Chapter 2

2.1 Classify each of the following signals as finite or infinite. For the finite signals, find the smallest integer N such that $x(k) = 0$ for $|k| > N$.

(a) $x(k) = \mu(k+5) - \mu(k-5)$
(b) $x(k) = \sin(.2\pi k)\mu(k)$
(c) $x(k) = \min(k^2 - 9, 0)\mu(k)$
(d) $x(k) = \mu(k)\mu(-k)/(1+k^2)$
(e) $x(k) = \tan(\sqrt{2}\pi k)[\mu(k) - \mu(k-100)]$
(f) $x(k) = \delta(k) + \cos(\pi k) - (-1)^k$
(g) $x(k) = k^{-k}\sin(.5\pi k)$

Solution

(a) finite, $N = 5$
(b) infinite
(c) finite, $N = 2$
(d) finite, $N = 1$
(e) finite, $N = 99$
(f) finite, $N = 0$
(g) infinite

2.2 Classify each of the following signals as causal or noncausal.

(a) $x(k) = \max\{k, 0\}$
(b) $x(k) = \sin(.2\pi k)\mu(-k)$
(c) $x(k) = 1 - \exp(-k)$
(d) $x(k) = \text{mod}(k, 10)$
(e) $x(k) = \tan(\sqrt{2}\pi k)[\mu(k) + \mu(k-100)]$
(f) $x(k) = \cos(\pi k) + (-1)^k$
(g) $x(k) = \sin(.5\pi k)/(1+k^2)$

Solution

(a) causal

(b) noncausal

(c) noncausal

(d) noncausal

(e) causal

(e) causal

(f) noncausal

2.3 Classify each of the following signals as periodic or aperiodic. For the periodic signals, find the period, M.

(a) $x(k) = \cos(.02\pi k)$

(b) $x(k) = \sin(.1k)\cos(.2k)$

(c) $x(k) = \cos(\sqrt{3}k)$

(d) $x(k) = \exp(j\pi/8)$

(e) $x(k) = \text{mod}(k, 10)$

(f) $x(k) = \sin^2(.1\pi k)\mu(k)$

(g) $x(k) = j^{2k}$

Solution

(a) periodic, $M = 100$

(b) nonperiodic, $(\tau = 20\pi)$

(c) nonperiodic, $(\tau = 2\pi/\sqrt{3})$

(d) periodic, $M = 16$

(e) periodic, $M = 10$

(f) nonperodic, (causal)

(g) periodic, $M = 2$

2.4 Classify each of the following signals as bounded or unbounded.

(a) $x(k) = k\cos(.1\pi k)/(1 + k^2)$

(b) $x(k) = \sin(.1k)\cos(.2k)\delta(k - 3)$

(c) $x(k) = \cos(\pi k^2)$

(d) $x(k) = \tan(.1\pi k)[\mu(k) - \mu(k - 10]$

(e) $x(k) = k^2/(1 + k^2)$

(f) $x(k) = k\exp(-k)\mu(k)$

Solution

(a) bounded
(b) bounded
(c) bounded
(d) unbounded
(e) bounded
(f) bounded

2.5 For each of the following signals, determine whether or not it is bounded. For the bounded signals, find a bound, B_x.

(a) $x(k) = [1 + \sin(5\pi k)]\mu(k)$
(b) $x(k) = k(.5)^k \mu(k)$
(c) $x(k) = \left[\dfrac{(1+k)\sin(10k)}{1+(.5)^k}\right]\mu(k)$
(d) $x(k) = [1 + (-1)^k]\cos(10k)\mu(k)$

Solution

(a) bounded, $B_x = 1$
(b) The following are the first few values of $x(k)$.

k	$x(k)$
0	0
1	1/2
2	1/2
3	3/8
4	4/16
5	5/25

Thus $x(k)$ is bounded with $B_x = .5$.
(c) unbounded
(d) bounded, $B_x = 2$.

2.6 Consider the following sum of causal exponentials.

$$x(k) = [c_1 p_1^k + c_2 p_2^k]\mu(k)$$

(a) Using the inequalities in Appendix 2, show that

$$|x(k)| \quad \leq \quad |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k$$

(b) Show that $x(k)$ is absolutely summable if $|p_1| < 1$ and $|p_2| < 1$. Find an upper bound on $\|x\|_1$

(c) Suppose $|p_1| < 1$ and $|p_2| < 1$. Find an upper bound on the energy E_x.

Solution

(a) Using Appendix 2

$$\begin{aligned} |x(k)| &= |[c_1(p_1)^k + c_2(p_2)^k]\mu(k)| \\ &= |c_1(p_1)^k + c_2(p_2)^k| \cdot |\mu(k)| \\ &= |c_1(p_1)^k + c_2(p_2)^k| \\ &\leq |c_1(p_1)^k| + |c_2(p_2)^k| \\ &= |c_1| \cdot |p_1^k| + |c_2| \cdot |p_2^k]| \\ &= |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k \end{aligned}$$

(b) Suppose $|p_1| < 1$ and $|p_2| < 1$. Then using (a) and the geometric series in (2.2.14)

$$\begin{aligned} \|x\|_1 &= \sum_{k=-\infty}^{\infty} |x(k)| \\ &\leq \sum_{k=0}^{\infty} |c_1| \cdot |p_1|^k + |c_2| \cdot |p_2|^k \\ &= |c_1| \sum_{k=0}^{\infty} |p_1|^k + |c_2| \sum_{k=0}^{\infty} |p_2|^k \\ &= \frac{|c_1|}{1 - |p_1|} + \frac{|c_2|}{1 - |p_2|} \end{aligned}$$

(c) Using (b) and (2.2.7) through (2.2.9)

$$
\begin{aligned}
E_x &= \|x\|_2^2 \\
&\leq \|x\|_1^2 \\
&\leq \frac{|c_1|}{1-|p_1|} + \frac{|c_2|}{1-|p_2|}
\end{aligned}
$$

2.7 Find the average power of the following signals.

(a) $x(k) = 10$

(b) $x(k) = 20\mu(k)$

(c) $x(k) = \text{mod}(k, 5)$

(d) $x(k) = a\cos(\pi k/8) + b\sin(\pi k/8)$

(e) $x(k) = 100[\mu(k+10) - \mu(k-10)]$

(f) $x(k) = j^k$

Solution

Using (2.2.10)-(2.2.12) and Appendix 2

(a) $P_x = 100$

(b) $P_x = 400$

(c) $P_x = (1 + 4 + 9 + 16)/5 = 6$

(d)

$$
\begin{aligned}
[a\cos(\pi k/8) + b\sin(\pi k/8)]^2 &= a^2\cos^2(\pi k/8)) + 2ab\cos(\pi k/8)\sin(\pi k/i) + b^2\sin^2(\pi k/8) \\
&= a^2\left[\frac{1+\cos(\pi k/4)}{2}\right] + ab\sin(\pi k/4) + b^2\left[\frac{1-\cos(\pi k/4)}{2}\right]
\end{aligned}
$$

Thus

$$
P_x = \frac{a^2 + b^2}{2}
$$

(e) $P_x = 10^4$

(f)

$$\begin{aligned}
P_x &= \lim \frac{1}{2N+1} \sum_{k=-N}^{N} |j^k|^2 \\
&= \lim \frac{1}{2N+1} \sum_{k=-N}^{N} (|j|^k)^2 \\
&= \lim \frac{1}{2N+1} \sum_{k=-N}^{N} 1 \\
&= 1
\end{aligned}$$

2.8 Classify each of the following systems as linear or nonlinear.

(a) $y(k) = 4[y(k-1)+1]x(k)$
(b) $y(k) = 6kx(k)$
(c) $y(k) = -y(k-2) + 10x(k+3)$
(d) $y(k) = .5y(k) - 2y(k-1)$
(e) $y(k) = .2y(k-1) + x^2(k)$
(f) $y(k) = -y(k-1)x(k-1)/10$

Solution

(a) nonlinear (product term)
(b) linear
(c) linear
(d) linear
(e) nonlinear (input term)
(f) nonlinear (product term)

2.9 Classify each of the following systems as time-invariant or time-varying.

(a) $y(k) = [x(k) - 2y(k-1)]^2$
(b) $y(k) = \sin[\pi y(k-1)] + 3x(k-2)$
(c) $y(k) = (k+1)y(k-1) + \cos[.1\pi x(k)]$
(d) $y(k) = .5y(k-1) + \exp(-k/5)\mu(k)$
(e) $y(k) = \log[1 + x^2(k-2)]$

(f) $y(k) = kx(k-1)$

Solution

(a) time-invariant
(b) time-invariant
(c) time-varying
(d) time-varying
(e) time-invariant
(f) time-varying

2.10 Classify each of the following systems as causal or noncausal.

(a) $y(k) = [3x(k) - y(k-1)]^3$
(b) $y(k) = \sin[\pi y(k-1)] + 3x(k+1)$
(c) $y(k) = (k+1)y(k-1) + \cos[.1\pi x(k^2)]$
(d) $y(k) = .5y(k-1) + \exp(-k/5)\mu(k)$
(e) $y(k) = \log[1 + y^2(k-1)x^2(k+2)]$
(f) $h(k) = \mu(k+3) - \mu(k-3)$

Solution

(a) causal
(b) noncausal
(c) causal
(d) causal
(e) noncausal
(f) noncausal

2.11 Consider the following system that consists of a gain of A and a delay of d samples.

$$y(k) = Ax(k-d)$$

(a) Find the impulse response $h(k)$ of this system.
(b) Classify this system as FIR or IIR.

(c) Is this system BIBO stable? If so, find $\|h\|_1$.

(d) For what values of A and d is this a passive system?

(e) For what values of A and d is this an active system?

(f) For what values of A and d is this a lossless system?

Solution

(a) $h(k) = A\delta(k-d)$

(b) FIR

(c) Yes, it is BIBO stable with $\|h\|_1 = |A|$.

(d)

$$\begin{aligned} E_y &= \sum_{k=-\infty}^{\infty} y^2(k) \\ &= \sum_{k=-\infty}^{\infty} [Ax(k-d)]^2 \\ &= A^2 \sum_{k=-\infty}^{\infty} x^2(k-d) \\ &= A^2 \sum_{i=-\infty}^{\infty} x^2(i) \quad , \quad i = k-d \\ &= A^2 E_x \end{aligned}$$

This is a passive system for $|A| < 1$.

(e) This is an active system for $|A| > 1$

(f) This is a lossless system for $|A| = 1$

2.12 Consider the following linear time-invariant discrete-time system S.

$$y(k) - y(k-2) = 2x(k)$$

(a) Find the characteristic polynomial of S and express it in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$.

(c) Find the zero-input response when $y(-1) = 4$ and $y(-2) = -1$.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - 1 \\ &= (z-1)(z+1) \end{aligned}$$

(b)

$$\begin{aligned} y_{zi}(k) &= c_1(p_1)^k + c_2(p_2)^k \\ &= c_1 + c_2(-1)^k \end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$\begin{aligned} c_1 - c_2 &= 4 \\ c_1 + c_2 &= -1 \end{aligned}$$

Adding the equations yields $2c_1 = 3$ or $c_1 = 1.5$. Subtracting the first equation from the second yields $2c_2 = -5$ or $c_2 = -2.5$.. Thus the zero-input response is

$$y_{zi}(k) \quad = 1.5 - 2.5(-1)^k$$

√ **2.13** Consider the following linear time-invariant discrete-time system S.

$$y(k) \quad = \quad 1.8y(k-1) - .81y(k-2) - 3x(k-1)$$

(a) Find the characteristic polynomial $a(z)$ and express it in factored form.
(b) Write down the general form of the zero-input response, $y_{zi}(k)$.
(c) Find the zero-input response when $y(-1) = 2$ and $y(-2) = 2$.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - 1.8z + .81 \\ &= (z - .9)^2 \end{aligned}$$

(b)

$$\begin{aligned} y_{zi}(k) &= (c_1 + c_2 k)p^k \\ &= (c_1 + c_2 k).9^k \end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$\begin{aligned} (c_1 - c_2).9^{-1} &= 2 \\ (c_1 - 2c_2).9^{-2} &= 2 \end{aligned}$$

or

$$\begin{aligned} c_1 - c_2 &= 1.8 \\ c_1 - 2c_2 &= 1.62 \end{aligned}$$

Subtracting the second equation from the first yields $c_2 = .18$. Subtracting the second equation from two times the first yields $c_1 = 1.98$. Thus the zero-input response is

$$y_{zi}(k) = (1.98 + .18k).9^k$$

2.14 Consider the following linear time-invariant discrete-time system S.

$$y(k) = -.64y(k-2) + x(k) - x(k-2)$$

(a) Find the characteristic polynomial $a(z)$ and express it in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$, expressing it as a real signal.

(c) Find the zero-input response when $y(-1) = 3$ and $y(-2) = 1$.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 + .64 \\ &= (z - .8j)(z + .8j) \end{aligned}$$

(b) In polar form the roots are $z = .8\exp(\pm j\pi/2)$. Thus

$$\begin{aligned} y_{zi}(k) &= r^k[c_1 \cos(k\theta) + c_2 \sin(k\theta)] \\ &= .8^k[c_1 \cos(k\pi/2) + c_2 \sin(\pi k/2)] \end{aligned}$$

(c) Evaluating part (b) at the two initial conditions yields

$$\begin{aligned} .8^{-1}c_2(-1) = 3 \\ .8^{-2}c_1(-1) = 1 \end{aligned}$$

Thus $c_2 = -3(.8)$ and $c_1 = -1(.64)$. Hence the zero-input response is

$$y_{zi}(k) \quad = -(.8)^k[.64\cos(\pi k/2) + 2.4\sin(\pi k/2)]$$

2.15 Consider the following linear time-invariant discrete-time system S.

$$y(k) - 2y(k-1) + 1.48y(k-2) - .416y(k-3) \quad = \quad 5x(k)$$

(a) Find the characteristic polynomial $a(z)$. Using the MATLAB function *roots*, express it in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$.

(c) Write the equations for the unknown coefficient vector $c \in R^3$ as $Ac = y_0$, where $y_0 = [y(-1), y(-2), y(-3)]^T$ is the initial condition vector.

Solution

(a)

$$a(z) = z^3 - 2z^2 + 1.48z - .416$$

```
a = [1 -2 1.48 -.416]
r = roots(a)
```

$$a(z) = (z - .8)(z - .6 - .4j)(z - .6 + .4j)$$

(b) The complex roots in polar form are $p_{2,3} = r\exp(\pm j\theta)$ where

$$\begin{aligned} r &= \sqrt{.6^2 + .4^2} \\ &= .7211 \\ \theta &= \arctan(\pm .4/.6) \\ &= \pm .588 \end{aligned}$$

Thus the form of the zero-input response is

$$\begin{aligned} y_{zi}(k) &= c_1(p_1)^k + r^k[c_2\cos(k\theta) + c_3\sin(k\theta)] \\ &= c_1(.8)^k + .7211^k[c_2\cos(.588k) + c_3\sin(.588k)] \end{aligned}$$

(c) Let $c \in R^3$ be the unknown coefficient vector, and $y_0 = [y(-1), y(-2), y(-3)]^T$. Then $Ac = y_0$ or

$$\begin{bmatrix} .8^{-1} & .7211^{-1}\cos(-.588) & .7211^{-1}\sin(-.588) \\ .8^{-2} & .7211^{-2}\cos[-2(.588)] & .7211^{-2}\sin[-2(.588)] \\ .8^{-3} & .7211^{-3}\cos[-3(.588)] & .7211^{-3}\sin[-3(.588)] \end{bmatrix} c = y_0$$

2.16 Consider the following linear time-invariant discrete-time system S.

$$y(k) - .9y(k-1) = 2x(k) + x(k-1)$$

(a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.

(b) Write down the general form of the zero-state response, $y_{zs}(k)$, when the input is $x(k) = 3(.4)^k\mu(k)$.

(c) Find the zero-state response.

Solution

(a)

$$\begin{aligned} a(z) &= z - .9 \\ b(z) &= 2z + 1 \end{aligned}$$

(b)

$$\begin{aligned} y_{zs}(k) &= [d_0(p_0)^k + d_1(p_1)^k]\mu(k) \\ &= [d_0(.4)^k + d_1(.9)^k]\mu(k) \end{aligned}$$

(c)

$$\begin{aligned} d_0 &= \left.\frac{Ab(z)}{a(z)}\right|_{z=p_0} \\ &= \frac{3[2(.4)+1]}{.4-.9} \\ &= \frac{5.4}{-.5} \\ &= -10.8 \\ d_1 &= \left.\frac{A(z-p_1)b(z)}{(z-p_0)a(z)}\right|_{z=p_1} \\ &= \frac{3[2(.9)+1]}{.5} \\ &= \frac{8.4}{.5} \\ &= 16.8 \end{aligned}$$

Thus the zero-state response is

$$y_{zs}(k) = [-10.8(.4)^k + 16.8(.9)^k]\mu(k)$$

2.17 Consider the following linear time-invariant discrete-time system S.

$$y(k) = y(k-1) - .24y(k-2) + 3x(k) - 2x(k-1)$$

(a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.

(b) Suppose the input is the unit step, $x(k) = \mu(k)$. Write down the general form of the zero-state response, $y_{zs}(k)$.

(c) Find the zero-state response to the unit step input.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - z + .24 \\ b(z) &= 3z - 2 \end{aligned}$$

(b) The factored form of $a(z)$ is

$$a(z) = (z - .6)(z - .4)$$

Thus the form of the zero-state response to a unit step input is

$$y_{zs}(k) = [d_0 + d_1(.6)^k + d_2(.4)^k]\mu(k)$$

(c)

$$\begin{aligned}
d_0 &= \left.\frac{Ab(z)}{a(z)}\right|_{z=p_0} \\
&= \frac{3-2}{(1-.6)(1-.4)} \\
&= \frac{1}{.24} \\
&= 4.167 \\
d_1 &= \left.\frac{A(z-p_1)b(z)}{(z-p_0)a(z)}\right|_{z=p_1} \\
&= \frac{3(.6)-2}{(.6-1)(.6-.4)} \\
&= \frac{-.2}{-.08} \\
&= 2.5 \\
d_2 &= \left.\frac{A(z-p_2)b(z)}{(z-p_0)a(z)}\right|_{z=p_2} \\
&= \frac{3(.4)-2}{(.4-1)(.4-.6)} \\
&= \frac{-.8}{-.12} \\
&= 6.667
\end{aligned}$$

Thus the zero-state response is

$$y_{zs}(k) = [4.167 + 2.5(.6)^k + 6.667(.4)^k]\mu(k)$$

2.18 Consider the following linear time-invariant discrete-time system S.

$$y(k) = y(k-1) - .21y(k-2) + 3x(k) + 2x(k-2)$$

(a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$. Express $a(z)$ in factored form.

