the direction field, the solutions eventually become oscillatory.

(c) The integrating factor is $\mu(t) = e^{\int (1/t)\,dt} = e^{\ln t} = t$, so $(ty)' = 3t\cos 2t$, and integration by parts yields the general solution $y(t) = (3/4t)\cos 2t + (3/2)\sin 2t + c/t$, in which c is an arbitrary constant. As t becomes large, all solutions converge to the function $y_1(t) = 3(\sin 2t)/2$.

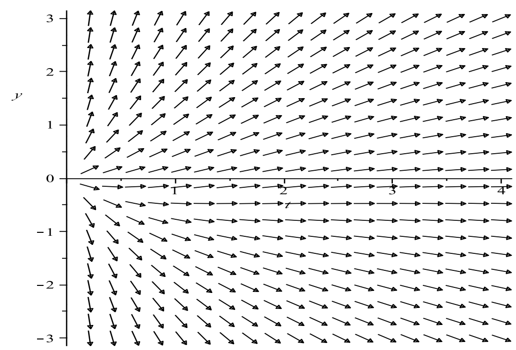
5.(a)



(b) If $y(0) > -3$, solutions eventually have positive slopes, and hence increase without bound. If $y(0) \le -3$, solutions have negative slopes and decrease without bound.

(c) The integrating factor is $\mu(t) = e^{-\int 2dt} = e^{-2t}$. The differential equation can be written as $e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$, that is, $(e^{-2t}y)' = 3e^{-t}$. Integration of both sides of the equation results in the general solution $y(t) = -3e^t + c\,e^{2t}$. It follows that all solutions will increase exponentially if $c > 0$ and will decrease exponentially if $c \le 0$. Letting $c = 0$ and then $t = 0$, we see that the boundary of these behaviors is at $y(0) = -3$.

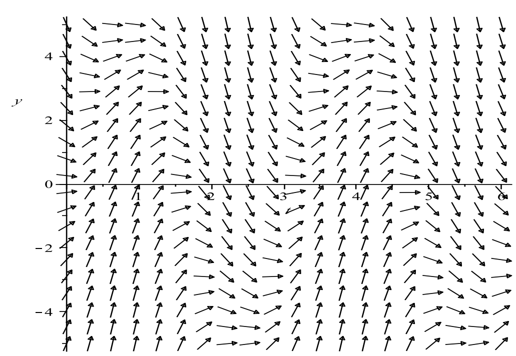
6.(a)



(b) For $y > 0$, the slopes are all positive, and hence the corresponding solutions increase without bound. For $y < 0$, almost all solutions have negative slopes, and hence solutions tend to decrease without bound.

(c) First divide both sides of the equation by t $(t > 0)$. From the resulting standard form, the integrating factor is $\mu(t) = e^{-\int (1/t)\,dt} = 1/t$. The differential equation can be written as $y'/t - y/t^2 = t\,e^{-t}$, that is, $(y/t)' = t\,e^{-t}$. Integration leads to the general solution $y(t) = -te^{-t} + c\,t$. For $c \neq 0$, solutions diverge, as implied by the direction field. For the case $c = 0$, the specific solution is $y(t) = -te^{-t}$, which evidently approaches zero as $t \to \infty$.

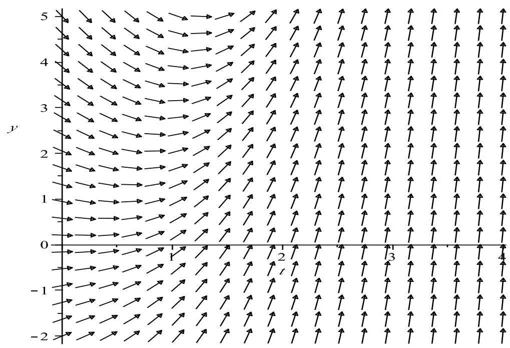
SS 7.(a)



(b) The solutions appear to be oscillatory.

(c) The integrating factor is $\mu(t) = e^t$, so $(e^t y)' = 5e^t \sin 2t$. To integrate the right side we can integrate by parts (twice), use an integral table or use a symbolic computational software to find $y(t) = \sin 2t - 2\cos 2t + c\,e^{-t}$. It is evident that all solutions converge to the specific solution $y_0(t) = \sin 2t - 2\cos 2t$.

8.(a)



(b) All solutions eventually have positive slopes, and hence increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/2}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t^2/2$, that is, $(e^{t/2}\,y/2)' = 3t^2/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t^2 - 12t + 24 + c\,e^{-t/2}$. It follows that all solutions converge to the specific solution $3t^2 - 12t + 24$.

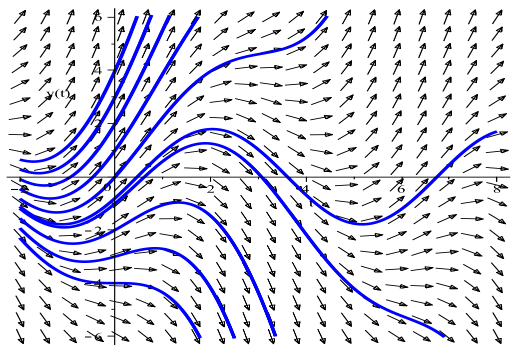
SS 9. $\mu(t) = e^{-t}$, so $(e^{-t}y)' = 2te^t$ and thus $e^{-t}y = 2\int te^t\,dt + c = 2(te^t - \int e^t\,dt) + c = 2(te^t - e^t) + c$. Thus $y(t) = 2(t-1)e^{2t} + ce^t$, so setting $t = 0$ we have $1 = -2 + c$, or $c = 3$. Hence $y(t) = 2(t-1)e^{2t} + 3e^t$.

10. The integrating factor is $\mu(t) = e^{2t}$. After multiplying both sides by $\mu(t)$, the equation can be written as $(e^{2t}\,y)' = t$. Integrating both sides of the equation results in the general solution $y(t) = t^2e^{-2t}/2 + c\,e^{-2t}$. Invoking the specified condition, we require that $e^{-2}/2 + c\,e^{-2} = 0$. Hence $c = -1/2$, and the solution to the initial value problem is $y(t) = (t^2 - 1)e^{-2t}/2$.

11. The integrating factor is $\mu(t) = e^{\int (2/t)\,dt} = t^2$. Multiplying both sides by $\mu(t)$, the equation can be written as $(t^2\,y)' = \cos t$. Integrating both sides of the equation results in the general solution $y(t) = \sin t/t^2 + c\,t^{-2}$. Substituting $t = \pi$ and setting the value equal to zero gives $c = 0$. Hence the specific solution is $y(t) = \sin t/t^2$.

SS 12. $\mu(t) = e^{\int (t+1)/t\,dt} = e^{1+\ln t} = te^t$.

SS 13.(a)

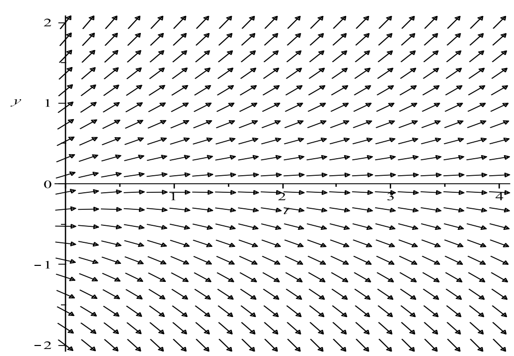


The solutions appear to diverge from an apparent oscillatory solution. From the direction field, the critical value of the initial condition seems to be $a_0 = -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, solutions decrease without bound.

(b) The integrating factor is $\mu(t) = e^{-t/2}$, so $(e^{-t/2}y)' = 2e^{-t/2}\cos t$. Integrating (see comments in Problem 7) and dividing by $e^{-t/2}$ yields the general solution $y(t) = (8\sin t - 4\cos t)/5 + c\,e^{t/2}$. Thus $y(0) = -4/5 + c = a$, or $c = a + 4/5$ and $y(t) = -4\cos t/5 + 8\sin t/5 + (a + 4/5)e^{t/2}$.

(c) If $a + 4/5 = 0$, then the solution is oscillatory for all t, while if $a + 4/5 \neq 0$, the solution is unbounded as $t \to \infty$. Thus $a_0 = -4/5$.

14.(a)

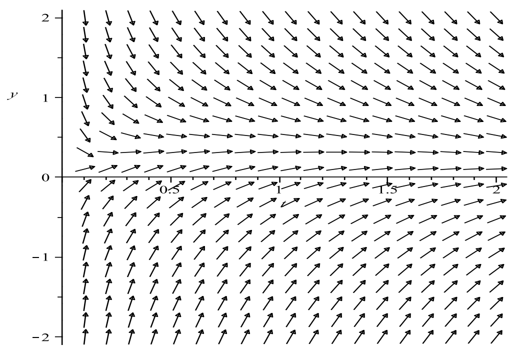


Solutions appear to grow infinitely large in absolute value, with signs depending on the initial value $y(0) = a_0$. The direction field appears horizontal for $a_0 \approx -1/8$.

(b) Dividing both sides of the given equation by 3, the integrating factor is $\mu(t) = e^{-2t/3}$. Multiplying both sides of the original differential equation by $\mu(t)$ and integrating results in $y(t) = (2\,e^{2t/3} - 2\,e^{-\pi t/2} + a(4 + 3\pi)\,e^{2t/3})/(4 + 3\pi)$. The qualitative behavior of the solution is determined by the terms containing $e^{2t/3}$: $2\,e^{2t/3} + a(4 + 3\pi)\,e^{2t/3}$. The nature of the solutions will change when $2 + a(4 + 3\pi) = 0$. Thus the critical initial value is $a_0 = -2/(4 + 3\pi)$.

(c) In addition to the behavior described in part (a), when $y(0) = -2/(4+3\pi)$, the solution is $y(t) = (-2\,e^{-\pi t/2})/(4+3\pi)$, and that specific solution will converge to $y = 0$.

15.(a)

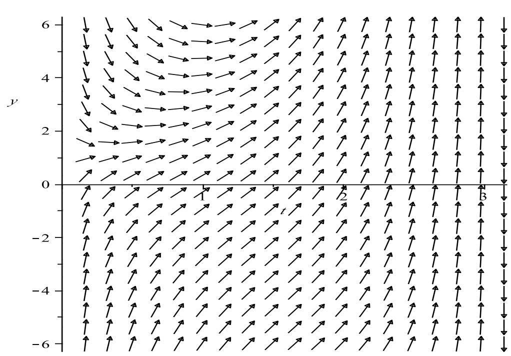


As $t \to 0$, solutions increase without bound if $y(1) = a > 0.4$, and solutions decrease without bound if $y(1) = a < 0.4$.

(b) The integrating factor is $\mu(t) = e^{\int (t+1)/t\,dt} = t\,e^t$. The general solution of the differential equation is $y(t) = t\,e^{-t} + c\,e^{-t}/t$. Since $y(1) = a$, we have that $1 + c = ae$. That is, $c = ae - 1$. Hence the solution can also be expressed as $y(t) = t\,e^{-t} + (ae-1)\,e^{-t}/t$. For small values of t, the second term is dominant. Setting $ae - 1 = 0$, the critical value of the parameter is $a_0 = 1/e$.

(c) When $a = 1/e$, the solution is $y(t) = t\,e^{-t}$, which approaches 0 as $t \to 0$.

SS 16.(a)



(b) $\mu(t) = e^{\int \cos t/\sin t\,dt} = e^{\ln \sin t} = \sin t$ and thus $(y \sin t)' = e^t$. Hence $y \sin t = e^t + c$ or $y = (e^t + c)/\sin t$. Setting $t = 1$ and $y = a$ we get $c = a\sin 1 - e$ so $y(t) = (e^t - e + a\sin 1)/\sin t$. If $y(t)$ is to remain finite as $t \to 0$ the numerator, $e^t - e + a\sin 1$, must approach 0 as $t \to 0$ and hence $a_0 = (e-1)/\sin 1$.

(c) Using a_0 we have $y(t) = (e^t - 1)/\sin t$, which approaches 1 as $t \to 0$, using L'Hospital's rule.

17. The integrating factor is $\mu(t) = e^{\int (1/2)\,dt} = e^{t/2}$. Therefore the general solution is $y(t) = (4\cos t + 8\sin t)/5 + c\,e^{-t/2}$. Invoking the initial condition, the specific solution is $y(t) = (4\cos t + 8\sin t - 9\,e^{-t/2})/5$. Differentiating, it follows that $y'(t) = (-4\sin t + 8\cos t + 4.5\,e^{-t/2})/5$ and $y''(t) = (-4\cos t - 8\sin t - 2.25\,e^{-t/2})/5$. Setting $y'(t) = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the first stationary point. Since $y''(t_1) < 0$, the first stationary point in a local maximum. The coordinates of the point are $(1.3643\,, 0.82008)$.

18. The integrating factor is $\mu(t) = e^{\int (2/3)\,dt} = e^{2t/3}$, and the differential equation can be written as $(e^{2t/3}\,y)' = e^{2t/3} - t\,e^{2t/3}/2$. The general solution is $y(t) = (21 - 6t)/8 + c\,e^{-2t/3}$. Imposing the initial condition, we have $y(t) = (21 - 6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Since the solution is smooth, the desired intersection will be a point of tangency. Taking the derivative, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = (3/2)\ln[(21 - 8y_0)/9]$. Substituting into the solution, the respective value at the stationary point is $y(t_1) = 3/2 + (9/4)\ln 3 - (9/8)\ln(21 - 8y_0)$. Setting this result equal to zero, we obtain the required initial value $y_0 = (21 - 9\,e^{4/3})/8 \approx -1.643$.

19.(a) The integrating factor is $\mu(t) = e^{t/4}$, and the differential equation can be written as $(e^{t/4}\,y)' = 3\,e^{t/4} + 2\,e^{t/4}\cos 2t$. After integration, we get that the general solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t)/65 + ce^{-t/4}$. Invoking the initial condition, $y(0) = 0$, the specific solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t - 788\,e^{-t/4})/65$. As $t \to \infty$, the exponential term will decay, and the solution will oscillate about an average value of 12, with an amplitude of $8/\sqrt{65}$.

(b) Solving $y(t) = 12$, we obtain the desired value $t \approx 10.0658$.

SS 20. $(e^{-t}y)' = e^{-t} + 3e^{-t}\sin t$ so $e^{-t}y = -e^{-t} - 3e^{-t}(\sin t + \cos t)/2 + c$ or $y(t) = -1 - 3(\sin t + \cos t)/2 + ce^{t}$. Thus $y(0) = -1 - 3/2 + c = y_0$ or $c = y_0 + 5/2$. Now, if $y(t)$ is to remain bounded as $t \to \infty$, we must have $c = 0$ so that $y_0 = -5/2$.

21. The integrating factor is $\mu(t) = e^{-3t/2}$, and the differential equation can be written as $(e^{-3t/2}\,y)' = 3t\,e^{-3t/2} + 2\,e^{-t/2}$. The general solution is $y(t) = -2t - 4/3 - 4\,e^{t} + c\,e^{3t/2}$. Imposing the initial condition, $y(t) = -2t - 4/3 - 4\,e^{t} + (y_0 + 16/3)\,e^{3t/2}$. Now as $t \to \infty$, the term containing $e^{3t/2}$ will dominate the solution. Its sign will determine the divergence properties. Hence the critical value of the initial condition is $y_0 = -16/3$. The corresponding solution, $y(t) = -2t - 4/3 - 4\,e^{t}$, will also decrease without bound.

Note on Problems 24, 26, and 27:

Let $g(t)$ be given, and consider the function $y(t) = y_1(t) + g(t)$, in which $y_1(t) \to 0$ as $t \to \infty$. Differentiating, $y'(t) = y_1'(t) + g'(t)$. Letting a be a constant, it follows that $y'(t) + ay(t) = y_1'(t) + ay_1(t) + g'(t) + ag(t)$. Note that the hypothesis on the function $y_1(t)$ will be satisfied, if $y_1'(t) + ay_1(t) = 0$. That is, $y_1(t) = c\,e^{-at}$. Hence $y(t) = c\,e^{-at} + g(t)$, which is a solution of the equation $y' + ay = g'(t) + ag(t)$. For convenience, choose $a = 1$.

SS 22. Write the first term of Eq.(47) as $(\int_0^t e^{s^2/4}\,ds)/e^{t^2/4}$. In applying L'Hospital's rule, the derivative of the numerator term is $e^{t^2/4}$ by the Fundamental Theorem of Calculus. The derivative of the denominator is $(t/2)e^{t^2/4}$ and thus the limit of both terms in Eq.(47) is 0 as $t \to \infty$.

SS 23. $\mu(t) = e^{at}$ so the differential equation can be written as $(e^{at}y)' = be^{at}e^{-\lambda t} = be^{(a-\lambda)t}$. If $a \neq \lambda$, then integration and solution for y yields $y = [b/(a-\lambda)]e^{-\lambda t} + ce^{-at}$. Then $\lim_{t\to\infty} y$ is zero since both λ and a are positive numbers. If $a = \lambda$, then the differential equation becomes $(e^{at}y)' = b$, which yields $y = (bt + c)e^{-\lambda t}$ as the solution. L'Hospital's rule gives

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{bt + c}{e^{\lambda t}} = \lim_{t\to\infty} \frac{b}{\lambda e^{\lambda t}} = 0.$$

24. Here $g(t) = 3$, and we consider the linear equation $y' + y = 3$. The integrating factor is $\mu(t) = e^t$, and the differential equation can be written as $(e^t\,y)' = 3e^t$. The general solution is $y(t) = 3 + c\,e^{-t}$.

SS 25. There is no unique answer for this situation. One possible answer is to assume $y(t) = ce^{-2t} + 3 - t$ (which satisfies the given condition), then $y'(t) = -2ce^{-2t} - 1$. Eliminating ce^{-2t} between the two equations yields $y' + 2y = 5 - 2t$.

26. Here $g(t) = 2t - 5$. Consider the linear equation $y' + y = 2 + 2t - 5$. The integrating factor is $\mu(t) = e^t$, and the differential equation can be written as $(e^t\,y)' = (2t - 3)e^t$. The general solution is $y(t) = 2t - 5 + c\,e^{-t}$.

27. $g(t) = 4 - t^2$. Consider the linear equation $y' + y = 4 - 2t - t^2$ .The integrating factor is $\mu(t) = e^t$, and the equation can be written as $(e^t\,y)' = (4 - 2t - t^2)e^t$. The general solution is $y(t) = 4 - t^2 + c\,e^{-t}$.

28.(a) Differentiating y and using the fundamental theorem of calculus we obtain that $y' = Ae^{-\int p(t)dt} \cdot (-p(t))$, and then $y' + p(t)y = 0$.

(b) Differentiating y we obtain that

$$y' = A'(t)e^{-\int p(t)dt} + A(t)e^{-\int p(t)dt} \cdot (-p(t)).$$

If this satisfies the differential equation then

$$y' + p(t)y = A'(t)e^{-\int p(t)dt} = g(t)$$

and the required condition follows.

(c) Let us denote $\mu(t) = e^{\int p(t)dt}$. Then clearly $A(t) = \int \mu(t)g(t)dt$, and after substitution $y = \int \mu(t)g(t)dt \cdot (1/\mu(t))$, which is just Eq. (33).

SS 29. By Problem 28, $y(t) = A(t)e^{-\int(-2)\,dt} = A(t)e^{2t}$. Differentiating $y(t)$ and substituting into the differential equation yields $A'(t) = t^2$ since the terms involving $A(t)$ add to zero. Thus $A(t) = t^3/3 + c$, which substituted into $y(t)$ yields the solution.

30. We assume a solution of the form $y = A(t)e^{-\int (1/t)\,dt} = A(t)e^{-\ln t} = A(t)t^{-1}$, where $A(t)$ satisfies $A'(t) = 3t\cos 2t$. This implies that

$$A(t) = \frac{3\cos 2t}{4} + \frac{3t\sin 2t}{2} + c$$

and the solution is

$$y = \frac{3\cos 2t}{4t} + \frac{3\sin 2t}{2} + \frac{c}{t}.$$

2.2

Problems 1 through 16 follow the pattern of the examples worked in this section. The first eight problems, however, do not have an initial condition, so the integration constant c cannot be found.

SS 1. Write the equation in the form $y dy = x^2 dx$. Integrating the left side with respect to y and the right side with respect to x yields $y^2/2 = x^3/3 + C$, or $3y^2 - 2x^3 = c$.

2. The differential equation may be written as $y^{-2}dy = -\sin x\, dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-1} = \cos x + c$. That is, $(c - \cos x)y = 1$, in which c is an arbitrary constant. Solving for the dependent variable, explicitly, $y(x) = 1/(c - \cos x)$.

3. Write the differential equation as $\cos^{-2} 2y\, dy = \cos^2 x\, dx$, which also can be written as $\sec^2 2y\, dy = \cos^2 x\, dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $\tan 2y = \sin x \cos x + x + c$.

SS 4. We need $x \neq 0$ and $|y| < 1$ for this problem to be defined. Separating the variables we get $(1 - y^2)^{-1/2}\, dy = x^{-1}\, dx$. Integrating each side yields $\arcsin y = \ln|x| + c$, so $y = \sin(\ln|x| + c)$, $x \neq 0$ (note that $|y| < 1$). Also, $y = \pm 1$ satisfy the differential equation, since both sides are zero.

5. The differential equation may be written as $(y + e^y)dy = (x - e^{-x})dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $y^2 + 2\,e^y = x^2 + 2\,e^{-x} + c$.

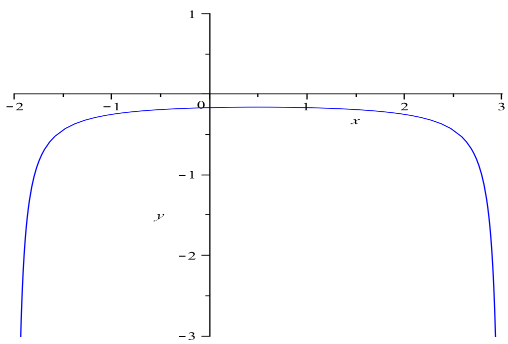
6. Write the differential equation as $(1 + y^2)dy = x^2\, dx$. Integrating both sides of the equation, we obtain the relation $y + y^3/3 = x^3/3 + c$.

SS 7. Write the differential equation as $y^{-1}\, dy = x^{-1}\, dx$. Integrating both sides of the equation, we obtain the relation $\ln|y| = \ln|x| + c$. Solving for y explicitly gives $y = kx$. Note that k may be positive or negative due to the absolute values in the integrated equation.

SS 8. Write the differential equation as $y\, dy = -x\, dx$. Integrating both sides of the equation, we obtain the relation $(1/2)y^2 = -(1/2)x^2 + c$. The explicit form of the solution is $y(x) = \pm\sqrt{x^2 + c}$. The initial condition would then be used to determine whether the positive of negative solution is to be used for a specific initial value problem.

9.(a) The differential equation is separable, with $y^{-2}dy = (1 - 2x)dx$. Integration yields $-y^{-1} = x - x^2 + c$. Substituting $x = 0$ and $y = -1/6$, we find that $c = 6$. Hence the specific solution is $y = 1/(x^2 - x - 6)$.

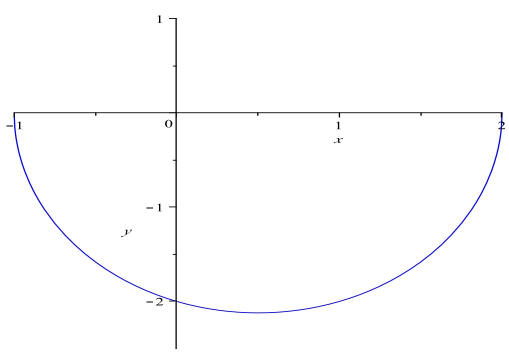
(b)



(c) Note that $x^2 - x - 6 = (x + 2)(x - 3)$. Hence the solution becomes singular at $x = -2$ and $x = 3$, so the interval of existence is $(-2, 3)$.

SS 10.(a) Separating the variables we get $ydy = (1 - 2x)dx$, so $y^2/2 = x - x^2 + c$. Setting $x = 1$ and $y = -2$ we have $c = 2$ and thus $y^2 = 2x - 2x^2 + 4$ or $y(x) = -\sqrt{2x - 2x^2 + 4}$. The negative square root must be used since $y(1) = -2$.

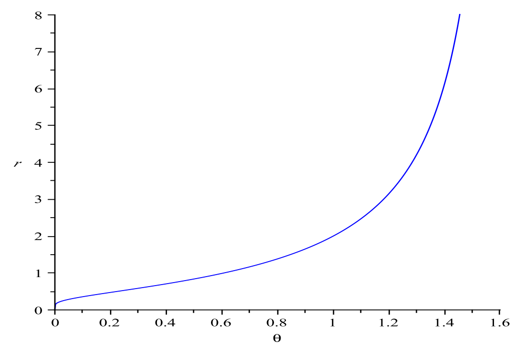
(b)



(c) Rewriting $y(x)$ as $y(x) = -\sqrt{2(2 - x)(x + 1)}$, we see that y is defined for $-1 \le x \le 2$. However, since y' does not exist for $x = -1$ or $x = 2$, the solution is valid only for the open interval $-1 < x < 2$. The interval of existence is $(-1, 2)$.

12.(a) Write the differential equation as $r^{-2}dr = \theta^{-1}\,d\theta$. Integrating both sides of the equation results in the relation $-r^{-1} = \ln\,\theta + c$. Imposing the condition $r(1) = 2$, we obtain $c = -1/2$. The explicit form of the solution is $r = 2/(1 - 2\ln\,\theta)$.

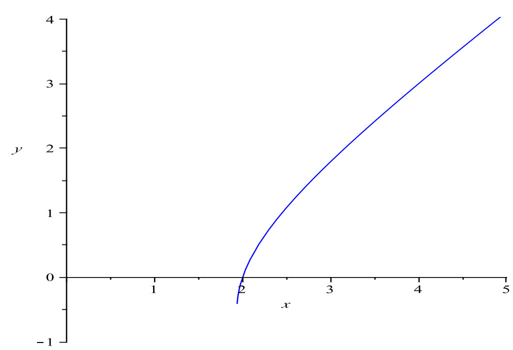
(b)



(c) Clearly, the solution makes sense only if $\theta > 0$. Furthermore, the solution becomes singular when $\ln\theta = 1/2$, that is, $\theta = \sqrt{e}$.

