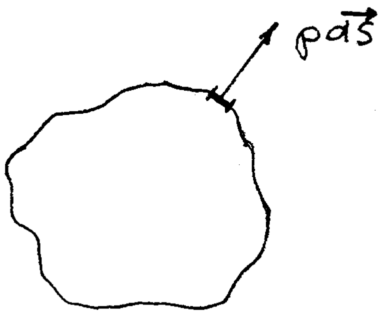
CHAPTER 2

2.1



$\vec{F} = -\oiint_S p\,d\vec{S}$

If p = constant = p_∞

$\vec{F} = -p_\infty \oiint_S p\,d\vec{S}$ (1)

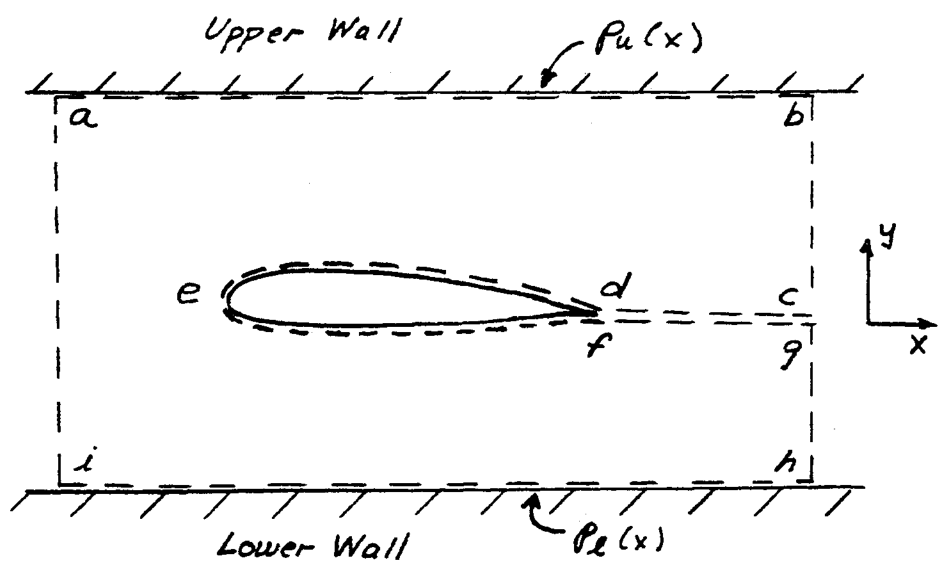
However, the integral of the surface vector over a closed surface is zero, i.e.,

$\oiint_S d\vec{S} = 0$

Hence, combining Eqs. (1) and (2), we have

$\boxed{\vec{F} = 0}$

2.2



Denote the pressure distributions on the upper and lower walls by $p_u(x)$ and $p_\ell(x)$ respectively. The walls are close enough to the model such that p_u and p_ℓ are not necessarily equal to p_∞. Assume that faces ai and bh are far enough upstream and downstream of the model such that

$p = p_\infty$ and $v = 0$ and ai and bh.

Take the y-component of Eq. (2.66)

$$L = -\oiint_S (\rho\vec{V}\cdot d\vec{S})\, v - \iint_{abhi} (p\,d\vec{S})_y$$

The first integral = 0 over all surfaces, either because $\vec{V}\cdot d\vec{s} = 0$ or because $v = 0$. Hence

$$L' = -\iint_{abhi} (p\,d\vec{S})_y = -\left[\int_a^b p_u\,dx - \int_i^h p_\ell\,dx\right]$$

Minus sign because y-component is in downward Direction.

Note: In the above, the integrals over ia and bh cancel because $p = p_\infty$ on both faces. Hence

$$\boxed{L' = \int_i^h p_\ell\,dx - \int_a^b p_u\,dx}$$

2.3 $$\frac{dy}{dx} = \frac{v}{u} = \frac{cy/(x^2+y^2)}{cx/(x^2+y^2)} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ell n\, y = \ell n\, x + c_1 = \ell n\,(c_2 x)$$

$$\boxed{y = c_2 x}$$

The streamlines are straight lines emanating from the origin. (This is the velocity field and streamline pattern for a source, to be discussed in Chapter 3.)