(b) Write down the general form of the zero-input response, $y_{zi}(k)$.

(c) Find the zero-input response when the initial condition is $y(-1) = 1$ and $y(-2) = -1$.

(d) Write down the general form of the zero-state response when the input is $x(k) = 2(.5)^{k-1}\mu(k)$.

(e) Find the zero-state response using the input in (d).

(f) Find the complete response using the initial condition in (c) and the input in (d).

Solution

(a)

$$\begin{aligned} a(z) &= z^2 - z + .21 \\ &= (z - .3)(z - .7) \\ b(z) &= 3z^2 + 2 \end{aligned}$$

(b) The general form of the zero-input response is

$$\begin{aligned} y_{zi}(k) &= c_1(p_1)^k + c_2(p_2)^k \\ &= c_1(.3)^k + c_2(.7)^k \end{aligned}$$

(c) Using (b) and applying the initial conditions yields

$$\begin{aligned} c_1(.3)^{-1} + c_2(.7)^{-1} &= 1 \\ c_1(.3)^{-2} + c_2(.7)^{-2} &= -1 \end{aligned}$$

Clearing the denominators,

$$\begin{aligned} .7c_1 + .3c_2 &= .21 \\ .49c_1 + .09c_2 &= -.0441 \end{aligned}$$

Subtracting the second equation from seven times the first equation yields $2.01c_2 = 1.51$. Subtracting .3 times the first equation from the second yields $.28c_1 = -.127$. Thus the zero-input response is

$$y_{zi}(k) = -.454(.3)^k + .751(.7)^k$$

(d) First note that

$$\begin{aligned} x(k) &= 2(.5)^{k-1}\mu(k) \\ &= 4(.5)^k\mu(k) \end{aligned}$$

The general form of the zero-state response is

$$y_{zs}(k) = [d_0(.5)^k + d_1(.3)^k + d_2(.7)^k]\mu(k)$$

(e)

$$\begin{aligned} d_0 &= \left.\frac{Ab(z)}{a(z)}\right|_{z=p_0} \\ &= \frac{4[3(.5)^2+2]}{(.5-.3)(.5-.7)} \\ &= \frac{4(2.75)}{-.04} \\ &= -275 \\ d_1 &= \left.\frac{A(z-p_1)b(z)}{(z-p_0)a(z)}\right|_{z=p_1} \\ &= \frac{4[3(.3)^2+2]}{(.3-.5)(.3-.7)} \\ &= \frac{4(2.27)}{.08} \\ &= 113.5 \\ d_2 &= \left.\frac{A(z-p_2)b(z)}{(z-p_0)a(z)}\right|_{z=p_2} \\ &= \frac{4[3(.7)^2+2]}{(.7-.5)(.7-.3)} \\ &= \frac{4(2.63)}{.08} \\ &= 131.5 \end{aligned}$$

Thus the zero-state response is

$$y_{zs}(k) = [-275(.5)^k + 113.5(.3)^k + 131.5(.7)^k]\mu(k)$$

(f) By superposition, the complete response is

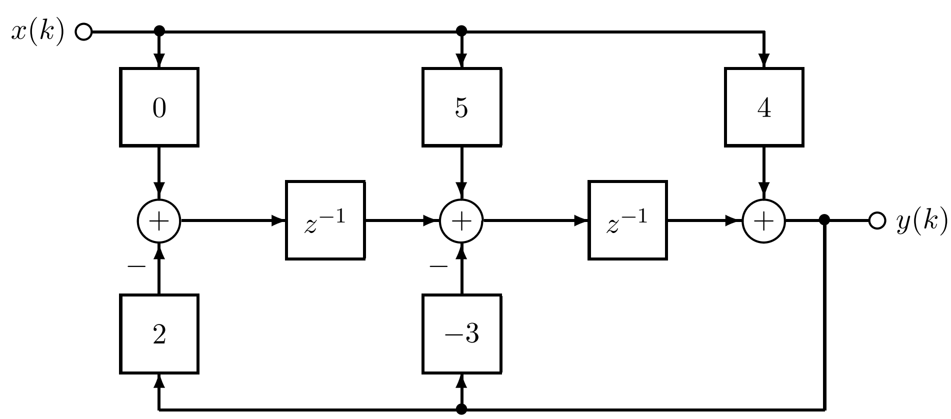
$$
\begin{aligned}
y(k) &= y_{zi}(k) + y_{zs}(k) \\
&= -.454(.3)^k + .751(.7)^k + [-275(.5)^k + 113.5(.3)^k + 131.5(.7)^k]\mu(k)
\end{aligned}
$$

2.19 Consider the following linear time-invariant discrete-time system S. Sketch a block diagram of this IIR system.

$$
y(k) = 3y(k-1) - 2y(k-2) + 4x(k) + 5x(k-1)
$$

Solution

$$
\begin{aligned}
a &= [1, -3, 2] \\
b &= [4, 5, 0]
\end{aligned}
$$



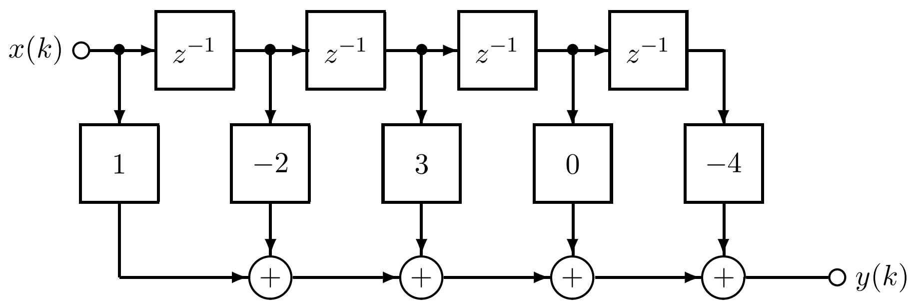
Problem 2.19

2.20 Consider the following linear time-invariant discrete-time system S. Sketch a block diagram of this FIR system.

$$
y(k) = x(k) - 2x(k-1) + 3x(k-2) - 4x(k-4)
$$

Solution

$$\begin{aligned} a &= [1,0,0] \\ b &= [1,-2,3,0,-4] \end{aligned}$$



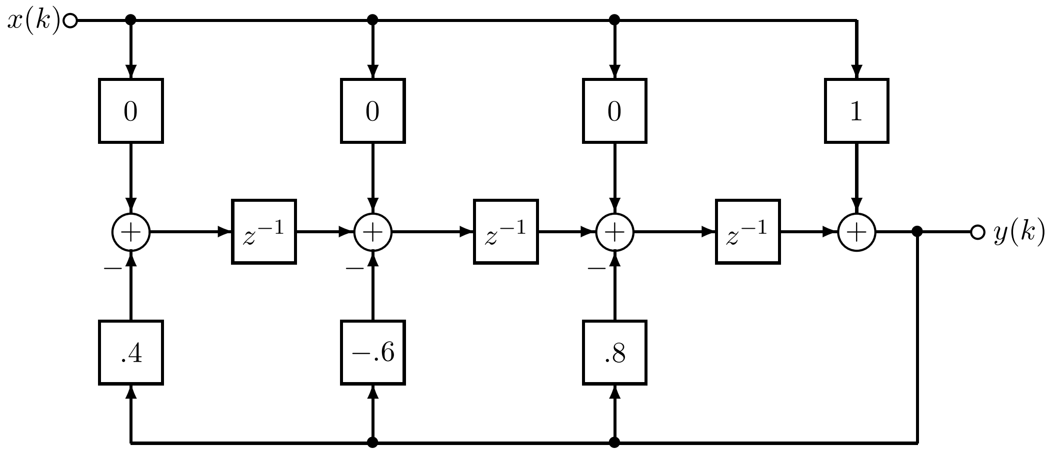
Problem 2.20

2.21 Consider the following linear time-invariant discrete-time system S called an *auto-regressive* system. Sketch a block diagram of this system.

$$y(k) = x(k) - .8y(k-1) + .6y(k-2) - .4y(k-3)$$

Solution

$$\begin{aligned} a &= [1,.8,-.6,.4] \\ b &= [1,0,0,0] \end{aligned}$$



Problem 2.21

2.22 Consider the block diagram shown in Figure 2.32.

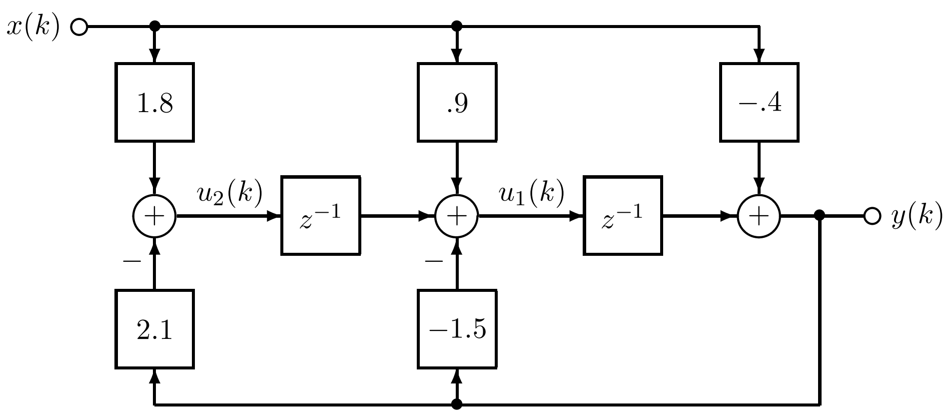


Figure 2.32 A Block Diagram of the System in Problem 2.22

(a) Write a single difference equation description of this system.
(b) Write a system of difference equations for this system for $u_i(k)$ for $1 \le i \le 2$ and $y(k)$.

Solution

(a) By inspection of Figure 2.32

$$y(k) \quad = \quad -.4x(k) + .9x(k-1) + 1.8x(k-2) + 1.5y(k-1) - 2.1y(k-2)$$

(b) The equivalent system of equations is

$$\begin{aligned} u_2(k) &= 1.8x(k) - 2.1y(k) \\ u_1(k) &= .9x(k) + 1.5y(k) + u_2(k-1) \\ y(k) &= -.4x(k) + u_1(k-1) \end{aligned}$$

2.23 Consider the following linear time-invariant discrete-time system S.

$$y(k) = .6y(k-1) + x(k) - .7x(k-1)$$

(a) Find the characteristic polynomial and the input polynomial.
(b) Write down the form of the impulse response, $h(k)$.
(c) Find the impulse response.

Solution

(a)

$$\begin{aligned} a(z) &= z - .6 \\ b(z) &= z - .7 \end{aligned}$$

(b)

$$h(k) = d_0\delta(k) + d_1(.6)^k\mu(k)$$

(c)

$$\begin{aligned} d_0 &= \left.\frac{b(z)}{a(z)}\right|_{z=0} \\ &= \frac{-.7)}{-.6)} \\ &= 1.167 \\ d_1 &= \left.\frac{(z-p_1)b(z)}{za(z)}\right|_{z=p_1} \\ &= \frac{.6-.7)}{.6)} \\ &= -.167 \end{aligned}$$

Thus the impulse response is

$$h(k) = 1.167\delta(k) - .167(.6)^k\mu(k)$$

2.24 Consider the following linear time-invariant discrete-time system S.

$$y(k) = -.25y(k-2) + x(k-1)$$

(a) Find the characteristic polynomial and the input polynomial.
(b) Write down the form of the impulse response, $h(k)$.
(c) Find the impulse response. Use the identities in Appendix 2 to express $h(k)$ in real form.

Solution

(a)

$$\begin{aligned} a(z) &= z^2 + .25 \\ b(z) &= z \end{aligned}$$

(b) First note that

$$a(z) = (z - .5j)(z + .5j)$$

Thus the form of the impulse response is

$$h(k) = d_0\delta(k) + [d_1(.5j)^k + d_2(-.5j)^k]\mu(k)$$

(c)

$$
\begin{aligned}
d_0 &= \left.\frac{b(z)}{a(z)}\right|_{z=0} \\
&= 0 \\
d_1 &= \left.\frac{(z-p_1)b(z)}{za(z)}\right|_{z=p_1} \\
&= \frac{.5j)}{.5j(j)} \\
&= -j \\
d_2 &= \left.\frac{(z-p_2)b(z)}{za(z)}\right|_{z=p_2} \\
&= \frac{-.5j)}{-.5j(-j)} \\
&= j
\end{aligned}
$$

Thus from Appendix 2 the impulse response is

$$
\begin{aligned}
h(k) &= [-j(.5j)^k + j(-.5j)^k]\mu(k) \\
&= 2\text{Re}[-j(.5j)^k]\mu(k) \\
&= -2\text{Re}[(.5)^k(j)^{k+1}]\mu(k) \\
&= -2(.5)^k\text{Re}\{[\exp(j\pi/2)]^{k+1}\}\mu(k) \\
&= -2(.5)^k\text{Re}[\exp[j(k+1)\pi/2]\mu(k) \\
&= -2(.5)^k\cos[(k+1)\pi/2]\mu(k)
\end{aligned}
$$

2.25 Consider the following linear time-invariant discrete-time system S. Suppose $0 < m \le n$ and the characteristic polynomial $a(z)$ has simple nonzero roots.

$$
y(k) = \sum_{i=0}^{m} b_i x(k-i) - \sum_{i=1}^{n} a_i y(k-i)
$$

(a) Find the characteristic polynomial $a(z)$ and the input polynomial $b(z)$.

(b) Find a constraint on $b(z)$ that ensures that the impulse response $h(k)$ does not contain an impulse term.

Solution

(a)

$$\begin{aligned} a(z) &= z^n + a_1 z^{n-1} + \cdots + a_n \\ b(z) &= b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m} \end{aligned}$$

(b) The coefficient of the impulse term is

$$\begin{aligned} d_0 &= \left. \frac{b(z)}{a(z)} \right|_{z=0} \\ &= \frac{b(0)}{a(0)} \end{aligned}$$

Thus

$$\begin{aligned} d_0 \neq 0 &\Leftrightarrow b(0) \neq 0 \\ &\Leftrightarrow m = n \end{aligned}$$

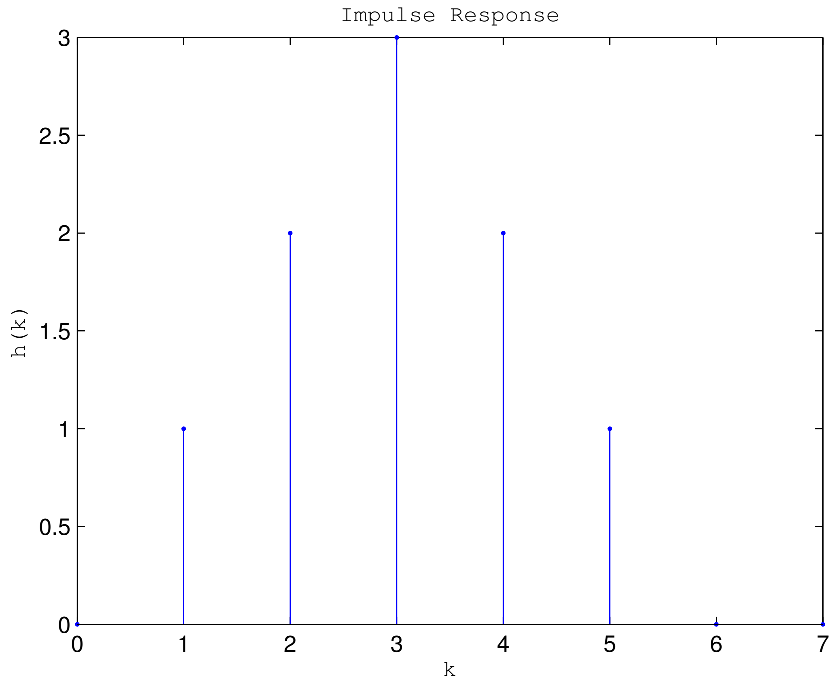
2.26 Consider the following linear time-invariant discrete-time system S. Compute and sketch the impulse response of this FIR system.

$$y(k) = u(k-1) + 2u(k-2) + 3u(k-3) + 2u(k-4) + u(k-5)$$

Solution

By inspection, the impulse response is

$$h(k) = [0, 1, 2, 3, 2, 1, 0, 0, \ldots]$$



Problem 2.26

2.27 Using the definition of linear convolution, show that for any signal $h(k)$

$$h(k) \star \delta(k) \quad = \quad h(k)$$

Solution

From Definition 2.3 we have

$$\begin{aligned} h(k) \star \delta(k) &= \sum_{i=-\infty}^{\infty} h(i)x(k-i) \\ &= \sum_{i=-\infty}^{\infty} h(i)\delta(k-i) \\ &= h(k) \end{aligned}$$

2.28 Use Definition 2.3 and the commutative property to show that the linear convolution operator is associative.

$$f(k) \star [g(k) \star h(k)] \quad = \quad [f(k) \star g(k)] \star h(k)$$

Solution

From Definition 2.3 we have

$$\begin{aligned} d_1(k) &= f(k) \star [g(k) \star h(k)] \\ &= \sum_{m=-\infty}^{\infty} f(m) \left[\sum_{i=-\infty}^{\infty} g(i) h(k-m-i) \right] \\ &= \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(m) g(i) h(k-m-i) \end{aligned}$$

Next, using the commutative property

$$\begin{aligned} d_2(k) &= [f(k) \star g(k)] \star h(k)] \\ &= h(k) \star [f(k) \star g(k)] \\ &= \sum_{i=-\infty}^{\infty} h(i) \left[\sum_{m=-\infty}^{\infty} f(m) g(k-i-m) \right] \\ &= \sum_{i=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(i) f(m) g(k-i-m) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(k-n-m) f(m) g(n) \quad , \quad n = k-i-m \\ &= \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} f(m) g(i) h(k-m-i) \quad , \quad i = n \end{aligned}$$

Thus $d_2(k) = d_1(k)$.