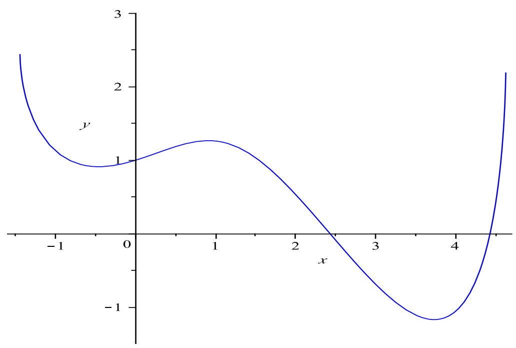
SS 14.(a) Separating variables and integrating yields $y + y^2 = x^2 + c$. Setting $y = 0$ when $x = 2$ yields $c = -4$ or $y^2 + y = x^2 - 4$. To solve for y complete the square on the left side by adding $1/4$ to both sides. This yields $y^2 + y + 1/4 = x^2 - 4 + 1/4$ or $(y + 1/2)^2 = x^2 - 15/4$. Taking the square root of both sides yields $y + 1/2 = \pm\sqrt{x^2 - 15/4}$, where the positive square root must be taken in order to satisfy the initial condition. Thus $y(x) = -1/2 + \sqrt{x^2 - 15/4}$, which is defined for $x^2 \geq 15/4$ or $x \geq \sqrt{15}/2$.

(b)



SS 15.(a) Separating variables gives $(2y - 5)dy = (3x^2 - e^x)dx$ and integration then gives $y^2 - 5y = x^3 - e^x + c$. Setting $x = 0$ and $y = 1$ we have $1 - 5 = 0 - 1 + c$, or $c = -3$ and thus $y^2 - 5y - (x^3 - e^x - 3) = 0$. Using the quadratic formula then gives $y(x) = 5/2 - \sqrt{x^3 - e^x + 13/4}$, where the negative square root is chosen so that $y(0) = 1$.

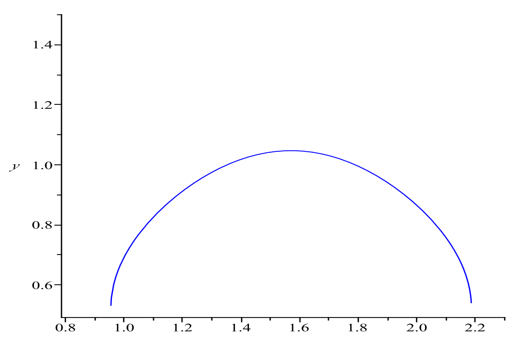
(b)



(c) The solution is valid for approximately $-1.45 < x < 4.63$. These values are found by estimating the roots of $4x^3 - 4e^x + 13 = 0$.

SS 16.(a) We start with $\cos 3y\, dy = -\sin 2x\, dx$ and integrate to get $(1/3)\sin 3y = (1/2)\cos 2x + c$. Setting $y = \pi/3$ when $x = \pi/2$ (from the initial condition) we find that $0 = -1/2 + c$ or $c = 1/2$, so that $(1/3)\sin 3y = (1/2)\cos 2x + 1/2 = \cos^2 x$ (using the appropriate trigonometric identity). To solve for y we must choose the branch that passes through the point $(\pi/2, \pi/3)$, so $y(x) = (\pi - \arcsin(3\cos^2 x))/3$.

(b)



(c) The solution in part (a) is defined only for $0 \le 3\cos^2 x \le 1$, or $-\sqrt{1/3} \le \cos x \le \sqrt{1/3}$. Taking the indicated square roots and then finding the inverse cosine of each side yields $0.9553 \le x \le 2.1863$, or $|x - \pi/2| \le 0.6155$, as the appropriate interval.

SS 17. We have $(3y^2 - 6y)dy = (1 + 3x^2)dx$ so that $y^3 - 3y^2 = x + x^3 - 2$, once the initial condition is used. From the differential equation, the integral curve will have a vertical tangent when $3y^2 - 6y = 0$, or $y = 0, 2$. For $y = 0$ we have $x^3 + x - 2 = 0$, which is satisfied for $x = 1$, which is the only zero of the function $w = x^3 + x - 2$. Likewise, for $y = 2$, $x = -1$. Thus the solution is valid on $|x| < 1$.

18. The differential equation can be written as $(3y^2 - 4)dy = 3x^2 dx$. Integrating both sides, we obtain $y^3 - 4y = x^3 + c$. Imposing the initial condition, the specific

solution is $y^3 - 4y = x^3 - 1$. Referring back to the differential equation, we find that $y' \to \infty$ as $y \to \pm 2/\sqrt{3}$. The respective values of the abscissas are $x \approx -1.276$, 1.598. Hence the solution is valid for $-1.276 < x < 1.598$.

SS 21.(a) By sketching the direction field or by using the differential equation we note that $y' < 0$ for $y > 4$ and y' approaches zero as y approaches 4. For $0 < y < 4$, $y' > 0$ and again approaches 0 as y approaches 4. Thus $\lim_{t\to\infty} y = 4$ if $y_0 > 0$. For $y_0 < 0$, $y' < 0$ for all y and hence y becomes negatively unbounded as t increases. If $y_0 = 0$, then $y' = 0$ for all t, so $y = 0$ for all t.

(b) Separating variables and using a partial fraction expansion we obtain that $(1/y - 1/(y-4))\,dy = (4/3)t\,dt$. Hence $\ln|y/(y-4)| = 2t^2/3 + c_1$, and therefore $|y/(y-4)| = e^{c_1}e^{2t^2/3} = ce^{2t^2/3}$, where c is positive. For $y(0) = y_0 = 0.5$, this gives us the equation $|0.5/(0.5-4)| = c$ and thus $c = 1/7$. Using this value for c and solving for y yields $y(t) = 4/(1 + 7e^{-2t^2/3})$. Setting this equal to 3.98 and solving for t yields $t = 3.29527$.

22.(a) Write the differential equation as $y^{-1}(4-y)^{-1}dy = t(1+t)^{-1}dt$. Integrating both sides of the equation, we obtain $\ln|y| - \ln|y-4| = 4t - 4\ln|1+t| + c$. Taking the exponential of both sides $|y/(y-4)| = c\,e^{4t}/(1+t)^4$. It follows that as $t \to \infty$, $|y/(y-4)| = |1 + 4/(y-4)| \to \infty$. That is, $y(t) \to 4$.

(b) Setting $y(0) = 2$, we obtain that $c = 1$. Based on the initial condition, the solution may be expressed as $y/(y-4) = -e^{4t}/(1+t)^4$. Note that $y/(y-4) < 0$, for all $t \geq 0$. Hence $y < 4$ for all $t \geq 0$. Referring back to the differential equation, it follows that y' is always positive. This means that the solution is monotone increasing. We find that the root of the equation $e^{4t}/(1+t)^4 = 399$ is near $t = 2.844$.

(c) Note the $y(t) = 4$ is an equilibrium solution. Examining the local direction field we see that if $y(0) > 0$, then the corresponding solutions converge to $y = 4$. Referring back to part (a), we have $y/(y-4) = [y_0/(y_0-4)]\,e^{4t}/(1+t)^4$, for $y_0 \neq 4$. Setting $t = 2$, we obtain $y_0/(y_0-4) = (3/e^2)^4 y(2)/(y(2)-4)$. Now since the function $f(y) = y/(y-4)$ is monotone for $y < 4$ and $y > 4$, we need only solve the equations $y_0/(y_0-4) = -399(3/e^2)^4$ and $y_0/(y_0-4) = 401(3/e^2)^4$. The respective solutions are $y_0 = 3.6622$ and $y_0 = 4.4042$.

SS 23. Separating variables yields $(cy+d)/(ay+b)\,dy = dx$. If $a \neq 0$ and $ay+b \neq 0$ then $dx = (c/a + (ad-bc)/(a(ay+b)))\,dy$. Integration then yields the desired answer.

SS 24. Separating variables yields $dQ/(a+bQ) = r dt$. If $b \neq 0$, then integrating gives $\ln(|a+bQ|)/b = rt + c$; solving for Q and applying the initial condition yield

$$Q(t) = \left(Q_0 + \frac{a}{b}\right)e^{brt} - \frac{a}{b}$$

As $t \to \infty$, $Q(t) \to \infty$ if $br > 0$ and $Q_0 + \frac{a}{b} > 0$, $Q(t) \to -\infty$ if $br > 0$ and $Q_0 + \frac{a}{b} < 0$, $Q(t) \to -\frac{a}{b}$ if $br < 0$ or if $br > 0$ and $Q_0 + \frac{a}{b} = 0$, and $Q(t) \to Q_0$ if $br = 0$. If $b = 0$ then $Q(t) = art + Q_0$ and $Q(t) \to \infty$ if $ar > 0$, $Q(t) \to -\infty$ if $ar < 0$, and $Q(t) \to Q_0$ if $ar = 0$.

SS 25.(a) $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y} = \dfrac{x\left(\frac{y}{x}-4\right)}{x\left(1-\frac{y}{x}\right)} = \dfrac{\frac{y}{x}-4}{1-\frac{y}{x}}$

(b) From $v = y/x$ we find that $y = xv$. Thus $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

(c) Combining the results of (a) and (b) produces $v + x\dfrac{dv}{dx} = \dfrac{v-4}{1-v}$, which may be solved to find that $x\dfrac{dv}{dx} = \dfrac{v-4}{1-v} - v = \dfrac{v-4-v(1-v)}{1-v} = \dfrac{v^2-4}{1-v}$.

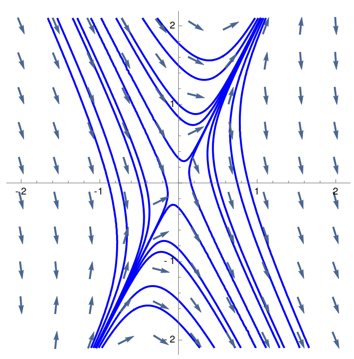
(d) Separating variables yields $(1-v)/(v^2-4)dv = dx/x$. A partial fraction expansion on the leftt side of the equation gives

$$\left(\frac{1}{4(v-2)} - \frac{3}{4(v+2)}\right)dv = \frac{dx}{x} \text{ or } \left(-\frac{1}{v-2} + \frac{3}{v+2}\right)dv = -\frac{4dx}{x}$$

Integration leads to $-\ln|v-2| + 3\ln|v+2| = -4\ln|x| + c$. Combining the logarithmic terms and exponentiating both sides of the equation produces the implicit relationship $|v+2|^3|v-2| = k/x^4$.

(e) Substituting $v = y/x$ in the implicit relationship given in (d) and then mutliplying each side by x^4 gives the implicit relationship $(y+2x)^3(y-2x) = k$.

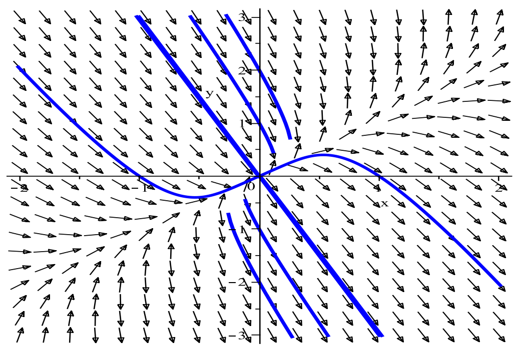
(f)



SS 28.(a) Observe that $(4y-3x)/(2x-y) = (4(y/x)-3)/(2-y/x)$. Hence the differential equation is homogeneous.

(b) Substituting $y = vx$ we get $v + xv' = (4v-3)/(2-v)$ which can be rewritten as $xv' = (v^2+2v-3)/(2-v)$. Note that $v = -3$ and $v = 1$ are solutions of this equation. For $v \neq 1$, -3 separating variables gives $(2-v)/((v+3)(v-1))\,dv = (1/x)\,dx$. Applying a partial fraction decomposition to the left side we obtain $(1/(4(v-1)) - 5/(4(v+3)))\,dv = (1/x)\,dx$, and upon integrating both sides we find that $(1/4)\ln|v-1| - (5/4)\ln|v+3| = \ln|x| + c$. Substituting for v and performing some algebraic manipulations we get the solution in the implicit form $|y-x| = c|y+3x|^5$. $v = 1$ and $v = -3$ yield $y = x$ and $y = -3x$, respectively, as solutions also.

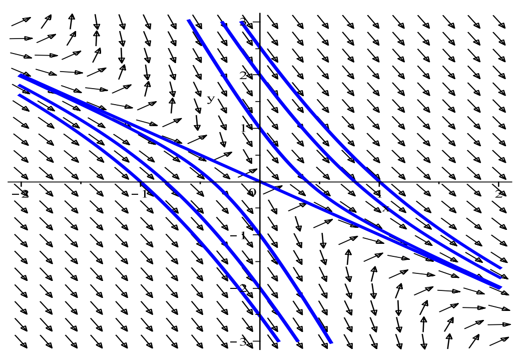
(c) The integral curves are symmetric with respect to the origin.



29.(a) Observe that $-(4x+3y)/(2x+y) = -2 - (y/x)\,[2+(y/x)]^{-1}$. Hence the differential equation is homogeneous.

(b) The substitution $y = x\,v$ results in $v + x\,v\,' = -2 - v/(2+v)$. The transformed equation is $v\,' = -(v^2+5v+4)/(2+v)x$. This equation is separable, with general solution $(v+4)^2\,|v+1| = c/x^3$. In terms of the original dependent variable, the solution is $(4x+y)^2\,|x+y| = c$.

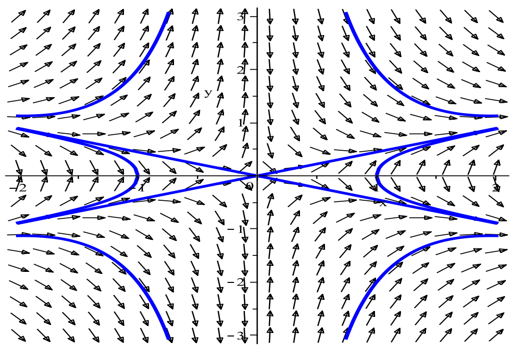
(c) The integral curves are symmetric with respect to the origin.



30.(a) The differential equation can be expressed as $y\,' = (1/2)(y/x)^{-1} - (3/2)(y/x)$. Hence the equation is homogeneous. The substitution $y = xv$ results in $xv\,' = (1-5v^2)/2v$. Separating variables, we have $2vdv/(1-5v^2) = dx/x$.

(b) Integrating both sides of the transformed equation yields $-(\ln|1-5v^2|)/5 = \ln|x| + c$, that is, $1 - 5v^2 = c/\,|x|^5$. In terms of the original dependent variable, the general solution is $5y^2 = x^2 - c/\,|x|^3$.

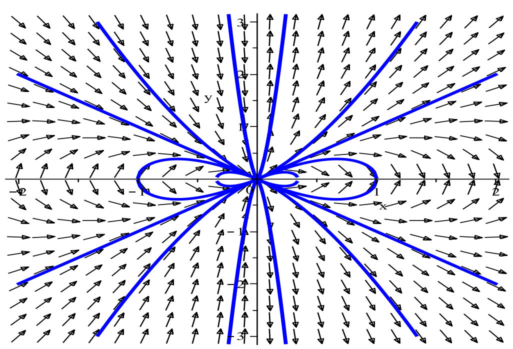
(c) The integral curves are symmetric with respect to the origin.



31.(a) The differential equation can be expressed as $y' = (3/2)(y/x) - (1/2)(y/x)^{-1}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (v^2-1)/2v$, that is, $2vdv/(v^2-1) = dx/x$.

(b) Integrating both sides of the transformed equation yields $\ln\left|v^2-1\right| = \ln|x| + c$, that is, $v^2 - 1 = c\,|x|$. In terms of the original dependent variable, the general solution is $y^2 = c\,x^2\,|x| + x^2$.

(c) The integral curves are symmetric with respect to the origin.



2.3

1. Let $Q(t)$ be the amount of dye in the tank at time t. Clearly, $Q(0) = 200$ g. The differential equation governing the amount of dye is $Q'(t) = -2Q(t)/200$. The solution of this separable equation is $Q(t) = Q(0)e^{-t/100} = 200e^{-t/100}$. We need the time T such that $Q(T) = 2$ g. This means we have to solve $2 = 200e^{-T/100}$ and we obtain that $T = -100\ln(1/100) = 100\ln 100 \approx 460.5$ min.

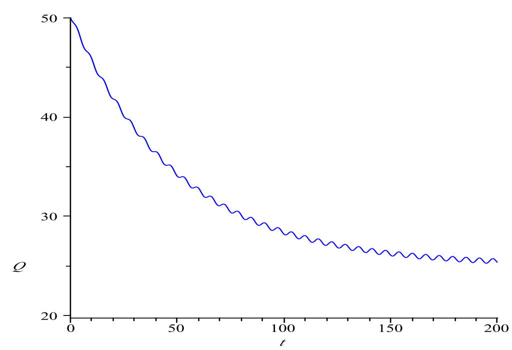
SS 2. Let $S(t)$ be the amount of salt that is present at any time t, then $S(0) = 0$ is the original amount of salt in the tank, 2γ is the amount of salt entering per minute, and $2(S/120)$ is the amount of salt leaving per minute (all amounts measured in grams). Thus $dS/dt = 2\gamma - 2S/120$, $S(0) = 0$. This is a linear equation, which has $e^{t/60}$ as its integrating factor. Thus the general solution is $S(t) = 120\gamma + ce^{-t/60}$. $S(0) = 0$ gives $c = -120\gamma$, so $S(t) = 120\gamma(1 - e^{-t/60})$ and hence $S(t) \to 120\gamma$ grams as $t \to \infty$.

3.(a) Let Q be the amount of salt in the tank. Salt enters the tank of water at a rate of $2\,(1/4)(1 + (1/2)\sin t) = 1/2 + (1/4)\sin t$ oz/min. It leaves the tank at a rate of $2\,Q/100$ oz/min. Hence the differential equation governing the amount of salt at any time is

$$\frac{dQ}{dt} = \frac{1}{2} + \frac{1}{4}\sin t - \frac{Q}{50}\,.$$

The initial amount of salt is $Q_0 = 50$ oz. The governing differential equation is linear, with integrating factor $\mu(t) = e^{t/50}$. Write the equation as $(e^{t/50}Q)' = e^{t/50}(1/2 + (1/4)\sin t)$. The specific solution is $Q(t) = 25 + (12.5\sin t - 625\cos t + 63150\,e^{-t/50})/2501$ oz.

(b)



(c) The amount of salt approaches a steady state, which is an oscillation of approximate amplitude 1/4 about a level of 25 oz.

4.(a) Using the Principle of Conservation of Energy, the speed v of a particle falling from a height h is given by

$$\frac{1}{2}mv^2 = mgh\,.$$

(b) The outflow rate is (outflow cross-section area)×(outflow velocity): $\alpha\, a\sqrt{2gh}$. At any instant, the volume of water in the tank is $V(h) = \int_0^h A(u)du$. The time rate of change of the volume is given by $dV/dt = (dV/dh)(dh/dt) = A(h)dh/dt$. Since the volume is decreasing, $dV/dt = -\alpha\, a\sqrt{2gh}$.

(c) With $A(h) = \pi$, $a = 0.01\,\pi$, $\alpha = 0.6$, the differential equation for the water level h is $\pi(dh/dt) = -0.006\,\pi\sqrt{2gh}$, with solution $h(t){=}0.000018gt^2 - 0.006\sqrt{2gh(0)}\,t{+}h(0)$. Setting $h(0) = 3$ and $g = 9.8$, $h(t) = 0.0001764\,t^2 - 0.046\,t + 3$, resulting in $h(t) = 0$ for $t \approx 130.4$ s.

5.(a) The equation governing the value of the investment is $dS/dt = r\,S$. The value of the investment, at any time, is given by $S(t) = S_0e^{rt}$. Setting $S(T) = 2S_0$, the required time is $T = \ln(2)/r$.

(b) For the case $r = .07$, $T \approx 9.9$ yr.

(c) Referring to part (a), $r = \ln(2)/T$. Setting $T = 8$, the required interest rate is to be approximately $r = 8.66\%$.

SS 6.(a) Set $S_0 = 0$ in Eq.(16) (or solve Eq.(15) with $S(0) = 0$).

(b) Set $r = 0.075$, $t = 40$ and $S(t) = \$1,000,000$ in the answer to part (a) and then solve for k.

(c) Set $k = \$2,000$, $t = 40$ and $S(t) = \$1,000,000$ in the answer to part (a) and then solve numerically for r.

SS 7. Let $S(t)$ be the amount of the loan remaining at time t, then $dS/dt = 0.1S - k$, $S(0) = \$8,000$. Solving this for $S(t)$ yields $S(t) = 8000e^{0.1t} - 10k(e^{0.1t} - 1)$. Setting $S = 0$ and substitution of $t = 3$ gives $k = \$3,086.64$ per year. For 3 years this totals $\$9,259.92$, so $\$1,259.92$ has been paid in interest.

8.(a) Using Eq.(15) we have $dS/dt - 0.005S = -(800 + 10t)$, $S(0) = 150,000$. Using an integrating factor and integration by parts we obtain that $S(t) = 560,000 - 410,000e^{0.005t} + 2000t$. Setting $S(t) = 0$ and solving numerically for t yields $t = 146.54$ months.

(b) The solution we obtained in part (a) with a general initial condition $S(0)=S_0$ is $S(t)=560,000 - 560,000e^{0.005t}+S_0e^{0.005t}+2000t$. Solving the equation $S(240) = 0$ yields $S_0 = 246,758$.

9.(a) Let $Q' = -r\,Q$. The general solution is $Q(t) = Q_0e^{-rt}$. Based on the definition of half-life, consider the equation $Q_0/2 = Q_0e^{-5730\,r}$. It follows that $-5730\,r = \ln(1/2)$, that is, $r = 1.2097 \times 10^{-4}$ per year.

(b) The amount of carbon-14 is given by $Q(t) = Q_0\,e^{-1.2097\times10^{-4}t}$.

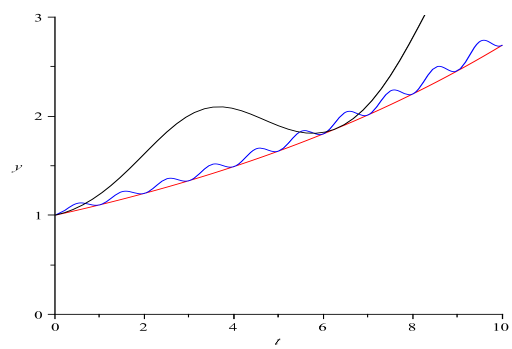
(c) Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097\times10^{-4}T}$. Solving for the decay time, the apparent age of the remains is approximately $T = 13,305$ years.

SS 10.(a) We have $(1/y)dy = (0.1 + 0.2\sin t)dt$, by separating variables, and thus $y(t) = ce^{0.1t-0.2\cos t}$. $y(0) = 1$ gives $c = e^{0.2}$, so $y(t) = e^{0.2+0.1t-0.2\cos t}$. Setting $y = 2$ yields $\ln 2 = 0.2 + 0.1\tau - 0.2\cos\tau$, which can be solved numerically to give $\tau = 2.9632$. If $y(0) = y_0$ then as above, $y(t) = y_0e^{0.2+0.1t-0.2\cos t}$. Thus if we set $y = 2y_0$ we get the same numerical equation for τ and hence the doubling time has not changed.

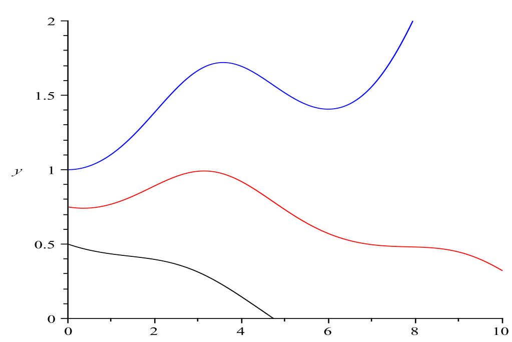
(b) The differential equation is $dy/dt = y/10$, with solution $y(t) = y(0)e^{t/10}$. The doubling time is given by $\tau = 10\ln 2 \approx 6.9315$.

(c) Consider the differential equation $dy/dt = (0.5 + \sin(2\pi t))\, y/5$. The equation is separable, with $(1/y)dy = (0.1 + \frac{1}{5}\sin(2\pi t))dt$. Integrating both sides, with respect to the appropriate variable, we obtain $\ln y = (\pi t - \cos(2\pi t))/10\pi + c$. Invoking the initial condition, the solution is $y(t) = e^{(1+\pi t - \cos(2\pi t))/10\pi}$. The doubling-time is $\tau \approx 6.3804$. The doubling time approaches the value found in part (b).

(d)



11.(a) The differential equation $dy/dt = r(t)\, y - k$ is linear, with integrating factor $\mu(t) = e^{-\int r(t)dt}$. Write the equation as $(\mu\, y)' = -k\, \mu(t)$. Integration of both sides yields the general solution $y = \left[-k \int \mu(\tau)d\tau + y_0\, \mu(0)\right] / \mu(t)$. In this problem, the integrating factor is $\mu(t) = e^{(\cos\, t - t)/5}$.



(b) The population becomes extinct, if $y(t^*) = 0$, for some $t = t^*$. Referring to part (a), we find that $y(t^*) = 0$ when

$$\int_0^{t^*} e^{(\cos\, \tau - \tau)/5} d\tau = 5\, e^{1/5} y_c.$$

It can be shown that the integral on the left hand side increases monotonically, from zero to a limiting value of approximately 5.0893. Hence extinction can happen only if $5\, e^{1/5} y_0 < 5.0893$. Solving $5e^{1/5} y_c = 5.0893$ yields $y_c = 0.8333$.