2.4 $$\frac{dy}{dx} = \frac{v}{u} = -\frac{x}{y}$$

$$y\,dy = -x\,dx$$

$y^2 = -x^2 + \text{const}$

$x^2 + y^2 = \text{const.}$

The streamlines are concentric with their centers at the origin. (This is the velocity field and streamline pattern for a vortex, to be discussed in Chapter 3.)

2.5 From inspection, since there is no radial component of velocity, the streamlines must be circular, with centers at the origin. To show this more precisely,

$$u = - V_\theta \sin = - cr \frac{y}{r} = - cy$$

$$v = V_\theta \cos \theta = cr \frac{x}{r} = cx$$

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{x}{y}$$

$$\boxed{y^2 + x^2 = \text{const.}}$$

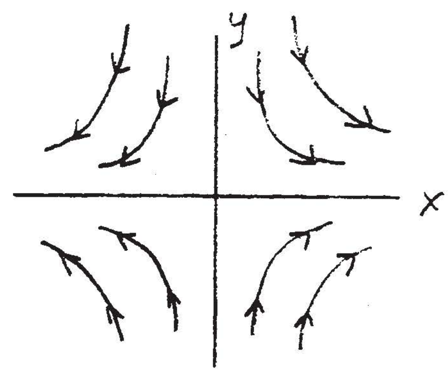
This is the equation of a circle with the center at the origin. (This velocity field corresponds to solid body rotation.)

2.6 $$\frac{dy}{dx} = \frac{v}{u} = -\frac{y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\ell n\, y = x\ \ell n\, x + c_1$$

$$y = c_2/x$$



The streamlines are hyperbolas.

2.7 (a) $\frac{1}{\delta v}\frac{D(\delta v)}{Dt} = \nabla \cdot \vec{V}$

In polar coordinates: $\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r\, V_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$

Transformation: $x = r \cos\theta$

$y = r \sin\theta$

$V_r = u \cos\theta + v \sin\theta$

$V_\theta = - u \sin\theta + v \cos\theta$

$$u = \frac{cx}{(x^2 + y^2)} = \frac{cr \cos\theta}{r^2} = \frac{c \cos\theta}{r}$$

$$v = \frac{cy}{(x^2 + y^2)} = \frac{cr \sin\theta}{r^2} = \frac{c \sin\theta}{r}$$

$$V_r = \frac{c}{r} \cos^2\theta + \frac{c}{r} \sin^2\theta = \frac{c}{r}$$

$$V_\theta = -\frac{c}{r} \cos\theta \sin\theta + \frac{c}{r} \cos\theta \sin\theta = 0$$

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(c) + \frac{1}{r}\frac{\partial(0)}{\partial \theta} = 0$$

(b) From Eq. (2.23)

$$\nabla \times \vec{V} = e_z \left[\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} \right]$$

$$\nabla \times V = e_z [0 + 0 - 0] = \boxed{0}$$

The flowfield is irrotational.

2.8 $$u = \frac{cy}{(x^2 + y^2)} = \frac{cr \sin\theta}{r^2} = \frac{c \sin\theta}{r}$$

$$v = \frac{-cx}{(x^2 + y^2)} = \frac{cr \cos\theta}{r^2} = -\frac{c \cos\theta}{r}$$

$$V_r = \frac{c}{r} \cos\theta \sin\theta - \frac{c}{r} \cos\theta \sin\theta = 0$$

$$V_\theta = -\frac{c}{r} \sin^2\theta - \frac{c}{r} \cos^2\theta = -\frac{c}{r}$$

(a) $$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(0) + \frac{1}{r}\frac{\partial(-c/r)}{\partial \theta} = 0 + 0 = \boxed{0}$$

(b) $\nabla \times \vec{V} = \vec{e}_z \left[\frac{\partial(-c/r)}{\partial r} - \frac{c}{r^2} - \frac{1}{r}\frac{\partial(0)}{\partial \theta} \right]$

$$= \vec{e}_z \left[\frac{c}{r^2} - \frac{c}{r_2} - 0 \right]$$

$\nabla \times \vec{V} = \boxed{0}$ <u>except</u> at the origin, where r = 0. The flowfield is singular at the origin.

