Chapter 2: Probability, Statistics, and Traffic Theories

P2.1. A random number generator produces numbers between 1 and 99. If the current value of the random variable is 45, then what is the probability that the next randomly generated value for the same random variable, will also be 45. Explain clearly.

[Solution]

Let the random variable be X such that $1 \le X \le 99$.
Current value of $X = 45$.
Since the value of generation of random numbers is independent of the previous generation, the probability of generating 45 again is the same as for the first time. Hence,

$$P(X = 45) = \frac{1}{99}.$$

P2.2. A random digit generator on a computer is activated three times consecutively to simulate a random three-digit number.

(a) How many random three-digit numbers are possible?
(b) How many numbers will begin with the digit 2?
(c) How many numbers will end with the digit 9?
(d) How many numbers will begin with the digit 2 and end with the digit 9?
(e) What is the probability that a randomly formed number ends with 9 given that it begins with a 2?

[Solution]

(a) 900
(b) 100
(c) 90
(d) 10
(e) 1/10

P2.3. A snapshot of the traffic pattern in a cell with 10 users of a wireless system is given as follows:

User Number	1	2	3	4	5	6	7	8	9	10
Call Initiation Time	0	2	0	3	1	7	4	2	5	1
Call Holding Time	5	7	4	8	6	2	1	4	3	2

(a) Assuming the call setup/connection and call disconnection time to be zero, what is the average duration of a call?
(b) What is the minimum number of channels required to support this sequence of calls?
(c) Show the allocation of channels to different users for part (b) of the Problem.
(d) Given the number of channels obtained in part (b), for what fraction of time are the channels utilized?

[Solution]

(a) Average duration of a call is

$$\frac{5+7+4+8+6+2+1+4+3+2}{10}=4.2.$$

(b) By plotting the number of calls by all users, we can determine how many users need to have a channel simultaneously. This gives us the minimum number of channels required to support the sequence of calls as 6.

(c) Allocation of channels to various users is

Channel number	1	2	3	4	5	6
User number	1	3	5	10	2	8
User number (allocated to same channel)	9	7		4		6

(d) Total duration of the calls is 42.
Total amount of time channels are available is $11\times 6=66$.
Therefore fraction of time channels are used is $\frac{42}{66}=0.6363$.

P2.4. A department survey found that 4 of 10 graduate students use CDMA cell phone service. If 3 graduate students are selected at random, what is the probability that 3 graduate students use CDMA cell phones?

[Solution]

$$\frac{C_4^3}{C_{10}^3}=\frac{2}{60}=0.0333.$$

P2.5. There are three red balls and seven white balls in box A, and six red balls and four white balls in box B. After throwing a die, if the number on the die is 1 or 6, then pick a ball from box A. Otherwise, if any other number appears (i.e., 2, 3, 4, or 5), then pick a ball from box B. Selected ball has to put it back before proceeding further. Answer the followings:

(a) What is the probability that the selected ball is red?
(b) What is the probability a white ball is picked up in two successive selections?

[Solution]

(a) Probability of selecting the box × probability of selecting a red ball is equal to

$$=\frac{2}{6}\times\frac{3}{10}\times\frac{4}{6}\times\frac{6}{10}=0.5.$$

(b) Since we are replacing the balls, the occurrence of two successive white balls is independent.
Thus probability of picking two white balls in succession is equal to

$$=\left(\frac{2}{6}\times\frac{7}{10}+\frac{4}{6}\times\frac{4}{10}\right)\left(\frac{2}{6}\times\frac{7}{10}+\frac{4}{6}\times\frac{4}{10}\right)=\frac{1}{4}=0.25.$$

P2.6. Consider an experiment consisting of tossing two true dice. Let *X*, *Y*, and *Z* be the numbers shown on the first die, the second die, and total of both dice, respectively. Find P ($X \le 1$, $Z \le 2$) and P ($X \le 1$) P ($Z \le 2$) to show that *X* and *Y* are not independent.

[Solution]

$$P\left(X \le 1, Z \le 1\right) = \frac{1}{36} \approx 0.02778.$$

$$\mathrm{P}\left(X \le 1\right) P\left(Z \le 1\right) = \frac{1}{6} \times \frac{1}{36} = 0.00463$$

Since P ($X \le 1$, $Z \le 1$) and P ($X \le 1$) P ($Z \le 1$) are not equal, they are not independent.

P2.7. The following table shows the density of the random variable X.

x	1	2	3	4	5	6	7	8
p(x)	0.03	0.01	0.04	0.3	0.3	0.1	0.07	?