(c) Repeating the argument in part (b), it follows that $y(t^*) = 0$ when

$$\int_0^{t^*} e^{(\cos\, \tau - \tau)/5} d\tau = \frac{1}{k}\, e^{1/5} y_c.$$

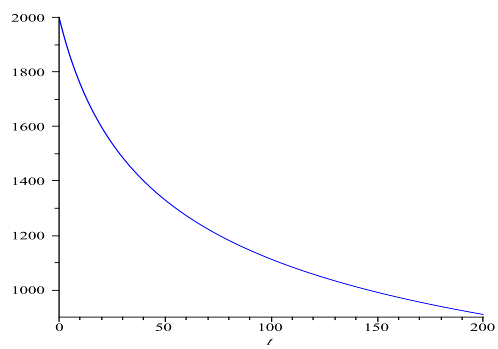
Hence extinction can happen only if $e^{1/5}y_0/k < 5.0893$, so $y_c = 4.1667\,k$.

(d) Evidently, y_c is a linear function of the parameter k.

SS 12. If T is the temperature of the coffee at any time t, then $dT/dt = -k(T-70)$, $T(0) = 200$, $T(1) = 190$. The solution of this linear equation will involve k (the cooling rate) and the integration constant c. Use $T(0) = 200$ to find c and then use $T(1) = 190$ to evaluate k.

13.(a) The solution of the governing equation satisfies $u^3 = u_0^3/(\,3\,\alpha\,u_0^3\,t+1\,)$. With the given data, it follows that $u(t) = 2000/\sqrt[3]{6\,t/125+1}$.

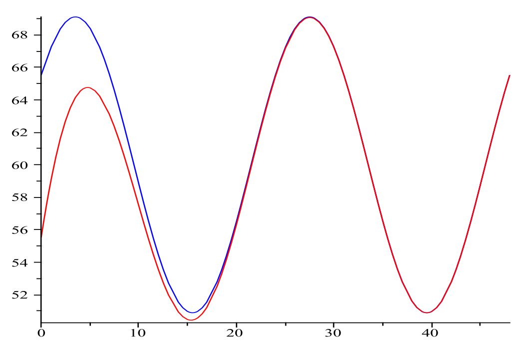
(b)



(c) Numerical evaluation results in $u(t) = 600$ for $t \approx 750.77$ s.

SS 14.(a) Eq.(i) is a linear equation with the integrating factor e^{kt}. Thus $(e^{kt}u)' = k(T_0 + T_1\cos\omega t)e^{kt}$ and hence $e^{kt}u = T_0e^{kt} + kT_1\int \cos\omega t e^{kt}\,dt + c$. Evaluating the integral (by parts or by a symbolic software package) and dividing by e^{kt} yields $u(t) = T_0 + kT_1(k\cos\omega t + \omega\sin\omega t)/(k^2+\omega^2) + ce^{-kt}$. Note that the last term approaches zero as $t \to \infty$ for any initial condition, and that the rest of the solution oscillates about $u(t) = T_0$.

(b) $R \approx 9°$F, $\tau \approx 3.5$h.



(c) Recall that $R\cos(\omega(t-\tau)) = R\cos\omega t\cos\omega\tau + R\sin\omega t\sin\omega\tau$. Comparing this with the oscillatory portion of the above solution we have $R\cos\omega\tau = k^2T_1/(k^2+\omega^2)$ and $R\sin\omega\tau = k\omega T_1/(k^2+\omega^2)$ since these are the coefficients of $\cos\omega t$ and $\sin\omega t$, respectively. By squaring and adding we find $R^2 = k^2T_1^2/(k^2+\omega^2)$ and by dividing we find $\tan\omega\tau = \omega/k$.

SS 15.(a) The required differential equation is $dQ/dt = kr + P - Q(t)r/V$, since kr is the rate of water pollutant entering the lake, P is the rate of pollutant entering directly and $Q(t)r/V$ is the rate at which the pollutant leaves the lake. The initial condition is $Q(0) = Vc_0$. Since $c = Q(t)/V$, the initial value problem may be rewritten as $Vc'(t) = kr + P - rc$, $c(0) = c_0$, which has the solution $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. It is easy to see that $\lim_{t\to\infty} c(t) = k + P/r$.

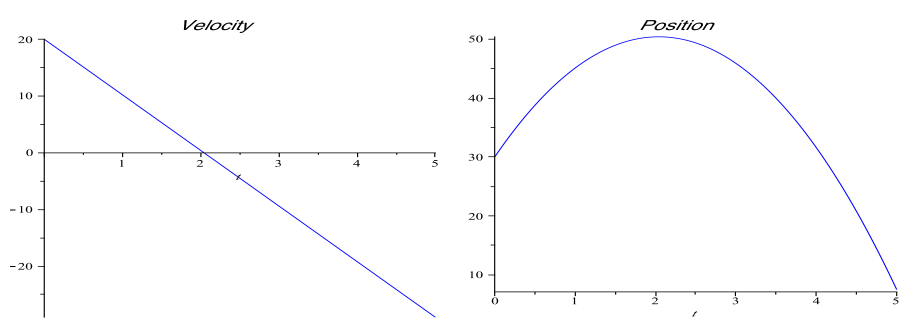
(b) $c(t) = c_0\, e^{-rt/V}$. The reduction times are $T_{50} = V\ln 2/r$ and $T_{10} = V\ln 10/r$.

(c) The reduction times are $T_S = (12,200)\ln 10/65.2 = 430.85$ years; for Lake Michigan, $T_M = (4,900)\ln 10/158 = 71.4$ years; $T_E = (460)\ln 10/175 = 6.05$ years; and $T_O = (16,000)\ln 10/209 = 17.63$ years.

SS 16.(a) If we measure x positively upward from the ground, then Eq.(4) of Section 1.1 becomes $mdv/dt = -mg$, since there is no air resistance. Thus the initial value problem for $v(t)$ is $dv/dt = -g$, $v(0) = 20$, which gives $v(t) = 20 - gt$. Since $dx/dt = v(t)$ we get $x(t) = 20t - (g/2)t^2 + c$. Then $x(0) = 30$ gives $c = 30$ and thus $x(t) = 20t - (g/2)t^2 + 30$. At the maximum height $v(t_m) = 0$ and thus $t_m = 20/9.8 = 2.04$ sec, which when substituted in the equation for $x(t)$ yields the maximum height.

(b) The ball hits the ground when $x(t) = 0$, solving this equation gives $t = 5.2$ sec.

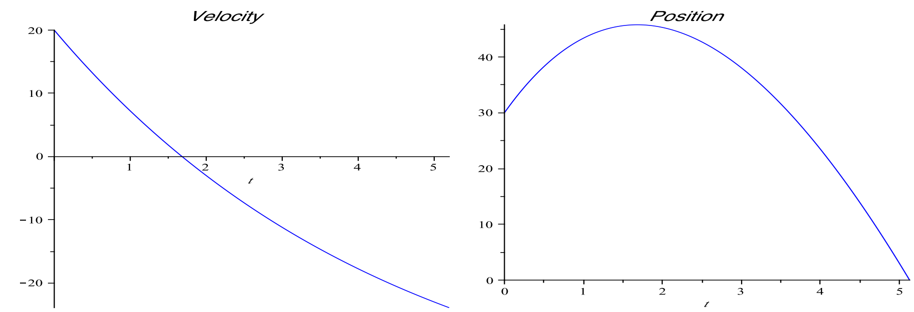
(c)



SS 17.(a) The differential equation for the motion is $m\,dv/dt = -v/30 - mg$. Given the initial condition $v(0) = 20$ m/s , the solution is $v(t) = -44.1 + 64.1\,e^{-t/4.5}$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.683$ s . Integrating $v(t)$, the position is given by $x(t) = 318.45 - 44.1\,t - 288.45\,e^{-t/4.5}$. Hence the maximum height is $x(t_1) = 45.78$ m .

(b) Setting $x(t_2) = 0$, the ball hits the ground at $t_2 = 5.128$ s .

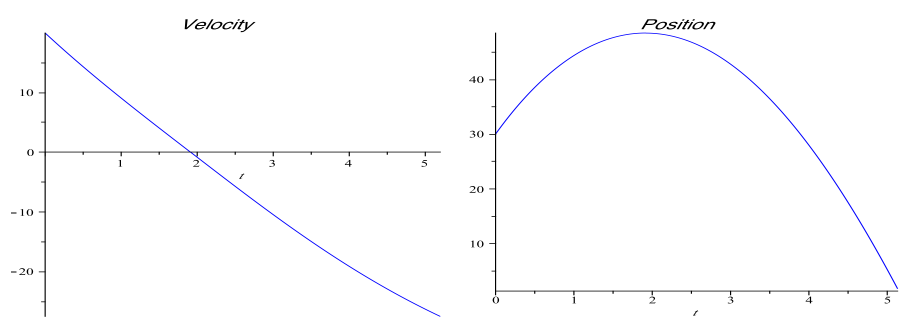
(c)



18.(a) The differential equation for the upward motion is $mdv/dt = -\mu v^2 - mg$, in which $\mu = 1/1325$. This equation is separable, with $m/(\mu v^2 + mg)\,dv = -dt$. Integrating both sides and invoking the initial condition, $v(t) = 44.133\,\tan(0.425 - 0.222\,t)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.916$ s. Integrating $v(t)$, the position is given by $x(t) = 198.75\,\ln[\cos(0.222\,t - 0.425)] + 48.57$. Therefore the maximum height is $x(t_1) = 48.56$ m.

(b) The differential equation for the downward motion is $m\,dv/dt = +\mu v^2 - mg$. This equation is also separable, with $m/(mg - \mu\,v^2)\,dv = -dt$. For convenience, set $t = 0$ at the top of the trajectory. The new initial condition becomes $v(0) = 0$. Integrating both sides and invoking the initial condition, we obtain $\ln((44.13 - v)/(44.13 + v)) = t/2.25$. Solving for the velocity, $v(t) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, we obtain $x(t) = 99.29\ln(e^{t/2.25}/(1 + e^{t/2.25})^2) + 186.2$. To estimate the duration of the downward motion, set $x(t_2) = 0$, resulting in $t_2 = 3.276$ s. Hence the total time that the ball spends in the air is $t_1 + t_2 = 5.192$ s.

(c)



19.(a) Measure the positive direction of motion upward. The equation of motion is given by $mdv/dt = -k\,v - mg$. The initial value problem is $dv/dt = -kv/m - g$, with $v(0) = v_0$. The solution is $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Setting $v(t_m) = 0$, the maximum height is reached at time $t_m = (m/k)\ln[(mg + k\,v_0)/mg]$. Integrating the velocity, the position of the body is

$$x(t) = -mg\,t/k + \left[(\frac{m}{k})^2 g + \frac{m\,v_0}{k}\right](1 - e^{-kt/m}).$$

Hence the maximum height reached is

$$x_m = x(t_m) = \frac{m\,v_0}{k} - g(\frac{m}{k})^2 \ln\left[\frac{mg + k\,v_0}{mg}\right].$$

(b) Recall that for $\delta \ll 1$, $\ln(1+\delta) = \delta - \delta^2/2 + \delta^3/3 - \delta^4/4 + \ldots$.

(c) The dimensions of the quantities involved are $[k] = MT^{-1}$, $[v_0] = LT^{-1}$, $[m] = M$ and $[g] = LT^{-2}$. This implies that kv_0/mg is dimensionless.

SS 20.(a) As in Problem 19, $mdv/dt = -mg - kv$, $v(0) = v_0$.

(b) From part (a), $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. As $k \to 0$, this has the indeterminate form of $-\infty + \infty$. Thus rewrite $v(t)$ as

$$v(t) = (-mg + (v_0 k + mg)e^{-kt/m})/k,$$

which has the indeterminate form 0/0 as $k \to 0$. Using L'Hospital's rule,

$$\lim_{k\to 0} \frac{-mg + (k\,v_0 + mg)e^{-kt/m}}{k} = \lim_{k\to 0}[v_0 e^{-kt/m} - \frac{t}{m}(k\,v_0 + mg)e^{-kt/m}] = v_0 - gt.$$

(c)

$$\lim_{m\,\to 0}\left[-\frac{mg}{k} + (\frac{mg}{k} + v_0)e^{-kt/m}\right] = 0,$$

since $\lim_{m\,\to 0} e^{-kt/m} = 0$.

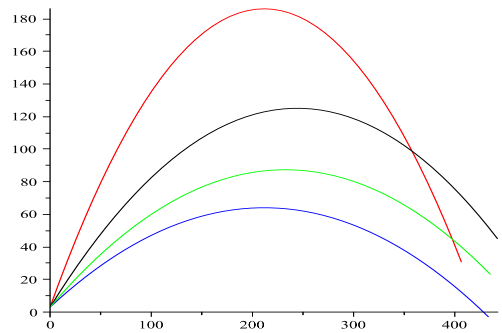
SS 21.(a) The equation of motion is $m(dv/dt) = w - R - B$ which, in this problem, is $(4/3)\pi a^3\rho(dv/dt) = (4/3)\pi a^3\rho g - 6\pi\mu a v - (4/3)\pi a^3\rho' g$. The limiting velocity occurs when $dv/dt = 0$.

(b) Since the droplet is motionless, $v = dv/dt = 0$, we have the equation of motion $0 = (4/3)\pi a^3\rho g - Ee - (4/3)\pi a^3\rho' g$, where ρ is the density of the oil and ρ' is the density of air. Solving for e yields the answer.

SS 22.(a) We obtain these by solving the given differential equations with the initial conditions $v(0) = u\cos A$ and $w(0) = u\sin A$.

(b) From part (a) $dx/dt = v = u\cos A$ and hence $x(t) = tu\cos A + d_1$. Since $x(0) = 0$, we have $d_1 = 0$ and $x(t) = tu\cos A$. Likewise, $dy/dt = w = -gt + u\sin A$ and therefore $y(t) = -gt^2/2 + tu\sin A + d_2$. Since $y(0) = h$ we have $d_2 = h$ and $y(t) = -gt^2/2 + tu\sin A + h$.

(c)



(d) Let t_w be the time the ball reaches the wall. Then $x(t_w)) = L = t_w u \cos A$ and thus $t_w = L/(u \cos A)$. For the ball to clear the wall $y(t_w) \geq H$ and thus (setting $t_w = L/(u \cos A)$, $g = 32$ and $h = 3$ in y) we get $-16L^2/(u^2 \cos^2 A) + L \tan A + 3 \geq H$.

(e) Setting $L = 350$ and $H = 10$ we get $-161.98/\cos^2 A + 350 \tan A \geq 7$ or $7 \cos^2 A - 350 \cos A \sin A + 161.98 \leq 0$. This can be solved numerically or by plotting the left side as a function of A and finding where the zero crossings are.

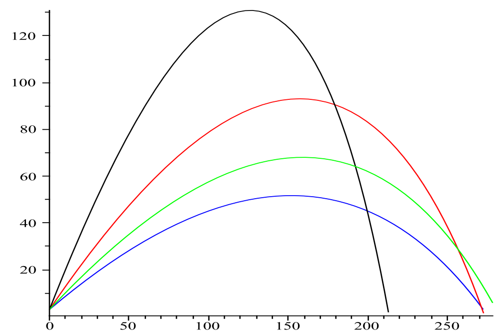
(f) Setting $L = 350$ and $H = 10$ in the answer to part (d) yields the equation $-16(350)^2/(u^2 \cos^2 A) + 350 \tan A = 7$, where we have chosen the equality sign since we want to just clear the wall. Solving for u^2, we obtain that in this case $u^2 = 1,960,000/(175 \sin 2A - 7 \cos^2 A)$. Now u will have a minimum when the denominator has a maximum. Thus $350 \cos 2A + 7 \sin 2A = 0$, or $\tan 2A = -50$, which yields $A = 0.7954$ rad and $u = 106.89$ ft/sec.

23.(a) Both equations are linear and separable. Initial conditions: $v(0) = u \cos A$ and $w(0) = u \sin A$. We obtain the solutions $v(t) = (u \cos A)e^{-rt}$ and $w(t) = -g/r + (u \sin A + g/r)e^{-rt}$.

(b) Integrating the solutions in part (a), and invoking the initial conditions, the coordinates are $x(t) = u \cos A(1 - e^{-rt})/r$ and

$$y(t) = -\frac{gt}{r} + \frac{g + ur \sin A + hr^2}{r^2} - (\frac{u}{r} \sin A + \frac{g}{r^2})e^{-rt}.$$

(c)



(d) Let T be the time that it takes the ball to go 350 ft horizontally. Then from above, $e^{-T/5} = (u \cos A - 70)/u \cos A$. At the same time, the height of the ball is given by

$$y(T) = -160T + 803 + 5u \sin A - \frac{(800 + 5u \sin A)(u \cos A - 70)}{u \cos A}.$$

Hence A and u must satisfy the equality

$$800 \ln \left[\frac{u \cos A - 70}{u \cos A} \right] + 803 + 5u \sin A - \frac{(800 + 5u \sin A)(u \cos A - 70)}{u \cos A} = 10$$

for the ball to touch the top of the wall. To find the optimal values for u and A, consider u as a function of A and use implicit differentiation in the above equation to find that

$$\frac{du}{dA} = -\frac{u(u^2 \cos A - 70u - 11200 \sin A)}{11200 \cos A}.$$

Solving this equation simultaneously with the above equation yields optimal values for u and A: $u \approx 145.3\,\text{ft/s}$, $A \approx 0.644\,\text{rad}$.

24.(a) Solving equation (i), $y'(x) = \left[(k^2 - y)/y\right]^{1/2}$. The positive answer is chosen, since y is an increasing function of x.

(b) Let $y = k^2 \sin^2 t$. Then $dy = 2k^2 \sin t \cos t dt$. Substituting into the equation in part (a), we find that

$$\frac{2k^2 \sin t \cos t dt}{dx} = \frac{\cos t}{\sin t}.$$

Hence $2k^2 \sin^2 t dt = dx$.

(c) Setting $\theta = 2t$, we further obtain $k^2 \sin^2(\theta/2)\, d\theta = dx$. Integrating both sides of the equation and noting that $t = \theta = 0$ corresponds to the origin, we obtain the solutions $x(\theta) = k^2(\theta - \sin \theta)/2$ and (from part (b)) $y(\theta) = k^2(1 - \cos \theta)/2$.

(d) Note that $y/x = (1 - \cos \theta)/(\theta - \sin \theta)$. Setting $x = 1$, $y = 2$, the solution of the equation $(1 - \cos \theta)/(\theta - \sin \theta) = 2$ is $\theta \approx 1.401$. Substitution into either of the expressions yields $k \approx 2.193$.

2.4

SS 1. If the equation is written in the form of Eq.(1), then $p(t) = \ln t/(t-3)$ and $g(t) = 2t/(t-3)$. These are defined and continuous on the intervals $(0,3)$ and $(3,\infty)$, but since the initial point is $t = 1$, the solution will be continuous on $0 < t < 3$.

2. The function $\tan t$ is discontinuous at odd multiples of $\pi/2$. Since $\pi/2 < \pi < 3\pi/2$, the initial value problem has a unique solution on the interval $(\pi/2, 3\pi/2)$.

SS 3. $p(t) = 2t/(4-t^2)$ and $g(t) = 3t^2/(4-t^2)$, which have discontinuities at $t = \pm 2$. Since $t_0 = -3$, the solution will be continuous on $-\infty < t < -2$.

4. The function $\ln t$ is defined and continuous on the interval $(0\,,\infty)$. At $t = 1$, $\ln t = 0$, so the normal form of the differential equation has a singularity there. Also, $\cot t$ is not defined at integer multiples of π, so the initial value problem will have a solution on the interval $(1, \pi)$.

SS 5. Theorem 2.4.2 guarantees a unique solution to the differential equation through any point (t_0, y_0) such that $t_0^2 + y_0^2 < 1$ since $\partial f/\partial y = -y/(1-t^2-y^2)^{1/2}$ is defined and continuous only for $1 - t^2 - y^2 > 0$. Note also that $f = (1-t^2-y^2)^{1/2}$ is defined and continuous in this region as well as on the boundary $t^2 + y^2 = 1$. The boundary can't be included in the final region due to the discontinuity of $\partial f/\partial y$ there.

6. The function $f(t\,,y)$ is discontinuous along the coordinate axes, and on the hyperbola $t^2 - y^2 = 1$. Furthermore,

$$\frac{\partial f}{\partial y} = \frac{\pm 1}{y(1-t^2+y^2)} - 2\frac{y\,\ln|ty|}{(1-t^2+y^2)^2}$$

has the same points of discontinuity.

7. $f(t\,,y)$ is continuous everywhere on the plane. The partial derivative $\partial f/\partial y$ is also continuous everywhere.

SS 8. In this case $f = (1+t^2)/(y(3-y))$, and then $\partial f/\partial y = (1+t^2)/(y(3-y)^2) - (1+t^2)/(y^2(3-y))$, which are both continuous everywhere except for $y = 0$ and $y = 3$.

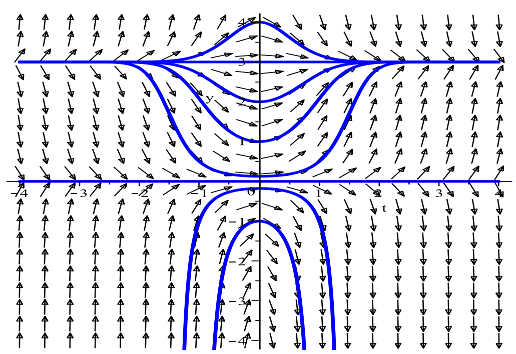
SS 9. The differential equation can be written as $ydy = -4tdt$, so $y^2/2 = -2t^2 + c$, or $y^2 = c - 4t^2$. The initial condition then yields $y_0^2 = c$, so that $y^2 = y_0^2 - 4t^2$ or $y = \pm\sqrt{y_0^2 - 4t^2}$, which is defined for $4t^2 < y_0^2$ or $|t| < |y_0|/2$. Note that $y_0 \neq 0$ since Theorem 2.4.2 does not hold there.

10. The equation is separable, with $dy/y^2 = 2t\,dt$. Integrating both sides, the solution is given by $y(t) = y_0/(1 - y_0 t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \leq 0$, solutions are defined for all t.

11. The equation is separable, with $dy/y^3 = -\,dt$. Integrating both sides and invoking the initial condition, $y(t) = y_0/\sqrt{2y_0^2 t + 1}$. Solutions exist as long as $2y_0^2 t + 1 > 0$, that is, $2y_0^2 t > -1$. If $y_0 \neq 0$, solutions exist for $t > -1/2y_0^2$. If $y_0 = 0$, then the solution $y(t) = 0$ exists for all t.

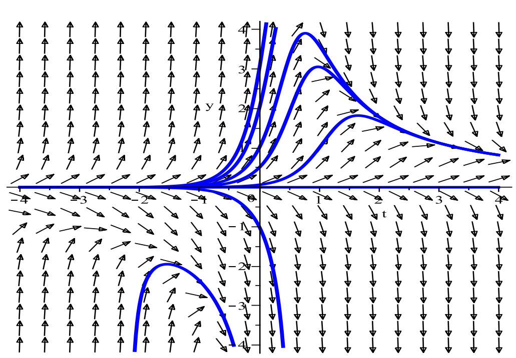
12. The function $f(t\,, y)$ is discontinuous along the straight lines $t = -1$ and $y = 0$. The partial derivative $\partial f/\partial y$ is discontinuous along the same lines. The equation is separable, with $y\,dy = t^2\,dt/(1+t^3)$. Integrating and invoking the initial condition, the solution is $y(t) = \left[(2/3)\ln\left|1+t^3\right| + y_0^2\right]^{1/2}$. Solutions exist as long as $(2/3)\ln\left|1+t^3\right| + y_0^2 \geq 0$, that is, $y_0^2 \geq -(2/3)\ln\left|1+t^3\right|$. For all y_0 (it can be verified that $y_0 = 0$ yields a valid solution, even though Theorem 2.4.2 does not guarantee one), solutions exist as long as $\left|1+t^3\right| \geq e^{-3y_0^2/2}$. From above, we must have $t > -1$. Hence the inequality may be written as $t^3 \geq e^{-3y_0^2/2} - 1$. It follows that the solutions are valid for $(e^{-3y_0^2/2} - 1)^{1/3} < t < \infty$.

SS 13.



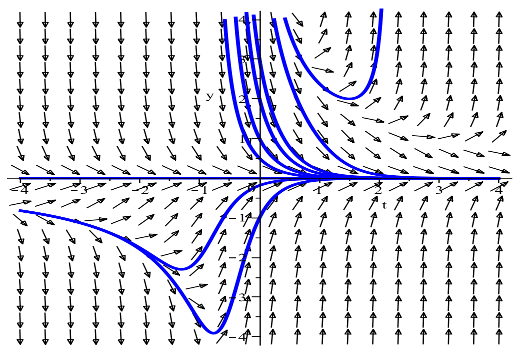
From the direction field (or the given differential equation) it is noted that for $t > 0$ and $y < 0$ that $y' < 0$, so $y \to -\infty$ for $y_0 < 0$. Likewise, for $0 < y_0 < 3$, $y' > 0$ and $y' \to 0$ as $y \to 3$, so $y \to 3$ for $0 < y_0 < 3$ and for $y_0 > 3$, $y' < 0$ and again $y' \to 0$ as $y \to 3$, so $y \to 3$ for $y_0 > 3$. For $y_0 = 3$, $y' = 0$ and $y = 3$ for all t and for $y_0 = 0$, $y' = 0$ and $y = 0$ for all t.

14.