2.29 Use Definition 2.3 to show that the linear convolution operator is distributive.

$$f(k) \star [g(k) + h(k)] \quad = \quad f(k) \star g(k) + f(k) \star h(k)$$

Solution

$$\begin{aligned} d(k) &= f(k) \star [g(k) + h(k)] \\ &= \sum_{i=-\infty}^{\infty} f(i)[g(k-i) + h(k-i)] \\ &= \sum_{i=-\infty}^{\infty} f(i)g(k-i) + f(i)h(k-i)] \\ &= \sum_{i=-\infty}^{\infty} f(i)g(k-i) + \sum_{i=-\infty}^{\infty} f(i)h(k-i)] \\ &= f(k) \star g(k) + f(k) \star h(k) \end{aligned}$$

2.30 Suppose $h(k)$ and $x(k)$ are defined as follows.

$$\begin{aligned} h &= [2, -1, 0, 4]^T \\ x &= [5, 3, -7, 6]^T \end{aligned}$$

(a) Let $y_c(k) = h(k) \circ x(k)$. Find the circular convolution matrix $C(x)$ such that $y_c = C(x)h$.
(b) Use $C(x)$ to find $y_c(k)$.

Solution

(a) Using (2.7.9) and Example 2.14 as a guide, the 4×4 circular convolution matrix is

$$\begin{aligned} C(x) &= \begin{bmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(3) \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & -7 & 3 \\ 3 & 5 & 6 & -7 \\ -7 & 3 & 5 & 6 \\ 6 & -7 & 3 & 5) \end{bmatrix} \end{aligned}$$

(b) Using (2.7.10) and the results from part (a)

$$
\begin{aligned}
y_c &= C(x)h \\
&= \begin{bmatrix} 5 & 6 & -7 & 3 \\ 3 & 5 & 6 & -7 \\ -7 & 3 & 5 & 6 \\ 6 & -7 & 3 & 5) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 16 \\ -27 \\ 7 \\ 39 \end{bmatrix}
\end{aligned}
$$

This can be verified using the DSP Companion function *f_conv*.

2.31 Suppose $h(k)$ and $x(k)$ are the following signals of length L and M, respectively.

$$
\begin{aligned}
h &= [3, 6, -1]^T \\
x &= [2, 0, -4, 5]^T
\end{aligned}
$$

(a) Let h_z and x_z be zero-padded versions of $h(k)$ and $x(k)$ of length $N = L + M - 1$. Construct h_z and x_z.

(b) Let $y_c(k) = h_z(k) \circ x_z(k)$. Find the circular convolution matrix $C(x_z)$ such that $y_c = C(x_z)h_z$.

(c) Use $C(x_z)$ to find $y_c(k)$.

(d) Use $y_c(k)$ to find the linear convolution $y(k) = h(k) \star x(k)$ for $0 \le k < N$.

Solution

(a) Here

$$
\begin{aligned}
N &= L + M - 1 \\
&= 3 + 4 - 1 \\
&= 6
\end{aligned}
$$

Thus the zero-padded versions of $h(k)$ and $x(k)$ are

$$\begin{aligned} h_z &= [3, 6, -1, 0, 0, 0]^T \\ x_z &= [2, 0, -4, 5, 0, 0]^T \end{aligned}$$

(b) Using (2.7.9) and the results from part (a), the $N \times N$ circular convolution matrix is

$$\begin{aligned} C(x_z) &= \begin{bmatrix} x_z(0) & x_z(5) & x_z(4) & x_z(3) & x_z(2) & x_z(1) \\ x_z(1) & x_z(0) & x_z(5) & x_z(4) & x_z(3) & x_z(2) \\ x_z(2) & x_z(1) & x_z(0) & x_z(5) & x_z(4) & x_z(3) \\ x_z(3) & x_z(2) & x_z(1) & x_z(0) & x_z(5) & x_z(4) \\ x_z(4) & x_z(3) & x_z(2) & x_z(1) & x_z(0) & x_z(5) \\ x_z(5) & x_z(4) & x_z(3) & x_z(2) & x_z(1) & x_z(0) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 & 5 & -4 & 0 \\ 0 & 2 & 0 & 0 & 5 & -4 \\ -4 & 0 & 2 & 0 & 0 & 5 \\ 5 & -4 & 0 & 2 & 0 & 0 \\ 0 & 5 & -4 & 0 & 2 & 0 \\ 0 & 0 & 5 & -4 & 0 & 2 \end{bmatrix} \end{aligned}$$

(c) Using (2.7.9), the circular convolution of $h_z(k)$ with $x_z(k)$ is

$$\begin{aligned} y_z(k) &= C(x_z)h_z \\ &= \begin{bmatrix} 2 & 0 & 0 & 5 & -4 & 0 \\ 0 & 2 & 0 & 0 & 5 & -4 \\ -4 & 0 & 2 & 0 & 0 & 5 \\ 5 & -4 & 0 & 2 & 0 & 0 \\ 0 & 5 & -4 & 0 & 2 & 0 \\ 0 & 0 & 5 & -4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 12 \\ -14 \\ -9 \\ 34 \\ -5 \end{bmatrix} \end{aligned}$$

(d) Using (2.7.14) and the results of part (c), the linear convolution $y(k) = h(k) \star x(k)$ is

$$\begin{aligned} y(k) &= h_z(k) \circ x_z(k) \\ &= C(x_z)h_z \\ &= [6, 12, -14, -9, 34, -5]^T \end{aligned}$$

This can be verified using the DSP Companion function *f_conv*.

2.32 Consider a linear discrete-time system S with input x and output y. Suppose S is driven by an input $x(k)$ for $0 \le k < L$ to produce a zero-state output $y(k)$. Use deconvolution to find the impulse response $h(k)$ for $0 \le k < L$ if $x(k)$ and $y(k)$ are as follows.

$$\begin{aligned} x &= [2, 0, -1, 4]^T \\ y &= [6, 1, -4, 3]^T \end{aligned}$$

Solution

Using (2.7.15) and Example 2.16 as a guide

$$\begin{aligned} h(0) &= \frac{y(0)}{x(0)} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$

Applying (2.7.18) with $k = 1$ yields

$$\begin{aligned} h(1) &= \frac{y(1) - h(0)x(1)}{x(0)} \\ &= \frac{1 - 3(0)}{2} \\ &= .5 \end{aligned}$$

Applying (2.7.18) with $k = 2$ yields

$$\begin{aligned} h(2) &= \frac{y(2) - h(0)x(2) - h(1)x(1)}{x(0)} \\ &= \frac{-4 - 3(-1) - .5(0)}{2} \\ &= -.5 \end{aligned}$$

Finally, applying (2.7.18) with $k = 3$ yields

$$\begin{aligned} h(3) &= \frac{y(3) - h(0)x(3) - h(1)x(2) - h(2)x(1)}{x(0)} \\ &= \frac{3 - 3(4) - .5(-1) + .5(0)}{2} \\ &= -4.25 \end{aligned}$$

Thus the impulse response of the discrete-time system is

$$h(k) = [3, .5, -.5, -4.25]^T \quad , \quad 0 \le k < 4$$

This can be verified using the DSP Companion function *f_conv*.

2.33 Suppose $x(k)$ and $y(k)$ are the following finite signals.

$$\begin{aligned} x &= [5, 0, -4]^T \\ y &= [10, -5, 7, 4, -12]^T \end{aligned}$$

(a) Write the polynomials $x(z)$ and $y(z)$ whose coefficient vectors are x and y, respectively. The leading coefficient corresponds to the highest power of z.

(b) Using long division, compute the quotient polynomial $q(z) = y(z)/x(z)$.

(c) Deconvolve $y(k) = h(k) \star x(k)$ to find $h(k)$, using (2.7.15) and (2.7.18). Compare the result with $q(z)$ from part (b).

Solution

(a)

$$\begin{aligned} x(z) &= 5z^2 - 4 \\ y(z) &= 10z^4 - 5z^3 + 7z^2 + 4z - 12 \end{aligned}$$

(b)

$$
\begin{array}{r|l}
 & 2z^2 - z + 3 \\
\cline{2-2}
5z^2 - 4 & 10z^4 - 5z^3 + 7z^2 + 4z - 12 \\
 & \underline{10z^4 - 0z^3 - 8z^2} \\
 & \qquad -5z^3 + 15z^2 + 4z \\
 & \qquad \underline{-5z^3 - 0z^2 + 4z} \\
 & \qquad\qquad 15z^2 + 0z - 12 \\
 & \qquad\qquad \underline{15z^2 + 0z - 12} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Thus the quotient polynomial is

$$q(z) = 2z^2 - z + 3$$

(c) Using (2.7.15) and Example 2.16 as a guide

$$
\begin{aligned}
q(0) &= \frac{y(0)}{x(0)} \\
&= \frac{-12}{-4} \\
&= 3
\end{aligned}
$$

Applying (2.7.18) with $k = 1$ yields

$$
\begin{aligned}
q(1) &= \frac{y(1) - q(0)x(1)}{x(0)} \\
&= \frac{4 - 3(0)}{-4} \\
&= -1
\end{aligned}
$$

Applying (2.7.18) with $k = 2$ yields

$$
\begin{aligned}
q(2) &= \frac{y(2) - q(0)x(2) - q(1)x(1)}{x(0)} \\
&= \frac{7 - 3(5) - (-1)0}{-4} \\
&= 2
\end{aligned}
$$

Thus $q = [2, -1, 3]$ and the quotient polynomial is

$$q(z) = 2z^2 - z + 3$$

This can be verified using the MATLAB function *deconv*.

2.34 Some books use the following alternative way to define the linear cross-correlation of an L point signal $y(k)$ with an M-point signal $x(k)$. Using a change of variable, show that this is equivalent to Definition 2.5

$$r_{yx}(k) = \frac{1}{L}\sum_{n=0}^{L-1-k} y(n+k)x(n)$$

Solution

Consider the change of variable $i = n + k$. Then $n = i - k$ and

$$\begin{aligned} r_{yx}(k) &= \frac{1}{L}\sum_{n=0}^{L-1-k} y(n+k)x(n)\Big|_{i=n+k} \\ &= \frac{1}{L}\sum_{i=k}^{L-1} y(i)x(i-k) \end{aligned}$$

Since $x(n) = 0$ for $n < 0$, the lower limit of the sum can be changed to zero without affecting the result. Thus,

$$r_{yx}(k) = \frac{1}{L}\sum_{i=0}^{L-1} y(i)x(i-k) \quad , \quad 0 \le k < L$$

This is identical to Definition 2.5.

2.35 Suppose $x(k)$ and $y(k)$ are defined as follows.

$$\begin{aligned} x &= [5, 0, -10]^T \\ y &= [1, 0, -2, 4, 3]^T \end{aligned}$$

(a) Find the linear cross-correlation matrix $D(x)$ such that $r_{yx} = D(x)y$.

(b) Use $D(x)$ to find the linear cross-correlation $r_{yx}(k)$.

(c) Find the normalized linear cross-correlation $\rho_{yx}(k)$.

Solution

(a) Using (2.8.2) and Example 2.18 as a guide, the linear cross-correlation matrix is

$$\begin{aligned} D(x) &= \frac{1}{5}\begin{bmatrix} x(0) & x(1) & x(2) & 0 & 0 \\ 0 & x(0) & x(1) & x(2) & 0 \\ 0 & 0 & x(0) & x(1) & x(2) \\ 0 & 0 & 0 & x(0) & x(1) \\ 0 & 0 & 0 & 0 & x(0) \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 5 & 0 & -10 & 0 & 0 \\ 0 & 5 & 0 & -10 & 0 \\ 0 & 0 & 5 & 0 & -10 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(b) Using (2.8.3) and the results from part (a)

$$
\begin{aligned}
r_{yx} &= D(x)y \\
&= \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \\ 4 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ -8 \\ -8 \\ 4 \\ 3 \end{bmatrix}
\end{aligned}
$$

This can be verified using the DSP Companion function *f_corr*.

(c) Using (2.8.5) we have $L = 5$ and $M = 3$. Also from Definition 2.5

$$
\begin{aligned}
r_{yy}(0) &= \frac{1}{L}\sum_{i=0}^{L-1} y^2(i) \\
&= \frac{1+0+4+16+9}{5} \\
&= 6 \\
r_{xx}(0) &= \frac{1}{M}\sum_{i=0}^{M-1} x^2(i) \\
&= \frac{25+0+100}{3} \\
&= 41.67
\end{aligned}
$$

Finally, from (4.49) the normalized cross-correlation of $x(k)$ with $y(k)$ is

$$
\begin{aligned}
\rho_{yx}(k) &= \frac{r_{yx}(k)}{\sqrt{(M/L)r_{xx}(0)r_{yy}(0)}} \\
&= \frac{r_{yx}(k)}{\sqrt{.6(6)41.67}} \\
&= [.408, -.653, -.653, .327, .245]^T
\end{aligned}
$$

This can be verified using the DSP Companion function *f_corr*.

√ 2.36 Suppose $y(k)$ is as follows.

$$y = [5, 7, -2, 4, 8, 6, 1]^T$$

(a) Construct a 3-point signal $x(k)$ such that $r_{yx}(k)$ reaches its peak positive value at $k = 3$ and $|x(0)| = 1$.
(b) Construct a 4-point signal $x(k)$ such that $r_{yx}(k)$ reaches its peak negative value at $k = 2$ and $|x(0)| = 1$.

Solution

(a) Recall that the cross-correlation $r_{yx}(k)$ measures the degree which $x(k)$ is similar to a subsignal of $y(k)$. In order for $r_{yx}(k)$ to reach its maximum positive value at $k = 3$, one must have maximum positive correlation starting at $k = 3$. Thus for some positive constant α it is necessary that

$$\begin{aligned} x &= \alpha[y(3), y(4), y(5)]^T \\ &= \alpha[4, 8, 6]^T \end{aligned}$$

The constraint, $|x(0)| = 1$, implies that the positive scale factor must be $\alpha = 1/4$. Thus

$$x = [1, 2, 1.5]^T$$

(b) In order for $r_{yx}(k)$ to reach its maximum negative value at $k = 2$, one must have maximum negative correlation starting at $k = 2$. Thus for some positive constant α we need

$$\begin{aligned} x &= -\alpha[y(2), y(3), y(4), y(5)]^T \\ &= \alpha[2, -4, -8, -6]^T \end{aligned}$$

The constraint, $|x(0)| = 1$, implies that the positive scale factor must be $\alpha = 1/2$. Thus

$$x = [1, -2, -4, -3]^T$$

The answers to (a) and (b) can be verified using the DSP Companion function *f_corr*.

2.37 Suppose $x(k)$ and $y(k)$ are defined as follows.

$$\begin{aligned} x &= [4, 0, -12, 8]^T \\ y &= [2, 3, 1, -1]^T \end{aligned}$$

(a) Find the circular cross-correlation matrix $E(x)$ such that $c_{yx} = E(x)y$.

(b) Use $E(x)$ to find the circular cross-correlation $c_{yx}(k)$.

(c) Find the normalized circular cross-correlation $\sigma_{yx}(k)$.

Solution

(a) Using Definition 2.6, $c_{yx}(k)$ is just $1/N$ times the dot product of y with x rotated right by k samples. Thus the kth row of $E(x)$ is the vector x rotated right by k samples.

$$\begin{aligned} E(x) &= \frac{1}{4}\begin{bmatrix} x(0) & x(1) & x(2) & x(3) \\ x(3) & x(0) & x(1) & x(2) \\ x(2) & x(3) & x(0) & x(1) \\ x(1) & x(2) & x(3) & x(0) \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 4 & 0 & -12 & 8 \\ 8 & 4 & 0 & -12 \\ -12 & 8 & 4 & 0 \\ 0 & -12 & 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & 1 & 0 & -3 \\ -3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 1 \end{bmatrix} \end{aligned}$$

(b) Using Definition 2.6 and the results from part (a)

$$\begin{aligned} c_{yx} &= E(x)y \\ &= \begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & 1 & 0 & -3 \\ -3 & 2 & 1 & 0 \\ 0 & -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 10 \\ 1 \\ -8 \end{bmatrix} \end{aligned}$$

This can be verified using the DSP Companion function *f_corr*.

(c) Using (2.8.7), $N = 4$. Also from Definition 2.6

$$\begin{aligned}
c_{yy}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} y^2(i) \\
&= \frac{4+9+1+1}{4} \\
&= 3.75 \\
c_{xx}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\
&= \frac{16+0+144+64}{4} \\
&= 56
\end{aligned}$$

Finally, from (2.8.7) the normalized circular cross-correlation of $y(k)$ with $x(k)$ is

$$\begin{aligned}
\sigma_{yx}(k) &= \frac{c_{yx}(k)}{\sqrt{c_{xx}(0)c_{yy}(0)}} \\
&= \frac{c_{yx}(k)}{\sqrt{3.75(56)}} \\
&= [-.207, .690, .069, -.552]^T
\end{aligned}$$

This can be verified using the DSP Companion function *f_corr*.

2.38 Suppose $y(k)$ is as follows.

$$y = [8, 2, -3, 4, 5, 7]^T$$

(a) Construct a 6-point signal $x(k)$ such that $\sigma_{yx}(2) = 1$ and $|x(0)| = 6$.

(b) Construct a 6-point signal $x(k)$ such that $\sigma_{yx}(3) = -1$ and $|x(0)| = 12$.

Solution

(a) Recall that normalized circular cross-correlation, $-1 \leq \sigma_{yx}(k) \leq 1$, measures the degree which a rotated version of a signal $x(k)$ is similar to the signal $y(k)$. In order for $\sigma_{yx}(k)$ to reach its maximum positive value at $k = 2$, one must have maximum positive correlation starting at $k = 2$. Thus for some positive constant α it is necessary that

$$\begin{aligned} x &= \alpha[y(2), y(3), y(4), y(5), y(0), y(1)]^T \\ &= \alpha[-3, 4, 5, 7, 8, 2]^T \end{aligned}$$

The constraint, $|x(0)| = 6$, implies that the positive scale factor must be $\alpha = 2$. Thus

$$x = [-6, 8, 10, 14, 16, 4]^T$$

Because y and x are of the same length, this will result is $\sigma_{yx}(2) = 1$ which can be verified by using the DSP Companion function *f_corr*.

(b) In order for $\sigma_{yx}(k)$ to reach its maximum negative value at $k = 3$, one must have maximum negative correlation starting at $k = 3$. Thus for some positive constant α

$$\begin{aligned} x &= -\alpha[y(3), y(4), y(5), y(0), y(1), y(2)]^T \\ &= \alpha[4, 5, 7, 8, 2, -3]^T \end{aligned}$$

The constraint, $|x(0)| = 12$, implies that the positive scale factor must be $\alpha = 3$. Thus

$$x = [12, 15, 21, 24, 6, -9]^T$$

Because y and x are of the same length, this will result is $\sigma_{yx}(3) = -1$ which can be verified by using the DSP Companion function *f_corr*.