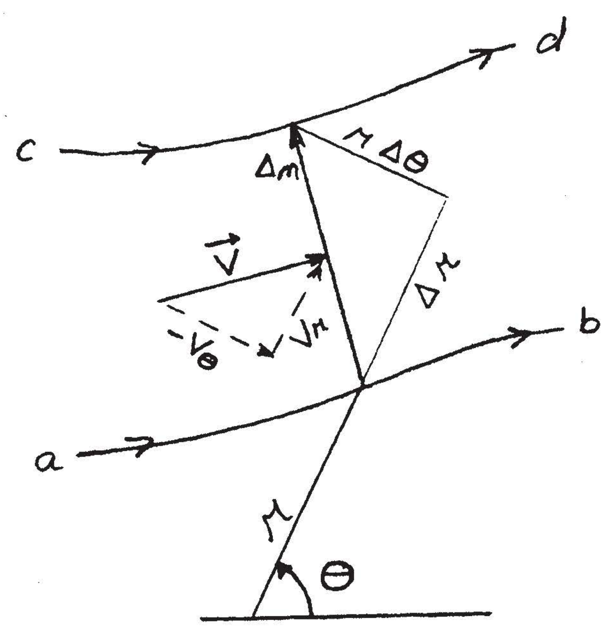
2.9 $V_r = 0.$ $\quad V_\theta = c\,r$

$$\nabla \times \vec{V} = \vec{e}_z \left[\frac{\partial(c/r)}{\partial r} + \frac{cr}{r} - \frac{1}{r}\frac{\partial(0)}{\partial \theta} \right]$$

$$= \vec{e}_z\,(c + c - 0) = 2c\,\vec{e}_z$$

The vorticity is <u>finite</u>. The flow is <u>not</u> irrotational; it is <u>rotational</u>.

2.10



Mass flow between streamlines = $\Delta \bar{\psi}$

$$\Delta \bar{\psi} = \rho V \Delta n$$

$$\Delta \bar{\psi} = (-\rho V_\theta) \Delta r + \rho V_r (r\theta)$$

Let cd approach ab

$$d\psi = -\rho V_\theta dr + \rho r V_r d\theta \qquad (1)$$

Also, since $\bar{\psi} = \bar{\psi}(r,\theta)$, from calculus

$$d\bar{\psi} = \frac{\partial \bar{\psi}}{\partial r} dr + \frac{\partial \bar{\psi}}{\partial \theta} d\theta \qquad (2)$$

Comparing Eqs. (1) and (2)

$$-\rho V_\theta = \frac{\partial \bar{\psi}}{\partial r}$$

and

$$\rho r V_r = \frac{\partial \bar{\psi}}{\partial \theta}$$

or:

$$\rho V_r = \frac{1}{r}\frac{\partial \bar{\psi}}{\partial \theta}$$

$$\rho V_\theta = -\frac{\partial \bar{\psi}}{\partial r}$$

2.11 $u = cx = \frac{\partial \psi}{\partial y} : \psi = cxy + f(x)$ (1)

$$v = -cy = -\frac{\partial \psi}{\partial x} : \psi = cxy + f(y) \qquad (2)$$

Comparing Eqs. (1) and (2), f(x) and f(y) = constant

$$\boxed{\psi = c\,x\,y + \text{const.}} \tag{3}$$

$$u = cx = \frac{\partial \psi}{\partial x} \;:\; \phi = cx^2 + f(y) \tag{4}$$

$$v = -\,cy = \frac{\partial \psi}{\partial y} \;:\; \phi = -\,cy^2 + f(x) \tag{5}$$