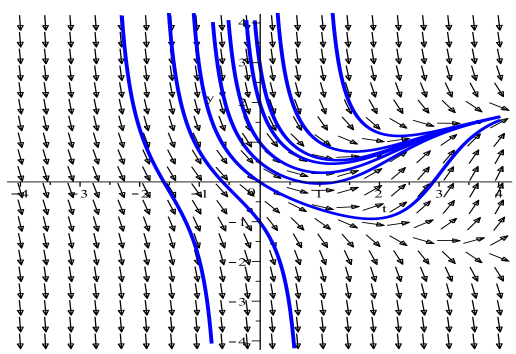
Based on the direction field, and the differential equation, for $y_0 < 0$, the slopes eventually become negative, and hence solutions tend to $-\infty$. For $y_0 > 0$, solutions increase without bound if $t_0 < 0$. Otherwise, the slopes eventually become negative, and solutions tend to zero. Furthermore, $y_0 = 0$ is an equilibrium solution. Note that slopes are zero along the curves $y = 0$ and $ty = 3$.

15.



For initial conditions (t_0, y_0) satisfying $ty < 3$, the respective solutions all tend to zero. For $y_0 \leq 9$, the solutions tend to 0; for $y_0 > 9$, the solutions tend to ∞. Also, $y_0 = 0$ is an equilibrium solution.

16.



Solutions with $t_0 < 0$ all tend to $-\infty$. Solutions with initial conditions (t_0, y_0) to the right of the parabola $t = 1 + y^2$ asymptotically approach the parabola as $t \to \infty$. Integral curves with initial conditions above the parabola (and $y_0 > 0$) also approach the curve. The slopes for solutions with initial conditions below the parabola (and $y_0 < 0$) are all negative. These solutions tend to $-\infty$.

17.(a) No. There is no value of $t_0 \geq 0$ for which $(2/3)(t - t_0)^{2/3}$ satisfies the condition $y(1) = 1$.

(b) Yes. Let $t_0 = 1/2$ in Eq.(19).

(c) For $t_0 > 0$, $|y(2)| \leq (4/3)^{3/2} \approx 1.54$.

SS 18.(a) For $y_1 = 1 - t$, $y_1' = -1$, so substitution into the differential equation gives $-1 = (-t + \sqrt{t^2 + 4(1-t)})/2 = (-t + \sqrt{(t-2)^2})/2 = (-t + |t-2|)/2$. By the definition of the absolute value, the right side is -1 if $t - 2 \geq 0$. Setting $t = 2$ in y_1 we get $y_1(2) = -1$, as required by the initial condition. For $y_2 = -t^2/4$, $y_2' = -t/2$ so substitution into the differential equation yields $-t/2 = (-t + \sqrt{t^2 + 4(-t^2/4)})/2 = -t/2$ which is valid for all t values. Also, $y_2(2) = -1$.

(b) By Theorem 2.4.2 we are guaranteed a unique solution only where $f(t, y) = (-t + \sqrt{t^2 + 4y})/2$ and $f_y(t, y) = 1/\sqrt{t^2 + 4y}$ are continuous. In this case the initial point $(2, -1)$ lies in the region $t^2 + 4y \leq 0$, so $\partial f/\partial y$ is not continuous and hence the theorem is not applicable and there is no contradiction.

(c) For $y = ct + c^2$ follow the steps of part (a). If $y = y_2(t)$ then we must have $ct + c^2 = -t^2/4$ for all t, which is not possible since c is a constant.

SS 19.(a) $\phi(t) = e^{2t}$ gives $\phi'(t) = 2e^{2t}$ so $\phi' - 2\phi = 0$. $\phi(t) = ce^{2t}$ gives $\phi'(t) = 2ce^{2t}$, so $\phi' - 2\phi = 0$.

(b) $\phi(t) = t^{-1}$ gives $\phi'(t) = -t^{-2}$ so $\phi' + \phi^2 = 0$. $\phi(t) = ct^{-1}$ gives $\phi'(t) = -ct^{-2}$, so $\phi' + \phi^2 \neq 0$ unless $c = 0$ or $c = 1$.

20. The assumption is $\phi'(t) + p(t)\phi(t) = 0$. But then $c\phi'(t) + p(t)c\phi(t) = 0$ as well.

SS 21. $(y_1(t)+y_2(t))' + p(t)(y_1(t) + y_2(t)) = y_1'(t) + p(t)y_1(t) + y_2'(t) + p(t)y_2(t) = 0 + g(t)$.

22.(a) Recalling Eq.(33) in Section 2.1,

$$y = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)\, ds + \frac{c}{\mu(t)}.$$

It is evident that $y_1(t) = 1/\mu(t)$ and $y_2(t) = (1/\mu(t)) \int_{t_0}^{t} \mu(s)g(s)\, ds$.

(b) By definition, $1/\mu(t) = e^{-\int p(t)dt}$. Hence $y_1' = -p(t)/\mu(t) = -p(t)y_1$. That is, $y_1' + p(t)y_1 = 0$.

(c) $y_2' = (-p(t)/\mu(t)) \int_0^t \mu(s)g(s)\, ds + \mu(t)g(t)/\mu(t) = -p(t)y_2 + g(t)$. This implies that $y_2' + p(t)y_2 = g(t)$.

SS 23.(a) For $n = 1$, we have $y' + (p(t) - q(t))y = 0$, which is linear. Thus Eq.(3) gives $y(t) = c\mu^{-1}(t) = ce^{-\int (p(t)-q(t))\, dt}$, since $g(t) = 0$.

(b) Let $v = y^{1-n}$, then $dv/dt = (1-n)y^{-n}dy/dt$, so $dy/dt = (1/(1-n))y^n dv/dt$, for $n \neq 1$. Substituting into the differential equation yields $(1/(1-n))y^n dv/dt + p(t)y = q(t)y^n$, or $v' + (1-n)p(t)y^{1-n} = (1-n)q(t)$, which is $v' + (1-n)p(t)v = (1-n)q(t)$, which is a linear differential equation for v.

SS 24. Here $n = 2$, so $v = y^{-1}$ and $dv/dt = -y^{-2}dy/dt$. Thus the differential equation becomes $-y^{-2}dv/dt - ry = -ky^2$ or $dv/dt + rv = k$. Hence $\mu(t) = e^{rt}$ and $v = (k/r) + ce^{-rt}$. Setting $v = 1/y$ then yields the solution.

25. Since $n = 3$, set $v = y^{-2}$. It follows that $v' = -2y^{-3}y'$ and $y' = -(y^3/2)v'$. Substitution into the differential equation yields $-(y^3/2)v' - \varepsilon y = -\sigma y^3$, which further results in $v' + 2\varepsilon v = 2\sigma$. The latter differential equation is linear, and can be written as $(ve^{2\varepsilon t})' = 2\sigma e^{2\varepsilon t}$. The solution is given by $v(t) = \sigma/\varepsilon + ce^{-2\varepsilon t}$. Converting back to the original dependent variable, $y = \pm v^{-1/2} = \pm(\sigma/\varepsilon + ce^{-2\varepsilon t})^{-1/2}$.

SS 26. Since $g(t)$ is continuous on the interval $0 \le t \le 1$ and hence we solve the initial value problem $y_1' + 2y_1 = 1$, $y_1(0) = 0$ on that interval to obtain $y_1 = 1/2 - (1/2)e^{-2t}$, $0 \le t \le 1$. For $1 < t$, $g(t) = 0$; and hence we solve $y_2' + 2y_2 = 0$ to obtain $y_2 = ce^{-2t}$, $1 < t$. The solution y of the original initial value problem must be continuous at $t = 1$ (since its derivative must exist) and hence we need c in y_2 so y_2 at 1 has the same value as y_1 at 1. Thus $ce^{-2} = 1/2 - e^{-2}/2$ or $c = (1/2)(e^2 - 1)$ and we obtain

$$y(t) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2t}, & 0 \le t \le 1 \\ \frac{1}{2}(e^2 - 1)e^{-2t}, & t > 1 \end{cases}.$$

and

$$y'(t) = \begin{cases} e^{-2t}, & 0 < t < 1 \\ (1 - e^2)e^{-2t}, & t > 1 \end{cases}.$$

Evaluating the two parts of y' at $t_0 = 1$ we see that they are different, and hence y' is not continuous at $t_0 = 1$.

27. The solution of the initial value problem $y_1' + 2y_1 = 0$, $y_1(0) = 1$ is $y_1(t) = e^{-2t}$. Therefore $y(1^-) = y_1(1) = e^{-2}$. On the interval $(1, \infty)$, the differential equation is $y_2' + y_2 = 0$, with $y_2(t) = ce^{-t}$. Therefore $y(1^+) = y_2(1) = ce^{-1}$. Equating the limits $y(1^-) = y(1^+)$, we require that $c = e^{-1}$. Hence the global solution of the initial value problem is

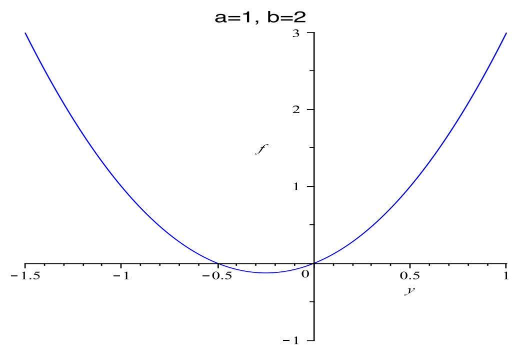
$$y(t) = \begin{cases} e^{-2t}, & 0 \le t \le 1 \\ e^{-1-t}, & t > 1 \end{cases}.$$

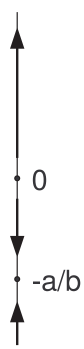
Note the discontinuity of the derivative

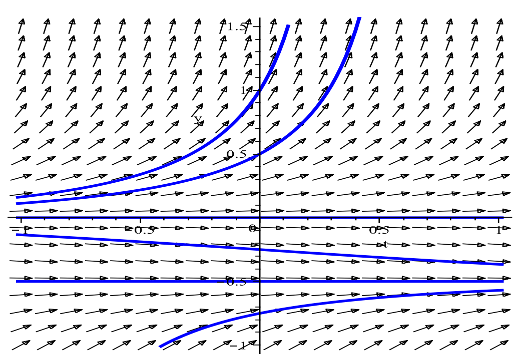
$$y'(t) = \begin{cases} -2e^{-2t}, & 0 < t < 1 \\ -e^{-1-t}, & t > 1 \end{cases}.$$

2.5

1.

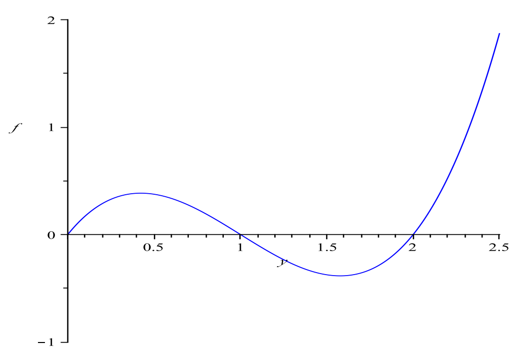


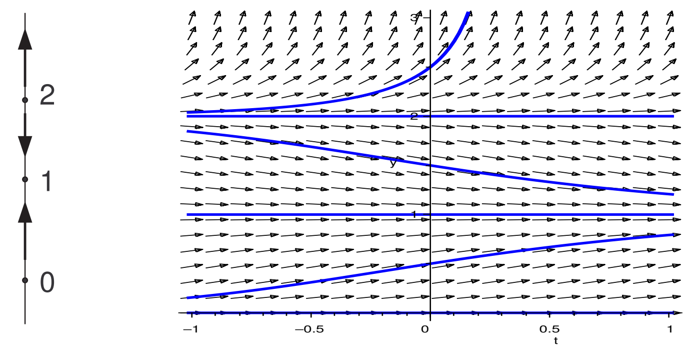




The equilibrium points are $y^* = -a/b$ and $y^* = 0$, and $y' > 0$ when $y > 0$ or $y < -a/b$, and $y' < 0$ when $-a/b < y < 0$, therefore the equilibrium solution $y = -a/b$ is asymptotically stable and the equilibrium solution $y = 0$ is unstable.

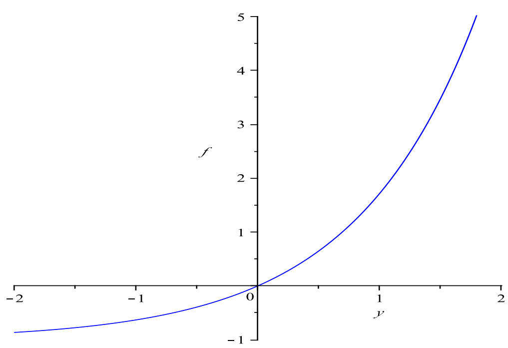
SS 2.

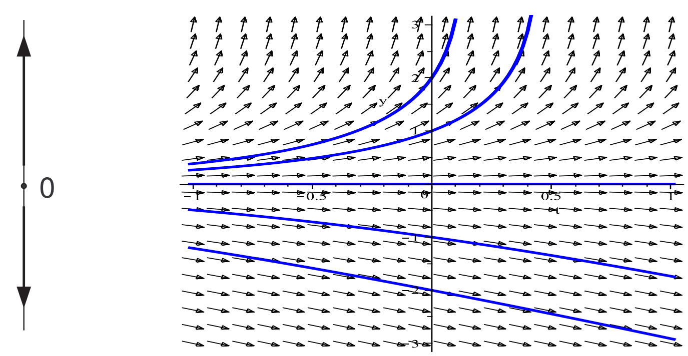




From the graph, $y' < 0$ when $1 < y < 2$ and $y' > 0$ when $0 < y < 1$ or $y > 2$, so the equilibrium solutions $y = 0$ and $y = 2$ are unstable, the equilibrium solution $y = 1$ is asymptotically stable.

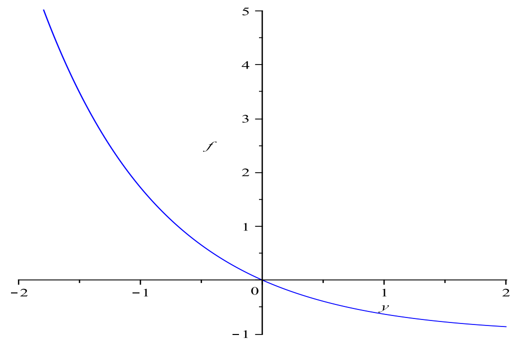
3.

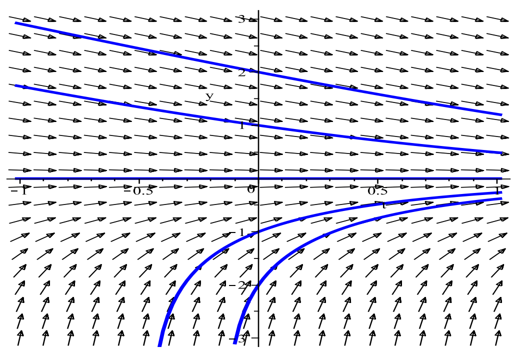




The only equilibrium point is $y^* = 0$, and $y' > 0$ when $y > 0$, $y' < 0$ when $y < 0$, hence the equilibrium solution $y = 0$ is unstable.

SS 4.

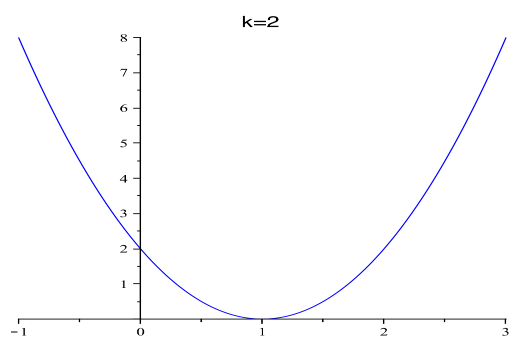




The only equilibrium point is $y^* = 0$, and $y' > 0$ when $y < 0$, $y' < 0$ when $y > 0$, hence the equilibrium solution $y = 0$ is asymptotically stable.

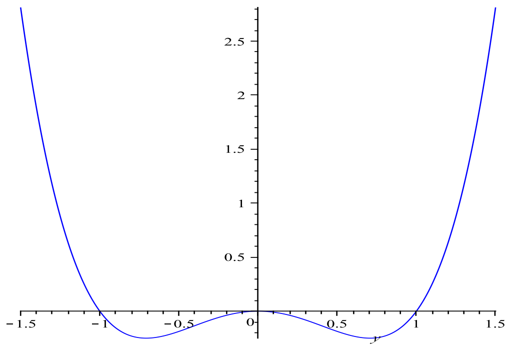
SS 5.(a) $f(y) = 0$ only when $y = 1$. Therefore, $y^* = 1$ is the only critical point.

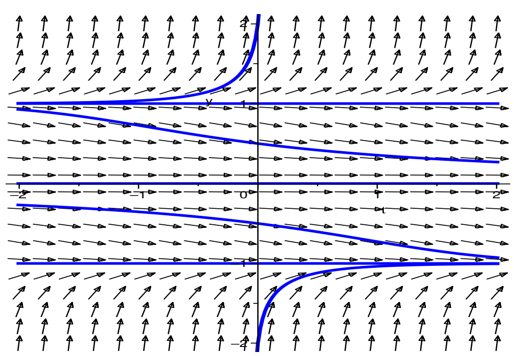
(b)



(c) Separate variables to get $dy/(1-y)^2 = kdt$. Integration yields $1/(1-y) = kt + c$, or $y = (kt + c - 1)/(kt + c)$. Setting $t = 0$ and $y(0) = y_0$ yields $y_0 = (c-1)/c$ or $c = 1/(1-y_0)$. Hence $y(t) = [y_0 + (1-y_0)kt]/[1 + (1-y_0)kt]$. If $y_0 < 1$, then $y \to 1$ as $t \to \infty$. If $y_0 > 1$, then the denominator will go to zero at some finite time $T = 1/(y_0 - 1)$. Therefore, the solution will go towards infinity at that time T.

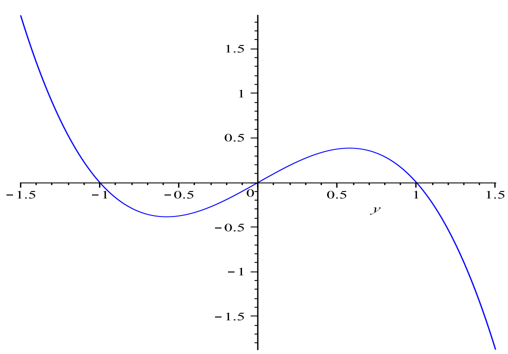
SS 6.

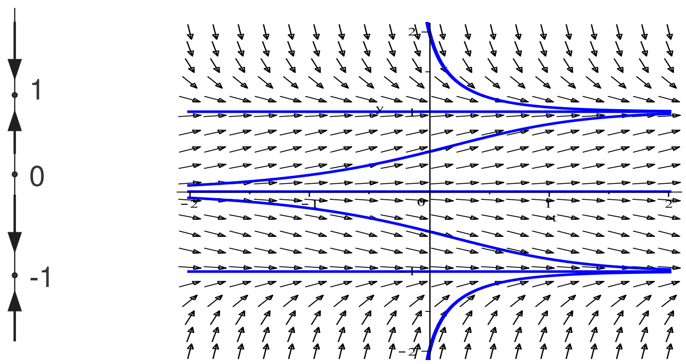




The critical points are $y = 0, \pm 1$. We have $y' > 0$ for $|y| > 1$ while $y' < 0$ for $|y| < 1$. Thus the equilibrium solution $y = -1$ is asymptotically stable, $y = 0$ is semistable and $y = 1$ is unstable.

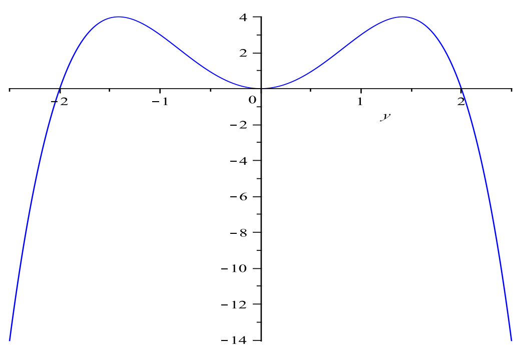
7.

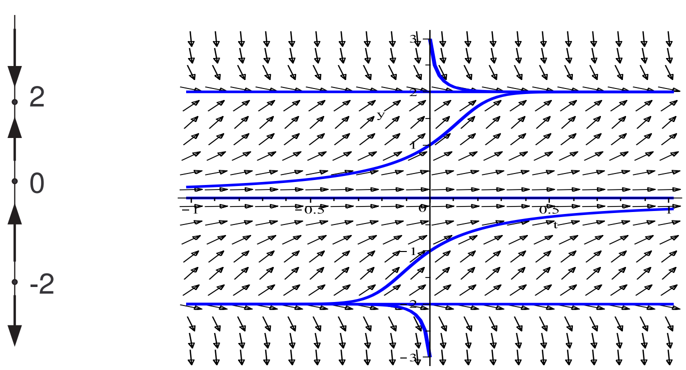




The equilibrium points are $y^* = 0, \pm 1$, and $y' > 0$ for $y < -1$ or $0 < y < 1$ and $y' < 0$ for $-1 < y < 0$ or $y > 1$. The equilibrium solution $y = 0$ is unstable, and the remaining two are asymptotically stable.

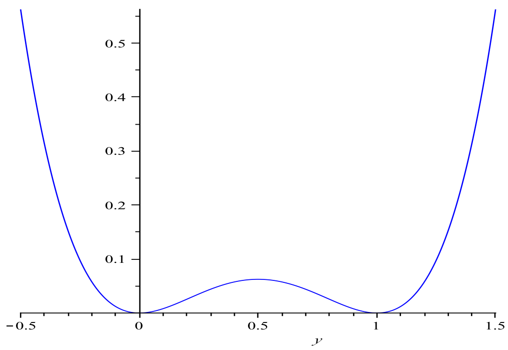
8.

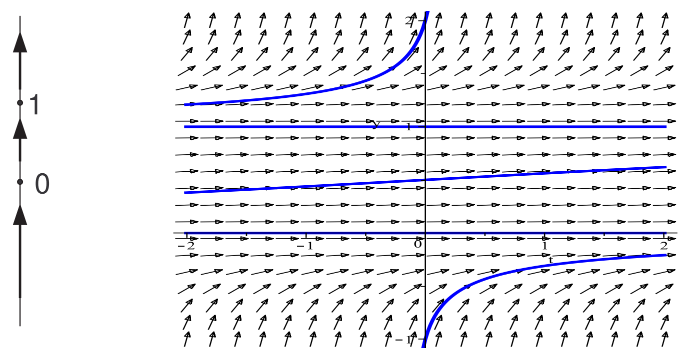




The equilibrium points are $y^* = 0, \pm 2$, and $y' < 0$ when $y < -2$ or $y > 2$, and $y' > 0$ for $-2 < y < 0$ or $0 < y < 2$. The equilibrium solutions $y = -2$ and $y = 2$ are unstable and asymptotically stable, respectively. The equilibrium solution $y = 0$ is semistable.

9.





The equilibrium points are $y^* = 0, 1$. $y' > 0$ for all y except $y = 0$ and $y = 1$. Both equilibrium solutions are semistable.

SS 10. Eq.(10) is $y/(1-(y/K)) = Ce^{rt}$. Clearing the denominator gives $y = Ce^{rt}(1 - (y/K)) = Ce^{rt} - Ce^{rt}(y/K)$. Thus $y + Ce^{rt}(y/K) = Ce^{rt}$, or $(1 + (C/K)e^{rt})y = Ce^{rt}$. This last equation may easily be solved for y to give $y = Ce^{rt}/(1 + (C/K)e^{rt})$. Applying the initial condition $y(0) = y_0$ gives $y_0 = C/(1 + (C/K))$, which may be solved for C to give $C = y_0/(1 - (y_0/K)) = Ky_0/(K - y_0)$. Using this last value of C in the solution for y gives $y(x) = ((Ky_0e^{rt}/(K - y_0))/(1 + (y_0e^{rt}/(K - y_0))$, which may be simplified to yield Eq.(11).

SS 11. To solve Eq.(12) for t, multiply each side of the equation by $(y_0/K) + [1 - (y_0/K)]e^{-rt}$ to obtain $y((y_0/K) + [1 - (y_0/K)]e^{-rt}) = y_0$, or $y_0/y = (y_0/K) + [1 - (y_0/K)]e^{-rt}$. Multiplying each side of this equation by K gives $(y_0K)/y = y_0 + (K - y_0)e^{-rt}$, which may be solved for e^{-rt} to find that

$$e^{-rt} = \frac{(y_0K)/y - y_0}{K - y_0} = \frac{(y_0/y) - (y_0/K)}{1 - (y_0/K)} = \frac{(y_0/K)[1 - (y/K)]}{(y/K)[1 - (y_0/K)]}$$

as in the text. Taking logarithms and dividing by $-r$ gives

$$t = -\frac{1}{r}\ln\frac{(y_0/K)[1 - (y/K)]}{(y/K)[1 - (y_0/K)]}$$

as given in Eq.(13).

SS 12. To locate the time at which the solution given in Eq.(15) reaches its vertical asymptote, determine when the denominator of the solution is zero. Solving $y_0 + (T - y_0)e^{rt} = 0$ for e^{rt} gives $e^{rt} = y_0/(y_0 - T)$, where the fact that $y_0 > T$ ensures that $y_0/(y_0 - T) > 0$. Thus the equation $e^{rt} = y_0/(y_0 - T)$ has a solution, which is $t = (1/r)\ln(y_0/(y_0 - T))$ as given in Eq.(16).

SS 13. To find the inflection points of a solution y to the differential equation $y' = f(y)$, first compute $y'' = f'(y)(dy/dt) = f'(y)f(y)$. Thus possible inflection points occur at solutions to $f'y) = 0$ or $f(y) = 0$. The function $f(y) = -r(1 - (y/T))(1 - (y/K))y = 0$ at $y = 0$, $y = T$, and $y = K$, but Figure 2.5.7 shows that y'' does not change sign at any of these points, so they are not inflection points. To consider the points at which $f'(y) = 0$, note that $f(y) = (-r/(KT))(y - T)(y - K)y$, so $f'(y) = (-r/(KT))((y - K)y + (y - T)y + (y - T)(y - K)) = (-r/KT)(3y^2 - 2(K + T) + KT)$. Setting $f'(y) = 0$ and using the quadratic formula gives that $f'(y) = 0$ when $y = (K + T \pm \sqrt{K^2 - KT + T^2})/3$ as given in Eq.(18).