2.39 Let $x(k)$ be an N-point signal with average power P_x.

(a) Show that $r_{xx}(0) = c_{xx}(0) = P_x$

(b) Show that $\rho_{xx}(0) = \sigma_{xx}(0) = 1$

Solution

(a) The average power of $x(k)$ is

$$P_x = \frac{1}{N}\sum_{k=0}^{N-1} x^2(k)$$

From Definition 2.5, the auto-correlation of an N-point signal is

$$\begin{aligned} r_{xx}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} x(i)x(i-0) \\ &= \frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\ &= P_x \end{aligned}$$

From Definition 2.6, the circular auto-correlation of an N-point signal with periodic extension $x_p(k)$ is

$$\begin{aligned} c_{xx}(0) &= \frac{1}{N}\sum_{i=0}^{N-1} x(i)x_p(i-0) \\ &= \frac{1}{N}\sum_{i=0}^{N-1} x(i)x_p(i) \\ &= \frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\ &= P_x \end{aligned}$$

(b) From (2.8.5), the normalized auto-correlation of an N-point signal is

$$\begin{aligned} \rho_{xx}(0) &= \frac{r_{xx}(0)}{\sqrt{(N/N)r_{xx}(0)r_{xx}(0)}} \\ &= 1 \end{aligned}$$

From (2.8.7), the normalized circular auto-correlation of an N-point signal is

$$\begin{aligned}\sigma_{xx}(0) &= \frac{c_{xx}(0)}{\sqrt{c_{xx}(0)c_{xx}(0)}} \\ &= 1\end{aligned}$$

2.40 This problem establishes the normalized circular cross-correlation inequality, $|\sigma_{yx}(k)| \leq 1$. Let $x(k)$ and $y(k)$ be sequences of length N where $x_p(k)$ is the periodic extension of $x(k)$.

(a) Consider the signal $u(i,k) = ay(i) + x_p(i-k)$ where a is arbitrary. Show that

$$\frac{1}{N}\sum_{i=0}^{N-1}[ay(i) + x_p(i-k)]^2 = a^2 c_{yy}(0) + 2ac_{yx}(k) + c_{xx}(0) \geq 0$$

(b) Show that the inequality in part (a) can be written in matrix form as

$$[a, 1]\begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0)\end{bmatrix}\begin{bmatrix} a \\ 1\end{bmatrix} \geq 0$$

(c) Since the inequality in part (b) holds for any a, the 2×2 coefficient matrix $C(k)$ is positive semi-definite, which means that $\det[C(k)] \geq 0$. Use this fact to show that

$$c_{yx}^2(k) \leq c_{xx}(0)c_{yy}(0) \quad , \quad 0 \leq k < N$$

(d) Use the results from part (c) and the definition of normalized cross-correlation to show that

$$-1 \leq \sigma_{yx}(k) \leq 1 \quad , \quad 0 \leq k < N$$

Solution

(a)

$$\begin{aligned}
\frac{1}{N}\sum_{i=0}^{N-1} u^2(i,k) &= \frac{1}{N}\sum_{i=0}^{N-1}[ay(i)+x_p(i-k)]^2 \\
&= \frac{1}{N}\sum_{i=0}^{N-1} a^2y^2(i)+2ay(i)x_p(i-k)+x_p^2(i-k) \\
&= \frac{a^2}{N}\sum_{i=0}^{N-1} y^2(i)+\frac{2a}{N}\sum_{i=0}^{N-1} y(i)x_p(i-k)+\frac{1}{N}\sum_{i=0}^{N-1} x_p^2(i-k) \\
&= a^2c_{yy}(0)+2ac_{yx}(k)+\frac{1}{N}\sum_{i=0}^{N-1} x^2(i) \\
&= a^2c_{yy}(0)+2ac_{yx}(k)+c_{xx}(0) \\
&\geq 0
\end{aligned}$$

(b)

$$\begin{aligned}
[a,1]\begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0) \end{bmatrix}\begin{bmatrix} a \\ 1 \end{bmatrix} &= [a,1]\begin{bmatrix} ac_{yy}(0)+c_{yx}(k) \\ ac_{yx}(k)+c_{xx}(0) \end{bmatrix} \\
&= a^2c_{yy}(0)+ac_{yx}(k)+ac_{yx}(k)+c_{xx}(0) \\
&= a^2c_{yy}(0)+2ac_{yx}(k)+c_{xx}(0)
\end{aligned}$$

(c) The coefficient matrix $C(k)$ from part (b) is positive semi-definite and therefore $\det[C(k)] \geq 0$. But

$$\begin{aligned}
\det[C(k)] &= \det\left\{\begin{bmatrix} c_{yy}(0) & c_{yx}(k) \\ c_{yx}(k) & c_{xx}(0) \end{bmatrix}\right\} \\
&= c_{yy}(0)c_{xx}(0)-c_{yx}^2(k) \\
&\geq 0
\end{aligned}$$

Thus

$$c_{yx}^2(k) \leq c_{xx}(0)c_{yy}(0) \quad , \quad 0 \leq k < N$$

(d) Using (2.8.7) and the results from part (c)

$$\begin{aligned}
|\sigma_{yx}(k)| &= \left| \frac{c_{yx}(k)}{\sqrt{c_{xx}(0)c_{yy}(0)}} \right| \\
&= \left| \sqrt{\frac{c_{yx}^2(k)}{c_{xx}(0)c_{yy}(0)}} \right| \\
&\leq 1
\end{aligned}$$

Thus

$$-1 \leq \sigma_{yx}(k) \leq 1 \quad , \quad 0 \leq k < N$$

2.41 Consider the following FIR system.

$$y(k) = \sum_{i=0}^{5} (1+i)^2 x(k-i)$$

Let $x(k)$ be a bounded input with bound B_x. Show that $y(k)$ is bounded with bound $B_y = cB_x$. Find the minimum scale factor, c.

Solution

$$\begin{aligned}
|y(k)| &= \left| \sum_{i=0}^{5} (1+i)^2 x(k-i) \right| \\
&\leq \sum_{i=0}^{5} |(1+i)^2 x(k-i)| \\
&= \sum_{i=0}^{5} |(1+i)^2| \cdot |x(k-i)| \\
&\leq B_x \sum_{i=0}^{5} |(1+i)^2| \\
&= \|h\|_1 B_x
\end{aligned}$$

Here

$$
\begin{aligned}
\|h\|_1 &= \sum_{i=0}^{5}(1+i)^2 \\
&= 1+4+9+16+25+36 \\
&= 93
\end{aligned}
$$

Thus

$$
B_y = 93B_x
$$

2.42 Consider a linear time-invariant discrete-time system S with the following impulse response. Find conditions on A and p that guarantee that S is BIBO stable.

$$
h(k) = Ap^k\mu(k)
$$

Solution

The system S is BIBO stable if an only if $\|h\|_1 < \infty$. Here

$$
\begin{aligned}
\|h\|_1 &= \sum_{k=-\infty}^{\infty} |h(k)| \\
&= \sum_{k=0}^{\infty} Ap^k \\
&= A\sum_{k=0}^{\infty} p^k \\
&= \frac{A}{1-p} \quad , \quad |p| < 1
\end{aligned}
$$

Thus S is BIBO stable if and only if $|p| < 1$. There is no constraint on A.

2.43 From Proposition 2.1, a linear time-invariant discrete-time system S is BIBO stable if and only if the impulse response $h(k)$ is absolutely summable, that is, $\|h\|_1 < \infty$. Show that $\|h\|_1 < \infty$ is necessary for stability. That is, suppose that S is stable but $h(k)$ is not absolutely summable. Consider the following input, where $h^*(k)$ denotes the complex conjugate of $h(k)$ (Proakis and Manolakis,1992).

$$x(k) = \begin{cases} \dfrac{h^*(k)}{|h(k)|} & , \quad h(k) \neq 0 \\ 0 & , \quad h(k) = 0 \end{cases}$$

(a) Show that $x(k)$ is bounded by finding a bound B_x.

(b) Show that S is not is BIBO stable by showing that $y(k)$ is unbounded at $k = 0$.

Solution

(a) Since $x(0) = 0$ when $h(k) = 0$, consider the case when $h(k) \neq 0$.

$$\begin{aligned} |x(k)| &= \left| \frac{h^*(k)}{|h(k)|} \right| \\ &= \frac{|h^*(k)|}{|h(k)|} \\ &= \frac{|h(k)|}{|h(k)|} \\ &= 1 \end{aligned}$$

Thus $x(k)$ is bounded with $B_x = 1$.

(b)

$$\begin{aligned}
|y(0)| &= |h(k) \star x(k)|_{k=0} \\
&= \left| \sum_{i=-\infty}^{\infty} h(i)x(-i) \right| \\
&= \left| \sum_{i=-\infty}^{\infty} \frac{h(i)h^*(-i)}{|h(-i)|} \right| \\
&= \sum_{i=-\infty}^{\infty} \frac{|h(i)| \cdot |h^*(-i)|}{|h(-i)|} \\
&= \sum_{i=-\infty}^{\infty} |h(i)| \\
&= \|h\|_1 \\
&= \infty
\end{aligned}$$

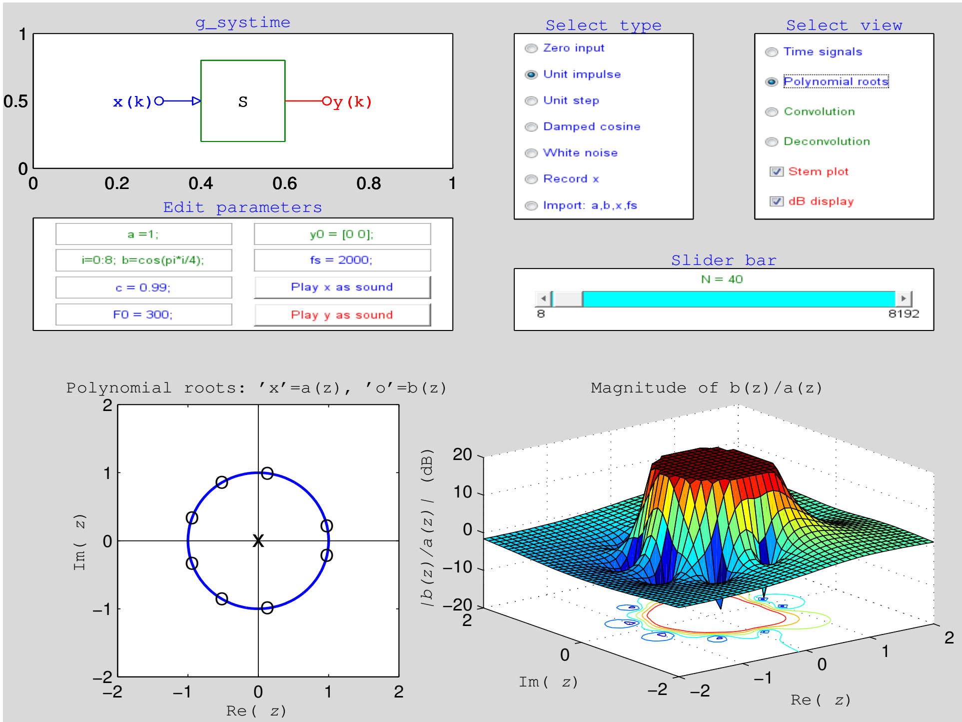
2.44 Consider the following discrete-time system. Use GUI module *g_systime* to simulate this system. Hint: You can enter the b vector in the edit box by using two statements on one line: i=0:8; b=cos(pi*i/4)

$$y(k) = \sum_{i=0}^{8} \cos(\pi i/4)x(k-i)$$

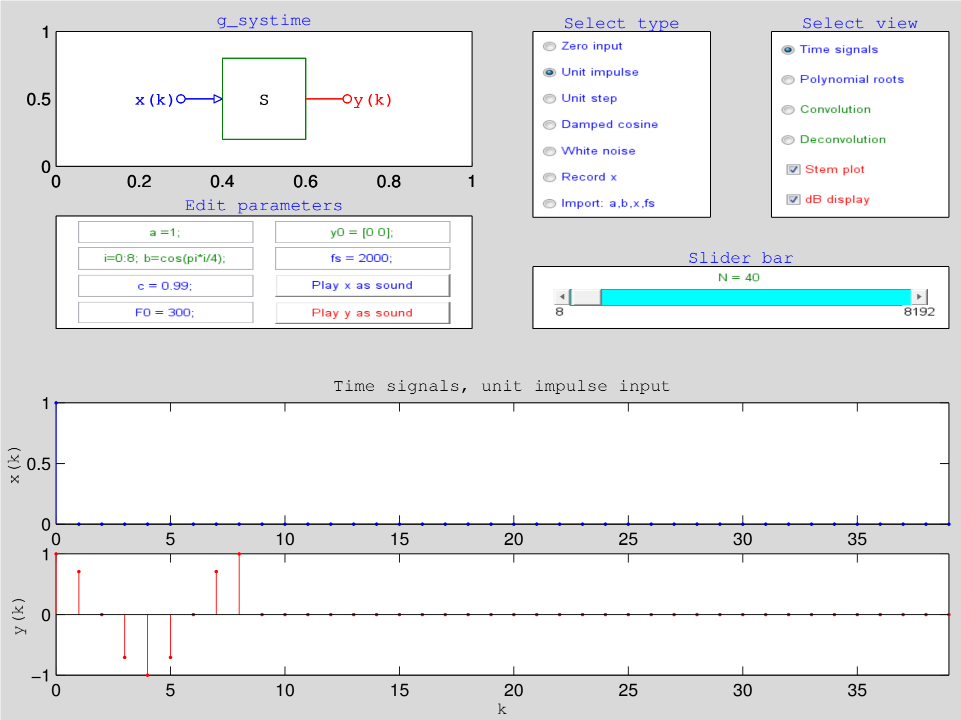
(a) Plot the polynomial roots

(b) Plot and the impulse response using $N = 40$.

Solution



Problem 2.44 (a) Polynomial Roots

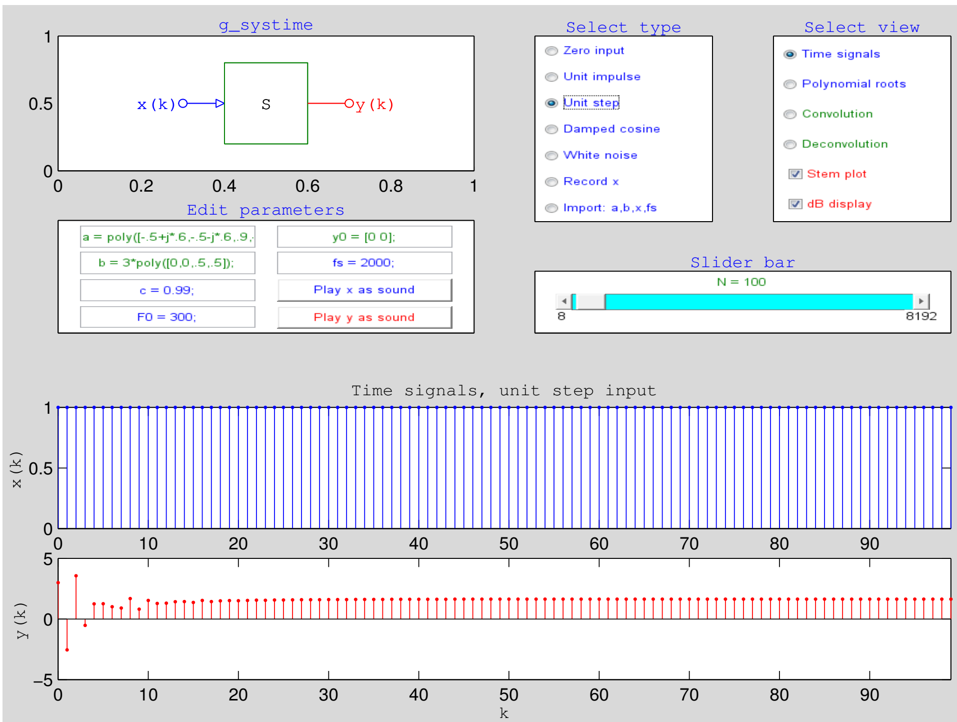


Problem 2.44 (b) Impulse Response

2.45 Consider a discrete-time system with the following characteristic and input polynomials. Use GUI module *g_systime* to plot the step response using $N = 100$ points. The MATLAB *poly* function can be used to specify coefficient vectors a and b in terms of their roots, as discussed in Section 2.9.

$$
\begin{aligned}
a(z) &= (z + .5 \pm j.6)(z - .9)(z + .75) \\
b(z) &= 3z^2(z - .5)^2
\end{aligned}
$$

Solution



Problem 2.45 Step Response

√ 2.46 Consider the following linear discrete-time system.

$$y(k) = 1.7y(k-2) - .72y(k-4) + 5x(k-2) + 4.5x(k-4)$$

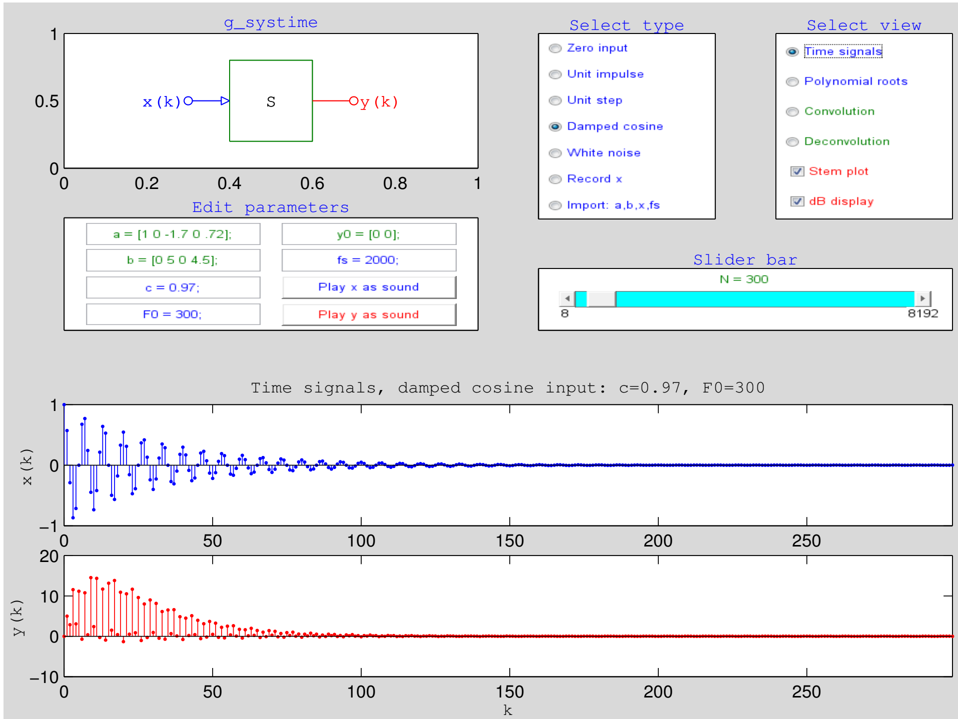
Use GUI module *g_systime* to plot the following damped cosine input and the zero-state response to it using $N = 30$. To determine F_0, set $2\pi F_0 kT = .3\pi k$ and solve for F_0/f_s where $T = 1/f_s$.

$$x(k) = .97^k \cos(.3\pi k)$$

Solution

$$2\pi F_0 kT = .3\pi k$$

Thus $2F_0T = .3$ or $F_0 = .15f_s$. If $f_s = 2000$, then $F_0 = 300$.