Comparing Eqs. (4) and (5), $f(y) = -\,cy^2$ and $f(x) = cx^2$

$$\boxed{\phi = c\,(x^2 - y^2)} \tag{6}$$

Differentiating Eq. (3) with respect to x, holding ψ = const.

$$0 = cx\,\frac{dy}{dx} + cy$$

or,

$$\left(\frac{dy}{dx}\right)_{\psi=\text{const}} = -\,y/x \tag{7}$$

Differentiating Eq. (6) with respect to x, holding ϕ = const.

$$0 = 2\,c\,x - 2\,c\,y\,\frac{dy}{dx}$$

or,

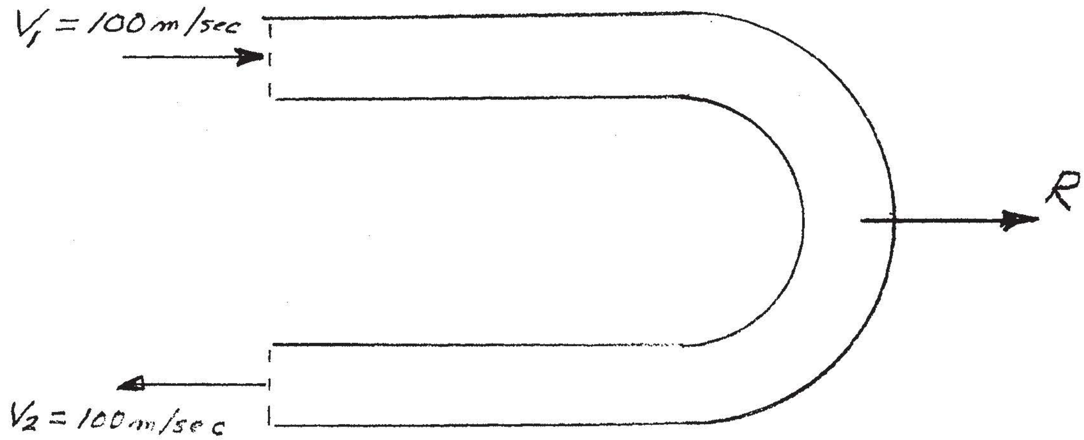
$$\left(\frac{dy}{dx}\right)_{\phi=\text{const}} = x/y \tag{8}$$

Comparing Eqs. (7) and (8), we see that

$$\left(\frac{dy}{dx}\right)_{\psi=\text{const}} = -\,\frac{1}{\left(\frac{dy}{dx}\right)_{\phi=\text{const}}}$$

Hence, lines of constant ψ are perpendicular to lines of constant ϕ.

2.12. The geometry of the pipe is shown below.



As the flow goes through the U-shape bend and is turned, it exerts a net force R on the internal surface of the pipe. From the symmetric geometry, R is in the horizontal direction, as shown, acting to the right. The equal and opposite force, -R, exerted by the pipe on the flow is the mechanism that reverses the flow velocity. The cross-sectional area of the pipe inlet is $\pi d^2/4$ where d is the inside pipe diameter. Hence, $A = \pi d^2/4 = \pi(0.5)^2/4 = 0.196\text{m}^2$. The mass flow entering the pipe is

$$\dot{m} = \rho_1 A V_1 = (1.23)(0.196)(100) = 24.11 \text{ kg/sec.}$$

Applying the momentum equation, Eq. (2.64) to this geometry, we obtain a result similar to Eq. (2.75), namely

$$R = -\oiint (\rho \mathbf{V} \cdot \mathbf{dS})\, \mathbf{V} \qquad (1)$$

Where the pressure term in Eq. (2.75) is zero because the pressure at the inlet and exit are the same values. In Eq. (1), the product (ρ **V** · **dS**) is negative at the inlet (**V** and **dS** are in opposite directions), and is positive at the exit (**V** and **dS**) are in the same direction). The magnitude of ρ **V** · **dS** is simply the mass flow, $\dot{m}$. Finally, at the inlet V_1 is to the right, hence it is in the positive x-direction. At the exit, V_2 is to the left, hence it is in the negative x-direction. Thus, $V_2 = -V_1$. With this, Eq. (1) is written as