SS 14. If $f'(y_1) < 0$ then the slope of f is negative at y_1 and thus $f(y) > 0$ for $y < y_1$ and $f(y) < 0$ for $y > y_1$ since $f(y_1) = 0$. Hence y_1 is an asymptotically stable critical point. A similar argument will yield the result for $f'(y_1) > 0$.

SS 17.(a) Consider the change of variable $u = \ln(y/K)$. Differentiating both sides with respect to t, $u' = y'/y$. Substitution into the Gompertz equation yields $u' = -ru$, with solution $u = u_0 e^{-rt}$. It follows that $\ln(y/K) = \ln(y_0/K)e^{-rt}$. This implies that $y/K = e^{\ln(y_0/K)e^{-rt}}$.

(b) Given $K = 80.5 \times 10^6$, $y_0/K = 0.25$ and $r = 0.71$ per year, $y(2) = 57.58 \times 10^6$.

(c) Solving for t,

$$t = -\frac{1}{r}\ln\left[\frac{\ln(y/K)}{\ln(y_0/K)}\right].$$

Setting $y(\tau) = 0.75K$, the corresponding time is $\tau \approx 2.21$ years.

SS 18.(a) The differential equation is $dV/dt = k - \alpha\pi r^2$. The volume of a cone of height L and radius r is given by $V = \pi r^2 L/3$ where $L = hr/a$ from symmetry. Solving for r yields the desired equation $dV/dt = k - \alpha\pi(3a/\pi h)^{2/3}V^{2/3}$.

(b) The equilibrium is given by the equation $k = \alpha\pi r^2$, which yields $r = \sqrt{k/\alpha\pi}$ and then $L = h\sqrt{k/\alpha\pi}/a$. By checking the graph of V' we obtain that this is an asymptotically stable equilibrium point.

(c) The equilibrium height must be less than h, or $\sqrt{k/\alpha\pi}/a < 1$.

SS 19.(a) If $E < r$, then the equilibrium points are given by $0 = r(1 - y/K)y - Ey = y(r - ry/K - E)$, which means that either $y = 0$ or $y = (r - E)K/r = (1 - E/r)K > 0$.

(b) $f'(y) = r - E - 2ry/K$, so $f'(0) = r - E >$ and 0 is an unstable equilibrium, while $f'((1 - E/r)K) = E - r < 0$ and $(1 - E/r)K$ is an asymptotically stable equilibrium.

(c) $Y = E(1 - E/r)K$.

(d) We have to solve $0 = dY/dE = K - 2EK/r$ to get $E = r/2$, and then $Y_m = rK/4$.

SS 20.(a) Setting $dy/dt = 0$ the quadratic formula yields the roots

$$y_{1,2} = \frac{r \pm \sqrt{r^2 - 4rh/K}}{2r/K} = \frac{K}{2}\left(1 \pm \sqrt{1 - \frac{4h}{rK}}\right),$$

which are real when $h < rK/4$. ($y_1 < y_2$ because of the minus sign in front of the square root.)

(b) The graph of the right side of the differential equation is a downward opening parabola, which implies that y_1 is unstable and y_2 is asymptotically stable. We can also use the derivative test of Problem 14.

(c) The graph of $f(y)$ is a downward opening parabola intersecting the horizontal axis at y_1 and y_2, so we know that between y_1 and y_2 the value of $y' = f(y)$ is positive, which implies that if $y_1 < y_0 < y_2$, then the solution is increasing towards y_2, and when $y_2 < y_0$, the solution is decreasing towards y_2 (because $y' = f(y)$ is negative there). Also, when $y_0 < y_1$, then $y' < 0$, so the solution will decrease and reach 0 in finite time.

(d) If $h > rK/4$ there are no critical points (see part (a)) and $dy/dt < 0$ for all t.

(e) We can see from part (a) that when $h = rK/4$, then $y_1 = y_2$. The graph of $f(y)$ intersects the horizontal axis at a single point of tangency in this case, and $y' = f(y)$ is negative for any other y value, giving the semistability result.

21.(a) The equilibrium points are at $y^* = 0$ and $y^* = 1$. Since $f'(y) = \alpha - 2\alpha y$, the equilibrium solution $y = 0$ is unstable and the equilibrium solution $y = 1$ is asymptotically stable.

(b) The differential equation is separable, with $[y(1-y)]^{-1}\,dy = \alpha\,dt$. Integrating both sides and invoking the initial condition, the solution is

$$y(t) = \frac{y_0\,e^{\alpha t}}{1 - y_0 + y_0\,e^{\alpha t}} = \frac{y_0}{y_0 + (1 - y_0)e^{-\alpha t}}.$$

It is evident that (independent of y_0) $\lim_{t \to -\infty} y(t) = 0$ and $\lim_{t \to \infty} y(t) = 1$.

22.(a) $y(t) = y_0\, e^{-\beta t}$.

(b) From part (a), $dx/dt = -\alpha x y_0 e^{-\beta t}$. Separating variables, $dx/x = -\alpha y_0 e^{-\beta t} dt$. Integrating both sides, the solution is $x(t) = x_0\, e^{-\alpha\, y_0 (1-e^{-\beta t})/\beta}$.

(c) As $t \to \infty$, $y(t) \to 0$ and $x(t) \to x_0\, e^{-\alpha\, y_0/\beta}$. Over a long period of time, the proportion of carriers vanishes. Therefore the proportion of the population that escapes the epidemic is the proportion of susceptibles left at that time, $x_0\, e^{-\alpha\, y_0/\beta}$.

SS 23.(a) Differentiating with respect to t, we obtain that the derivative is $z' = (nx' - xn')/n^2 = (-\beta nx - \mu nx + \nu\beta x^2 + \mu nx)/n^2 = -\beta x/n + \nu\beta x^2/n^2 = -\beta z + \nu\beta z^2 = -\beta z(1 - \nu z)$.

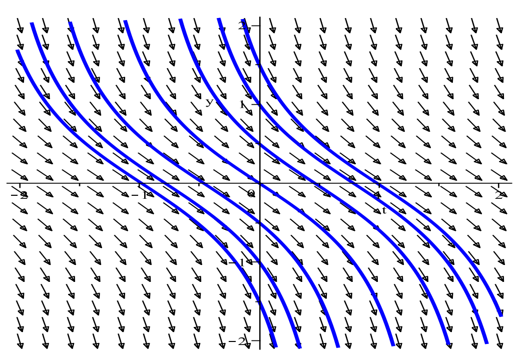
(b) First, we rewrite the equation as $z' + \beta z = \beta\nu z^2$. This is a Bernoulli equation with $n = 2$. Let $w = z^{1-n} = z^{-1}$. Then, our equation can be written as $w' - \beta w = -\beta\nu$. This is a linear equation with solution $w = \nu + ce^{\beta t}$. Then, using the fact that $z = 1/w$, we see that $z = 1/(\nu + ce^{\beta t})$. Finally, the initial condition $z(0) = 1$ implies $c = 1 - \nu$. Therefore, $z(t) = 1/(\nu + (1 - \nu)e^{\beta t})$.

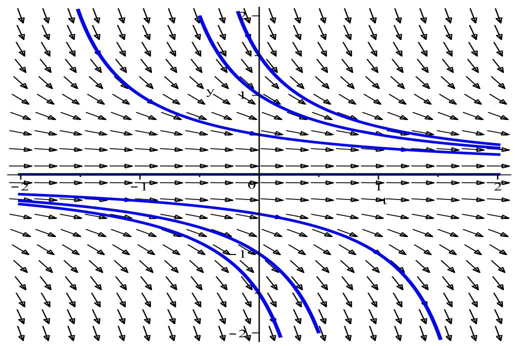
(c) Evaluating $z(20)$ for $\nu = \beta = 1/8$, we conclude that $z(20) = 0.0927$.

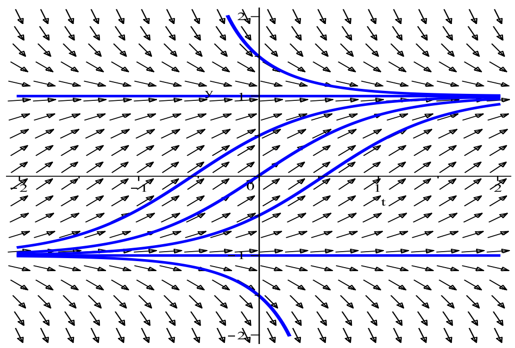
SS 24.(a) The critical points occur when $a - y^2 = 0$. If $a < 0$, there are no critical points. If $a = 0$, then $y^* = 0$ is the only critical point. If $a > 0$, then $y^* = \pm\sqrt{a}$ are the two critical points.

(b) We note that $f'(y) = -2y$. Therefore, $f'(\sqrt{a}) < 0$ which implies that $\sqrt{a}$ is asymptotically stable; $f'(-\sqrt{a}) > 0$ which implies $-\sqrt{a}$ is unstable; the behavior of f' around $y^* = 0$ implies that $y^* = 0$ is semistable.

(c) Below, we graph solutions in the case $a = -1$, $a = 0$ and $a = 1$ respectively.

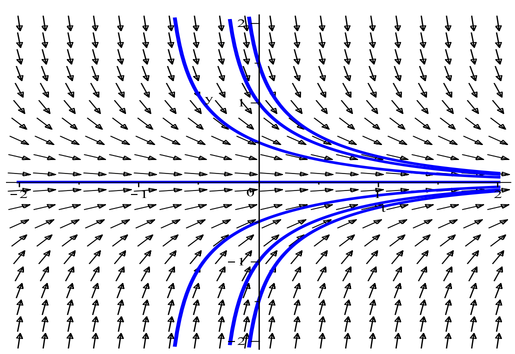


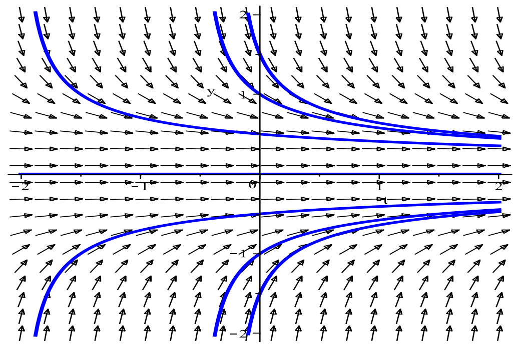


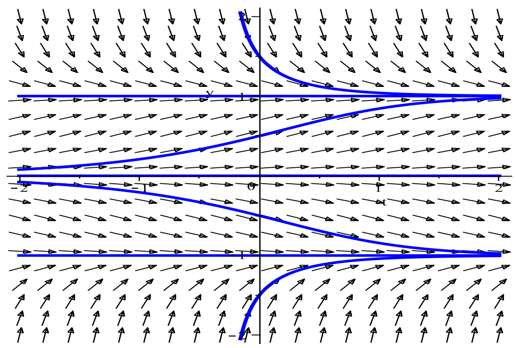


25.(a) For $a < 0$, the only critical point is at $y = 0$, which is asymptotically stable. For $a = 0$, the only critical point is at $y = 0$, which is asymptotically stable. For $a > 0$, the three critical points are at $y = 0$, $\pm\sqrt{a}$. The critical point at $y = 0$ is unstable, whereas the other two are asymptotically stable.

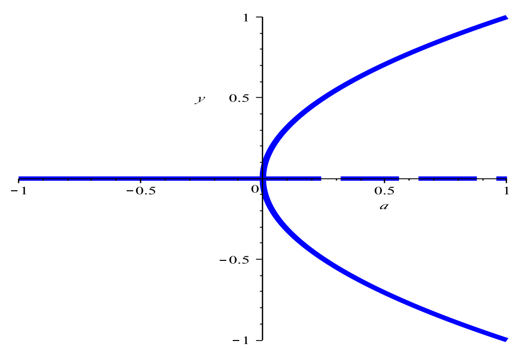
(b) Below, we graph solutions in the case $a = -1$, $a = 0$ and $a = 1$ respectively.





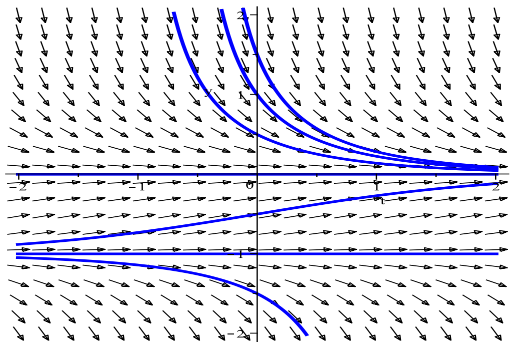


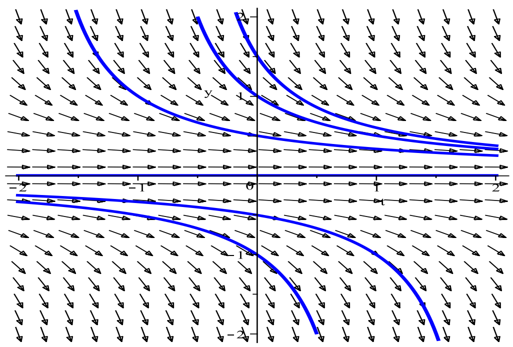
(c)

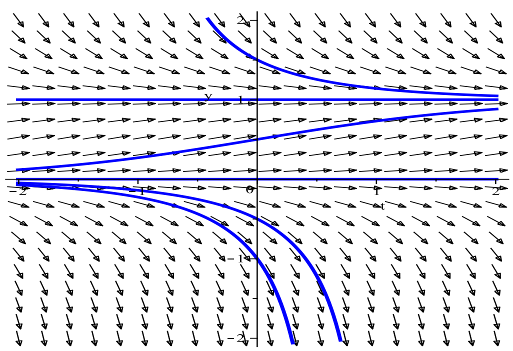


26.(a) $f(y) = y(a-y)$; $f'(y) = a - 2y$. For $a < 0$, the critical points are at $y = a$ and $y = 0$. Observe that $f'(a) > 0$ and $f'(0) < 0$. Hence $y = a$ is unstable and $y = 0$ asymptotically stable. For $a = 0$, the only critical point is at $y = 0$, which is semistable since $f(y) = -y^2$ is concave down. For $a > 0$, the critical points are at $y = 0$ and $y = a$. Observe that $f'(0) > 0$ and $f'(a) < 0$. Hence $y = 0$ is unstable and $y = a$ asymptotically stable.

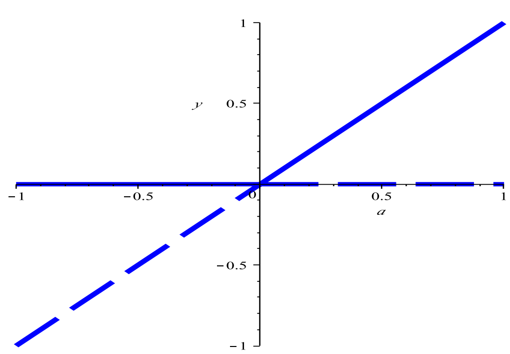
(b) Below, we graph solutions in the case $a = -1$, $a = 0$ and $a = 1$ respectively.







(c)



SS 27. (a) Since the critical points are $x^* = p, q$, we will look at their stability. Since $f'(x) = -\alpha q - \alpha p + 2\alpha x^2$, we see that $f'(p) = \alpha(2p^2 - q - p)$ and $f'(q) = \alpha(2q^2 - q - p)$. Now if $p > q$, then $-p < -q$, and, therefore, $f'(q) = \alpha(2q^2 - q - p) < \alpha(2q^2 - 2q) = 2\alpha q(q-1) < 0$ since $0 < q < 1$. Therefore, if $p > q$, $f'(q) < 0$, and, therefore, $x^* = q$ is asymptotically stable. Similarly, if $p < q$, then $x^* = p$ is asymptotically stable, and therefore, we can conclude that $x(t) \to \min\{p, q\}$ as $t \to \infty$.

The equation is separable. It can be solved by using partial fractions as follows. We can rewrite the equation as

$$\left(\frac{1/(q-p)}{p-x} + \frac{1/(p-q)}{q-x}\right) dx = \alpha dt,$$

which implies

$$\ln\left|\frac{p-x}{q-x}\right| = (p-q)\alpha t + C.$$

The initial condition $x_0 = 0$ implies $C = \ln|p/q|$, and, therefore,

$$\ln\left|\frac{q(p-x)}{p(q-x)}\right| = (p-q)\alpha t.$$

Applying the exponential function and simplifying, we conclude that

$$x(t) = \frac{pq(e^{(p-q)\alpha t} - 1)}{pe^{(p-q)\alpha t} - q} = \frac{pq(e^{\alpha(q-p)t} - 1)}{qe^{\alpha(q-p)t} - p}.$$

(b) In this case, $x^* = p$ is the only critical point. Since $f(x) = \alpha(p-x)^2$ is concave up, we conclude that $x^* = p$ is semistable. Further, if $x_0 = 0$, we can conclude that $x \to p$ as $t \to \infty$. This equation is separable. Its solution is given by $x(t) = p^2\alpha t/(p\alpha t + 1)$.

2.6

1. $M(x,y) = 2x + 3$ and $N(x,y) = 2y - 2$. Since $M_y = N_x = 0$, the equation is exact. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) = x^2 + 3x + h(y)$. Now $\psi_y = h'(y)$, and equating with N results in the possible function $h(y) = y^2 - 2y$. Hence $\psi(x,y) = x^2 + 3x + y^2 - 2y$, and the solution is defined implicitly as $x^2 + 3x + y^2 - 2y = c$.

2. $M(x,y) = 2x + 4y$ and $N(x,y) = 2x - 2y$. Note that $M_y \neq N_x$, and hence the differential equation is not exact.

SS 3. Here $M(x,y) = 3x^2 - 2xy + 2$ and $N(x,y) = 6y^2 - x^2 + 3$. Since $M_y = -2x = N_x$, the equation is exact. Since $\psi_x = M = 3x^2 - 2xy + 2$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = x^3 - x^2 y + 2x + h(y)$. Then $\psi_y = -x^2 + h'(y) = N = 6y^2 - x^2 + 3$ implies $h'(y) = 6y^2 + 3$. Therefore, $h(y) = 2y^3 + 3y$ and $\psi(x,y) = x^3 - x^2 y + 2x + 2y^3 + 3y = c$.

SS 4. Here $M(x,y) = ax + by$ and $N(x,y) = bx + cy$. Since $M_y = b = N_x$, the equation is exact. Since $\psi_x = M = ax + by$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = ax^2/2 + bxy + h(y)$. Then $\psi_y = bx + h'(y) = N = bx + cy$ implies $h'(y) = cy$. Therefore, $h(y) = cy^2/2$ and $\psi(x,y) = ax^2 + 2bxy + cy^2 = k$.

5. Write the equation as $(ax - by)dx + (bx - cy)dy = 0$. Now $M(x,y) = ax - by$ and $N(x,y) = bx - cy$. Since $M_y \neq N_x$, the differential equation is not exact.

SS 6. Here $M(x,y) = ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x$ and $N(x,y) = xe^{xy}\cos(2x) - 3$. Since $M_y = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2xe^{xy}\sin(2x) = N_x$, the equation is exact. If we try to find $\psi(x,y)$ by integrating $M(x,y)$ with respect to x we must integrate by parts. Instead we find $\psi(x,y)$ by integrating $N(x,y)$ with respect to y to obtain $\psi(x,y) = e^{xy}\cos(2x) - 3y + g(x)$. Then we find $g(x)$ by differentiating $\psi(x,y)$ with respect to x and setting it equal to $M(x,y)$, resulting in $g'(x) = 2x$ or $g(x) = x^2$. As before, the implicit solution is $\psi(x,y) = e^{xy}\cos(2x) + x^2 - 3y = c$.

7. $M(x,y) = y/x + 6x$ and $N(x,y) = \ln x - 2$. Since $M_y = N_x = 1/x$, the given equation is exact. Integrating N with respect to y , while holding x constant, results in $\psi(x,y) = y \ln x - 2y + h(x)$. Differentiating with respect to x, $\psi_x = y/x + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = 6x$, and hence $h(x) = 3x^2$. Therefore the solution is defined implicitly as $3x^2 + y \ln x - 2y = c$.

SS 8. Here $M(x,y) = x/(x^2+y^2)^{3/2}$ and $N(x,y) = y/(x^2+y^2)^{3/2}$. Since $M_y = N_x$, the equation is exact. Since $\psi_x = M = x/(x^2+y^2)^{3/2}$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = -1/(x^2+y^2)^{1/2} + h(y)$. Then $\psi_y = y/(x^2+y^2)^{3/2} + h'(y) = N = y/(x^2+y^2)^{3/2}$ implies $h'(y)=0$. Therefore, $h(y)=0$ and $\psi(x,y) = -1/(x^2+y^2)^{1/2} = c$ or $x^2 + y^2 = k$. We can observe that as long as $x^2 + y^2 \neq 0$, we can simplify the equation by multiplying both sides by $(x^2+y^2)^{3/2}$. This gives the (simpler) exact equation $xdx + ydy = 0$, whose solution is the same as the above.

9. $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. Since $M_y = N_x = -1$, the equation is exact. Integrating M with respect to x , while holding y constant, yields $\psi(x,y) = x^2 - xy + h(y)$. Now $\psi_y = -x + h'(y)$. Equating ψ_y with N results in $h'(y) = 2y$, and hence $h(y) = y^2$. Thus $\psi(x,y) = x^2 - xy + y^2$, and the solution is given implicitly as $x^2 - xy + y^2 = c$. Invoking the initial condition $y(1) = 3$, the specific solution is $x^2 - xy + y^2 = 7$. The explicit form of the solution is $y(x) = (x + \sqrt{28 - 3x^2})/2$. Hence the solution is valid as long as $3x^2 \leq 28$.

SS 10. Here $M(x,y) = 9x^2 + y - 1$ and $N(x,y) = -4y + x$. Therefore, $M_y = N_x = 1$ which implies that the equation is exact. Integrating M with respect to x, we conclude that $\psi = 3x^3 + xy - x + h(y)$. Then $\psi_y = x + h'(y) = N = -4y + x$ implies $h'(y) = -4y$. Therefore, $h(y) = -2y^2$ and we get $\psi = 3x^3 + xy - x - 2y^2 = c$. The initial condition $y(1) = 0$ implies $c = 2$. Therefore, $3x^3 + xy - x - 2y^2 = 2$. Solving for y using the quadratic formula, we get $y = [x - (24x^3 + x^2 - 8x - 16)^{1/2}]/4$. Using a numerical process the square root term is positive for $x > 0.9846$.

SS 11. Here $M(x,y) = xy^2 + bx^2y$ and $N(x,y) = x^3 + x^2y$. Therefore, $M_y = 2xy + bx^2$ and $N_x = 3x^2 + 2xy$. In order for the equation to be exact, we need $b = 3$. Taking this value for b, we integrating M with respect to x. We conclude that $\psi = x^2y^2/2 + x^3y + h(y)$. Then $\psi_y = x^2y + x^3 + h'(y) = N = x^3 + x^2y$ implies $h'(y) = 0$. Therefore, $h(y) = c$ and $\psi(x,y) = x^2y^2/2 + x^3y = c$. That is, the solution is given implicitly as $x^2y^2 + 2x^3y = k$.

12. $M(x,y) = y\,e^{2xy} + x$ and $N(x,y) = bx\,e^{2xy}$. Note that $M_y = e^{2xy} + 2xy\,e^{2xy}$, and $N_x = b\,e^{2xy} + 2bxy\,e^{2xy}$. The given equation is exact, as long as $b = 1$. Integrating N with respect to y, while holding x constant, results in $\psi(x,y) = e^{2xy}/2 + h(x)$. Now differentiating with respect to x, $\psi_x = y\,e^{2xy} + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = x$, and hence $h(x) = x^2/2$. We conclude that $\psi(x,y) = e^{2xy}/2 + x^2/2$. Hence the solution is given implicitly as $e^{2xy} + x^2 = c$.

13. Note that ψ is of the form $\psi(x\,,y) = f(x) + g(y)$, since each of the integrands is a function of a single variable. It follows that $\psi_x = f'(x)$ and $\psi_y = g'(y)$. That is, $\psi_x = M(x\,,y_0)$ and $\psi_y = N(x_0\,,y)$. Furthermore,

$$\frac{\partial^2\psi}{\partial x\partial y}(x_0\,,y_0) = \frac{\partial M}{\partial y}(x_0\,,y_0) \text{ and } \frac{\partial^2\psi}{\partial y\partial x}(x_0\,,y_0) = \frac{\partial N}{\partial x}(x_0\,,y_0)\,,$$

based on the hypothesis and the fact that the point (x_0, y_0) is arbitrary, $\psi_{xy} = \psi_{yx}$ and $M_y(x,y) = N_x(x,y)$.

14. Observe that $(M(x))_y = (N(y))_x = 0$.

SS 15. Here $M(x,y) = x^2y^3$ and $N(x,y) = x + xy^2$. Therefore, $M_y = 3x^2y^2$ and $N_x = 1 + y^2$. We see that the equation is not exact. Now, multiplying the equation by $\mu(x,y) = 1/xy^3$, the equation becomes $xdx + (1 + y^2)/y^3dy = 0$. Now we see that for this equation $M = x$ and $N = (1 + y^2)/y^3$. Therefore, $M_y = 0 = N_x$. Integrating M with respect to x, we see that $\psi = x^2/2 + h(y)$. Further, $\psi_y = h'(y) = N = (1 + y^2)/y^3 = 1/y^3 + 1/y$. Therefore, $h(y) = -1/2y^2 + \ln y$ and we conclude that the solution of the equation is given implicitly by $x^2 - 1/y^2 + 2\ln y = c$ and $y = 0$.