Problem 2.46 Input and Output

2.47 Consider the following linear discrete-time system.

$$y(k) = -.4y(k-1) + .19y(k-2) - .104y(k-3) + 6x(k) - 7.7x(k-1) + 2.5x(k-2)$$

Create a MAT-file called *prob2_47* that contains $fs = 100$, the appropriate coefficient vectors a and b, and the following input samples, where $v(k)$ is white noise uniformly distributed over $[-.2, .2]$. Uniform white noise can be generated with the MATLAB function *rand*.

$$x(k) = k\exp(-k/50) + v(k) \quad , \quad 0 \le k < 500$$

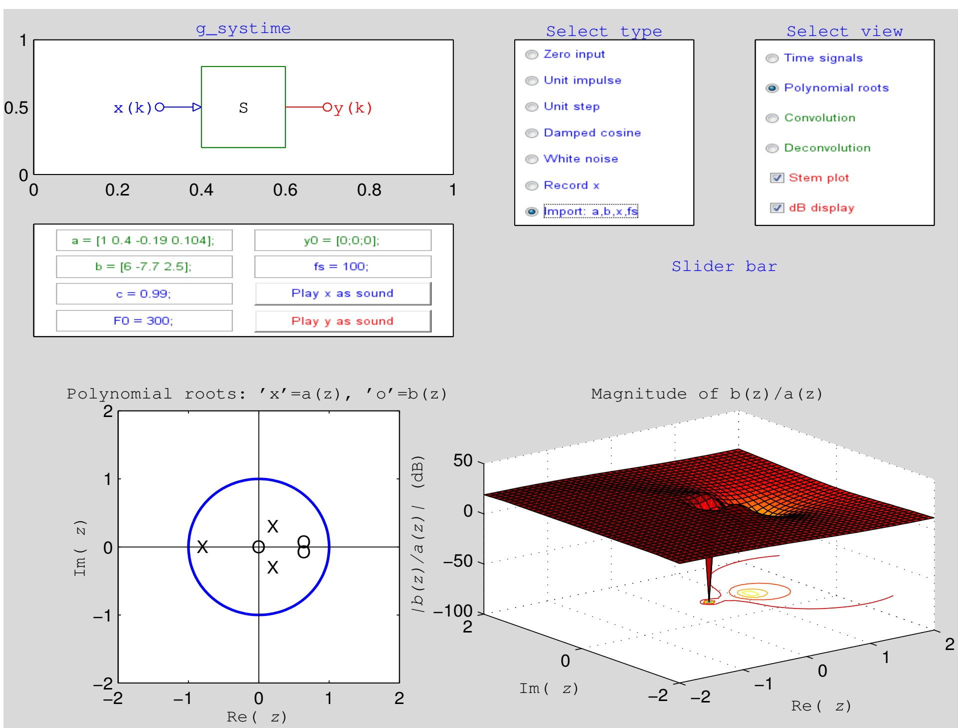
(a) Print the MATLAB program used to create *prob2_47.mat*.

(b) Use GUI module *g_systime* and the Import option to plot the roots of the characteristic polynomial and the input polynomial.

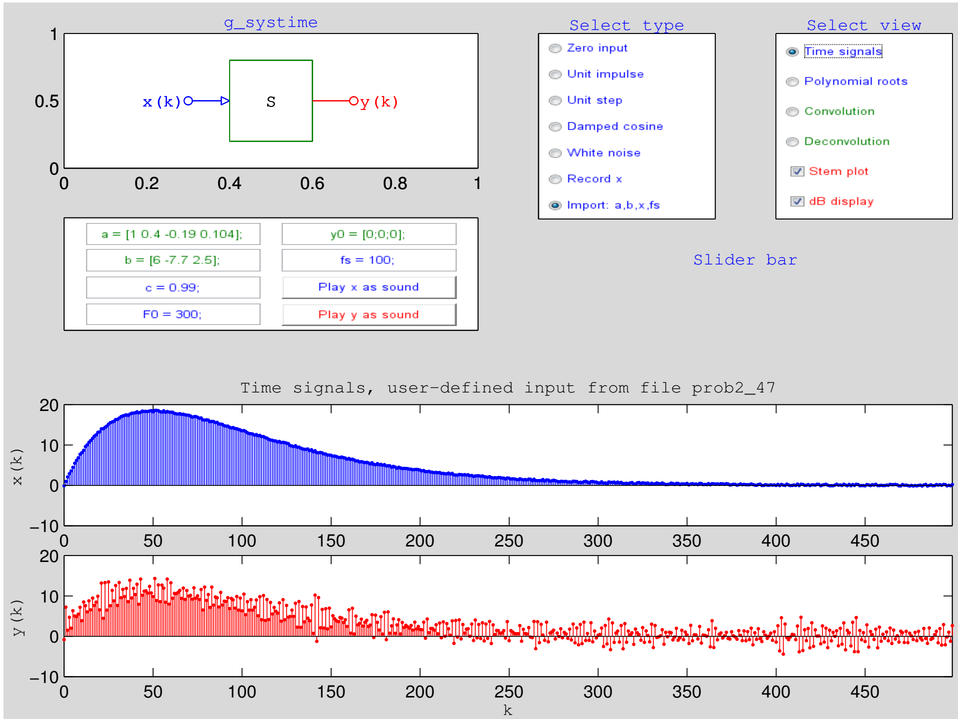
(c) Plot the zero-state response on the input $x(k)$.

Solution

(a)
```
% Problem 2.47

f_header('Problem 2.47: Create MAT file')
fs = 100;
a = [1 .4 -.19 .104]
b = [6 -7.7 2.5];
N = 500;
v = -.2 + .4*rand(1,N);
k = 0:N-1;
x = k .* exp(-k/50) + v;
save prob2_47 fs a b x
what
```



Problem 2.47 (b) Polynomial Roots



Problem 2.47 (c) Input and Output

2.48 Consider the following discrete-time system, which is a narrow band *resonator* filter with sampling frequency of $f_s = 800$ Hz.

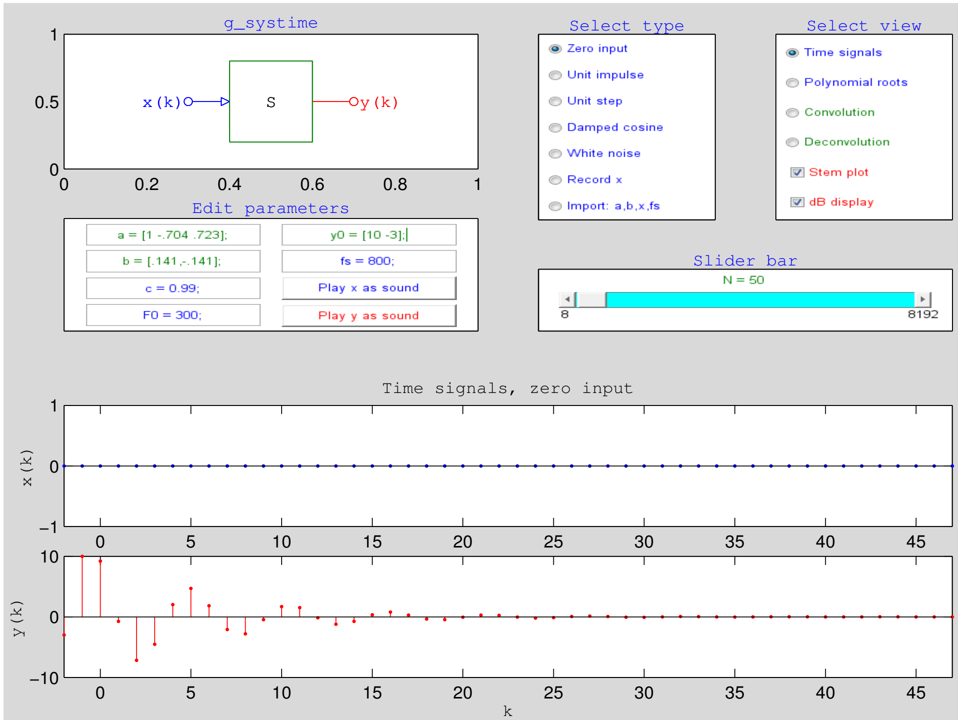
$$y(k) \quad = \quad .704y(k-1) - .723y(k-2) + .141x(k) - .141x(k-2)$$

Use GUI module *g_systime* to find the zero-input response for the following initial conditions. In each chase plot $N = 50$ points.

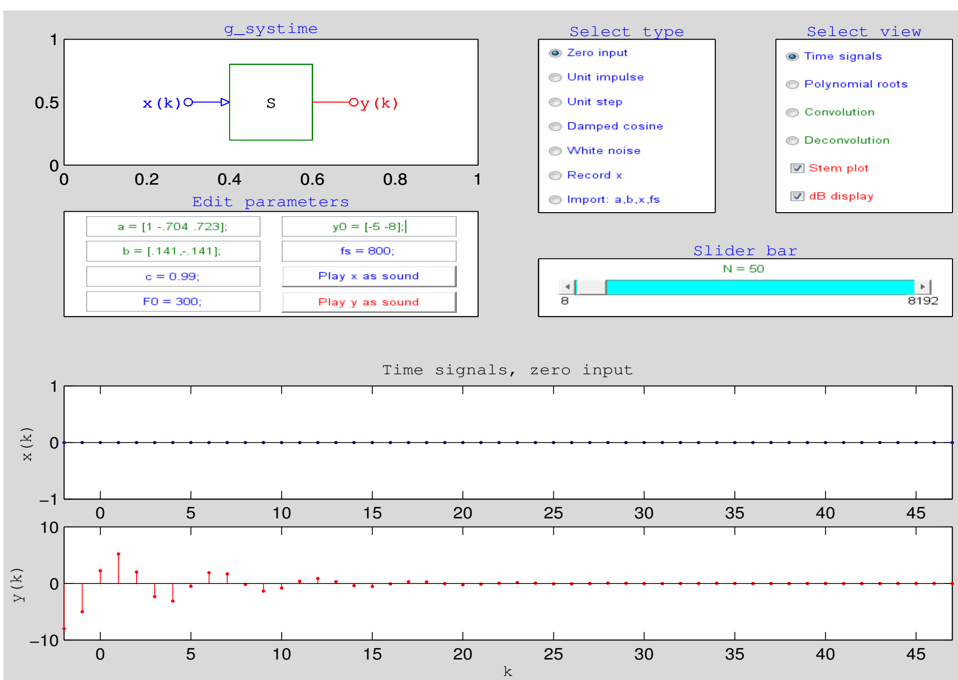
(a) $y_0 = [10, -3]^T$

(b) $y_0 = [-5, -8]^T$

Solution



Problem 2.48 (a) Zero-input Response



Problem 2.48 (b) Zero-input Response

2.49 Consider the following discrete-time system, which is a *notch* filter with sampling interval $T = 1/360$ sec.

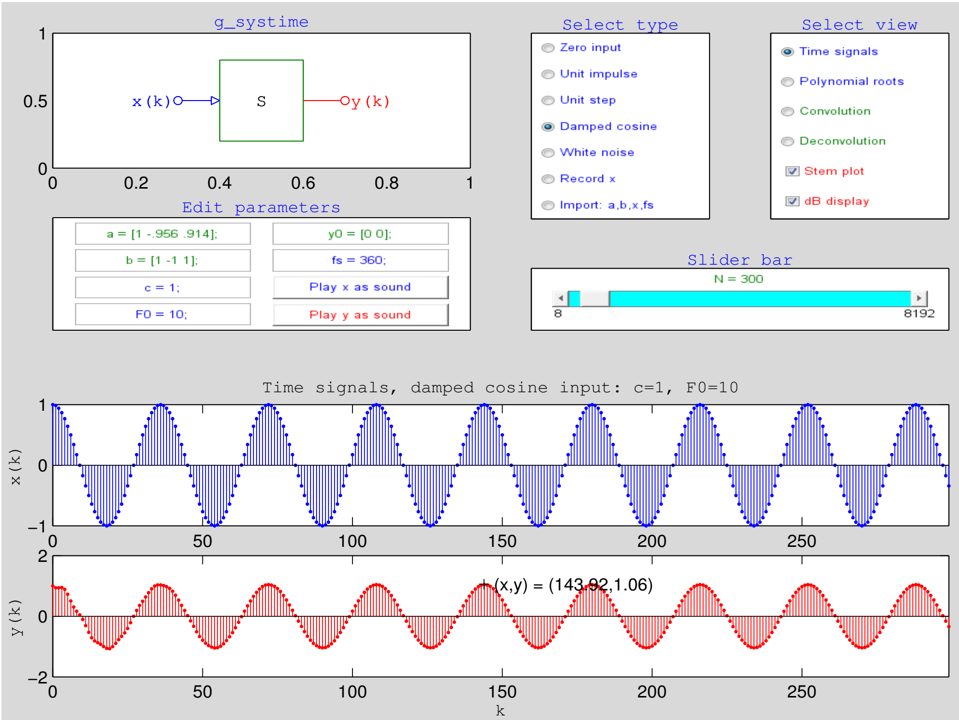
$$y(k) = .956y(k-1) - .914y(k-2) + x(k) - x(k-1) + x(k-2)$$

Use GUI module *g_systime* to find the output corresponding to the sinusoidal input $x(k) = \cos(2\pi F_0 kT)\mu(k)$. Do the following cases. Use the caliper option to estimate the steady state amplitude in each case.

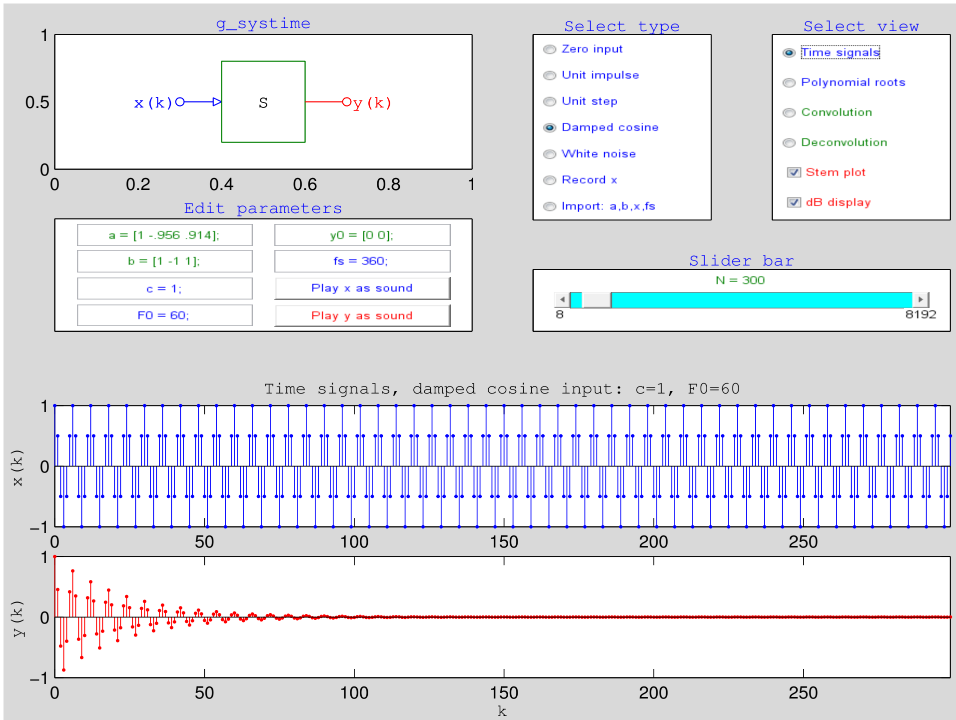
(a) Plot the output when $F_0 = 10$ Hz.

(b) Plot the output when $F_0 = 60$ Hz.