(a) Find $p(8)$
(b) Find the table for CDF $F(x)$
(c) Find $P(3 \le X \le 5)$
(d) Find $P(X \le 4)$ and $P(X < 4)$. Are the probabilities the same?
(e) Find $F(-3)$ and $F(10)$

[Solution]

(a) $p(8) = 0.15$

x	1	2	3	4	5	6	7	8
p(x)	0.03	0.01	0.04	0.3	0.3	0.1	0.07	0.15

(b) $F(x)$ is given by

x	1	2	3	4	5	6	7	8
p(x)	0.03	0.04	0.08	0.38	0.68	0.78	0.85	1.0

(c) P ($3 \le X \le 5$) = 0.04 + 0.3 + 0.3 = 0.68
(d) $P(X \le 4) = 0.38$ and $P(X < 4) = 0.08$
Thus they are not equal.
(e) $F(-3) = 0$ and $F(10) = 1$

P2.8. The density for *X* is given in the table of Problem 7.

(a) Find $E[X]$
(b) Find $E[X^2]$
(c) Find $\mathrm{Var}[X]$
(d) Find the standard deviation for *X*.

[Solution]

(a)

$$\begin{aligned} E[X] &= 1 \times 0.03 + 2 \times 0.01 + 3 \times 0.04 + 4 \times 0.3 + 5 \times 0.3 + 6 \times 0.1 \\ &\quad + 7 \times 0.07 + 8 \times 0.15 \\ &= 5.16. \end{aligned}$$

(b) $E\left[X^2\right] = \sum_{x=1}^{8} x^2 p(x) = 29.36.$

(c) Var[X] = 2.7344.

(d) Standard deviation = 1.65.

P2.9. Find the probability when

(a) $k = 2$ and $\lambda = 0.01$ for Poisson distribution.

(b) $p = 0.01$ and $k = 2$ for geometric distribution.

(c) Repeat (b) when binomial distribution is used and $n = 10$.

[Solution]

(a) $p_k = \dfrac{\lambda^k e^{-\lambda}}{k!}$, when $k = 2$, $\lambda = 0.01$, Probability = 0.0000495025.

(b) $p_k = p(1-p)^k$, when $k = 2$, $\lambda = 0.01$, Probability = 0.0000495025.

(c) $p_k = \begin{pmatrix} k \\ n \end{pmatrix} p^k (1-p)^{n-k}$ when $n = 10$, $k = 2$, $\lambda = 0.01$, Probability = 0.00415235.

P2.10. Find the distribution function of the maximum of a finite set of independent random variables $\{X_1, X_2, ..., X_n\}$, where Xi has distribution function F_{X_i}. What is this distribution when X_i is exponential with a mean of $\dfrac{1}{\mu_i}$.

[Solution]

$$Y = \max\{X_1, X_2, \ldots, X_n\}$$

$$\begin{aligned} P\ (Y \le y) &= P\left(\max\{X_1, X_2, \ldots, X_n\} \le y\right) \\ &= P\left(X_1 \le y, X_2 \le y, \ldots, X_n \le y\right) \\ &= \prod_{i=1}^{n} P\left(X_i \le y\right) \\ &= \prod_{i=1}^{n} F_{X_i}(y). \end{aligned}$$

When X_i is exponential with mean $\dfrac{1}{\mu_i}$,

$$F_{X_i}(y) = 1 - e^{-\mu_i y}.$$

Therefore, the CDF of Y is given by

$$F_Y(y) = \prod_{i=1}^{n} F_{X_i}(y)$$
$$= \prod_{i=1}^{n} \left(1 - e^{-\mu_i y}\right).$$

P2.11. The number of calls arrive under a particular time in a cell has been established to be a Poisson distribution. The average number of calls arriving in a cell in 1 millisecond is 5. What is the probability that 8 calls arrive in a cell in a given millisecond?

[Solution]

Since λ = 5,

$$p_k = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{5^8 e^{-5}}{8!} = 0.065278.$$

P2.12. Given that the number of arrivals of data packet in the receiver follows a Poisson distribution on which arrival rate is 10 arrivals per second. What is the probability that the number of arrivals is more than 8 but less than 11 during a time of interval of 2 seconds?