SS 16. We see that $M_y = (x + 2)\cos y$ while $N_x = \cos y$. Therefore, $M_y \neq N_x$. However, multiplying the equation by the given integrating factor $\mu(x,y) = xe^x$, this becomes $(x^2 + 2x)e^x \sin y dx + x^2e^x \cos y dy = 0$. Now we see that for this equation $M_y = (x^2 + 2x)e^x \cos y = N_x$. To solve this exact equation it is easiest to integrate (the new) N with respect to y to get $\psi(x,y) = x^2e^x \sin y + g(x)$. Finding ψ_x and setting it equal to (the new) M yields $g'(x) = 0$, which implies that the solution of the equation is given implicitly by $x^2e^x \sin y = c$.

SS 17. Suppose μ is an integrating factor which will make the equation exact. Then multiplying the equation by μ, we have $\mu M dx + \mu N dy = 0$. Then we need $(\mu M)_y = (\mu N)_x$. That is, we need $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. Then we rewrite the equation as $\mu(N_x - M_y) = \mu_y M - \mu_x N$. Suppose μ does not depend on x. Then $\mu_x = 0$. Therefore, $\mu(N_x - M_y) = \mu_y M$. Using the assumption that $(N_x - M_y)/M = Q(y)$, we can find an integrating factor μ by choosing μ which satisfies $\mu_y/\mu = Q$. We conclude that $\mu(y) = \exp \int Q(y)\, dy$ is an integrating factor of the differential equation.

SS 18. Since $(M_y - N_x)/N = 3$ is a function of x only, we know that $\mu = e^{3x}$ is an integrating factor for this equation. Multiplying the equation by μ, we obtain the equation $e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0$. Then $M_y = e^{3x}(3x^2 + 2x + 3y^2) = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = (x^2y + y^3/3)e^{3x} + h(y)$. Then $\psi_y = (x^2 + y^2)e^{3x} + h'(y) = N = e^{3x}(x^2 + y^2)$. Therefore, $h'(y) = 0$ and we conclude that the solution is given implicitly by $(3x^2y + y^3)e^{3x} = c$.

SS 19. Since $(M_y - N_x)/N = -1$ is a function of x only, we know that $\mu = e^{-x}$ is an integrating factor for this equation. Multiplying the equation by μ, we obtain the equation $(e^{-x} - e^x - ye^{-x})dx + e^{-x}dy = 0$. Then $M_y = -e^{-x} = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = -e^{-x} - e^x + ye^{-x} + h(y)$. Then $\psi_y = e^{-x} + h'(y) = N = e^{-x}$. Therefore, $h'(y) = 0$ and we conclude that the solution is given implicitly by $-e^{-x} - e^x + ye^{-x} = c$. Alternatively, we might recognize that $y' - y = e^{2x} - 1$ is a linear first order equation which can be solved as in Section 2.1.

SS 20. Since $(N_x - M_y)/M = 1/y$ is a function of y only, we know by Problem 17 that $\mu(y) = e^{\int 1/y\, dy} = y$ is an integrating factor for this equation. Multiplying the equation by μ, we obtain $ydx + (x - y\sin y)dy = 0$. Then for this equation, $M_y = 1 = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = xy + h(y)$. Then $\psi_y = x + h'(y) = N = x - y\sin y$. Therefore, $h'(y) = -y\sin y$ which implies that $h(y) = -\sin y + y\cos y$, and we conclude that the solution is given implicitly by $xy - \sin y + y\cos y = c$.

21. The equation is not exact, since $N_x - M_y = 2y - 1$. However, $(N_x - M_y)/M = (2y - 1)/y$ is a function of y alone. Hence there exists $\mu = \mu(y)$, which is a solution of the differential equation $\mu' = (2 - 1/y)\mu$. The latter equation is separable, with $d\mu/\mu = 2 - 1/y$. One solution is $\mu(y) = e^{2y - \ln y} = e^{2y}/y$. Now rewrite the given ODE as $e^{2y}dx + (2x\, e^{2y} - 1/y)dy = 0$. This equation is exact, and it is easy to see that $\psi(x, y) = x\, e^{2y} - \ln |y|$. Therefore the solution of the given equation is defined implicitly by $x\, e^{2y} - \ln |y| = c$.

22. Multiplying both sides of the ODE by $\mu = [xy(2x + y)]^{-1}$, the given equation is equivalent to $\left[(3x + y)/(2x^2 + xy)\right] dx + \left[(x + y)/(2xy + y^2)\right] dy = 0$. Rewrite the differential equation as

$$\left[\frac{2}{x} + \frac{2}{2x + y}\right] dx + \left[\frac{1}{y} + \frac{1}{2x + y}\right] dy = 0\,.$$

It is easy to see that $M_y = N_x$. Integrating M with respect to x, while keeping y constant, results in $\psi(x, y) = 2\ln |x| + \ln |2x + y| + h(y)$. Now taking the partial derivative with respect to y, $\psi_y = (2x + y)^{-1} + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 1/y$, and hence $h(y) = \ln |y|$. Therefore $\psi(x, y) = 2\ln |x| + \ln |2x + y| + \ln |y|$, and the solution of the given equation is defined implicitly by $2x^3y + x^2y^2 = c$.

2.7

SS 1. The Euler formula is $y_{n+1} = y_n + h(3 + t_n - y_n) = (1 - h)y_n + h(3 + t_n)$.

(a) 1.2, 1.39, 1.571, 1.7439

(b) 1.1975, 1.38549, 1.56491, 1.73658

(c) 1.19631, 1.38335, 1.56200, 1.73308

(d) The differential equation is linear with solution $y(t) = 2 + t - e^{-t}$. The values are 1.19516, 1.38127, 1.55918, 1.72968.

2. The Euler formula is given by $y_{n+1} = y_n + h(2y_n - 1) = (1 + 2h)y_n - h$.

(a) 1.1, 1.22, 1.364, 1.5368

(b) 1.105, 1.23205, 1.38578, 1.57179

(c) 1.10775, 1.23873, 1.39793, 1.59144

(d) The differential equation is linear with solution $y(t) = (1 + e^{2t})/2$. The values are 1.1107, 1.24591, 1.41106, 1.61277.

SS 3. The Euler formula is $y_{n+1} = y_n + h(0.5 - t_n + 2y_n) = (1 + 2h)y_n + h(0.5 - t_n)$.

(a) 1.25, 1.54, 1.878, 2.2736

(b) 1.26, 1.5641, 1.92156, 2.34359

(c) 1.26551, 1.57746, 1.94586, 2.38287

(d) The differential equation is linear with solution $y(t) = 0.5t + e^{2t}$. The values are 1.2714, 1.59182, 1.97212, 2.42554.

SS 4. The Euler formula is $y_{n+1} = y_n + h(3\cos(t_n) - 2y_n) = (1 - 2h)y_n + 3h\cos(t_n)$.

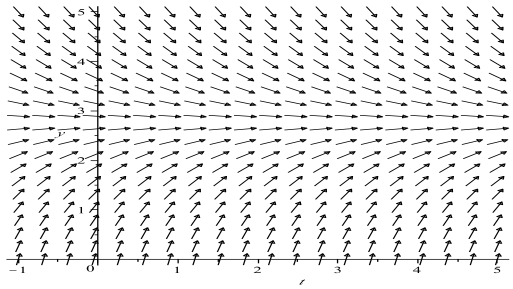
(a) 0.3, 0.538501, 0.724821, 0.866458

(b) 0.284813, 0.513339, 0.693451, 0.831571

(c) 0.277920, 0.501813, 0.678949, 0.815302

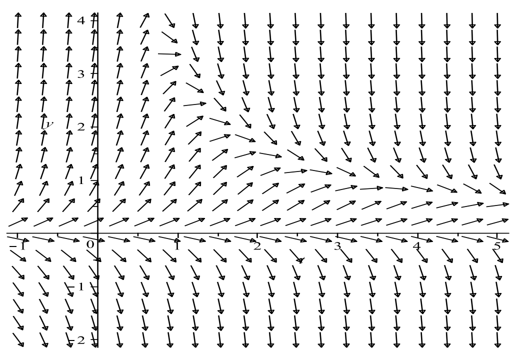
(d) The differential equation is linear with solution $y(t) = (6\cos(t) + 3\sin(t) - 6e^{-2t})/5$. The values are 0.271428, 0.490897, 0.665142, 0.799729.

5.



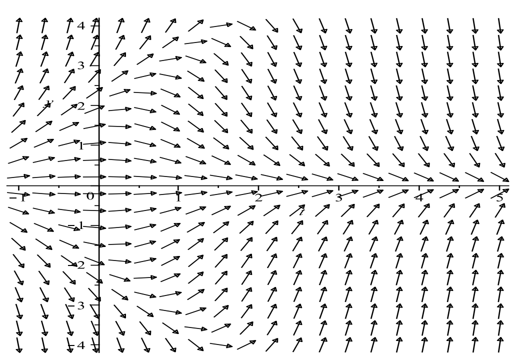
All solutions seem to converge to $y = 25/9$.

SS 6.



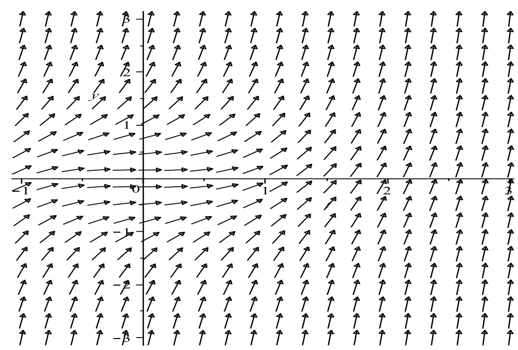
Solutions with $y(0) > 0$ appear to converge to a specific function, while solutions with $y(0) < 0$ decrease without bound. $y = 0$ is an equilibrium solution.

7.



Solutions with initial conditions $|y(0)| > 2.5$ seem to diverge. On the other hand, solutions with initial conditions $|y(0)| < 2.5$ seem to converge to zero. Also, $y = 0$ is an equilibrium solution.

SS 8.



All solutions seem to diverge.

9. The Euler formula is $y_{n+1} = y_n - 3h\sqrt{y_n} + 5h$. The initial value is $y_0 = 2$.

(a) 2.30800, 2.49006, 2.60023, 2.66773, 2.70939, 2.73521

(b) 2.30167, 2.48263, 2.59352, 2.66227, 2.70519, 2.73209

(c) 2.29864, 2.47903, 2.59024, 2.65958, 2.70310, 2.73053

(d) 2.29686, 2.47691, 2.58830, 2.65798, 2.70185, 2.72959

10. The Euler formula is $y_{n+1} = (1 + 3h)y_n - ht_ny_n^2$. The initial value is $(t_0, y_0) = (0, 0.5)$.

(a) 1.70308, 3.06605, 2.44030, 1.77204, 1.37348, 1.11925

(b) 1.79548, 3.06051, 2.43292, 1.77807, 1.37795, 1.12191

(c) 1.84579, 3.05769, 2.42905, 1.78074, 1.38017, 1.12328

(d) 1.87734, 3.05607, 2.42672, 1.78224, 1.38150, 1.12411

SS 11. The Euler formula is $y_{n+1} = y_n + h3t_n^2/(3y_n^2 - 4)$ with initial value $(t_0, y_0) = (1, 0)$.

(a) -0.166134, -0.410872, -0.804660, 4.15867

(b) -0.174652, -0.434238, -0.889140, -3.09810

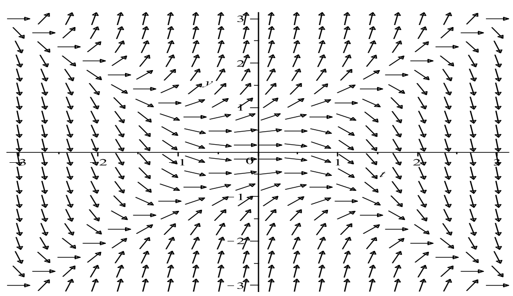
(c) There are two factors that explain the large differences. From the differential equation, the slope of y, y', becomes very large for values of y near -1.155. Also, the slope changes sign at $y = -1.155$. Thus for part (a), $y(1.7) = y_7 = -1.178$, which is close to -1.155 and the slope y' here is large and positive, creating the large change in $y_8 = y(1.8)$. For part (b), $y(1.65) = -1.125$, resulting in a large negative slope, which yields $y(1.7) = -3.133$. The slope at this point is now positive and the remainder of the solutions grow to -3.098 for the approximation to $y(1.8)$.

SS 12. The Euler formula is $y_{n+1} = y_n + h(t_n^2 + y_n^2)$ with $(t_0, y_0) = (0, 1)$. For the four step sizes given, the approximate values for $y(0.8)$ are 3.5078, 4.2013, 4.8004 and 5.3428. Thus, since these changes are still rather large, it is hard to give an estimate other than $y(0.8)$ is at least 5.3428. By using $h = 0.005$, 0.0025 and 0.001, we find further approximate values of $y(0.8)$ to be 5.576, 5.707 and 5.790. Thus a better estimate now is for $y(0.8)$ to be between 5.8 and 6. No reliable estimate is obtainable for $y(1)$, which is consistent with the direction field of Problem 8.

SS 13.(a) See the direction field in Problem 7 above.

(b) The Euler formula is $y_{n+1} = y_n + h(-t_n y_n + 0.1y_n^3)$. For $y_0 < 2.37$, the solutions seem to converge, while the solutions seem to diverge if $y_0 > 2.38$. We conclude that $2.37 < \alpha_0 < 2.38$.

14.(a)



(b) The iteration formula is $y_{n+1} = y_n + h\,y_n^2 - h\,t_n^2$. The critical value α_0 appears to be between 0.67 and 0.68. For $y_0 > \alpha_0$, the iterations diverge.

15.(a) The ODE is linear, with general solution $y(t) = t + ce^t$. Invoking the specified initial condition, $y(t_0) = y_0$, we have $y_0 = t_0 + ce^{t_0}$. Hence $c = (y_0 - t_0)e^{-t_0}$. Thus the solution is given by $\phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

(b) The Euler formula is $y_{n+1} = (1 + h)y_n + h - h\,t_n$. Now set $k = n + 1$.

(c) We have $y_1 = (1 + h)y_0 + h - ht_0 = (1 + h)y_0 + (t_1 - t_0) - ht_0$. Rearranging the terms, $y_1 = (1 + h)(y_0 - t_0) + t_1$. Now suppose that $y_k = (1+h)^k(y_0 - t_0) + t_k$, for some $k \geq 1$. Then $y_{k+1} = (1 + h)y_k + h - ht_k$. Substituting for y_k, we find that

$$y_{k+1} = (1 + h)^{k+1}(y_0 - t_0) + (1 + h)t_k + h - ht_k = (1 + h)^{k+1}(y_0 - t_0) + t_k + h.$$

Noting that $t_{k+1} = t_k + h$, the result is verified.

(d) Substituting $h = (t - t_0)/n$, with $t_n = t$, $y_n = (1 + (t - t_0)/n)^n(y_0 - t_0) + t$. Taking the limit of both sides, and using the fact that $\lim_{n\to\infty}(1 + a/n)^n = e^a$, pointwise convergence is proved.

16. The exact solution is $y(t) = e^t$. The Euler formula is $y_{n+1} = (1+h)y_n$. It is easy to see that $y_n = (1+h)^n y_0 = (1+h)^n$. Given $t > 0$, set $h = t/n$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(1+t/n)^n = e^t$.

SS 17. Using Eq.(8) we have $y_{n+1} = y_n + h(2y_n - 1) = (1+2h)y_n - h$. Setting $n+1 = k$ (and hence $n = k-1$) this becomes $y_k = (1+2h)y_{k-1} - h$, for $k = 1, 2, \ldots$. Since $y_0 = 1$, we have $y_1 = 1 + 2h - h = 1 + h = (1+2h)/2 + 1/2$, and hence $y_2 = (1+2h)y_1 - h = (1+2h)^2/2 + (1+2h)/2 - h = (1+2h)^2/2 + 1/2$. Furthermore, $y_3 = (1+2h)y_2 - h = (1+2h)^3/2 + (1+2h)/2 - h = (1+2h)^3/2 + 1/2$. Continuing in this fashion (or using induction) we obtain $y_k = (1+2h)^k/2 + 1/2$. For fixed $t > 0$ choose $h = t/k$. Then substitute for h in the last formula to obtain $y_k = (1 + 2t/k)^k/2 + 1/2$. Letting $k \to \infty$ we find (see hint for Problem 15(d)) that $y(t) = \lim_{k\to\infty} y_k = e^{2t}/2 + 1/2$, which is the exact solution.

2.8

SS 1. Let $s = t - 1$ and $w(s) = y(t(s)) - 2$, then when $t = 1$ and $y = 2$ we have $s = 0$ and $w(0) = 0$. Also, $dw/ds = (dw/dt)(dt/ds) = (d/dt)(y-2)(dt/ds) = dy/dt$ (since $t = s+1$) and hence $dw/ds = (s+1)^2 + (w+2)^2$, upon substitution into the given differential equation.

2. Let $z = y - 3$ and $\tau = t + 1$. It follows that $dz/d\tau = (dz/dt)(dt/d\tau) = dz/dt$. Furthermore, $dz/dt = dy/dt = 1 - y^3$. Hence $dz/d\tau = 1 - (z+3)^3$. The new initial condition is $z(0) = 0$.

3.(a) The approximating functions are defined recursively by

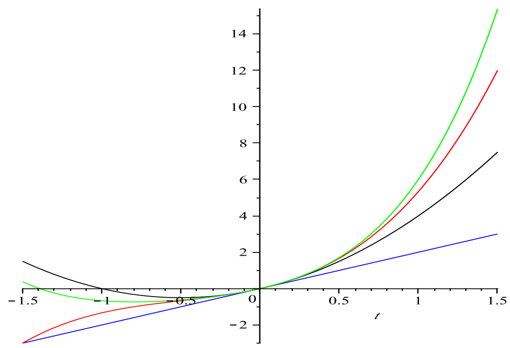
$$\phi_{n+1}(t) = \int_0^t 2\left[\phi_n(s) + 1\right] ds\,.$$

Setting $\phi_0(t) = 0$, $\phi_1(t) = 2t$. Continuing, $\phi_2(t) = 2t^2 + 2t$, $\phi_3(t) = 4t^3/3 + 2t^2 + 2t$, $\phi_4(t) = 2t^4/3 + 4t^3/3 + 2t^2 + 2t$, ... Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n} 2^k t^k/k!$ and use mathematical induction to verify this form for $\phi_n(t)$. First, let $n = 1$, then $\phi_n(t) = 2t$, so it is certainly true for $n = 1$. Then, using Eq.(7) again we have

$$\phi_{n+1}(t) = \int_0^t 2\left[\phi_n(s) + 1\right] ds = \int_0^t 2\left[\sum_{k=1}^{n} \frac{2^k}{k\,!} s^k + 1\right] ds = \sum_{k=1}^{n+1} \frac{2^k}{k\,!} t^k,$$

and we have verified our conjecture.

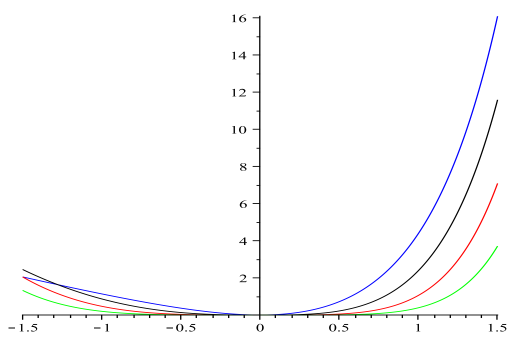
(b)



(c) Recall from calculus that $e^{at} = 1 + \sum_{k=1}^{\infty} a^k t^k/k!$. Thus

$$\phi(t) = \sum_{k=1}^{\infty} \frac{2^k}{k\,!} t^k = e^{2t} - 1\,.$$

(d)



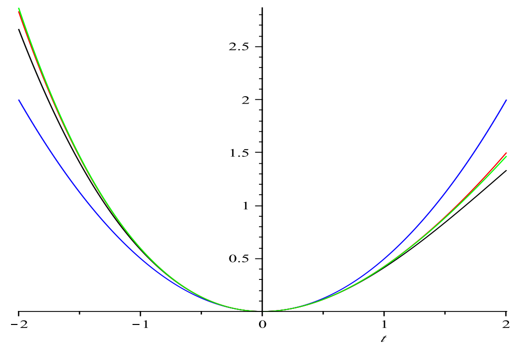
From the plot it appears that ϕ_4 is a good estimate for $|t| < 1/2$.

4.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[-\phi_n(s)/2 + s\right] ds\,.$$

Setting $\phi_0(t) = 0$, $\phi_1(t) = t^2/2$. Continuing, $\phi_2(t) = t^2/2 - t^3/12$, $\phi_3(t) = t^2/2 - t^3/12 + t^4/96$, $\phi_4(t) = t^2/2 - t^3/12 + t^4/96 - t^5/960$, ... Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n} 4(-1/2)^{k+1} t^{k+1}/(k+1)!$ and use mathematical induction to verify this form for $\phi_n(t)$.

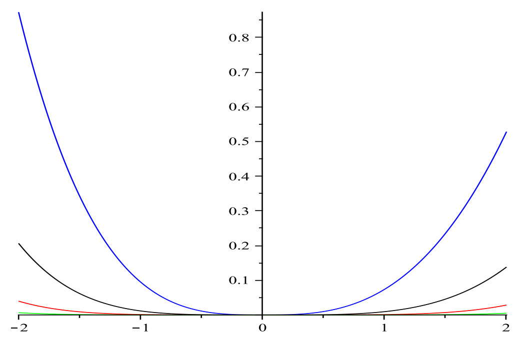
(b)



(c) Recall from calculus that $e^{at} = 1 + \sum_{k=1}^{\infty} a^k t^k / k!$. Thus

$$\phi(t) = \sum_{k=1}^{\infty} 4 \frac{(-1/2)^{k+1}}{k+1\,!} t^{k+1} = 4e^{-t/2} + 2t - 4\,.$$

(d)



From the plot it appears that ϕ_4 is a good estimate for $|t| < 2$.

SS 5.(a) The approximating functions are defined recursively by

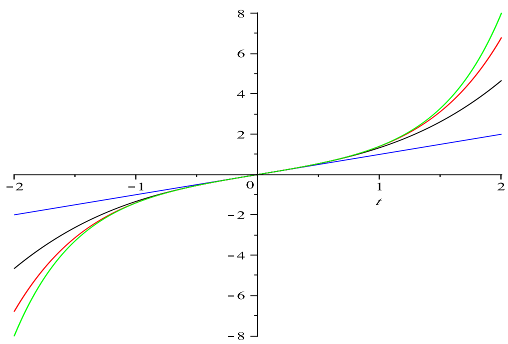
$$\phi_{n+1}(t) = \int_0^t [s\phi_n(s) + 1]\, ds\,.$$

Setting $\phi_0(t) = 0$, $\phi_1(t) = t$. Continuing, $\phi_2(t) = t + t^3/3$, $\phi_3(t) = t + t^3/3 + t^5/(3 \cdot 5)$, $\phi_4(t) = t + t^3/3 + t^5/(3 \cdot 5) + t^7/(3 \cdot 5 \cdot 7)$, ... Based upon these we conjecture that $\phi_n(t) = \sum_{k=1}^{n} t^{2k-1}/(1 \cdot 3 \cdot 5 \cdots (2k-1))$ and use mathematical induction to verify this form for $\phi_n(t)$. First, let $n = 1$, then $\phi_n(t) = t$, so it is certainly true for $n = 1$. Then, using Eq.(7) again we have

$$\phi_{n+1}(t) = \int_0^t [s\phi_n(s) + 1]\, ds = \int_0^t [\sum_{k=1}^{n} s \frac{s^{2k-1}}{1 \cdot 3 \cdots (2k-1)} + 1]\, ds = \sum_{k=1}^{n+1} \frac{t^{2k-1}}{1 \cdot 3 \cdots (2k-1)},$$

and we have verified our conjecture.

(b)



(c) Using the identity $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t) + [\phi_3(t) - \phi_2(t)] + \ldots + [\phi_n(t) - \phi_{n-1}(t)]$, consider the series $\phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)]$. Fix any t value now. We use the Ratio Test to prove the convergence of this series:

$$\left|\frac{\phi_{k+1}(t) - \phi_k(t)}{\phi_k(t) - \phi_{k-1}(t)}\right| = \left|\frac{\frac{t^{2k+1}}{1\cdot 3\cdots(2k+1)}}{\frac{t^{2k-1}}{1\cdot 3\cdots(2k-1)}}\right| = \frac{|t|^2}{2k+1}.$$

The limit of this quantity is 0 for any fixed t as $k \to \infty$, and we obtain that $\phi_n(t)$ is convergent for any t.