Solution



Problem 2.49 (a) $F_0 = 10$ **Hz**

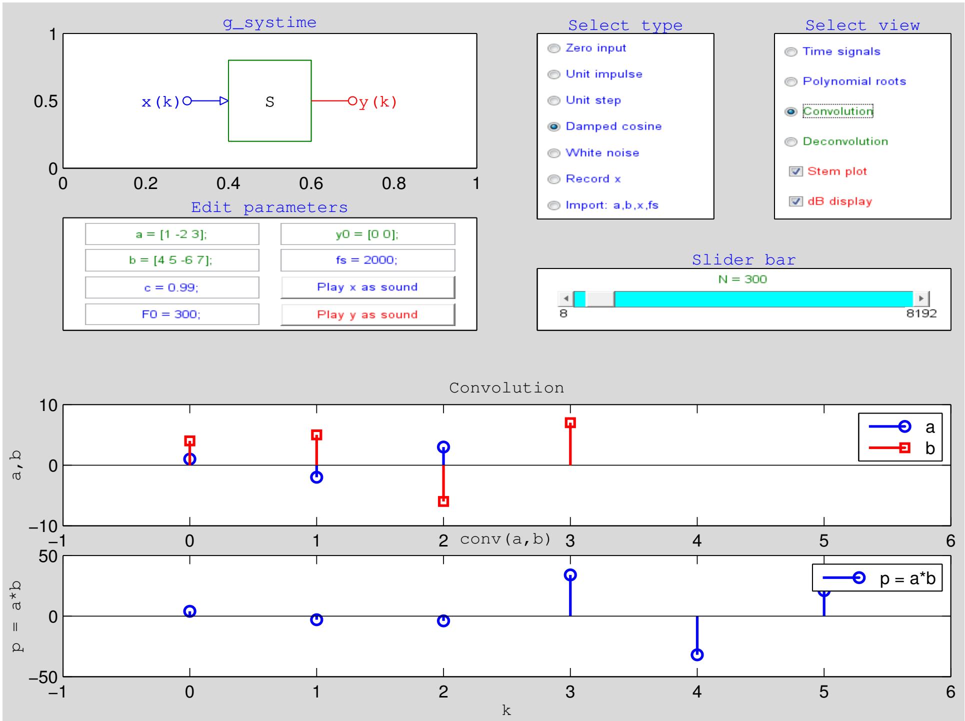


Problem 2.49 (b) $F_0 = 60$ **Hz**

2.50 Consider the following two polynomials. Use *g_systime* to compute, plot, and Export to a data file the coefficients of the product polynomial $c(z) = a(z)b(z)$. Then Import the saved file and display the coefficients of the product polynomial.

$$
\begin{aligned}
a(z) &= z^2 - 2z + 3 \\
b(z) &= 4z^3 + 5z^2 - 6z + 7
\end{aligned}
$$

Solution



Problem 2.50 Polynomial Multiplication

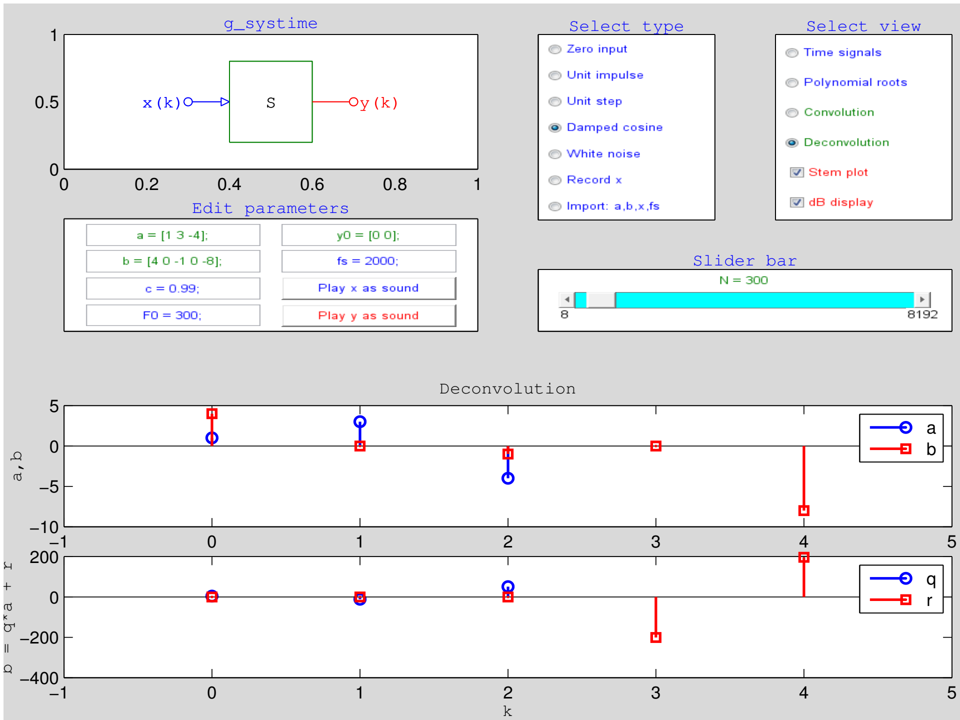
```
product =
     4    -3    -4    34   -32    21
```

2.51 Consider the following two polynomials. Use *g_systime* to compute, plot, and Export to a data file the coefficients of the quotient polynomial $q(z)$ and the remainder polynomial $r(z)$ where $b(z) = q(z)a(z) + r(z)$. Then Import the saved file and display the coefficients of the quotient and remainder polynomials.

$$\begin{aligned} a(z) &= z^2 + 3z - 4 \\ b(z) &= 4z^4 - z^2 - 8 \end{aligned}$$

Solution

```
quotient =
     4   -12    51
remainder =
     0     0     0  -201   196
```



Problem 2.51 Polynomial Division

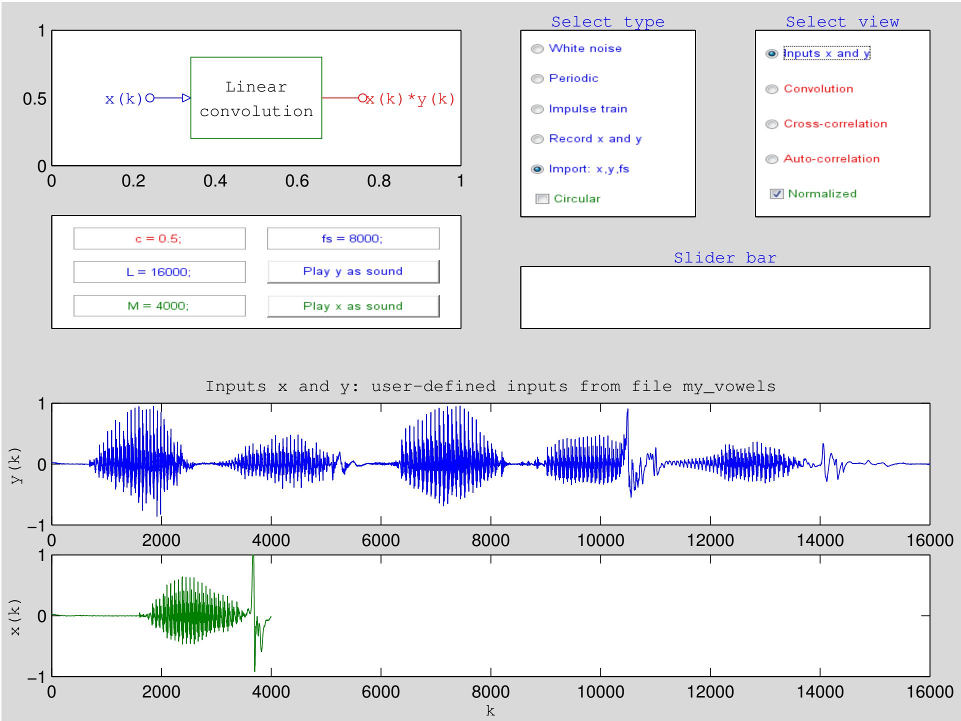
√ 2.52 Use the GUI module *g_correlate* to record the sequence of vowels "A","E","I", "O","U" in y. Play y to make sure you have a good recording of all five vowels. Then record the vowel "O" in x. Play x back to make sure you have a good recording of "O" that sounds similar to the "O" in y. Export the results to a MAT-file named *my_vowels*.

(a) Plot the inputs x and y showing the vowels.

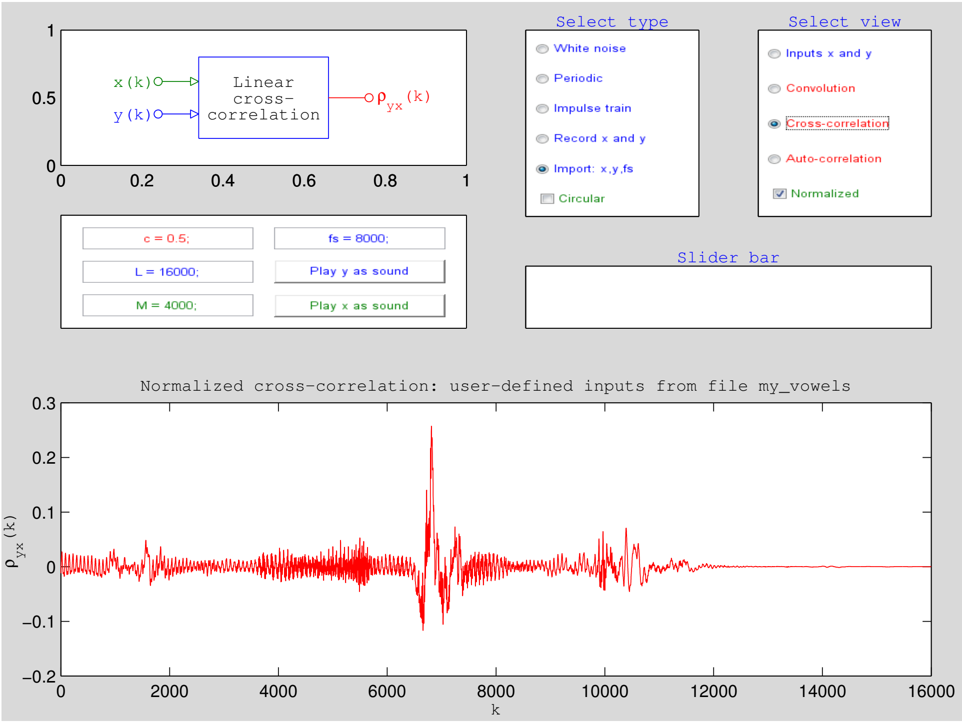
(b) Plot the normalized cross-correlation of y with x using the *Caliper* option to mark the peak which should show the location of x in y.

(c) Based on the plots in (a), estimate the lag d_1 that would be required to get the "O" in x to align with the "O" in y. Compare this with the peak location d_2 in (b). Find the percent error relative to the estimated lag d_1. There will be some error due to the overlap of x with adjacent vowels and co-articulation effects in creating y.

Solution



Problem 2.52 (a) The Vowels A, E, I, O, U



Problem 2.52 (b) Normalized Cross-correlation of x with y

(c) From part (a), the start of O in x is approximately $o_x = 9000$, and the start of O in y is approximately $o_y = 1800$. Thus the translation of y required to get a match with x is

$$\begin{aligned} d_1 &= o_x - o_y \\ &\approx 9000 - 1800 \\ &= 7200 \end{aligned}$$

The peak in part (b) is at $d_2 = 6807$. Thus the percent error in finding the location of O in x is

$$\begin{aligned} E &= \frac{100(d_2 - d_1)}{d_1} \\ &= \frac{100(6807 - 7200)}{7200} \\ &= -5.46\ \% \end{aligned}$$

2.53 The file *prob2_53.mat* contains two signals, x and y, and their sampling frequency, fs. Use the GUI module *g_correlate* to Import x, y, and fs.

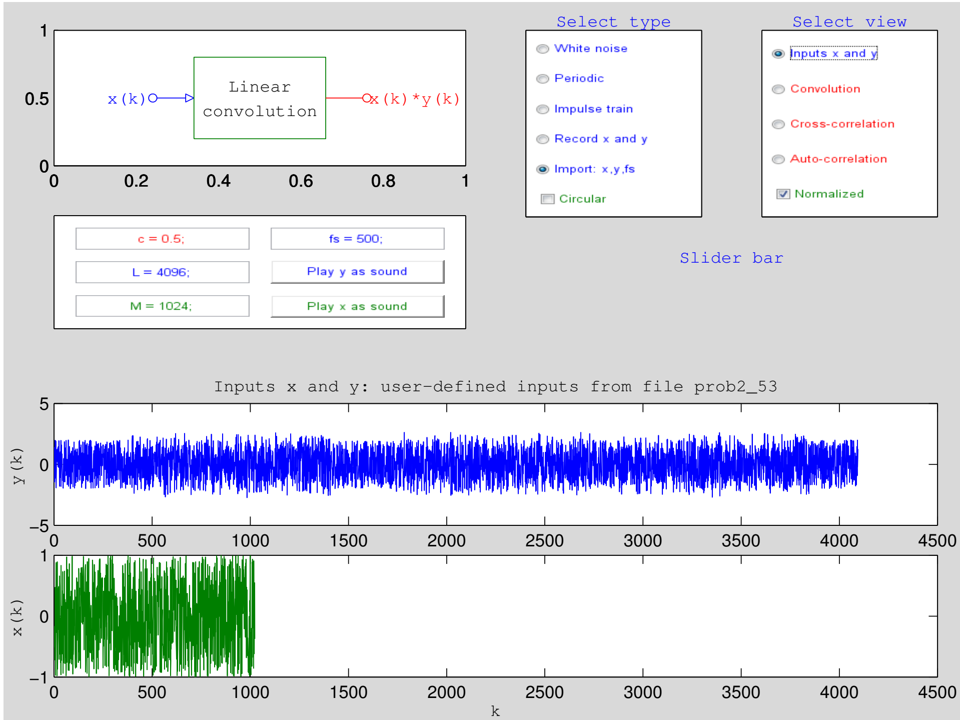
(a) Plot $x(k)$ and $y(k)$.

(b) Plot the normalized linear cross-correlation $\rho_{yx}(k)$. Does $y(k)$ contain any scaled and shifted versions of $x(k)$? Determine how many, and use the Caliper option to estimate the locations of $x(k)$ within $y(k)$.

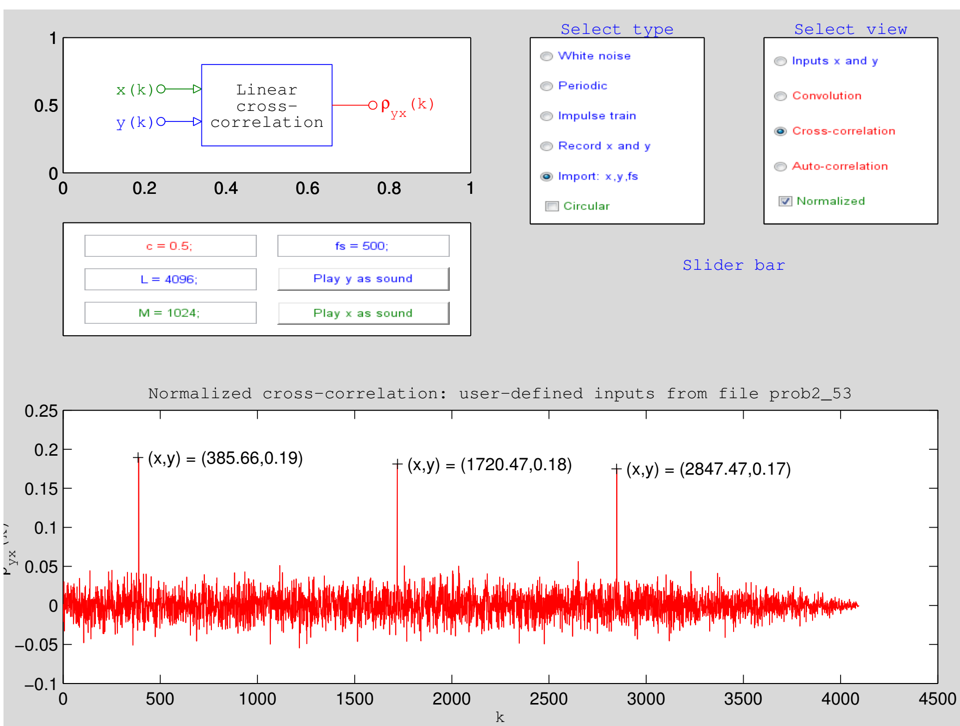
Solution

From the plot of $\rho_{xy}(k)$, there are three scaled and shifted versions of $y(k)$ within $x(k)$. They are located at

$$k = [388, 1718, 2851]$$



Problem 2.53 (a)



Problem 2.53 (b)

2.54 Consider the following discrete-time system.

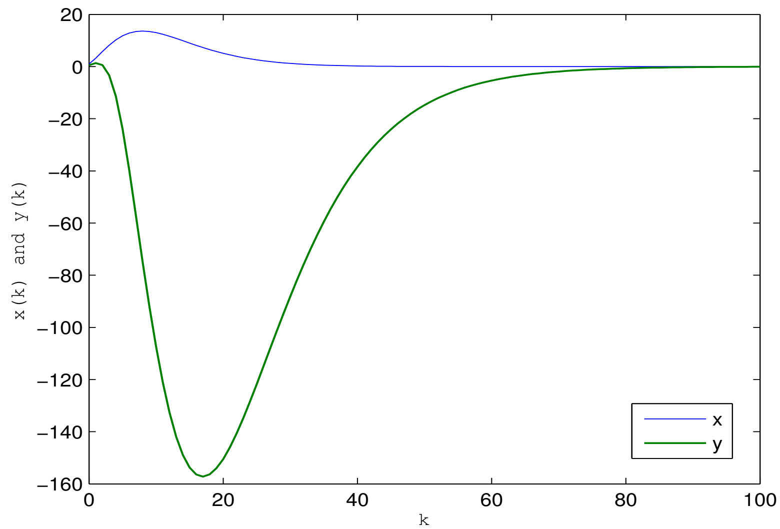
$$y(k) = .95y(k-1) + .035y(k-2) - .462y(k-3) + .351y(k-4) + .5x(k) - .75x(k-1) - 1.2x(k-2) + .4x(k-3) - 1.2x(k-4)$$

Write a MATLAB program that uses *filter* and *plot* to compute and plot the zero-state response of this system to the following input. Plot both the input and the output on the same graph.

$$x(k) = (k+1)^2(.8)^k\mu(k) \quad , \quad 0 \le k \le 100$$

Solution

```
% Problem 2.54

% Initialize

f_header('Problem 2.54')
a = [1 -.95 -.035 .462 -.351]
b = [.5 -.75 -1.2 .4 -1.2]
N =101;
k = 0 : N-1;
x = (k+1).^2 .* (.8).^k;

% Find zero-state response

y = filter (b,a,x);

% Plot input and output

figure
h = plot (k,x,k,y);
set (h(2),'LineWidth',1.0)
f_labels ('','k','x(k) and y(k)')
legend ('x','y')
f_wait
```



Problem 2.54 Input and Zero-State Response

2.55 Consider the following discrete-time system.

$$a(z) = z^4 - .3z^3 - .57z^2 + .115z + .0168$$
$$b(z) = 10(z + .5)^3$$

This system has four simple nonzero roots. Therefore the zero-input response consists of a sum of the following four natural mode terms.

$$y_{zi}(k) = c_1 p_1^k + c_2 p_2^k + c_3 p_3^k + c_4 p_4^k$$

The coefficients can be determined from the initial condition

$$y_0 = [y(-1), y(-2), y(-3), y(-4)]^T$$

Setting $y_{zi}(-k) = y(-k)$ for $1 \le k \le 4$ yields the following linear algebraic system in the coefficient vector $c = [c_1, c_2, c_3, c_4]^T$.

$$\begin{bmatrix} p_1^{-1} & p_2^{-1} & p_3^{-1} & p_4^{-1} \\ p_1^{-2} & p_2^{-2} & p_3^{-2} & p_4^{-2} \\ p_1^{-3} & p_2^{-3} & p_3^{-3} & p_4^{-3} \\ p_1^{-4} & p_2^{-4} & p_3^{-4} & p_4^{-4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = y_0$$

Write a MATLAB program that uses *roots* to find the roots of the characteristic polynomial and then solves this linear algebraic system for the coefficient vector c using the MATLAB left division or \ operator when the initial condition is y_0. Print the roots and the coefficient vector c. Use *stem* to plot the zero-input response $y_{zi}(k)$ for $0 \leq k \leq 40$.