[Solution]

The number of arrivals during a time interval of 2 seconds follows a Poisson distribution with mean 10 × 2 = 20 arrivals/sec. Hence,

$$\begin{aligned}P(8<k<11) &= P(k=9, k=10)\\ &= P(k=9)+P(k=10)\\ &= \frac{20^9 e^{-20}}{9!} + \frac{20^{10} e^{-20}}{10!}\\ &= 0.00872446.\end{aligned}$$

P2.13. In a wireless office environment, all calls are made between 8 am and 5 pm over the period of 24 hours. Assuming the number of calls to be uniformly distributed between 8 am and 5 pm, find the pdf of the number of calls over the 24 hour period. Also, determine the CDF and the variance of the call distribution.

[Solution]

Since

$$f(x) = \begin{cases} \frac{1}{9}, & x \in [8,17] \\ 0, & x \in [0,8] \cup (17,24] \end{cases},$$

the CDF is given by

$$F(x)=\begin{cases}0, & x\in[0,8]\\ \dfrac{x-8}{9}, & x\in[8,17]\\ 1, & x\in[17,24]\end{cases},$$

Therefore,

$$\begin{aligned}E[X]&=\int_8^{17} xf(x)dx\\ &=\int_8^{17} x\frac{1}{9}dx\\ &=\frac{25}{2}\\ &=12.5,\\ E\left[X^2\right]&=\int_8^{17} x^2 f(x)dx\\ &=\int_8^{17} x^2\frac{1}{9}dx\\ &=163,\\ Var(X)&=E\left[X\quad\right]-\left(E[X]\right)^2\\ &=6.75.\end{aligned}$$

P2.14. A gambler has a regular coin and a two-headed coin in his packet. The probability of selecting the two-head coin is given as p = 2/3. He select a coin and flips it n = 2 times and obtains heads both the times. What is the probability that the two-headed coin is picked both the times?

[Solution]

Probability is equal to

$$\frac{2}{3}\times 1\times\frac{2}{3}\times 1=\frac{4}{9}\approx 0.4444.2$$

P2.15. A Poisson process exhibits a memoryless property and is of great importance in traffic analysis. Prove that this property is exhibited by all Poisson processes. Explain clearly every step of your analytical proof.

[Solution]

In order to prove that Poisson process is memoryless, we need to prove the following equation:

$$P\left(X>t+\delta \middle| X>\delta\right)=P\left(X>t\right).$$

Thus,

$$\begin{aligned}
P\left(X > t+\delta \middle| X > \delta\right) &= \frac{P\left(X > t+\delta \cap X > \delta\right)}{P\left(X > \delta\right)} \\
&= \frac{P\left(X > t+\delta\right)}{P\left(X > \delta\right)} \\
&= \frac{1-P\left(X \le t+\delta\right)}{1-P\left(X \le \delta\right)} \\
&= 1-P\left(X \le t\right) \\
&= P\left(X > t\right).
\end{aligned}$$

P2.16. What should be a relationship between call arrival rate and service rate when a cellular system is in a steady state? Explain clearly.

[Solution]

When the system reaches the steady state, the call arrival rate should be less or equal to the service rate, otherwise the system will be unstable.

P2.17. Consider a cellular system with an infinite number of channels. In such a system, all arriving calls begin receiving service immediately. The average call holding time is $\dfrac{1}{n\mu}$ when there are n calls in the system. Draw a state transition diagram for this system and develop expressions for the following:

(a) Steady-state probability P_n of n calls in the system.
(b) Steady-state probability P_0 of no calls in the system.
(c) Average number of calls in the system, L_S.
(d) Average dwell time, W_S.
(e) Average queue length, L_Q.

[Solution]

The system model is M/M/∞/∞ with average arrival rate of λ and average service rate of μ/n.

Probability P_n for the system with n calls can be found through the system balance equations as follows:

$$\lambda P_i = \left(i+1\right)\mu P_{i+1}, \qquad 0 \le i \le \infty.$$

Thus, we have

$$P_i = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i P_0, \qquad 0 \le \mathrm{i} \le \infty.$$

Using normalized condition

$$\sum_{i=0}^{\infty} P_i = 1,$$

we have

$$P_0 = \left[\sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{\lambda}{\mu} \right)^i \right]^{-1}$$
$$= e^{-\frac{\lambda}{\mu}}.$$

(a) Steady-state probability P_n of n calls in the system is given by

$$P_n = \frac{1}{\mathrm{n}!} \left(\frac{\lambda}{\mu} \right)^n e^{-\frac{\lambda}{\mu}}, \quad 0 \le n \le \infty.$$

(b) Steady-state probability P_0 of no calls in the system is given by

$$P_0 = e^{-\frac{\lambda}{\mu}}.$$

(c) Average number of calls in the system L_S is given by

$$L_S = \sum_{i=1}^{\infty} i P_i$$
$$= \frac{\lambda}{\mu}.$$

(d) Average dwell time W_S is given by

$$W_S = \frac{L_S}{\lambda}$$
$$= \frac{1}{\mu}.$$

(e) Average queue length is equal to 0.