6.(a) The approximating functions are defined recursively by

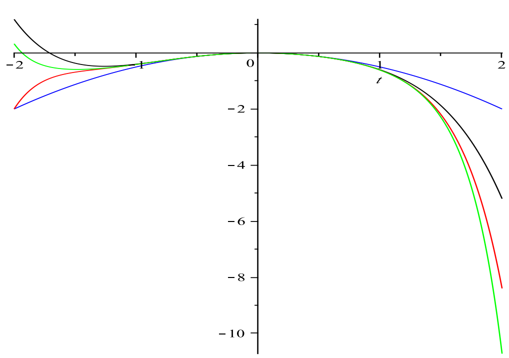
$$\phi_{n+1}(t) = \int_0^t \left[s^2\phi_n(s) - s\right] ds\,.$$

Set $\phi_0(t) = 0$. The iterates are given by $\phi_1(t) = -t^2/2$, $\phi_2(t) = -t^2/2 - t^5/10$, $\phi_3(t) = -t^2/2 - t^5/10 - t^8/80$, $\phi_4(t) = -t^2/2 - t^5/10 - t^8/80 - t^{11}/880, \ldots$. Upon inspection, it becomes apparent that

$$\phi_n(t) = -t^2\left[\frac{1}{2} + \frac{t^3}{2\cdot 5} + \frac{t^6}{2\cdot 5\cdot 8} + \ldots + \frac{(t^3)^{n-1}}{2\cdot 5\cdot 8\ldots[2+3(n-1)]}\right] =$$

$$= -t^2\sum_{k=1}^{n}\frac{(t^3)^{k-1}}{2\cdot 5\cdot 8\ldots[2+3(k-1)]}.$$

(b)



(c) Using the identity $\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + [\phi_3(t) - \phi_2(t)] + \ldots + [\phi_n(t) - \phi_{n-1}(t)]$, consider the series $\phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)]$. Fix any t value now. We use the Ratio Test to prove the convergence of this series:

$$\left|\frac{\phi_{k+1}(t) - \phi_k(t)}{\phi_k(t) - \phi_{k-1}(t)}\right| = \left|\frac{\frac{(-t^2)(t^3)^k}{2\cdot 5\cdots(2+3k)}}{\frac{(-t^2)(t^3)^{k-1}}{2\cdot 5\cdots(2+3(k-1))}}\right| = \frac{|t|^3}{2+3k}.$$

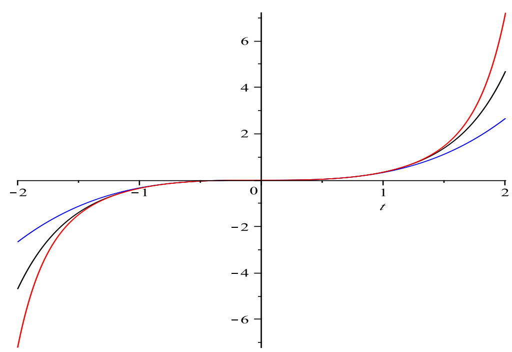
The limit of this quantity is 0 for any fixed t as $k \to \infty$, and we obtain that $\phi_n(t)$ is convergent for any t.

7.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2 + \phi_n^2(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t^3/3$, $\phi_2(t) = t^3/3 + t^7/63$, $\phi_3(t) = t^3/3 + t^7/63 + 2t^{11}/2079 + t^{15}/59535\,.$

(b)



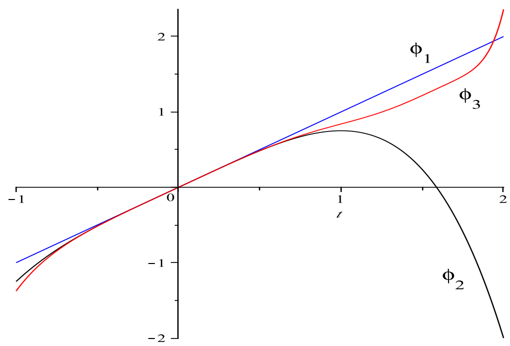
The iterates appear to be converging.

SS 8.(a) The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[1 - \phi_n^3(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t\,$, $\phi_2(t) = t - t^4/4\,$, $\phi_3(t) = t - t^4/4 + 3t^7/28 - 3t^{10}/160 + t^{13}/832.$

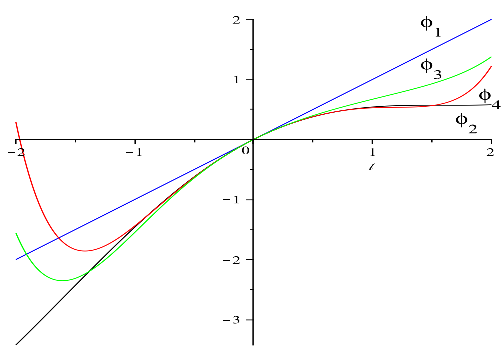
(b)



The approximations appear to be diverging.

SS 9.(a) First, recall that $\sin x = x - x^3/3! + x^5/5! + O(x^7)$. Now, for this problem, $\phi_1(t) = \int_0^t [1 - \sin\phi_0(s)]\,ds = t$ and we obtain that $\phi_2(t) = \int_0^t [1 - \sin s]\,ds = \int_0^t [1 - (s - s^3/3! + s^5/5! + O(s^7)]\,ds = t - t^2/2! + t^4/4! - t^6/6! + O(t^8)$. For ϕ_3 we need to find $\sin(\phi_2(t))$, which is given by $\sin(\phi_2(t)) = \phi_2(t) - \phi_2^3(t)/3! + \phi_2^5(t)/5! + O(t^7)$ $= (t - t^2/2! + t^4/4! - t^6/6!) - (t - t^2/2!)^3/3! + t^5/5! + O(t^7)$, where we have retained only the terms less than $O(t^7)$. Now use this in $\phi_3(t) = \int_0^t [1 - \sin(\phi_2(s))]\,ds$, which gives the desired answer up to $O(t^8)$.

(b)



10.(a) The approximating functions are defined recursively by

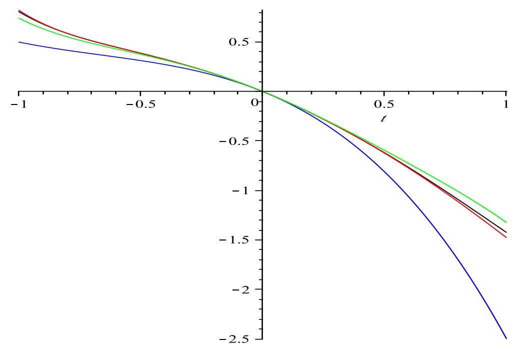
$$\phi_{n+1}(t) = \int_0^t \left[\frac{3s^2 + 4s + 2}{2(\phi_n(s) - 1)}\right] ds\,.$$

Note that $1/(2y - 2) = -(1/2)\sum_{k=0}^{6} y^k + O(y^7)$. For computational purposes, use the geometric series sum to replace the above iteration formula by

$$\phi_{n+1}(t) = -\frac{1}{2}\int_0^t \left[(3s^2 + 4s + 2)\sum_{k=0}^{6} \phi_n^k(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first four approximations are given by $\phi_1(t) = -t - t^2 - t^3/2$, $\phi_2(t) = -t - t^2/2 + t^3/6 + t^4/4 - t^5/5 - t^6/24 + \ldots$, $\phi_3(t) = -t - t^2/2 + t^4/12 - 3t^5/20 + 4t^6/45 + \ldots$, $\phi_4(t) = -t - t^2/2 + t^4/8 - 7t^5/60 + t^6/15 + \ldots$

(b)



The approximations appear to be converging to the exact solution, which can be found by separating the variables: $\phi(t) = 1 - \sqrt{1 + 2t + 2t^2 + t^3}$.

SS 11. Note that $\phi_n(0) = 0$ and $\phi_n(1) = 1$, for every $n \geq 1$. Let $a \in (0, 1)$. Then $\phi_n(a) = a^n$. Clearly, $\lim_{n\to\infty} a^n = 0$. Hence the assertion is true.

12.(a) $\phi_n(0) = 0$, for every $n \geq 1$. Let $a \in (0, 1]$. Then $\phi_n(a) = 2na\, e^{-na^2} = 2na/e^{na^2}$. Using l'Hospital's rule, $\lim_{z\to\infty} 2az/e^{az^2} = \lim_{z\to\infty} 1/ze^{az^2} = 0$. Hence $\lim_{n\to\infty} \phi_n(a) = 0$.

(b) $\int_0^1 2nx\, e^{-nx^2} dx = -e^{-nx^2}\Big|_0^1 = 1 - e^{-n}$. Therefore,

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)dx \neq \int_0^1 \lim_{n\to\infty} \phi_n(x)dx\,.$$

SS 13.(a) Recall that Eq.(9) states that $\phi(t) = \int_0^t 2s[1 + \phi(s)]\, ds$. Since $\phi(t) = \sum_{k=1}^{\infty} t^{2k}/k!$,

$$2s[1 + \phi(s)] = 2s \sum_{k=0}^{\infty} \frac{s^{2k}}{k!} = 2 \sum_{k=0}^{\infty} \frac{s^{2k+1}}{k!}$$

Integrating term-by-term,

$$\int_0^t 2s[1 + \phi(s)]\, ds = \int_0^t 2 \sum_{k=0}^{\infty} \frac{s^{2k+1}}{k!}\, ds = 2 \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^t s^{2k+1}\, ds$$
$$= 2 \sum_{k=0}^{\infty} \frac{1}{k!} \frac{t^{2k+2}}{2k+2} = \sum_{k=0}^{\infty} \frac{t^{2(k+1)}}{(k+1)!} = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = \phi(t)$$

and $\phi(t)$ is a solution of Eq.(9).

(b) Recall that the initial value problem in Eq.(8) is $y' = 2t(1+y), y(0) = 0$. Letting $y = \phi(t) = \sum_{k=1}^{\infty} t^{2k}/k!$,

$$y' = \phi'(x) = \sum_{k=1}^{\infty} \frac{2kt^{2k-1}}{k!} = 2\sum_{k=1}^{\infty} \frac{t^{2k-1}}{(k-1)!} = 2t\sum_{k=1}^{\infty} \frac{t^{2k-2}}{(k-1)!}$$
$$= 2t\sum_{k=0}^{\infty} \frac{t^{2k}}{k!} = 2t\left[1 + \sum_{k=1}^{\infty} \frac{t^{2k}}{k!}\right] = 2t[1+\phi(t)]$$

and $y(0) = \phi(0) = \sum_{k=1}^{\infty} 0 = 0$, so $y = \phi(t)$ satisfies the initial value problem $y' = 2t(1+y), y(0) = 0$.

(c) Since $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$,

$$\phi(t) = \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = -1 + \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} = -1 + \sum_{k=0}^{\infty} \frac{(t^2)^k}{k!} = -1 + e^{t^2}$$

(d) Separating the variables gives $(1/(1+y))\,dy = 2t\,dt$, and integration yields $\ln|1+y| = t^2 + c$. Applying the initial condition, $\ln 1 = 0 + c$, so $c = 0$ and $\ln|1+y| = t^2$. Solving for y first gives $|1+y| = e^{t^2}$, so $1 + y = \pm e^{t^2}$. In order for $y(0) = 0$ to be true, we choose $1 + y = e^{t^2}$, and thus $y = -1 + e^{t^2}$.

(e) Consider the first-order linear equation $y' - 2ty = 0$. The integrating factor will be $\mu(t) = e^{\int -2t\,dt} = e^{-t^2}$. Since multiplying the differential equation by $\mu(t)$ yields

$$\frac{d}{dt}\left(e^{-t^2} y\right) = 2te^{-t^2}$$

we have $e^{-t^2} y = \int 2te^{-t^2}\,dt = -e^{-t^2} + c$. The initial condition $y(0) = 0$ may now be applied to show that $c = 1$, and $y = e^{t^2}(-e^{-t^2} + 1) = -1 + e^{t^2}$.

14. Let t be fixed, such that $(t, y_1), (t, y_2) \in D$. Without loss of generality, assume that $y_1 < y_2$. Since f is differentiable with respect to y, the mean value theorem asserts that there exists $\xi \in (y_1, y_2)$ such that $f(t, y_1) - f(t, y_2) = f_y(t, \xi)(y_1 - y_2)$. This means that $|f(t, y_1) - f(t, y_2)| = |f_y(t, \xi)|\,|y_1 - y_2|$. Since, by assumption, $\partial f/\partial y$ is continuous in D, f_y attains a maximum K on any closed and bounded subset of D. Hence $|f(t, y_1) - f(t, y_2)| \le K\,|y_1 - y_2|$.

15. For a sufficiently small interval of t, $\phi_{n-1}(t)$, $\phi_n(t) \in D$. Since f satisfies a Lipschitz condition, $|f(t, \phi_n(t)) - f(t, \phi_{n-1}(t))| \le K\,|\phi_n(t) - \phi_{n-1}(t)|$. Here $K = \max|f_y|$.

16.(a) $\phi_1(t) = \int_0^t f(s, 0)ds$. Hence $|\phi_1(t)| \le \int_0^{|t|} |f(s, 0)|\,ds \le \int_0^{|t|} M ds = M\,|t|$, in which M is the maximum value of $|f(t, y)|$ on D.

(b) By definition, $\phi_2(t) - \phi_1(t) = \int_0^t [f(s\,,\phi_1(s)) - f(s\,,0)]\,ds$. Taking the absolute value of both sides, $|\phi_2(t) - \phi_1(t)| \le \int_0^{|t|} |[f(s\,,\phi_1(s)) - f(s\,,0)]|\,ds$. Based on the results in Problems 14 and 15,

$$|\phi_2(t) - \phi_1(t)| \le \int_0^{|t|} K\,|\phi_1(s) - 0|\,ds \le KM \int_0^{|t|} |s|\,ds\,.$$

Evaluating the last integral, we obtain that $|\phi_2(t) - \phi_1(t)| \le MK\,|t|^2/2$.

(c) Suppose that

$$|\phi_i(t) - \phi_{i-1}(t)| \le \frac{MK^{i-1}\,|t|^i}{i!}$$

for some $i \ge 1$. By definition,

$$\phi_{i+1}(t) - \phi_i(t) = \int_0^t [f(s, \phi_i(s)) - f(s, \phi_{i-1}(s))]\,ds\,.$$

It follows that

$$|\phi_{i+1}(t) - \phi_i(t)| \le \int_0^{|t|} |f(s, \phi_i(s)) - f(s, \phi_{i-1}(s))|\,ds$$

$$\le \int_0^{|t|} K\,|\phi_i(s) - \phi_{i-1}(s)|\,ds \le \int_0^{|t|} K \frac{MK^{\,i-1}\,|s|^i}{i!} ds =$$

$$= \frac{MK^{\,i}\,|t|^{i+1}}{(i+1)!} \le \frac{MK^{\,i}h^{i+1}}{(i+1)!}\,.$$

Hence, by mathematical induction, the assertion is true.

17.(a) Use the triangle inequality, $|a+b| \le |a| + |b|$.

(b) For $|t| \le h$, $|\phi_1(t)| \le Mh$, and $|\phi_n(t) - \phi_{n-1}(t)| \le MK^{\,n-1}h^n/(n\,!)$. Hence

$$|\phi_n(t)| \le M \sum_{i=1}^{n} \frac{K^{\,i-1}h^i}{i\,!} = \frac{M}{K} \sum_{i=1}^{n} \frac{(Kh)^i}{i\,!}\,.$$

(c) The sequence of partial sums in (b) converges to $M(e^{Kh} - 1)/K$. By the comparison test, the sums in (a) also converge. Since individual terms of a convergent series must tend to zero, $|\phi_n(t) - \phi_{n-1}(t)| \to 0$, and it follows that the sequence $|\phi_n(t)|$ is convergent.

18.(a) Let $\phi(t) = \int_0^t f(s\,,\phi(s))ds$ and $\psi(t) = \int_0^t f(s\,,\psi(s))ds$. Then by linearity of the integral, $\phi(t) - \psi(t) = \int_0^t [f(s\,,\phi(s)) - f(s\,,\psi(s))]\,ds$.

(b) It follows that $|\phi(t) - \psi(t)| \le \int_0^t |f(s\,,\phi(s)) - f(s\,,\psi(s))|\,ds$.

(c) We know that f satisfies a Lipschitz condition, $|f(t\,,y_1)-f(t\,,y_2)| \le K\,|y_1-y_2|$, based on $|\partial f/\partial y| \le K$ in D. Therefore,

$$|\phi(t)-\psi(t)| \le \int_0^t |f(s\,,\phi(s))-f(s\,,\psi(s))|\,ds \le \int_0^t K\,|\phi(s)-\psi(s)|\,ds.$$

2.9

1. Writing the equation for each $n \ge 0$, $y_1 = -0.9\,y_0$, $y_2 = -0.9\,y_1 = (-0.9^2)y_0$, $y_3 = -0.9\,y_2 = (-0.9)^3 y_0$ and so on, it is apparent that $y_n = (-0.9)^n\,y_0$. The terms constitute an alternating series, which converge to zero, regardless of y_0.

2. Write the equation for each $n \ge 0$, $y_1 = \sqrt{3}\,y_0$, $y_2 = \sqrt{4/2}\,y_1$, $y_3 = \sqrt{5/3}\,y_2, \ldots$ Upon substitution, we find that $y_2 = \sqrt{(4\cdot 3)/2}\,y_1$, $y_3 = \sqrt{(5\cdot 4\cdot 3)/(3\cdot 2)}\,y_0, \ldots$ It can be proved by mathematical induction, that

$$y_n = \frac{1}{\sqrt{2}}\sqrt{\frac{(n+2)!}{n!}}\;y_0 \;=\; \frac{1}{\sqrt{2}}\sqrt{(n+1)(n+2)}\;y_0\,.$$

This sequence is divergent, except for $y_0 = 0$.

3. Writing the equation for each $n \ge 0$, $y_1 = -y_0$, $y_2 = y_1$, $y_3 = -y_2$, $y_4 = y_3$, and so on. It can be shown that

$$y_n = \begin{cases} y_0, & \text{for } n = 4k \text{ or } n = 4k-1 \\ -y_0, & \text{for } n = 4k-2 \text{ or } n = 4k-3 \end{cases}$$

The sequence is convergent only for $y_0 = 0$.

SS 4. Writing the equation for each $n \ge 0$,

$$\begin{aligned} y_1 &= 0.5\,y_0 + 6 \\ y_2 &= 0.5\,y_1 + 6 = 0.5(0.5\,y_0+6)+6 = (0.5)^2 y_0 + 6 + (0.5)6 \\ y_3 &= 0.5\,y_2 + 6 = 0.5(0.5\,y_1+6)+6 = (0.5)^3 y_0 + 6\left[1+(0.5)+(0.5)^2\right] \\ &\;\vdots \\ y_n &= (0.5)^n y_0 + 12\,[1-(0.5)^n\;], \end{aligned}$$

which follows from Eq.(13) and (14). The sequence is convergent for all y_0, and in fact $y_n \to 12$.

5. Let y_n be the balance at the end of the nth month. Then $y_{n+1} = (1+r/12)y_n + 25$. We have $y_n = \rho^n[y_0 - 25/(1-\rho)] + 25/(1-\rho)$, in which $\rho = (1+r/12)$. Here r is the annual interest rate, given as 8%. Thus $y_{36} = (1.0066)^{36}\,[1000 + 12\cdot 25/r] - 12\cdot 25/r = \$2,283.63$.

6. Let y_n be the balance due at the end of the nth month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$. Here r is the annual interest rate and P is the monthly payment. The solution, in terms of the amount borrowed, is given by $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$, in which $\rho = (1 + r/12)$ and $y_0 = 8,000$. To figure out the monthly payment P, we require that $y_{36} = 0$. That is, $\rho^{36}[y_0 + P/(1-\rho)] = P/(1-\rho)$. After the specified amounts are substituted, we find that $P = \$258.14$.

7. Let y_n be the balance due at the end of the nth month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = .09$ and P is the monthly payment. The initial value of the mortgage is $y_0 = \$100,000$. Then the balance due at the end of the n-th month is $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$, where $\rho = (1 + r/12)$. In terms of the specified values, $y_n = (1.0075)^n[10^5 - 12P/r] + 12P/r$. Setting $n = 30 \cdot 12 = 360$, and $y_{360} = 0$, we find that $P = \$804.62$. For the monthly payment corresponding to a 20 year mortgage, set $n = 240$ and $y_{240} = 0$ to find that $P = \$899.73$. The total amount paid during the term of the loan is $360 \times 804.62 = \$289,663.20$ for the 30-year loan and is $240 \times 899.73 = \$215,935.20$ for the 20-year loan.

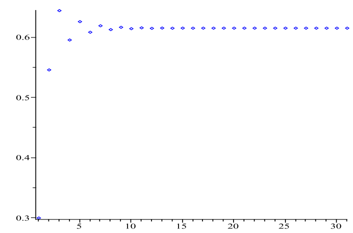
8. Let y_n be the balance due at the end of the nth month, with y_0 the initial value of the mortgage. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = 0.1$ and $P = \$1000$ is the maximum monthly payment. Given that the life of the mortgage is 20 years, we require that $y_{240} = 0$. The balance due at the end of the n-th month is $y_n = \rho^n[y_0 + P/(1-\rho)] - P/(1-\rho)$. In terms of the specified values for the parameters, the solution of $(1.00833)^{240}[y_0 - 12 \cdot 1000/0.1] = -12 \cdot 1000/0.1$ is $y_0 = \$103,624.62$.

SS 9. We must solve Eq.(14) numerically for ρ when $n = 240$, $y_{240} = 0$, $b = -\$900$ and $y_0 = \$95,000$. The result is $\rho = 1.0081$, so the monthly interest rate is $r = 0.81\%$, which is equivalent to an annual rate of 9.73%.

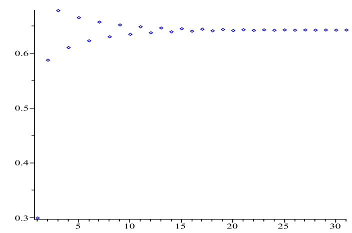
SS 10. Substituting Eq.(25), $u_n = (\rho-1)/\rho + v_n$ into Eq.(21) we get $(\rho-1)/\rho + v_{n+1} = \rho((\rho-1)/\rho + v_n)(1 - (\rho-1)/\rho - v_n)$, which after simplification turns into $v_{n+1} = -(\rho-1)/\rho + (\rho - 1 + \rho v_n)(1/\rho - v_n) = (1-\rho)/\rho + (\rho-1)/\rho - (\rho-1)v_n + v_n - \rho v_n^2 = (2-\rho)v_n - \rho v_n^2$, which is exactly what we wanted to prove.

SS 11.(a) For $u_0 = 0.2$, we have $u_1 = 3.2u_0(1-u_0) = 0.512$ and $u_2 = 3.2u_1(1-u_1) = 0.7995392$. Likewise, we get $u_3 = 0.51288406$, $u_4 = 0.7994688$, $u_5 = 0.51301899$, $u_6 = 0.7994576$ and $u_7 = 0.5130404$. Continuing, $u_{14} = u_{16} = 0.79945549$ and $u_{15} = u_{17} = 0.51304451$.

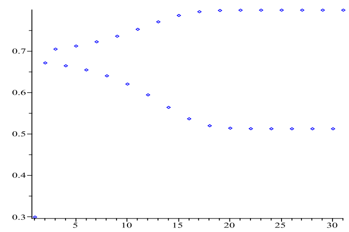
(b) The plots show the nature of solutions.



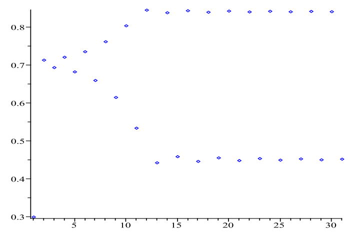
(a) $\rho = 2.6$



(b) $\rho = 2.8$

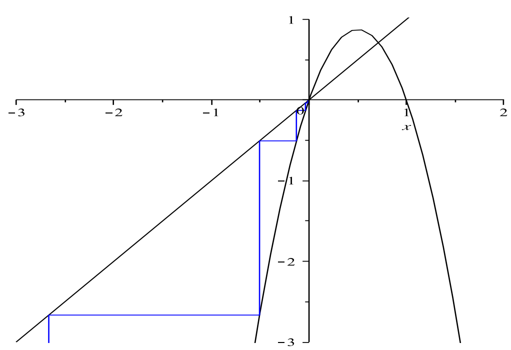


(c) $\rho = 3.2$

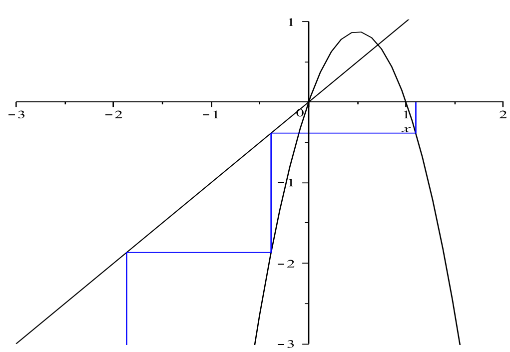


(d) $\rho = 3.4$

SS 12.(a) For example, take $\rho = 3.5$ and $u_0 = -0.01$:



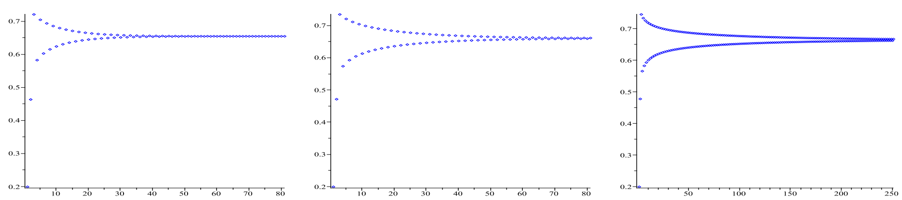
(b) For example, take $\rho = 3.5$ and $u_0 = 1.1$:



Clearly, $u_n \to -\infty$ as $n \to \infty$.

SS 13. For both parts of this problem a computer was used and an initial value of $u_0 = 0.2$ was chosen. Different initial values or different computer programs may need a slightly different number of iterations to reach the limiting value.

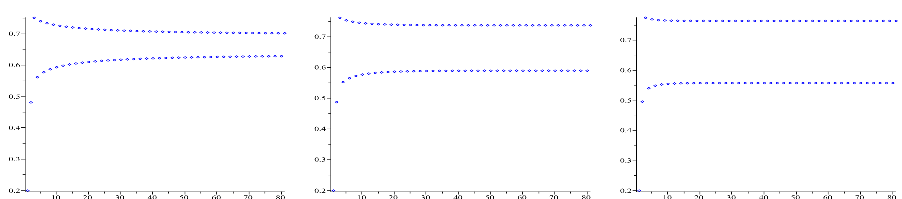
(a)



(a) $\rho = 2.9$ (b) $\rho = 2.95$ (c) $\rho = 2.99$

The limiting value of 0.65517 (to 5 decimal places) is reached after approximately 100 iterations for $\rho = 2.9$. The limiting value of 0.66102 (to 5 decimal places) is reached after approximately 200 iterations for $\rho = 2.95$. The limiting value of 0.66555 (to 5 decimal places) is reached after approximately 910 iterations for $\rho = 2.99$.

(b)



(a) $\rho = 3.01$ (b) $\rho = 3.05$ (c) $\rho = 3.1$

The solution oscillates between 0.63285 and 0.69938 after approximately 400 iterations for $\rho = 3.01$. The solution oscillates between 0.59016 and 0.73770 after approximately 130 iterations for $\rho = 3.05$. The solution oscillates between 0.55801 and 0.76457 after approximately 30 iterations for $\rho = 3.1$.