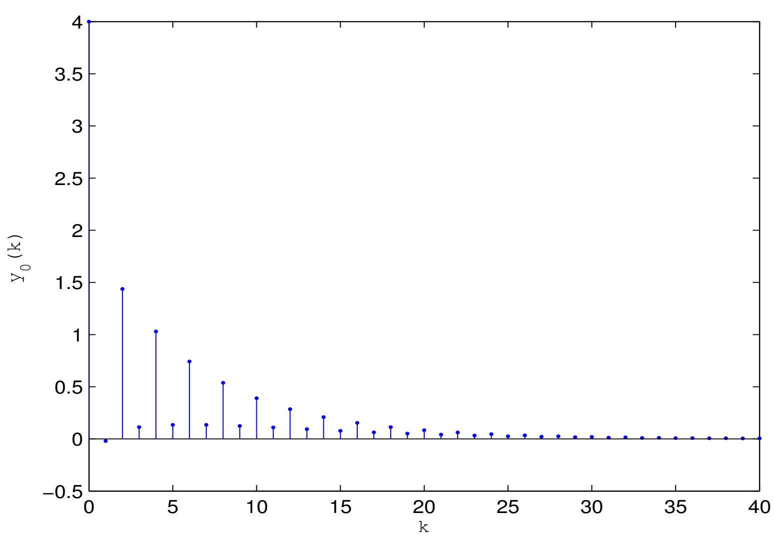
Solution

```
% Problem 2.55

% Initialize

f_header('Problem 2.55')
a = [1 -.3 -.57 .115 .0168]
y = [2 -1 0 3]'
n = 4;

% Construct coefficient matrix

p = roots(a)
A = zeros(n,n);
for i = 1 : n
    for k = 1 : n
       A(i,k) = p(k)^(-i);
   end
end

% Find coefficient vector c

c = A \ y

% Compute zero-input response

N =41;
k = 0 : N-1;
y_0 = zeros(1,N);
for i = 1 : n
```

```
    y_0 = y_0 + c(i) .^ k;
end

% Plot it

figure
stem (k,y_0,'filled','.')
f_labels ('','k','y_0(k)')
f_wait
```

Program Output:

```
p =
   -.7000
    .8000
    .3000
   -.1000
c =
   -.8195
    .8720
   -.0742
    .0013
```



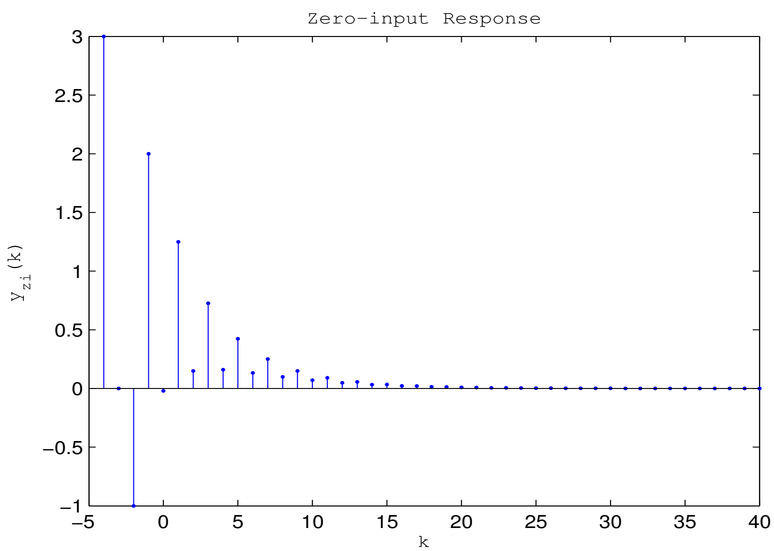
Problem 2.55 Zero-Input Response to Initial Condition

√ 2.56 Consider the discrete-time system in Problem 2.55. Write a MATLAB program that uses the DSP Companion function *f_filter0* to compute the zero-input response to the following initial condition. Use *stem* to plot the zero-input response $y_{zi}(k)$ for $-4 \le k \le 40$.

$$y_0 = [y(-1), y(-2), y(-3), y(-4)]^T$$

Solution

```
% Problem 2.56

% Initialize

f_header('Problem 2.56')
a = [1 -.3 -.57 .115 .0168]
b = 10*poly([-.5,-.5,-.5])
y0 = [2 -1 0 3]'
n = 4;

% Solve system

N = 41;
x = zeros(1,N);
y_zi = f_filter0(b,a,x,y0);

% Plot it

figure
k = [-n : N-1];
stem (k,y_zi,'filled','.')
f_labels ('Zero-input Response','k','y_{zi}(k)')
f_wait
```



Problem 2.56 Zero-input Response

2.57 Consider the following running average filter.

$$y(k) = \frac{1}{10} \sum_{i=0}^{9} x(k-i) \quad , \quad 0 \le k \le 100$$

Write a MATLAB program that performs the following tasks.

(a) Use *filter* and *plot* to compute and plot the zero-state response to the following input, where $v(k)$ is a random white noise uniformly distributed over $[-.1, .1]$. Plot $x(k)$ and $y(k)$ below one another. Uniform white noise can be generated using the MATLAB function *rand*.

$$x(k) \quad = \quad \exp(-k/20)\cos(\pi k/10)\mu(k) + v(k)$$

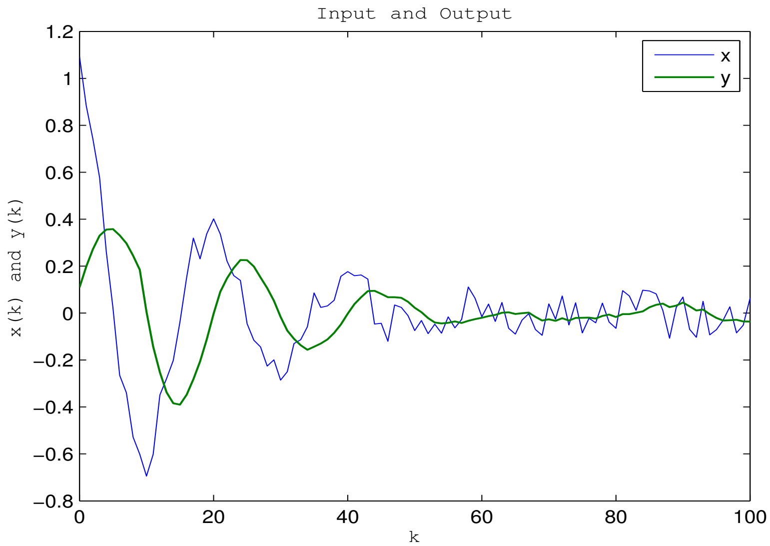
(b) Add a third curve to the graph in part (a) by computing and plotting the zero-state response using *conv* to perform convolution.

Solution

The transfer function of this FIR filter is

$$H(z) = .1 \sum_{i=0}^{9} z^{-i}$$

```
% Problem 2.57

% Initialize

f_header('Problem 2.57')
m = 9;
b = .1*ones(1,m+1);
a = 1;
N =101;
k = 0 : N-1;
c = .1;
x = exp(-k/20) .* cos(pi*k/10) + f_randu(1,N,-c,c);

% Find zero-state response

y = filter (b,a,x);

% Plot input and output

figure
h = plot (k,x,k,y);
set (h(2),'LineWidth',1.0)
f_labels ('Input and Output','k','x(k) and y(k)')
legend ('x','y')
f_wait
```



Problem 2.57 Running Average Filter of Order $m = 9$

2.58 Consider the following FIR filter. Write a MATLAB program that performs the following tasks.

$$y(k) = \sum_{i=0}^{20} \frac{(-1)^i x(k-i)}{10 + i^2}$$

(a) Use the function *filter* to compute and plot the impulse response $h(k)$ for $0 \le k < N$ where $N = 50$.

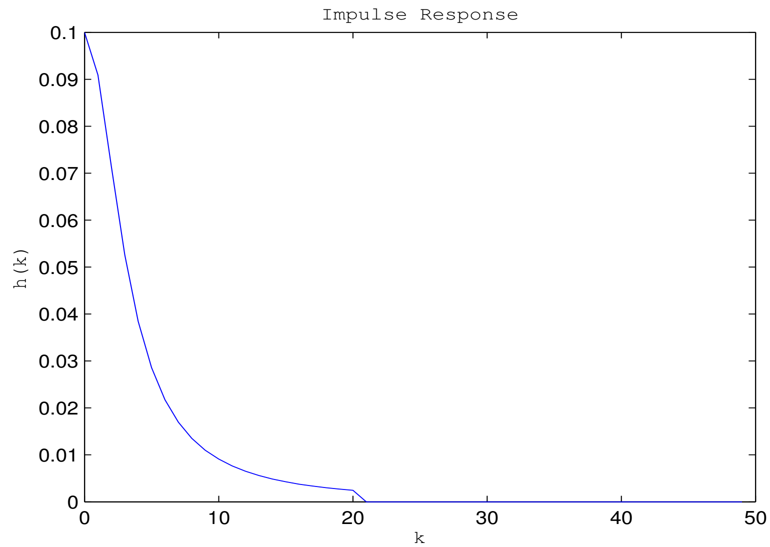
(b) Compute and plot the following periodic input.

$$x(k) = \sin(.1\pi k) - 2\cos(.2\pi k) + 3\sin(.3\pi k) \quad , \quad 0 \le k < N$$

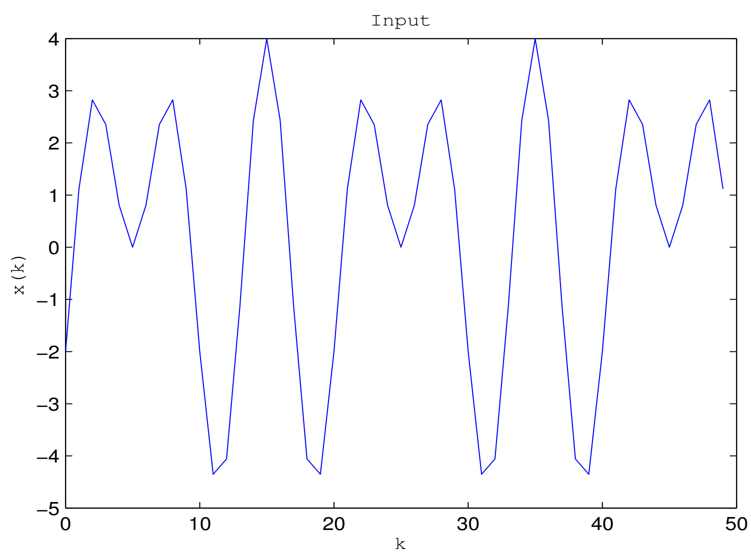
(c) Use *conv* to compute the zero-state response to the input $x(k)$ using convolution. Also compute the zero-state response to $x(k)$ using *filter*. Plot both responses on the same graph using a legend.

Solution

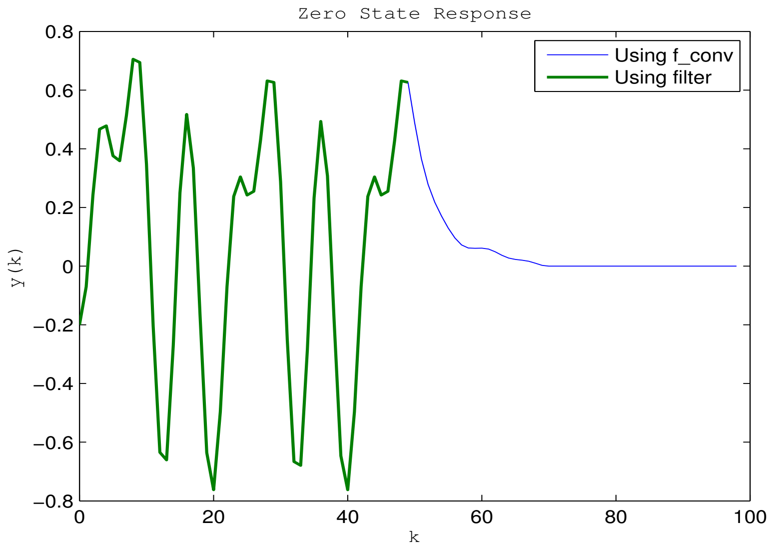
```
% Problem 2.58

% Construct filter

f_header('Problem 2.58')
i = 0 : 20;
b = (-1).^2 ./ (10 + i.^2);
a = 1;

% Construct input

N = 50;
k = 0 : N-1;
x = sin(.1*pi*k) - 2*cos(.2*pi*k) + 3*sin(.3*pi*k);

% Compute and plot impulse response

delta = [1,zeros(1,N-1)];
h = filter (b,a,delta);
figure
plot (k,h)
f_labels ('Impulse Response','k','h(k)')
f_wait

% Compute and plot zero-state response using convolution

figure
plot (k,x)
f_labels ('Input','k','x(k)')
f_wait
circ = 0;
y1 = f_conv (h,x,circ);
k1 = 0 : length(y1)-1;
y2 = filter (b,a,x);
k2 = 0 : N-1;
hp = plot (k1,y1,k2,y2);
set (hp(2),'LineWidth',1.5)
f_labels ('Zero State Response','k','y(k)')
legend ('Using f\_conv','Using filter')
f_wait
```



Problem 2.58 (a) Impulse Response



Problem 2.58 (b) Periodic Input



Problem 2.58 (c) Zero-State Response

2.59 Consider the following pair of signals.

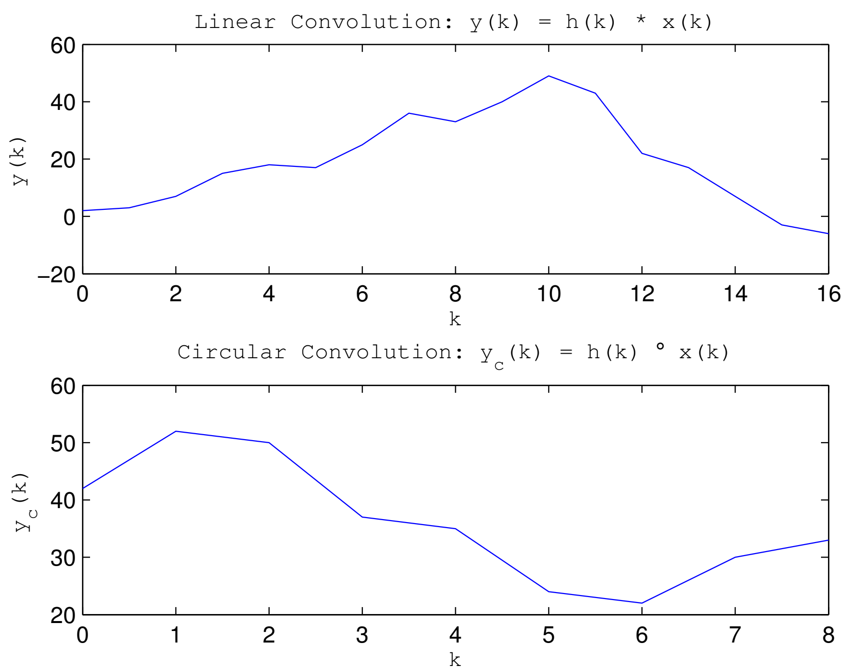
$$\begin{aligned} h &= [1, 2, 3, 4, 5, 4, 3, 2, 1]^T \\ x &= [2, -1, 3, 4, -5, 0, 7, 9, -6]^T \end{aligned}$$

Verify that linear convolution and circular convolution produce different results by writing a MATLAB program that uses the DSP Companion function *f_conv* to compute the linear convolution $y(k) = h(k) \star x(k)$ and the circular convolution $y_c(k) = h(k) \circ x(k)$. Plot $y(k)$ and $y_c(k)$ below one another on the same screen.

Solution

```
% Problem 2.59

% Initialize

f_header('Problem 2.59')
h = [1 2 3 4 5 4 3 2 1]
x = [2 -1 3 4 -5 0 7 9 -6]

% Compute convolutions
```

```
y = f_conv (h,x,0);
y_c = f_conv (h,x,1);

% Plot them

figure
subplot (2,1,1)
k = 0 : length(y)-1;
plot (k,y)
f_labels ('Linear Convolution: y(k) = h(k) * x(k)','k','y(k)')
subplot (2,1,2)
k = 0 : length(y_c)-1;
plot (k,y_c)
f_labels ('Circular Convolution: y_c(k) = h(k) \circ x(k)','k','y_c(k)')
f_wait
```



Problem 2.59 Linear and Circular Convolution

2.60 Consider the following pair of signals.

$$\begin{aligned} h &= [1, 2, 4, 8, 16, 8, 4, 2, 1]^T \\ x &= [2, -1, -4, -4, -1, 2]^T \end{aligned}$$

Verify that linear convolution can be achieved by zero padding and circular convolution by writing a MATLAB program that pads these signals with an appropriate number of zeros and uses the DSP Companion function *f_conv* to compare the linear convolution $y(k) = h(k) \star x(k)$ with the circular convolution $y_{zc}(k) = h_z(k) \circ x_z(k)$. Plot the following.

(a) The zero-padded signals $h_z(k)$ and $x_z(k)$ on the same graph using a legend.

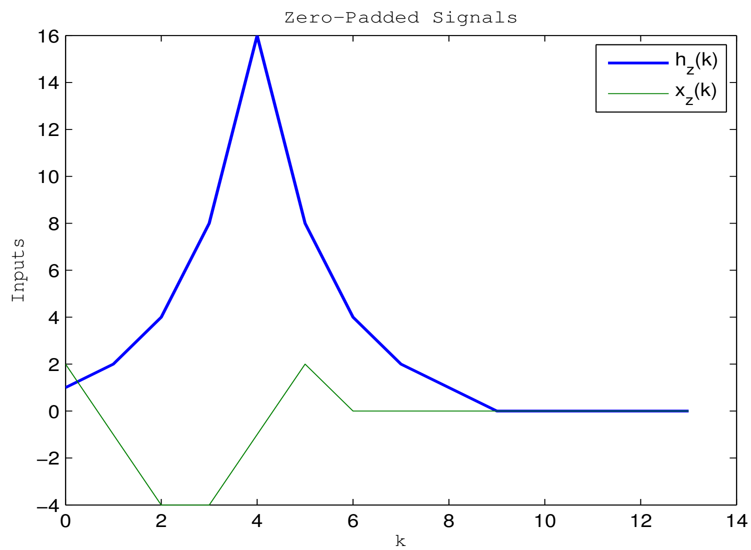
(b) The linear convolution $y(k) = h(k) \star x(k)$.