P2.18. Consider a cellular system in which each cell has only one channel (single server) and an infinite buffer for storage the calls. In this cellular system, call arrival rates are discouraged, that is, the call rate is only $\lambda/(n + 1)$ when there are n calls in the system. The interarrival times of calls are exponentially distributed. The call holding times are exponentially distributed with mean rate μ. Develop expressions for the following:

(a) Steady-state probability P_n of n calls in the system.
(b) Steady-state probability P_0 of no calls in the system.
(c) Average number of calls in the system, L_S.
(d) Average dwell time, W_S.
(e) Average queue length, L_Q.

[Solution]

The system model is M/M/1/∞ with average arrival rate of $\lambda/ (n + 1)$, and average service rate of μ.

Traffic intensity $\rho = \dfrac{\frac{\lambda}{n+1}}{\mu} = \dfrac{\lambda}{(n+1)\mu}.$

Probability P_n for the system with n calls can be found through the system balance equations as follows:

$$\lambda P_i = (i+1)\mu P_{i+1}, \qquad 0 \le i \le \infty.$$

Thus, we have

$$P_i = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i P_0, \qquad 0 \le \mathrm{i} \le \infty.$$

Using normalized condition

$$\sum_{i=0}^{\infty} P_i = 1,$$

we have

$$P_0 = \left[\sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\right]^{-1}$$
$$= e^{-\frac{\lambda}{\mu}}.$$

(a) Steady-state probability P_n of n calls in the system is given by

$$P_n = \frac{1}{\mathrm{n}!}\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}, \qquad 0 \le n \le \infty.$$

(b) Steady-state probability P_0 of no calls in the system is given by

$$P_0 = e^{-\frac{\lambda}{\mu}}.$$

(c) Average number of calls in the system L_S is given by

$$L_S = \sum_{i=1}^{\infty} iP_i$$
$$= \frac{\lambda}{\mu}.$$

(d) Average arrival rate is

$$\bar{\lambda} = \sum_{i=0}^{\infty} \frac{\lambda}{i+1} P_i$$
$$= \mu\left(1 - e^{-\frac{\lambda}{u}}\right).$$

Therefore, average dwell time W_S is given by

$$W_S = \frac{L_S}{\bar{\lambda}}$$
$$= \frac{\lambda}{\mu^2\left(1 - e^{-\frac{\lambda}{\mu}}\right)}.$$

(e) Since average queue waiting time is given by

$$W_Q = W_S - \frac{1}{\mu}$$

$$= \frac{\lambda - \mu + \mu e^{-\frac{\lambda}{\mu}}}{\mu^2\left(1 - e^{-\frac{\lambda}{\mu}}\right)},$$

average queue length is given by

$$L_Q = \bar{\lambda} W_Q$$

$$= \frac{\lambda - \mu + \mu e^{-\frac{\lambda}{\mu}}}{\mu}.$$

P2.19. In a transition diagram of M/M/5 model, write the state transition equations and find a relation for the system to be in each state.

[Solution]

Hint: The system model is M/M/5 with average arrival rate, and average service rate of

$$\mu = \begin{cases} n\mu, & n = 1, 2, 3, 4, \\ 5\mu, & n \geq 5, \end{cases}$$

The probability of *n* jobs in the system is given by

$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, & n = 1, 2, 3, 4, \\ \frac{1}{5!5^{n-5}}\left(\frac{\lambda}{\mu}\right)^n P_0, & n \geq 5, \end{cases}$$

P2.20. In the M/M/1/∞ queuing system, suppose λ and μ are doubled. How are L_s and W_s changed?

[Solution] L_S is the same as M/M/1/∞ and W_S is the helf of M/M/1/∞.

P2.21. If the number of mechanics in Example 2.4 is increased to 3, what is the impact on the performance parameters?

[Solution] It can be seen that the system is M/M/3/∞ queuing system with λ = 3/m, and the service rate $\mu = \frac{60}{10}$ = 6/m. Therfore, the offered load $\rho = \frac{\lambda}{3\mu} = \frac{1}{6}$.