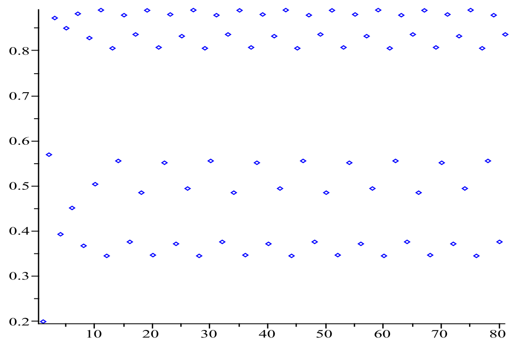
SS 14. For an initial value of 0.2 and $\rho = 3.448$ we have the solution oscillating between 0.4403086 and 0.8497146. After approximately 3570 iterations the eighth decimal place is still not fixed, though. For the same initial value and $\rho = 3.45$ the solution oscillates between the four values 0.43399155, 0.84746795, 0.44596778 and 0.85242779 after 3700 iterations. For $\rho = 3.449$ the solution is still varying in the fourth decimal place after 3570 iterations, but there appear to be four values.

15.(a) $\delta_2 = (\rho_2 - \rho_1)/(\rho_3 - \rho_2) = (3.449 - 3)/(3.544 - 3.449) = 4.7263$.

(b) diff$= (|\delta - \delta_2|/\delta) \cdot 100 = (|4.6692 - 4.7363|/4.6692) \cdot 100 \approx 1.22\%$.

(c) Assuming $(\rho_3 - \rho_2)/(\rho_4 - \rho_3) = \delta$, $\rho_4 \approx 3.5643$

(d) A period 16 solution appears near $\rho \approx 3.565$.



(e) Note that $(\rho_{n+1} - \rho_n) = \delta_n^{-1}(\rho_n - \rho_{n-1})$. With the assumption that $\delta_n = \delta$, we have $(\rho_{n+1} - \rho_n) = \delta^{-1}(\rho_n - \rho_{n-1})$, which is of the form $y_{n+1} = \alpha\, y_n$, $n \geq 3$. It follows that $(\rho_k - \rho_{k-1}) = \delta^{3-k}(\rho_3 - \rho_2)$ for $k \geq 4$. Then

$$\begin{aligned}\rho_n &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + (\rho_4 - \rho_3) + \ldots + (\rho_n - \rho_{n-1}) \\ &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[1 + \delta^{-1} + \delta^{-2} + \ldots + \delta^{3-n}\right] \\ &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[\frac{1 - \delta^{4-n}}{1 - \delta^{-1}}\right].\end{aligned}$$

Hence $\lim_{n\to\infty} \rho_n = \rho_2 + (\rho_3 - \rho_2)\left[\frac{\delta}{\delta - 1}\right]$. Substitution of the appropriate values yields

$$\lim_{n\to\infty} \rho_n = 3.5699$$

PROBLEMS

Before trying to find the solution of a differential equation, it is necessary to know its type. The student should first classify the differential equations before looking at this section, which identifies the type of each differential equation in Problems 1 through 24.

1. The equation is *linear*. It can be written in the form $y' + 2y/x = x^2$, and the integrating factor is $\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$. Multiplication by $\mu(x)$ yields $x^2 y' + 2yx = (yx^2)' = x^4$. Integration with respect to x and division by x^2 gives that $y = x^3/5 + c/x^2$.

SS 2. The equation is *separable*. Separating the variables gives the differential equation $(2 - \sin y)dy = (1 + \cos x)dx$, and after integration we obtain that the solution is $2y + \cos y - x - \sin x = c$.

SS 3. The equation is *exact*. Simplification gives $(2x + y)dx + (x - 3 - 3y^2)dy = 0$. We can check that $M_y = 1 = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2 + xy + g(y)$, then $\psi_y = x + g'(y) = x - 3 - 3y^2$, which means that $g'(y) = -3 - 3y^2$, so integrating with respect to y we obtain that $g(y) = -3y - y^3$. Therefore the solution is defined implicitly as $x^2 + xy - 3y - y^3 = c$. The initial condition $y(0) = 0$ implies that $c = 0$, so we conclude that the solution is $x^2 + xy - 3y - y^3 = 0$.

SS 4. The equation is *linear*. It can be written as $y' + (2x - 1)y = -3(2x - 1)$, and the integrating factor is e^{x^2-x}. Multiplication by this integrating factor and the subsequent integration gives the solution $ye^{x^2-x} = -3e^{x^2-x} + c$, which means that $y = -3 + ce^{x-x^2}$. (The equation is also *separable*.)

5. The equation is *exact*. Algebraic manipulations give the symmetric form of the equation, $(2xy + y^2 + 1)dx + (x^2 + 2xy)dy = 0$. We can check that $M_y = 2x + 2y = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2y + xy^2 + x + g(y)$, then $\psi_y = x^2 + 2xy + g'(y) = x^2 + 2xy$, so we get that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $x^2y + xy^2 + x = c$.

6. The equation is *linear*. It can be written in the form $y' + (1 + (1/x))y = 1/x$ and the integrating factor is $\mu(x) = e^{\int 1+(1/x)\,dx} = e^{x+\ln x} = xe^x$. Multiplication by $\mu(x)$ yields $xe^x y' + (xe^x + e^x)y = (xe^x y)' = e^x$. Integration with respect to x and division by xe^x shows that the general solution of the equation is $y = 1/x + c/(xe^x)$. The initial condition implies that $0 = 1 + c/e$, which means that $c = -e$ and the solution is $y = 1/x - e/(xe^x) = x^{-1}(1 - e^{1-x})$.

7. The equation is *linear*. It can be written in the form $y' + 2y/x = \sin x/x^2$ and the integrating factor is $\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$. Multiplication by $\mu(x)$ gives $x^2y' + 2xy = (x^2y)' = \sin x$, and after integration with respect to x and division by x^2 we obtain the general solution $y = (c - \cos x)/x^2$. The initial condition implies that $c = 4 + \cos 2$ and the solution becomes $y = (4 + \cos 2 - \cos x)/x^2$.

SS 8. The equation is *exact*. Simplification gives $(2xy + 1)dx + (x^2 + 2y)dy = 0$. We can check that $M_y = 2x = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2y + x + g(y)$, then $\psi_y = x^2 + g'(y) = x^2 + 2y$, which means that $g'(y) = 2y$, so we obtain that $g(y) = y^2$. Therefore the solution is defined implicitly as $x^2y + x + y^2 = c$.

SS 9. The equation is *separable*. Factoring the terms we obtain the differential equation $(x^2 + x - 1)ydx + x^2(y - 2)dy = 0$. We separate the variables by dividing this equation by yx^2 and obtain

$$(1 + \frac{1}{x} - \frac{1}{x^2})dx + (1 - \frac{2}{y})dy = 0.$$

Integration gives us the solution $x + \ln|x| + 1/x - 2\ln|y| + y = c$. We also have the solution $y = 0$ which we lost when we divided by y.

10. The equation is *exact*. It is easy to check that $M_y = 1 = N_x$. Integrating M with respect to x gives that $\psi(x,y) = x^3/3 + xy + g(y)$, then $\psi_y = x + g'(y) = x + e^y$, which means that $g'(y) = e^y$, so we obtain that $g(y) = e^y$. Therefore the solution is defined implicitly as $x^3/3 + xy + e^y = c$.

11. The equation is *exact*. We can check that $M_y = 1 = N_x$. Integrating M with respect to x gives that $\psi(x,y) = x^2/2 + xy + g(y)$, then $\psi_y = x + g'(y) = x + 2y$, which means that $g'(y) = 2y$, so we obtain that $g(y) = y^2$. Therefore the general solution is defined implicitly as $x^2/2 + xy + y^2 = c$. The initial condition gives us $c = 17$, so the solution is $x^2 + 2xy + 2y^2 = 34$.

12. The equation is *separable*. Separation of variables leads us to the equation

$$\frac{dy}{y} = \frac{1 - e^x}{1 + e^x}dx.$$

Note that $1 + e^x - 2e^x = 1 - e^x$. We obtain that

$$\ln|y| = \int \frac{1 - e^x}{1 + e^x}dx = \int 1 - \frac{2e^x}{1 + e^x}dx = x - 2\ln(1 + e^x) + \tilde{c}.$$

This means that $y = ce^x(1 + e^x)^{-2}$, which also can be written as $y = c/\cosh^2(x/2)$ after some algebraic manipulations.

13. The equation is *exact*. The symmetric form is $(-e^{-x}\cos y + e^{2y}\cos x)dx + (-e^{-x}\sin y + 2e^{2y}\sin x)dy = 0$. We can check that $M_y = e^{-x}\sin y + 2e^{2y}\cos x = N_x$. Integrating M with respect to x gives that $\psi(x,y) = e^{-x}\cos y + e^{2y}\sin x + g(y)$, then $\psi_y = -e^{-x}\sin y + 2e^{2y}\sin x + g'(y) = -e^{-x}\sin y + 2e^{2y}\sin x$, so we get that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $e^{-x}\cos y + e^{2y}\sin x = c$.

14. The equation is *linear*. The integrating factor is $\mu(x) = e^{-\int 3\,dx} = e^{-3x}$, which turns the equation into $e^{-3x}y' - 3e^{-3x}y = (e^{-3x}y)' = e^{-x}$. We integrate with respect to x to obtain $e^{-3x}y = -e^{-x} + c$, and the solution is $y = ce^{3x} - e^{2x}$ after multiplication by e^{3x}.

15. The equation is *linear*. The integrating factor is $\mu(x) = e^{\int 2\,dx} = e^{2x}$, which gives us $e^{2x}y' + 2e^{2x}y = (e^{2x}y)' = e^{-x^2}$. The antiderivative of the function on the right hand side can not be expressed in a closed form using elementary functions, so we have to express the solution using integrals. Let us integrate both sides of this equation from 0 to x. We obtain that the left hand side turns into

$$\int_0^x (e^{2s}y(s))'ds = e^{2x}y(x) - e^0 y(0) = e^{2x}y - 3.$$

The right hand side gives us $\int_0^x e^{-s^2}\,ds$. So we found that

$$y = e^{-2x}\int_0^x e^{-s^2}\,ds + 3e^{-2x}.$$

16. The equation is *exact*. Algebraic manipulations give us the symmetric form $(y^3+2y-3x^2)dx+(2x+3xy^2)dy=0$. We can check that $M_y=3y^2+2=N_x$. Integrating M with respect to x gives that $\psi(x,y)=xy^3+2xy-x^3+g(y)$, then $\psi_y=3xy^2+2x+g'(y)=2x+3xy^2$, which means that $g'(y)=0$, so we obtain that $g(y)=0$ is acceptable. Therefore the solution is $xy^3+2xy-x^3=c$.

17. The equation is *separable*, because $y'=e^{x+y}=e^xe^y$. Separation of variables yields the equation $e^{-y}dy=e^xdx$, which turns into $-e^{-y}=e^x+c$ after integration and we obtain the implicitly defined solution $e^x+e^{-y}=c$.

SS 18. The equation is *exact*. Algebraic manipulations give us the symmetric form $(2y^2+6xy-4)dx+(3x^2+4xy+3y^2)dy=0$. We can check that $M_y=4y+6x=N_x$. Integrating M with respect to x gives that $\psi(x,y)=2y^2x+3x^2y-4x+g(y)$, then $\psi_y=4yx+3x^2+g'(y)=3x^2+4xy+3y^2$, which means that $g'(y)=3y^2$, so we obtain that $g(y)=y^3$. Therefore the solution is $2xy^2+3x^2y-4x+y^3=c$.

19. The equation is *linear*. Division by t gives $y'+(1+(1/t))y=e^{2t}/t$, so the integrating factor is $\mu(t)=e^{\int(1+(1/t))dt}=e^{t+\ln t}=te^t$. The equation turns into $te^ty'+(te^t+e^t)y=(te^ty)'=e^{3t}$. Integration therefore leads to $te^ty=e^{3t}/3+c$ and the solution is $y=e^{2t}/(3t)+ce^{-t}/t$.

SS 20. The equation is *homogeneous*. (See Section 2.2, Problem 25) We can write the equation in the form $y'=y/x+e^{y/x}$. We substitute $u(x)=y(x)/x$, which means $y=ux$ and then $y'=u'x+u$. We obtain the equation $u'x+u=u+e^u$, which is a separable equation. Separation of variables gives us $e^{-u}du=(1/x)dx$, so after integration we obtain that $-e^{-u}=\ln|x|+c$ and then substituting $u=y/x$ back into this we get the implicit solution $e^{-y/x}+\ln|x|=c$.

SS 21. The equation can be made *exact* with an appropriate integrating factor. Algebraic manipulations give us the symmetric form $xdx-(x^2y+y^3)dy=0$. We can check that $(M_y-N_x)/M=2xy/x=2y$ depends only on y, which means we will be able to find an integrating factor in the form $\mu(y)$. This integrating factor is $\mu(y)=e^{-\int 2ydy}=e^{-y^2}$. The equation after multiplication becomes

$$e^{-y^2}xdx-e^{-y^2}(x^2y+y^3)dy=0.$$

This equation is exact now, as we can check that $M_y=-2ye^{-y^2}x=N_x$. Integrating M with respect to x gives that $\psi(x,y)=e^{-y^2}x^2/2+g(y)$, then $\psi_y=-e^{-y^2}x^2y+g'(y)=-e^{-y^2}(x^2y+y^3)$, which means that $g'(y)=-y^3e^{-y^2}$. We can integrate this expression by substituting $u=-y^2$, $du=-2ydy$. Integrating by parts, we obtain that

$$g(y) = -\int y^3 e^{-y^2} dy = -\int \frac{1}{2} u e^u du = -\frac{1}{2}(ue^u - e^u) + c =$$
$$-\frac{1}{2}(-y^2 e^{-y^2} - e^{-y^2}) + c.$$

Therefore the solution is defined implicitly as $e^{-y^2} x^2/2 - \frac{1}{2}(-y^2 e^{-y^2} - e^{-y^2}) = c$, or (after simplification) as $e^{-y^2}(x^2 + y^2 + 1) = c$. Remark: using the hint and substituting $u = x^2$ gives us $du = 2xdx$. The equation turns into $2(uy + y^3)dy = du$, which is a linear equation for u as a function of y. The integrating factor is e^{-y^2} and we obtain the same solution after integration.

22. The equation is *homogeneous*. (See Section 2.2, Problem 25) We can see that

$$y' = \frac{x+y}{x-y} = \frac{1 + (y/x)}{1 - (y/x)}.$$

We substitute $u = y/x$, which means also that $y = ux$ and then $y' = u'x + u = (1+u)/(1-u)$, which implies that

$$u'x = \frac{1+u}{1-u} - u = \frac{1+u^2}{1-u},$$

a separable equation. Separating the variables yields

$$\frac{1-u}{1+u^2} du = \frac{dx}{x},$$

and then integration gives $\arctan u - \ln(1+u^2)/2 = \ln|x| + c$. Substituting $u = y/x$ back into this expression and using that

$$-\ln(1 + (y/x)^2)/2 - \ln|x| = -\ln(|x|\sqrt{1 + (y/x)^2}) = -\ln(\sqrt{x^2 + y^2})$$

we obtain that the solution is $\arctan(y/x) - \ln(\sqrt{x^2 + y^2}) = c$.

23. The equation is *homogeneous*. (See Section 2.2, Problem 25) Algebraic manipulations show that it can be written in the form

$$y' = \frac{3y^2 + 2xy}{2xy + x^2} = \frac{3(y/x)^2 + 2(y/x)}{2(y/x) + 1}.$$

Substituting $u = y/x$ gives that $y = ux$ and then

$$y' = u'x + u = \frac{3u^2 + 2u}{2u + 1},$$

which implies that

$$u'x = \frac{3u^2 + 2u}{2u + 1} - u = \frac{u^2 + u}{2u + 1},$$

a separable equation. We obtain that $(2u+1)du/(u^2+u) = dx/x$, which in turn means that $\ln(u^2 + u) = \ln|x| + \tilde{c}$. Therefore, $u^2 + u = cx$ and then substituting $u = y/x$ gives us the solution $(y^2/x^3) + (y/x^2) = c$.

SS 24. This is a *Bernoulli* equation. (See Section 2.4, Problem 23) If we substitute $u = y^{-1}$, then $u' = -y^{-2}y'$, so $y' = -u'y^2 = -u'/u^2$ and the equation becomes

$-xu'/u^2 + (1/u) - e^{2x}/u^2 = 0$, and then $u' - u/x = -e^{2x}/x$, which is a linear equation. The integrating factor is $e^{-\int (1/x)dx} = e^{-\ln x} = 1/x$, and we obtain that $(u'/x) - (u/x^2) = (u/x)' = -e^{2x}/x^2$. The integral of the function on the right hand side can not be expressed in a closed form using elementary functions, so we have to express the solution using integrals. Let us integrate both sides of this equation from 1 to x. We obtain that the left hand side turns into

$$\int_1^x (u(s)/s)'ds = (u(x)/x) - (u(1)/1) = \frac{1}{yx} - \frac{1}{y(1)} = \frac{1}{yx} - 1/2.$$

The right hand side gives us $-\int_1^x [e^{2s}/s^2]\,ds$. So we find that

$$1/y = -x\int_1^x [e^{2s}/s^2]\,ds + (x/2).$$

25. Let y_1 be a solution, i.e. $y_1' = q_1 + q_2 y_1 + q_3 y_1^2$. Now let $y = y_1 + (1/v)$ also be a solution. Differentiating this expression with respect to t and using that y is also a solution we obtain $y' = y_1' - (1/v^2)v' = q_1 + q_2 y + q_3 y^2 = q_1 + q_2(y_1 + (1/v)) + q_3(y_1 + (1/v))^2$. Now using that y_1 was also a solution we get that $-(1/v^2)v' = q_2(1/v) + 2q_3(y_1/v) + q_3(1/v^2)$, which, after some simple algebraic manipulations turns into $v' = -(q_2 + 2q_3 y_1)v - q_3$.

SS 26.(a) Using the idea of Problem 25, we obtain that $y = t + (1/v)$, and v satisfies the differential equation $v' = -1$. This means that $v = -t + c$ and then $y = t + (c-t)^{-1}$.

(b) Using the idea of Problem 25, we set $y = (1/t) + (1/v)$, and then v satisfies the differential equation $v' = -1 - (v/t)$. This is a linear equation with integrating factor $\mu(t) = t$, and the equation turns into $tv' + v = (tv)' = -t$, which means that $tv = -t^2/2 + c$, so $v = -(t/2) + (c/t)$ and $y = (1/t) + (1/v) = (1/t) + 2t/(2c - t^2)$.

(c) Using the idea of Problem 25, we set $y = \sin t + (1/v)$. Then v satisfies the differential equation $v' = -\tan t v - 1/(2\cos t)$. This is a linear equation with integrating factor $\mu(t) = 1/\cos t$, which turns the equation into

$$v'/\cos t + v\sin t/\cos^2 t = (v/\cos t)' = -1/(2\cos^2 t).$$

Integrating this we obtain that $v = c\cos t - (1/2)\sin t$, and the solution is $y = \sin t + (c\cos t - (1/2)\sin t)^{-1}$.

27.(a) The equation is $y' = (1-y)(x+by) = x + (b-x)y - by^2$. We set $y = 1 + (1/v)$ and differentiate: $y' = -v^{-2}v' = x + (b-x)(1 + (1/v)) - b(1 + (1/v))^2$, which, after simplification, turns into $v' = (b+x)v + b$.

(b) When $x = at$, the equation is $v' - (b + at)v = b$, so the integrating factor is $\mu(t) = e^{-bt - at^2/2}$. This turns the equation into $(v\mu(t))' = b\mu(t)$, so $v\mu(t) = \int b\mu(t)dt$, and then $v = (b\int \mu(t)dt)/\mu(t)$.

28. Substitute $v=y'$, then $v'=y''$. The equation turns into $t^2v' + 2tv = (t^2v)' = 1$, which yields $t^2v = t + c_1$, so $y' = v = (1/t) + (c_1/t^2)$. Integrating this expression gives us the solution $y = \ln t - (c_1/t) + c_2$.

29. Set $v = y'$, then $v' = y''$. The equation with this substitution is $tv' + v = (tv)' = 1$, which gives $tv = t + c_1$, so $y' = v = 1 + (c_1/t)$. Integrating this expression yields the solution $y = t + c_1 \ln t + c_2$.

30. Set $v = y'$, so $v' = y''$. The equation is $v' + tv^2 = 0$, which is a separable equation. Separating the variables we obtain $dv/v^2 = -tdt$, so $-1/v = -t^2/2 + c$, and then $y' = v = 2/(t^2 + c_1)$. Now depending on the value of c_1, we have the following possibilities: when $c_1 = 0$, then $y = -2/t + c_2$, when $0 < c_1 = k^2$, then $y = (2/k)\arctan(t/k) + c_2$, and when $0 > c_1 = -k^2$ then

$$y = (1/k)\ln|(t-k)/(t+k)| + c_2.$$

We also divided by $v = y'$ when we separated the variables, and $v = 0$ (which is $y = c$) is also a solution.

31. Substitute $v = y'$ and $v' = y''$. The equation is $2t^2v' + v^3 = 2tv$. This is a *Bernoulli* equation (See Section 2.4, Problem 19), so the substitution $z = v^{-2}$ yields $z' = -2v^{-3}v'$, and the equation turns into $2t^2v'v^3 + 1 = 2t/v^2$, i.e. into $-2t^2z'/2 + 1 = 2tz$, which in turn simplifies to $t^2z' + 2tz = (t^2z)' = 1$. Integration yields $t^2z = t + c$, which means that $z = (1/t) + (c/t^2)$. Now $y' = v = \pm\sqrt{1/z} = \pm t/\sqrt{t + c_1}$ and another integration gives

$$y = \pm\frac{2}{3}(t - 2c_1)\sqrt{t + c_1} + c_2.$$

The substitution also loses the solution $v = 0$, i.e. $y = c$.

SS 32. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. The equation turns into $yv'v + v^2 = 0$, where the differentiation is with respect to y now. This is a separable equation, separation of variables yields $-dv/v = dy/y$, and then $-\ln v = \ln y + \tilde{c}$, so $v = 1/(cy)$. Now this implies that $y' = 1/(cy)$, where the differentiation is with respect to t. This is another separable equation and we obtain that $cydy = 1dt$, so $cy^2/2 = t + d$ and the solution is defined implicitly as $y^2 = c_1t + c_2$.

33. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v + y = 0$, where the differentiation is with respect to y. This is a separable equation which simplifies to $vdv = -ydy$. We obtain that $v^2/2 = -y^2/2 + c$, so $y' = v(y) = \pm\sqrt{c - y^2}$. We separate the variables again to get $dy/\sqrt{c - y^2} = \pm dt$, so $\arcsin(y/\sqrt{c}) = t + d$, which means that $y = \sqrt{c}\sin(\pm t + d) = c_1 \sin(t + c_2)$.

34. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $yv'v - v^3 = 0$, where the differentiation is with respect to y. This separable equation gives us $dv/v^2 = dy/y$, which means that $-1/v = \ln|y| + c$, and then $y' = v = 1/(c - \ln|y|)$. We separate variables again to obtain $(c - \ln|y|)dy = dt$, and then integration yields the implicitly defined solution $cy - (y\ln|y| - y) = t + d$. Also, $y = c$ is a solution which we lost when we divided by $v = 0$.

SS 35. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v + v^2 = 2e^{-y}$, where the differentiation is with respect to y. This is a *Bernoulli* equation (See Section 2.4, Problem 23) and substituting $z = v^2$ we get that $z' = 2vv'$, which means that the equation reads $z' + 2z = 4e^{-y}$. The integrating factor is $\mu(y) = e^{2y}$, which turns the equation into $e^{2y}z' + 2e^{2y}z = (e^{2y}z)' = 4e^{y}$. Integration gives us $v^2 = z = 4e^{-y} + ce^{-2y}$. This implies that $y' = v = \pm e^{-y}\sqrt{c + 4e^{y}}$. Separation of variables now shows that $\pm e^{y}dy/\sqrt{c + 4e^{y}} = dt$. Integration and simplification gives $\pm(1/2)(c + 4e^{y})^{1/2} = t + d$. Algebraic manipulations then yield the implicitly defined solution $e^{y} = (t + c_2)^2 + c_1$.

SS 36. Suppose that $y' = v(y)$ and then $y'' = v'(y)v(y)$. The equation is $v^2v' = 2$, which gives us $v^3/3 = 2y + c$. Now plugging 0 in place of t gives that $2^3/3 = 2 \cdot 1 + c$ and we get that $c = 2/3$. This turns into $v^3 = 6y + 2$, i.e. $y' = (6y + 2)^{1/3}$. This separable equation gives us $(6y + 2)^{-1/3}dy = dt$, and integration shows that $(1/6)(3/2)(6y + 2)^{2/3} = t + d$. Again, plugging in $t = 0$ gives us $d = 1$ and the solution is $(6y + 2)^{2/3} = 4(t + 1)$. Solving for y here yields $y = (4/3)(t + 1)^{3/2} - 1/3$.

37. Set $v = y'$, then $v' = y''$. The equation with this substitution turns into the equation $(1 + t^2)v' + 2tv = ((1 + t^2)v)' = -3t^{-2}$. Integrating this we get that $(1 + t^2)v = 3t^{-1} + c$, and $c = -5$ from the initial conditions. This means that $y' = v = 3/(t(1 + t^2)) - 5/(1 + t^2)$. The partial fraction decomposition of the first expression shows that $y' = 3/t - 3t/(1 + t^2) - 5/(1 + t^2)$ and then another integration here gives us that $y = 3\ln t - (3/2)\ln(1 + t^2) - 5\arctan t + d$. The initial conditions identify $d = 2 + (3/2)\ln 2 + 5\pi/4$, and we obtained the solution.