(c) The zero-padded circular convolution $y_{zc}(k) = h_z(k) \circ x_z(k)$.

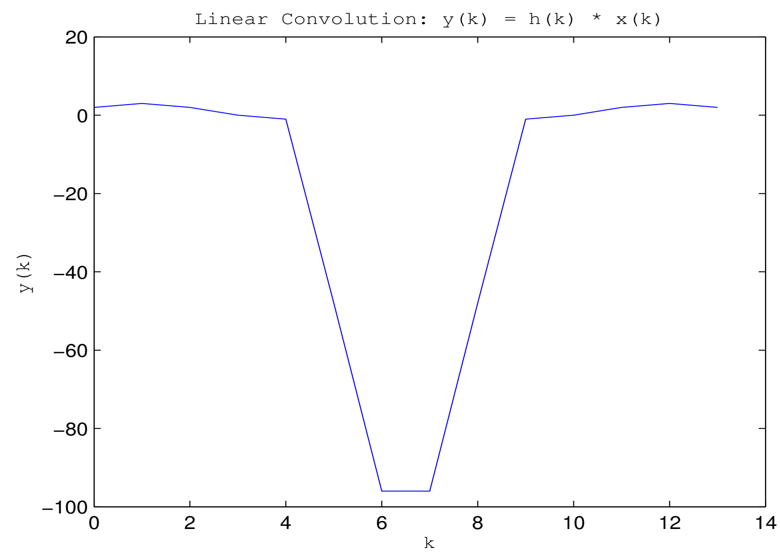
Solution

```
% Problem 2.60

% Initialize

f_header('Problem 2.60')
h = [1 2 4 8 16 8 4 2 1];
x = [2 -1 -4 -4 -1 2];

% Construct and plot zero-padded signals

L = length(h);
M = length(x);
h_z = [h, zeros(1,M-1)]
x_z = [x, zeros(1,L-1)]
figure
k = 0 : length(h_z)-1;
hp = plot (k,h_z,k,x_z);
set (hp(1),'LineWidth',1.5)
f_labels ('Zero-Padded Signals','k','Inputs')
legend ('h_z(k)','x_z(k)')
f_wait

% Compute and plot convolutions
```

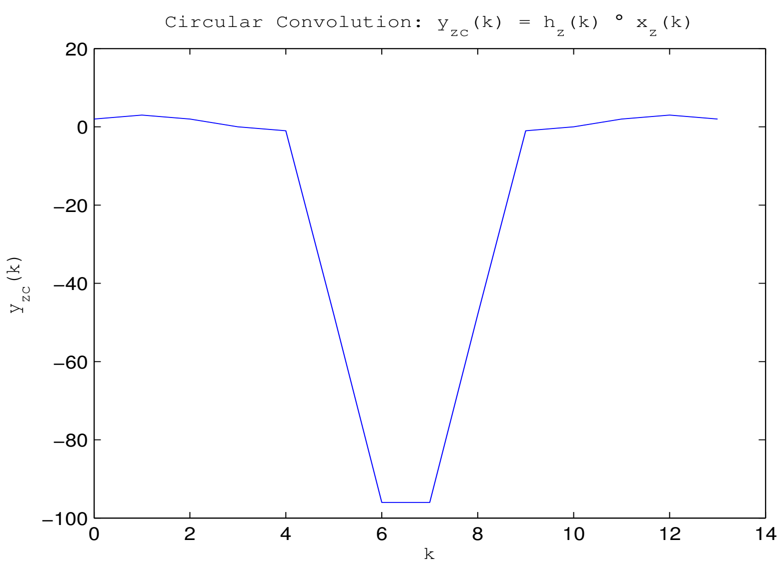
```
y = f_conv (h,x,0);
y_zc = f_conv (h_z,x_z,1);
figure
plot (k,y)
f_labels ('Linear Convolution: y(k) = h(k) * x(k)','k','y(k)')
f_wait
figure
plot (k,y_zc)
f_labels ('Circular Convolution: y_{zc}(k) = h_z(k) \circ x_z(k)','k','y_{zc}(k)')
f_wait
```



Problem 2.60 (a) Zero-padded Signals



Problem 2.60 (b) Linear Convolution



Problem 2.60 (c) Zero-padded Circular Convolution

$\boxed{2.61}$ Consider the following polynomials

$$
\begin{aligned}
a(z) &= z^4 + 4z^3 + 2z^2 - z + 3 \\
b(z) &= z^3 - 3z^2 + 4z - 1 \\
c(z) &= a(z)b(z)
\end{aligned}
$$

Let $a \in R^5$, $b \in R^4$ and $c \in R^8$ be the coefficient vectors of $a(z)$, $b(z)$ and $c(z)$, respectively.

(a) Find the coefficient vector of $c(z)$ by direct multiplication by hand.

(b) Write a MATLAB program that uses *conv* to find the coefficient vector of $c(z)$ by computing c as the linear convolution of a with b.

(c) In the program, show that a can be recovered from b and c by using the MATLAB function *deconv* to perform deconvolution.

Solution

```
% Problem 2.61

% Initialize

f_header('Problem 2.61')
a = [1 4 2 -1 3]
b = [1 -3 4 -1]

% Construct coefficient vector of product polynomial

c = conv (a,b)

% Recover coefficients of a from b and c

[a,r] = deconv (c,a)
```

(a) Using direct multiplication, $C(z) = A(z)B(z)$, we have

$$\begin{array}{rcl} A(z)B(z) & = & z^4 + 4z^3 + 2z^2 - z + 3 \\ & & \quad z^3 - 3z^2 + 4z - 1 \\ \hline & & z^7 + 4z^6 + 2z^5 - z^4 + 3z^3 \\ & & \quad -3z^6 - 12z^5 - 6z^4 + 3z^3 - 9z^2 \\ & & \qquad 4z^5 + 16z^4 + 8z^3 - 4z^2 + 12z \\ & & \qquad\quad -z^4 - 4z^3 - 2z^2 + z - 3 \\ \hline & & z^7 + z^6 - 6z^5 + 8z^4 + 10z^3 - 15z^2 + 13z - 3 \end{array}$$

Thus the coefficient vector of the product polynomial is

$$c = [1, 1, -6, 8, 10, -15, 13, -3]^T$$

(b) The program output for c using *conv* is

```
c =
         1     1    -6     8    10   -15    13    -3
```

(c) The program output for a using *deconv* is

```
a =
         1    -3     4    -1
```

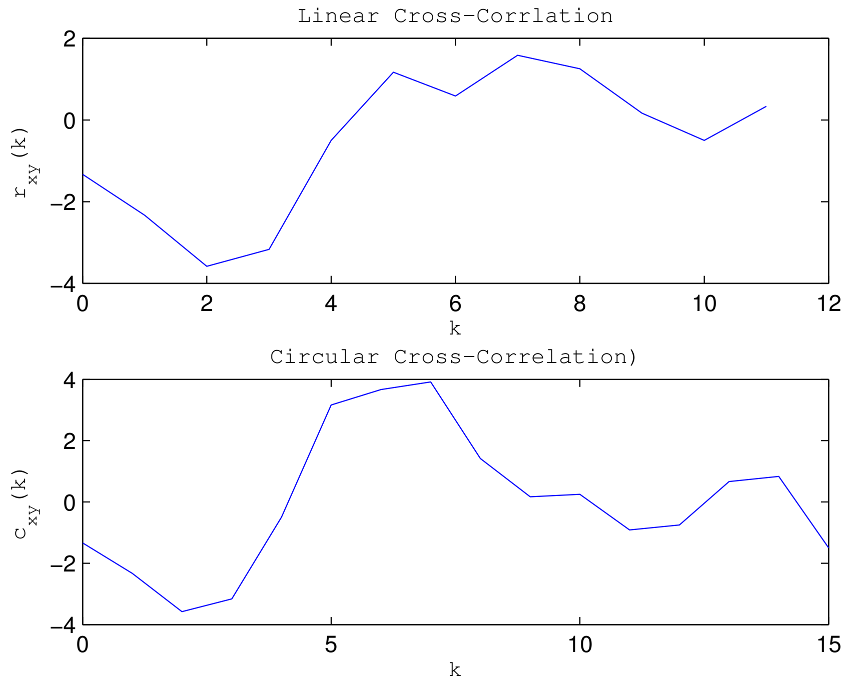
2.62 Consider the following pair of signals.

$$\begin{aligned} x &= [2, -4, 3, 7, 6, 1, 9, 4, -3, 2, 7, 8]^T \\ y &= [3, 2, 1, 0, -1, -2, -3, -2, -1, 0, 1, 2]^T \end{aligned}$$

Verify that linear cross-correlation and circular cross-correlation produce different results by writing a MATLAB program that uses the DSP Companion function *f_corr* to compute the linear cross-correlation, $r_{yx}(k)$, and the circular cross-correlation, $c_{yx}(k)$. Plot $r_{yx}(k)$ and $c_{yx}(k)$ below one another on the same screen.

Solution

```
% Problem 2.62

% Initialize

f_header('Problem 2.62')
x = [3 2 1 0 -1 -2 -3 -2 -1 0 1 2]
y = [2 -4 3 7 6 1 9 4 -3 2 7 8]

% Compute cross-correlations

r_xy = f_corr (x,y,0,0);
c_xy = f_corr (x,y,1,0);

% Plot them

figure
subplot (2,1,1)
k = 0 : length(r_xy)-1;
plot (k,r_xy)
f_labels ('Linear Cross-Correlation','k','r_{xy}(k)')
subplot (2,1,2)
k = 0 : length(c_xy)-1;
plot (k,c_xy)
f_labels ('Circular Cross-Correlation)','k','c_{xy}(k)')
f_wait
```



Problem 2.62 Linear and Circular Cross-Correlation

√ 2.63 Consider the following pair of signals.

$$\begin{aligned} y &= [1, 8, -3, 2, 7, -5, -1, 4]^T \\ x &= [2, -3, 4, 0, 5]^T \end{aligned}$$

Verify that linear cross-correlation can be achieved by zero-padding and circular cross-correlation by writing a MATLAB program that pads these signals with an appropriate number of zeros and uses the DSP Companion function *f_corr* to compute the linear cross-correlation $r_{yx}(k)$ and the circular cross-correlation $c_{y_z x_z}(k)$. Plot the following.

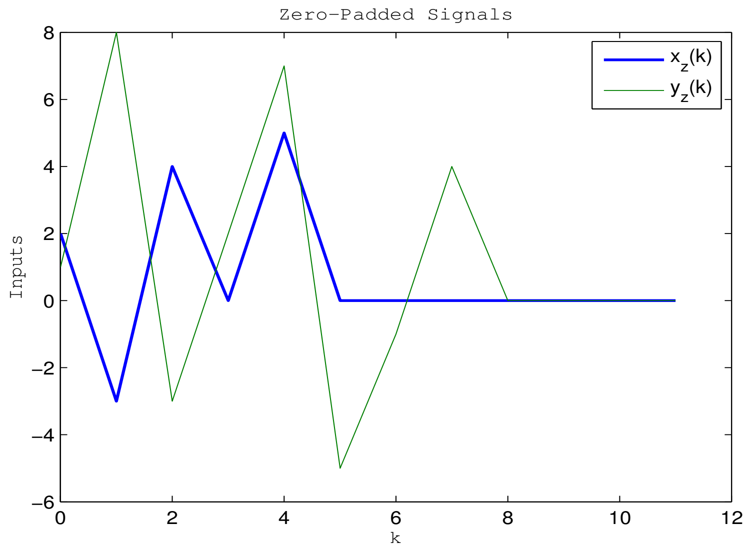
(a) The zero-padded signals $x_z(k)$ and $y_z(k)$ on the same graph using a legend.

(b) The linear cross-correlation $r_{yx}(k)$ and the scaled zero-padded circular cross-correlation $(N/L)c_{y_z x_z}(k)$ on the same graph using a legend.

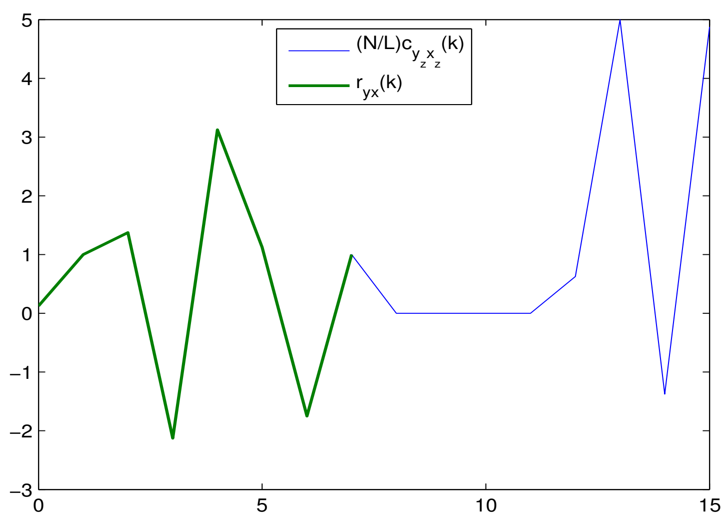
Solution

```
% Problem 2.63
```

```
% Initialize

f_header('Problem 2.63')
y = [1 8 -3 2 7 -5 -1 4]
x = [2 -3 4 0 5]

% Construct and plot zero-padded signals

L = length(y);
M = length(x);
x_z = [x, zeros(1,L-1)];
y_z = [y, zeros(1,M-1)];
figure
N = length(y_z);
k = 0 : N-1;
hp = plot (k,x_z,k,y_z);
set (hp(1),'LineWidth',1.5)
f_labels ('Zero-Padded Signals','k','Inputs')
legend ('x_z(k)','y_z(k)')
f_wait

% Compute and plot cross-correlations

r_yx = f_corr (y,x,0,0);
R_yx = (N/L)*f_corr (y_z,x_z,1,0);
kr = 0 : length(r_yx)-1;
kR = 0 : length(R_yx)-1;
figure
h = plot (kR,R_yx,kr,r_yx);
set (h(2),'LineWidth',1.5)
legend ('(N/L)c_{y_zx_z}(k)', 'r_{yx}(k)','Location','North')
f_wait
```



Problem 2.63 (a) Zero-Padded Signals



Problem 2.63 (b) Cross-Correlations

2.64 Consider the following pair of signals of length $N = 8$.

$$x = [2, -4, 7, 3, 8, -6, 5, 1]^T$$
$$y = [3, 1, -5, 2, 4, 9, 7, 0]^T$$

Write a MATLAB program that performs the following tasks.

(a) Use the DSP Companion function *f_corr* to compute the circular cross-correlation, $c_{yx}(k)$.

(b) Compute and print $u(k) = x(-k)$ using the periodic extension, $x_p(k)$.

(c) Verify that $c_{yx}(k) = [y(k) \circ x(-k)]/N$ by using the DSP Companion function *f_conv* to compute and plot the scaled circular convolution, $w(k) = [u(k) \circ x(k)]/N$. Plot $c_{yx}(k)$ and $w(k)$ below one another on the same screen.

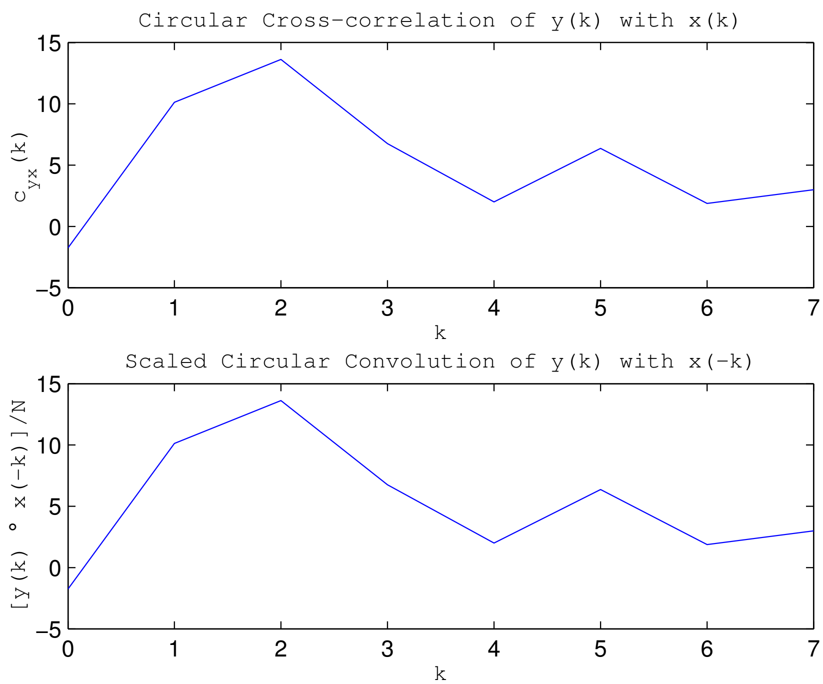
Solution

```
% Problem 2.64

% Initialize

f_header('Problem 2.64')
y = [3 1 -5 2 4 9 7 0]
x = [2 -4 7 3 8 -6 5 1]

% Compute and plot circular cross-correlation

c_yx = f_corr (y,x,1,0);

% Construct u(k) = x(-k) using periodic extension x_p(k)

N = length(x);
u = [x(1), x(N:-1:2)]

% Compute and plot scaled circular convolution

w = f_conv (y,u,1)/N;
figure
subplot(2,1,1)
kc = 0 : length(c_yx)-1;
plot (kc,c_yx)
f_labels ('Circular Cross-correlation of y(k) with x(k)','k','c_{yx}(k)')
```

```
subplot(2,1,2)
kw = 0 : length(w)-1;
plot (kw,w)
f_labels ('Scaled Circular Convolution of y(k) with x(-k)','k','[y(k) \circ x(-k)]/N')
f_wait
```

(b) The signal $u(k) = x(-k)$ using the periodic extension $x_p(k)$ is

```
u =
      2     1     5    -6     8     3     7    -4
```



Problem 2.64 (c) Scaled Circular Convolution