$$R = -[-\dot{m}\, V_1 + \dot{m}\, V_2] = \dot{m}\,(V_1 - V_2)$$

$$= \dot{m}\,[V_1 - (-V_1)] = \dot{m}\,(2V_1)$$

$$R = (24.11)(2)(100) = \boxed{4822 \text{ N}}$$

2.13 From Example 2.1, we have

$$u = V_\infty \left[1 + \frac{h}{\beta}\frac{2\pi}{\ell}\left(\cos\frac{2\pi x}{\ell}\right) e^{-2\pi\beta y/\ell}\right] \tag{2.35}$$

and

$$v = -V_\infty h \frac{2\pi}{\ell}\left(\sin\frac{2\pi x}{\ell}\right) e^{-2\pi\beta y/\ell} \tag{2.36}$$

Thus,

$$\frac{\partial\varphi}{\partial x} = u = V_\infty + \left(\frac{V_\infty h}{\beta}\right)\left(\frac{2\pi}{\ell}\right)\left(\cos\frac{2\pi x}{\ell}\right) e^{-2\pi\beta y/\ell} \tag{2.35a}$$

Integrating (2.35a) with respect to x, we have

$$\varphi = V_\infty x + \left(\frac{V_\infty h}{\beta}\right)\left(\frac{2\pi}{\ell}\right)\left(\sin\frac{2\pi x}{\ell}\right)\frac{1}{\left(\frac{2\pi}{\ell}\right)} e^{-2\pi\beta y/\ell} + f(y)$$

$$\varphi = V_\infty x + \frac{V_\infty h}{\beta}\left(\sin\frac{2\pi x}{\ell}\right) e^{-2\pi\beta y/\ell} + f(y) \tag{2.35b}$$

From (2.36)

$$\frac{\partial\varphi}{\partial y} = v = -V_\infty h \frac{2\pi}{\ell}\left(\sin\frac{2\pi x}{\ell}\right) e^{-2\pi\beta y/\ell} \tag{2.36a}$$

Integrating (2.36a) with respect to y, we have

$$\varphi = V_\infty\, h \left(\frac{2\pi}{\ell}\right)\left(\sin\frac{2\pi x}{\ell}\right)\left(e^{-2\pi\beta y/\ell}\right)\frac{1}{\left(\frac{2\pi\beta}{\ell}\right)} + f(x)$$

$$\varphi = \frac{V_\infty h}{\beta}\left(\sin\frac{2\pi x}{\ell}\right)\left(e^{-2\pi\beta y/\ell}\right) + f(x) \qquad (2.36b)$$

Comparing (2.35b) and (2.36b), which represent the same function for φ, we see in (2.36b) that $f(x) = V_\infty x$. So the velocity potential for the compressible subsonic flow over a wavy well is:

$$\boxed{\varphi = V_\infty x + \frac{V_\infty h}{\beta}\left(\sin\frac{2\pi x}{\ell}\right) e^{-2\pi\beta y/\ell}}$$

2.14 The equation of a streamline can be found from Eq. (2.118)

$$\frac{dy}{dx} = \frac{v}{u}$$

For the flow over the wavy wall in Example 2.1,

$$\frac{dy}{dx} = \frac{-V_\infty\, h\frac{2\pi}{\ell}\left(\sin\frac{2\pi x}{\ell}\right) e^{-2\pi\beta y/\ell}}{V_\infty\left[1+\frac{h}{\beta}\frac{2\pi}{\ell}\left(\cos\frac{2\pi x}{\ell}\right) e^{-2\pi\beta y/\ell}\right]}$$

As $y \to \infty$, then $e^{-2\pi\beta y/\ell} \to 0$. Thus,

$$\frac{dy}{dx} \to \frac{0}{V_\infty + 0} = 0$$

The slope is zero. Hence, the streamline at $y \to \infty$ is straight.